QED Contribution to electron and muon $g - 2$

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based on collaboration with
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M. Nio (RIKEN),
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INT Workshop INT-19-74W
Hadronic Contribution to $(g-2)_\mu$
University of Washington, Seattle
Anomalous magnetic moment of leptons

- **Electron $g-2$** is explained almost entirely by QED interaction between electron and photons. It has been the most stringent test of QED and the standard model.

<table>
<thead>
<tr>
<th></th>
<th>in units of $10^{-12}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_e($expr:HV08$)$</td>
<td>$1,159,652,180.73$ (28)</td>
</tr>
<tr>
<td>$a_e($theory$)$</td>
<td>$1,159,652,181.61$ (23)</td>
</tr>
<tr>
<td>QED: $e$ and $\gamma$</td>
<td>$1,159,652,177.14$ (23)</td>
</tr>
<tr>
<td>QED: $\mu$, $\tau$ contributions</td>
<td>$2.747,5720$ (14)</td>
</tr>
<tr>
<td>Hadronic</td>
<td>$1.693$ (12)</td>
</tr>
<tr>
<td>Weak</td>
<td>$0.03053$ (23)</td>
</tr>
</tbody>
</table>

- **Electron $g-2$** provides one of the most precise determination of the fine structure constant $\alpha$.

- **Muon $g-2$** is also dominated by QED contribution, which has been evaluated precisely for the on-going experiments.
Anomalous magnetic moment of electron

- The best measurement of the anomalous magnetic moment of electron obtained by Harvard group is:

  \[ a_e(HV08) = 1 159 652 180.73 (28) \times 10^{-12} \quad [0.24\text{ppb}] \]

  Hanneke, Fogwell, Gabrielse, PRL100, 120801 (2008)
  Hanneke, Fogwell Hoogerheide, Gabrielse, PRA83, 052122 (2011)

This result is 15-fold improvement over the previous measurement by the University of Washington group:

- For negative electron:
  \[ a_{e^-}(UW87) = 1 159 652 188.4 (43) \times 10^{-12} \quad [3.7\text{ppb}] \]

  Van Dyck, Schwinberg, Dehmelt, PRL59, 26 (1987)

- For positive electron:
  \[ a_{e^+}(UW87) = 1 159 652 187.9 (43) \times 10^{-12} \quad [3.7\text{ppb}] \]

- Further improvement of electron anomaly as well as new measurement of positron is ongoing.
Standard Model prediction of $a_e$

- Contributions to electron $g-2$ within the context of the standard model consist of:

$$a_e = a_e(QED) + a_e(Hadronic) + a_e(Weak)$$

- QED contribution is further divided according to its lepton-mass dependence through mass-ratio:

$$a_e(QED) = A_1 + A_2(m_e/m_\mu) + A_2(m_e/m_\tau) + A_3(m_e/m_\mu, m_e/m_\tau)$$

- Each contribution is evaluated by perturbation theory:

$$A_i = A_i^{(2)} \left( \frac{\alpha}{\pi} \right) + A_i^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + A_i^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + A_i^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + \cdots$$

These coefficients are calculated by using Feynman-diagram techniques.

<table>
<thead>
<tr>
<th># diagrams</th>
<th>w/o fermion loop</th>
<th>w/ fermion loop</th>
</tr>
</thead>
<tbody>
<tr>
<td>2nd</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>4th</td>
<td>7</td>
<td>6</td>
</tr>
<tr>
<td>6th</td>
<td>72</td>
<td>50</td>
</tr>
<tr>
<td>8th</td>
<td>891</td>
<td>518</td>
</tr>
<tr>
<td>10th</td>
<td>12,672</td>
<td>6536</td>
</tr>
</tbody>
</table>
## QED contribution: Summary

<table>
<thead>
<tr>
<th>Coefficient $A^{(2n)}_i$</th>
<th>Value (Error)</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A^{(2)}_1$</td>
<td>0.5</td>
<td>Schwinger 1948</td>
</tr>
<tr>
<td>$A^{(4)}_1$</td>
<td>$-0.328,478,965,579,193\ldots$</td>
<td>Petermann 1957, Sommerfield 1958</td>
</tr>
<tr>
<td>$A^{(4)}<em>2(m_e/m</em>\mu)$</td>
<td>$0.519,738,676,(24)\times10^{-6}$</td>
<td>Elend 1966</td>
</tr>
<tr>
<td>$A^{(4)}<em>2(m_e/m</em>\tau)$</td>
<td>$0.183,790,(25)\times10^{-8}$</td>
<td>Elend 1966</td>
</tr>
<tr>
<td>$A^{(6)}_1$</td>
<td>$1.181,241,456,587\ldots$</td>
<td>Laporta-Remiddi 1996, Kinoshita 1995</td>
</tr>
<tr>
<td>$A^{(6)}<em>2(m_e/m</em>\mu)$</td>
<td>$-0.737,394,164,(24)\times10^{-5}$</td>
<td>Samuel-Li, Laporta-Remiddi, Laporta</td>
</tr>
<tr>
<td>$A^{(6)}<em>2(m_e/m</em>\tau)$</td>
<td>$-0.658,273,(79)\times10^{-7}$</td>
<td>Samuel-Li, Laporta-Remiddi, Laporta</td>
</tr>
<tr>
<td>$A^{(6)}<em>3(m_e/m</em>\mu, m_e/m_\tau)$</td>
<td>$0.1909,(1)\times10^{-12}$</td>
<td>Passera 2007</td>
</tr>
<tr>
<td>$A^{(8)}_1$</td>
<td>$-1.912,245,764\ldots$</td>
<td>Laporta 2017, AHKN 2015</td>
</tr>
<tr>
<td>$A^{(8)}<em>2(m_e/m</em>\mu)$</td>
<td>$0.916,197,070,(37)\times10^{-3}$</td>
<td>Kurz et al 2014, AHKN 2012</td>
</tr>
<tr>
<td>$A^{(8)}<em>2(m_e/m</em>\tau)$</td>
<td>$0.742,92,(12)\times10^{-5}$</td>
<td>Kurz et al 2014, AHKN 2012</td>
</tr>
<tr>
<td>$A^{(8)}<em>3(m_e/m</em>\mu, m_e/m_\tau)$</td>
<td>$0.746,87,(28)\times10^{-6}$</td>
<td>Kurz et al 2014, AHKN 2012</td>
</tr>
<tr>
<td>$A^{(10)}_1$</td>
<td>$6.737,(159)$</td>
<td>AKN 2018,2019</td>
</tr>
<tr>
<td>$A^{(10)}<em>2(m_e/m</em>\mu)$</td>
<td>$-0.003,82,(39)$</td>
<td>AHKN 2012,2015</td>
</tr>
<tr>
<td>$A^{(10)}<em>2(m_e/m</em>\tau)$</td>
<td>$\mathcal{O}(10^{-5})$</td>
<td></td>
</tr>
<tr>
<td>$A^{(10)}<em>3(m_e/m</em>\mu, m_e/m_\tau)$</td>
<td>$\mathcal{O}(10^{-5})$</td>
<td></td>
</tr>
</tbody>
</table>

All terms up to 8th order are well-known. 10th order term is obtained numerically.
QED contribution: 8th order term

- 891 Feynman diagrams contribute to 8th order $A_1^{(8)}$ term.

- Laporta obtained near-analytic precise value upto 1100 digits.

---

Laporta, PLB772, 232 (2017)
QED contribution: 8th order term

- Mass-independent term $A_1^{(8)}$
  - Near-analytic result
    \[-1.9122457649264455741526471674 \ldots\]  
    Laporta, PLB772, 232 (2017)
  - Alternative semi-analytic result
    \[-1.87(12)\]  
    Marquad et al, arXiv:1708.07138
  - Numerical result
    \[-1.91298(84)\]  
    AHKN, PRL109, 111809 (2012); PRD91, 033006 (2015)

- Mass-dependent terms $A_2^{(8)}$ and $A_3^{(8)}$
  - Numerical evaluation.
  - Analytic calculation by the series expansion in mass-ratio $m_e/m_\ell \ll 1$.  
    Kurz et al. PRD93, 053017 (2016)

<table>
<thead>
<tr>
<th></th>
<th>Analytic</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2^{(8)}(m_e/m_\mu)$</td>
<td>$0.916197070\ (37) \times 10^{-3}$</td>
<td>$0.9222\ (66) \times 10^{-3}$</td>
</tr>
<tr>
<td>$A_2^{(8)}(m_e/m_\tau)$</td>
<td>$0.74292\ (12) \times 10^{-5}$</td>
<td>$0.738\ (12) \times 10^{-5}$</td>
</tr>
<tr>
<td>$A_3^{(8)}(m_e/m_\mu, m_e/m_\tau)$</td>
<td>$0.74687\ (28) \times 10^{-6}$</td>
<td>$0.7465\ (18) \times 10^{-6}$</td>
</tr>
</tbody>
</table>

- Now the 8th order term is well-known.
QED contribution: 10th order term

- 12,672 Feynman diagrams contribute to 10th order term. They are classified into 32 gauge invariant sets within 6 supersets.

Most difficult is Set V that consists of 6,354 diagrams w/o lepton loops.
Magnetic moment contribution

- Magnetic property of lepton can be studied through examining its scattering by a static magnetic field.
  
  The amplitude can be represented as:
  
  \[ e\bar{u}(p'') \left[ \gamma^\mu F_1(q^2) + \frac{i}{2m} \sigma^{\mu\nu} q_\nu F_2(q^2) \right] u(p') A_\mu^e(\bar{q}) \]

- The anomalous magnetic moment is the static limit of the magnetic form factor \( F_2(q^2) \):
  
  \[ a_\ell = F_2(0) = Z_2 M, \quad M = \lim_{q^2 \to 0} \text{Tr}(P_\nu(p, q)\Gamma^\nu) \]

  where \( \Gamma^\nu \) is the proper vertex function with the external lepton on the mass shell, and \( P_\nu(p, q) \) is the magnetic projection operator.
Numerical Approach

- A set of vertex diagrams $\Lambda$ obtained by inserting an external vertex into each lepton line of self-energy diagram $\Sigma$ can be related by Ward-Takahashi identity.

\[
\Lambda^\nu(p, q) \simeq -q_\mu \frac{\partial \Lambda^{\mu}(p, q)}{\partial q_\nu} \bigg|_{q\to 0} - \frac{\partial \Sigma(p)}{\partial p_\nu}.
\]

- Amplitude is given by an integral over loop momenta according to Feynman-Dyson rule. It is converted into Feynman parametric integral over $\{z_i\}$. Momentum integration is carried out analytically that yields

\[
M_G^{(2n)} = \left( -\frac{1}{4} \right)^n \Gamma(n-1) \int (dz)_G \left[ \frac{F_0}{U^2 V^{n-1}} + \frac{F_1}{U^3 V^{n-2}} + \cdots \right]
\]

- Integrand is expressed by a rational function of terms called building blocks, $U$, $V$, $B_{ij}$, $A_j$, and $C_{ij}$. Building blocks are given by functions of $\{z_i\}$, reflecting the topology of diagram, flow of momenta, etc.
Subtraction of UV Divergences

- UV divergence occurs when loop momenta in a subdiagram go to infinity. It corresponds to the region of Feynman parameter space $z_i \sim \mathcal{O}(\epsilon)$ for $i \in S$.

- In order to carry out subtraction numerically, the singularities are cancelled point-by-point on Feynman parameter space.

\[
M_g - L_S M_{g/S} \rightarrow \int (dz)_G \left[ m_g - K_S m_g \right]
\]

- The subtraction integrand $K_S m_g$ is derived from $m_g$ by simple power-counting rule called $K$-operation. (Cvitanović and Kinoshita, 1974)

- By construction, subtraction terms can be factorized into (UV-divergent part of) renormalization constant and lower-order magnetic part.

\[
\int (dz)_G \left[ K_S m_g \right] = L^\text{UV}_S M_{g/S}
\]

$L^\text{UV}_S$ is the leading UV-divergent part of $L_S$. 
IR subtraction Scheme

▶ A diagram may have IR divergence when some momenta of photon go to zero. It is really divergent by “enhancer” leptons that are close to on-shell by kinematical constraint.

▶ We adopt subtraction approach for these divergences point-by-point on Feynman parameter space.

▶ There are two types of sources of IR divergence in $M_g$ associated with a self-energy subdiagram. To handle these divergences, we introduce two subtraction operations:
  ▶ $R$-subtraction to remove the residual self-mass term
    \[
    \mathbb{R}_S M_g = \tilde{\delta} m_S M_g / S(i^*)
    \]
  ▶ $I$-subtraction to subtract remaining logarithmic IR divergence
    \[
    \mathbb{I}_S M_g = \tilde{L}_g / S(k) M_S
    \]
Amplitude as a finite integral

- Finite amplitude $\Delta M_g$ free from both UV and IR divergences is obtained by Feynman-parameter integral as:

$$\Delta M_g = \int (dz) \left[ F_g \right]$$

- $+ \sum_{f} \prod_{S \in f} (-K_S) F_g$: unrenormalized amplitude

- $+ \sum_{\tilde{f}} (-I_{S_i}) \cdots (-R_{S_j}) \cdots F_g$: UV subtraction terms

- $f$: Zimmermann's forests: combinations of UV divergent subdiagrams.

- $\tilde{f}$: annotated forests: combinations of self-energy subdiagrams with distinction of $I$-$R$-subtractions.

- $+ \sum_{\tilde{f}} (-I_{S_i}) \cdots (-R_{S_j}) \cdots F_g$: IR subtraction terms
Residual renormalization

- We adopt the standard on-shell renormalization to ensure that the coupling constant $\alpha$ and the electron mass $m_e$ are the ones measured by experiments.
- The sum of all these finite integrals defined by K-operation and I-/R-subtraction operations does not correspond to physical contribution to $g - 2$.
- The difference is adjusted by the step called the residual renormalization.

\[ a_e = M(\text{bare}) - \text{on-shell renormalization} \]

\[ = \left[ M(\text{bare}) - \text{UV subtr.} - \text{IR subtr.} \right] \]

Finite integral $\Delta M$

\[ + \left[ -\text{on-shell renorm.} + \text{UV subtr.} + \text{IR subtr.} \right] \]

finite residual renormalization
Deriving residual renormalization

- Sum up over 389 integrals of 10th order Set V, which requires analytic sum of $\sim 16,000$ symbolic terms.

- The physical contribution from 10th order Set V is given as:

\[
A_1^{(10)}[\text{Set V}] = \Delta M_{10}[\text{Set V}]
\]

\[
+ \Delta M_8(-7\Delta LB_2)
\]

\[
+ \Delta M_6\{-5\Delta LB_4 + 20(\Delta LB_2)^2\}
\]

\[
+ \Delta M_4\{-3\Delta LB_6 + 24\Delta LB_4\Delta LB_2 - 28(\Delta LB_2)^3 + 2\Delta L_2^* \Delta dm_4\}
\]

\[
+ M_2\{-\Delta LB_8 + 8\Delta LB_6\Delta LB_2 - 28\Delta LB_4(\Delta LB_2)^2
\]

\[
+ 4(\Delta LB_4)^2 + 14(\Delta LB_2)^4 + 2\Delta dm_6\Delta L_2^* \}
\]

\[
+ M_2\Delta dm_4\{-16\Delta L_2^* \Delta LB_2 + \Delta L_4^* - 2\Delta L_2^* \Delta dm_2^*\},
\]

- The terms with $\Delta$ are the finite $n$th order quantities.
  - $\Delta M_n$, $M_2$: finite magnetic moment.
  - $\Delta LB_n$: sum of vertex and wave-function renormalization constants.
  - $\Delta dm_n$: mass-renormalization constants.
  - $\Delta L_n^*$, $\Delta dm_n^*$: $^*$ denotes mass insertion.
We need to evaluate a large number of Feynman diagrams. It should be error-prone by writing numerical integration code for these huge integrals by hand. We developed an automated code-generating program. “genode$N$” takes a single-line information that represents a diagram, and generates numerical integration code in FORTRAN. These integrals are evaluated on computers using numerical integration routines.

AHKN, NPB740, 138 (2006); NPB796, 184 (2008)
Numerical integration

- Multi-dimensional integral
  - The amplitude is expressed as a $14 - 1$ dimensional integral for 10th order diagrams.
  - The integrands are huge. (approx. $O(10^5)$ FORTRAN lines for each integral.)

- Digit-deficiency problem
  - The point-by-point subtraction suffers from severe digit-deficiency problem by rounding-off of floating-point numbers.

  We employ extended numerical precision arithmetic using *double-double* and *quadruple-double* of *qd* library.


- Sharp peaks
  - Integrands have sharp peaks due to divergences, and therefore requires robust integration method.

  We employ VEGAS, an adaptive-iterative Monte-Carlo integration algorithm.

  A new version of VEGAS: https://github.com/gplepage/vegas
QED contribution: 10th order term

- Numerical evaluation of the complete 10th order contribution was reported in 2012 and an updated result was published in 2015. Latest value is:

\[ A_{1}^{(10)} = 6.737 \, (159) \]

- Contribution to \( A_{1}^{(10)} \) mainly comes from Set V that consists of 6354 vertex diagrams without closed lepton loops.

Recently, Volkov announced their preliminary result by an independent numerical method.

\[ A_{1}^{(10)}[\text{Set V}] = \begin{cases} 7.668 \, (159) \\ 6.782 \, (113) \end{cases} \]

AKN, Atoms, 7, 28 (2019)

Difference \(-0.89 \, (20) \) [4.5\( \sigma \)] does not affect seriously in the current precision.

- Mass-dependent term is also evaluated:

\[ A_{2}^{(10)}(m_e/m_\mu) = -0.003 \, 82 \, (39) \]

tau-lepton contribution is negligibly small for the current experimental precision.
Numerical checks of Set V integrals

- 13 integration variables in \([0, 1]^D\) are mapped to 14 Feynman parameters. Any mapping should yield the same result.

- As a cross check, we performed integrals with different mappings. They are regarded as independent evaluations.

- Numerical results are in good agreement.

List of results that exhibit relatively large differences:

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Expression</th>
<th>Results in 2015</th>
<th>Results in 2017</th>
<th>Difference</th>
<th>Weighted average</th>
</tr>
</thead>
<tbody>
<tr>
<td>X141</td>
<td>abbcadedec</td>
<td>-12.5567 (350)</td>
<td>-12.4879 (207)</td>
<td>-0.0688</td>
<td>-12.5057 (178)</td>
</tr>
<tr>
<td>X113</td>
<td>abaccddeebc</td>
<td>-4.3847 (322)</td>
<td>-4.4412 (176)</td>
<td>0.0565</td>
<td>-4.4282 (155)</td>
</tr>
<tr>
<td>X100</td>
<td>abacdcdeeb</td>
<td>-15.2919 (331)</td>
<td>-15.2360 (203)</td>
<td>-0.0559</td>
<td>-15.2513 (173)</td>
</tr>
<tr>
<td>X256</td>
<td>abccdeedba</td>
<td>-14.0405 (342)</td>
<td>-13.9856 (194)</td>
<td>-0.0549</td>
<td>-13.9990 (169)</td>
</tr>
<tr>
<td>X146</td>
<td>abbcdaedec</td>
<td>-2.2990 (335)</td>
<td>-2.2458 (202)</td>
<td>-0.0532</td>
<td>-2.2600 (173)</td>
</tr>
<tr>
<td>X075</td>
<td>abacbddeec</td>
<td>-8.1138 (340)</td>
<td>-8.0608 (195)</td>
<td>-0.0531</td>
<td>-8.0739 (169)</td>
</tr>
<tr>
<td>X144</td>
<td>abccdecdea</td>
<td>23.7239 (368)</td>
<td>23.6713 (189)</td>
<td>0.0526</td>
<td>23.6823 (168)</td>
</tr>
<tr>
<td>X252</td>
<td>abccdedeab</td>
<td>-10.9091 (343)</td>
<td>-10.8565 (179)</td>
<td>-0.0526</td>
<td>-10.8677 (158)</td>
</tr>
<tr>
<td>X236</td>
<td>abcbedceaa</td>
<td>2.0560 (180)</td>
<td>2.1072 (205)</td>
<td>-0.0512</td>
<td>2.0782 (135)</td>
</tr>
<tr>
<td>X325</td>
<td>abcdeceada</td>
<td>11.5958 (343)</td>
<td>11.5456 (198)</td>
<td>0.0503</td>
<td>11.5582 (172)</td>
</tr>
<tr>
<td>X158</td>
<td>abcdecda</td>
<td>0.4607 (329)</td>
<td>0.4106 (206)</td>
<td>0.0502</td>
<td>0.4247 (174)</td>
</tr>
</tbody>
</table>

AKN, PRD97, 036001 (2018)
Fine Structure Constant $\alpha$

- To obtain the theoretical prediction of $a_e$, we need a value of the fine-structure constant $\alpha$ determined independent of QED.

- Two high-precision values of $\alpha$ are obtained from the measurement of $h/m(X)$ of the Rb and Cs by the atom interferometer through the relation:

$$\alpha^{-1} = \left[ \frac{2R_\infty}{c} \frac{A_r(X)}{A_r(e)} \frac{h}{m(X)} \right]^{-1/2}$$

where

- $R_\infty$ the Rydberg constant
  - $R_\infty$ (MPQ) = 10 973 731.568 076 (096) $m^{-1}$
  - $R_\infty$ (Orsay) = 10 973 731.568 530 (140) $m^{-1}$

- $A_r(X)$ relative atomic mass of an atom $X$

- $m(X)$ mass of an atom $X$

It leads to

$$\alpha^{-1} (\text{Rb}) = 137.035 998 995 (85) \ [0.62\text{ppb}]$$

$$\alpha^{-1} (\text{Cs}) = 137.035 999 046 (27) \ [0.20\text{ppb}]$$
Theoretical Prediction of $a_e$

- Using $\alpha$(Cs) and including the hadronic and weak contributions, the theoretical prediction of $a_e$ becomes:

<table>
<thead>
<tr>
<th></th>
<th>mass-independent</th>
<th>mass-dependent</th>
<th>sum</th>
</tr>
</thead>
<tbody>
<tr>
<td>QED</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2nd</td>
<td>1 161 409 733.21 (23)</td>
<td>0</td>
<td>1 161 409 733.21 (23)</td>
</tr>
<tr>
<td>4th</td>
<td>−1 772 305.063 85 (70)</td>
<td>2.814 1613 (13)</td>
<td>−1 772 302.249 69 (70)</td>
</tr>
<tr>
<td>6th</td>
<td>14 804.203 6740 (88)</td>
<td>−0.093 240 76 (10)</td>
<td>14 804.110 4333 (88)</td>
</tr>
<tr>
<td>8th</td>
<td>−55.667 989 379 (44)</td>
<td>0.026 909 719 (35)</td>
<td>−55.641 079 660 (56)</td>
</tr>
<tr>
<td>10th</td>
<td>0.456 (11)</td>
<td>−0.000 258 (26)</td>
<td>0.455 (11)</td>
</tr>
<tr>
<td>$a_e$(QED)</td>
<td>1 159 652 177.14 (23)</td>
<td>2.747 5720 (14)</td>
<td>1 159 652 179.88 (23)</td>
</tr>
</tbody>
</table>

|        |                  |                |                   |
| Weak   |                  |                |                   |
| $a_e$(weak) |                      | 0.030 53 (23) |                   |

|        |                  |                |                   |
| Hadron |                  |                |                   |
| VP LO  |                      | 1.849 (10)     |                   |
| VP NLO |                      | −0.2213 (11)   |                   |
| VP NNLO|                      | 0.027 99 (17)  |                   |
| LbyL   |                      | 0.037 (5)      |                   |
| $a_e$(hadron) |                      | 1.693 (12) |                   |

|        |                  |                |                   |
| $a_e$(theory) |                      | 1 159 652 181.61 (23) |                   |
Theoretical Prediction of $a_e$

- We obtain the theoretical prediction of $a_e$ as

  \[ a_e(\text{theory: } \alpha(\text{Rb})) = 1\,159\,652\,182.037 \times 10^{-12} \]

  \[ a_e(\text{theory: } \alpha(\text{Cs})) = 1\,159\,652\,181.606 \times 10^{-12} \]

  where uncertainties are due to fine-structure constant $\alpha$, QED 10th order, and hadronic contribution.

- The measurement of $a_e$ is

  \[ a_e(\text{expt.}) = 1\,159\,652\,180.73 \times 10^{-12} \]

- The differences between theory and measurement are

  \[ a_e(\text{theory: } \alpha(\text{Rb})) - a_e(\text{expt.}) = 1.31 (77) \times 10^{-12} [1.7\sigma] \]

  \[ a_e(\text{theory: } \alpha(\text{Cs})) - a_e(\text{expt.}) = 0.88 (36) \times 10^{-12} [2.4\sigma] \]
Fine Structure Constant $\alpha$ from $a_e$

- From the measurement and the theory of electron $g-2$, the value of fine-structure constant can be determined.

Theoretical calculations

Experimental value

$$a_e = A^{(2)} \left( \frac{\alpha}{\pi} \right) + A^{(4)} \left( \frac{\alpha}{\pi} \right)^2 + A^{(6)} \left( \frac{\alpha}{\pi} \right)^3 + A^{(8)} \left( \frac{\alpha}{\pi} \right)^4 + A^{(10)} \left( \frac{\alpha}{\pi} \right)^5 + \cdots$$

+(small contributions)

- Newly obtained value of fine-structure constant is:

$$\alpha^{-1}(a_e) = 137.035\,999\,1496\,13(14)(330) \quad [0.24\text{ppb}]$$

AKN, Atoms, 7, 28 (2019)

- The differences in $\alpha$ from the atomic recoil determinations are

$$\alpha^{-1}(a_e) - \alpha^{-1}(\text{Rb}) = 0.155\,91 \times 10^{-6} \ [1.7\sigma],$$

$$\alpha^{-1}(a_e) - \alpha^{-1}(\text{Cs}) = 0.104\,43 \times 10^{-6} \ [2.4\sigma].$$
Muon $g-2$: QED contribution

- What distinguishes $a_e$(QED) and $a_\mu$(QED) is the mass-dependent component.

- Light lepton loop contribution yields large logarithmic enhancement involving a factor $\ln(m_e/m_\mu)$.

  - Vacuum polarization loop:
    
    $$\frac{2}{3} \ln(m_\mu/m_e) - \frac{5}{9} \approx 3.$$  

  - Light-by-light scattering loop:
    
    $$\frac{2}{3} \pi^2 \ln(m_\mu/m_e) \approx 35.$$  

  6th-order l-by-l effect is important.

  c.f. Aldins, Kinoshita, Brodsky, Dufner, PRL8, 441 (1969)

- Therefore, the sets of diagrams giving the leading contribution can be identified and were evaluated in the earlier stage. The entire contribution including non-leading diagrams have been evaluated.
Muon \( g - 2 \): QED contribution

- \( a_\mu \) (QED) is known up to 10th order. Their values contributing to mass-dependent terms are:

<table>
<thead>
<tr>
<th></th>
<th>( A_2(m_\mu/m_e) )</th>
<th>( A_2(m_\mu/m_\tau) )</th>
<th>( A_3(m_\mu/m_e, m_\mu/m_\tau) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>4th</td>
<td>1.094 258 3093 (76)</td>
<td>0.000 078 076 (11)</td>
<td>—</td>
</tr>
<tr>
<td>6th</td>
<td>22.868 379 98 (20)</td>
<td>0.000 360 671 (94)</td>
<td>0.000 527 738 (75)</td>
</tr>
<tr>
<td>8th</td>
<td>132.685 2 (60)</td>
<td>0.042 4941 (53)</td>
<td>0.062 722 (10)</td>
</tr>
<tr>
<td>10th</td>
<td>742.32 (86)</td>
<td>-0.0656 (45)</td>
<td>2.011 (10)</td>
</tr>
</tbody>
</table>

Elend, PL20, 682 (1966); Samuel and Li, PRD44, 3935 (1991); Li, Mendel and Samuel, PRD47, 1723 (1993)
Laporta, PLB312, 495 (1993); Kinoshita and Nio, PRD70, 113001 (2004); Kurz, Liu, Marquard, Steinhauser, NPB879, 1 (2014)
Laporta, PLB328, 522 (1994); Kinoshita and Nio, PRD73, 053007 (2006)
Time, Hayakawa, Kinoshita, Nio, Watanabe, PRD78, 053005 (2008)
Time, Asano, Hayakawa, Kinoshita, Nio, Watanabe, PRD81, 053009 (2010)
Time, Hayakawa, Kinoshita, Nio, PRD78, 113006 (2008); 82, 113004 (2010); 83, 053002 (2011)
83, 053003 (2011); 84, 053003 (2011); 85, 033007 (2012); 85, 093013 (2012)

- Together with the mass-independent term \( A_1 \), we obtain:

\[
a_\mu (\text{QED} : \alpha(\text{Cs})) = 116 \, 584 \, 718 \, . \, 931 \, (7) \, (17) \, (6) \, (100) \, (23) \, [104] \times 10^{-11}
\]

\[
a_\mu (\text{QED} : \alpha(a_e)) = 116 \, 584 \, 718 \, . \, 842 \, (7) \, (17) \, (6) \, (100) \, (28) \, [106] \times 10^{-11}
\]

(mass ratio)(8th)(10th)(12th)(\( \alpha \)) [combined]
In view of rather large values of $A_2(m_\mu/m_\ell)$, one might wonder how much the twelfth order contribution.

The leading contribution will come from three insertions of 2nd-order vacuum-polarization loop into the 6th-order light-by-light diagram. It is estimated as:

\[
\sim (6\text{th light-by-light}) \times (2\text{nd VP})^3 \times 10 \times \left(\frac{\alpha}{\pi}\right)^6
\]

\[
\sim 0.08 \times 10^{-11}.
\]

It is larger than the uncertainty of 10th order term. A crude evaluation may be desirable.
Summary

- QED contribution to electron $g-2$ up to 8th order has been firmly established.

- QED contribution of 10th order has been evaluated by extensive numerical calculation.

- QED contributions are now ready for the on-going new measurements of electron and position $g-2$, and muon $g-2$.

- Electron $g-2$ provides one of most precise determination of fine structure constant $\alpha$.
  It serves for new SI as a significant factor of the uncertainty of many physical constants.
Backup
Numerical Approach

Procedure:

Step 1. Find distinct set of Feynman diagrams.

Step 2. Construct amplitude in terms of Feynman parametric integral.

Step 3. Construct subtraction terms of UV divergence.
   • $K$-operation

Step 4. Construct subtraction terms of IR divergence.
   • $R$-subtraction of residual mass-renormalization.
   • $l$-subtraction of logarithmic IR divergences.

Step 5. Carry out residual renormalization to achieve the standard on-shell renormalization.

Step 6. Evaluate the finite amplitude by numerical integration.
Step-by-step example with 4th-order diagrams : Step 1

- Let us illustrate the steps by simpler case, e.g. 4th-order diagrams.
- There are 7 diagrams of 4th order; 6 of them have no closed lepton loop (q-type).
- They are WT-sumed into 2 self-energy-like diagrams, 4a and 4b.
Step 2: Amplitude

- Introduce Feynman parameters \( z_1, \ldots, z_5 \) to propagators:

![Diagram of Feynman parameters](image)

- Anomalous magnetic moment \( M_{4a} \) is converted analytically into the form:

\[
M_{4a} = \int (dz) \mathcal{F}_{4a} = \int (dz) \left[ \frac{E_0 + C_0}{U^2 V} + \frac{N_0 + Z_0}{U^2 V^2} + \frac{N_1 + Z_1}{U^3 V} \right]
\]

where integrand and building blocks are given as follows:

\[
(dz) = dz_1 dz_2 dz_3 dz_4 dz_5 \delta(1 - z_{12345})
\]
\[
B_{11} = z_{235}, \quad B_{12} = z_{35}, \quad B_{13} = -z_2,
\]
\[
B_{23} = z_{14}, \quad B_{22} = z_{1345}, \quad B_{33} = z_{124},
\]
\[
U = z_2 B_{12} + z_14 B_{11},
\]
\[
A_i = 1 - (z_1 B_{1i} + z_2 B_{2i} + z_3 B_{3i}) / U,
\]
\[
G = z_1 A_1 + z_2 A_2 + z_3 A_3, \quad V = z_{123} - G,
\]
\[
z_{ijk} \ldots = z_i + z_j + z_k + \cdots.
\]

\[
E_0 = 8(2A_1 A_2 A_3 - A_1 A_2 - A_1 A_3 - A_2 A_3)
\]
\[
C_0 = -24Z_4 Z_5 / U
\]
\[
N_0 = G(E_0 - 8(2A_2 - 1))
\]
\[
Z_0 = 8z_1(-A_1 + A_2 + A_3 + A_1 A_2 - A_1 A_3 - A_2 A_3) + 8z_2(1 - A_1 A_2 + A_1 A_3 - A_2 A_3 + 2A_1 A_2 A_3) + 8z_3(A_1 + A_2 - A_3 - A_1 A_2 + A_1 A_3 + A_2 A_3)
\]
\[
N_1 = 8G(B_{12}(2 - A_3) + 2B_{13}(1 - 2A_2) + B_{23}(2 - A_1))
\]
\[
Z_1 = -8z_1(B_{12}(1 - A_3) + B_{13} + B_{23} A_1) + 8z_2(B_{12}(1 - A_3) - 4B_{13} A_2 + B_{23}(1 - A_1)) - 8z_3(B_{12} A_3 + B_{13} + B_{23}(1 - A_1))
\]
Step 3: UV subtraction

- $M_{4a}$ is not well-defined — it has UV divergences when the loop momenta goes to infinity.

- This corresponds to a region of $z_i$’s when all $z_i$ on the loop vanish simultaneously.

- We prepare an integral which has the same UV divergent profile by $K$-operation, and perform subtraction point-by-point on the integrand.

Then the finite part of the anomalous magnetic moment $\Delta M_{4a}$ is obtained by the integral:

$$\Delta M_{4a} = \int (dz) \left[ F_{4a} - K_{12} F_{4a} - K_{23} F_{4a} \right]$$
**Step 4: IR subtraction**

- $M_{4b}$ has IR divergence as well, from vanishing of virtual photon momentum.
- This logarithmic IR divergence is handled by an integral which is constructed by $\mathcal{I}$-subtraction.
- Then the finite part of the anomalous magnetic moment $\Delta M_{4b}$ is obtained by the integral:

$$\Delta M_{4b} = \int (dz) \left[ \mathcal{F}_{4b} - K_{22} \mathcal{F}_{4b} - \Pi_{13} \mathcal{F}_{4b} \right]$$
Step 5: Residual renormalization

- Finite part of amplitude is given in terms of integral with appropriate UV and/or IR subtraction terms.

\[
\Delta M_{4a} = \int (dz) \left[ F_{4a} - i K_{12} F_{4a} - i K_{23} F_{4a} \right] \\
= M_{4a} - \hat{L}_2 M_2 - \tilde{L}_2 M_2
\]

\[
\Delta M_{4b} = \int (dz) \left[ F_{4b} - i K_{22} F_{4b} - \Pi_{13} F_{4b} \right] \\
= M_{4b} - (\delta m_2 M_2^* + \tilde{B}_2 M_2) - \tilde{\Phi}_2 M_2
\]

- Subtraction terms are analytically factorized into products of lower-order quantities.

- Standard on-shell renormalization is denoted by

\[
a^{(4)}[\text{q-type}] = M_{4a} - 2L_2 M_2 \\
+ M_{4b} - (\delta m_2 M_2^* + B_2 M_2)
\]

- By substitution, magnetic moment is given

\[
a^{(4)}[\text{q-type}] = (\Delta M_{4a} + \Delta M_{4b}) - \Delta L B_2 M_2
\]

where \(\Delta L B_2\) is finite part of \(L_2 + B_2\).