Nuclear Matrix Elements
NPLQCD

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Unphysical nuclei

- Nuclei with $A<5$
- QCD with unphysical quark masses
  - $m_\pi \sim 800$ MeV, $m_N \sim 1,600$ MeV
  - $m_\pi \sim 450$ MeV, $m_N \sim 1,200$ MeV

- Proton-proton fusion and tritium $\beta$-decay
  - [PRL 119, 062002 (2017)]

- Double $\beta$-decay
  - [PRL 119, 062003 (2017), PRD 96, 054505 (2017)]

- Gluon structure of light nuclei
  - [PRD 96 094512 (2017)]

- Scalar, axial and tensor MEs
  - [arXiv:1712.03221]

- Nuclear structure: magnetic moments, polarisabilities
  - [PRL 113, 252001 (2014), PRD 92, 114502 (2015)]

- First nuclear reaction: $np \rightarrow d\gamma$
  - [PRL 115, 132001 (2015)]
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Axial background field

Will’s talk: fixed magnetic field \( \rightarrow \) moments, polarisabilities
Axial MEs: fixed axial background field \( \rightarrow \) axial charges, other matrix elts.

Construct correlation functions from propagators modified in axial field

\[
S^{(q)}_\lambda(x, y) = S^{(q)}(x, y) + \lambda_q \int dz \, S^{(q)}(x, z) \gamma_3 \gamma_5 \, S^{(q)}(z, y)
\]

\[
C_{\lambda_u;\lambda_d}(t) = 
\]

Linear response \( \leftrightarrow \) axial matrix element

Axial background field

\[ C_{\lambda_u;\lambda_d}(t) = \]

\[ C_{\lambda_u;\lambda_d}(t) = \left\{ \begin{array}{l}
+ \lambda^3 \\
+ \lambda^2 \\
+ \lambda \\
+ 1
\end{array} \right. \]

Linear response gives axial matrix element

Implicit sum over current insertion times
Example: determination of the proton axial charge

\[ C_{\lambda_u;\lambda_d}(t) \bigg|_{\mathcal{O}(\lambda)} = \text{Implicit sum over current insertion times} \]

Time difference isolates matrix element part

\[
(C_{\lambda_u;\lambda_d}(t+1) - C_{\lambda_u;\lambda_d}(t)) \bigg|_{\mathcal{O}(\lambda)} = Z_0 e^{-M_p t} \langle p | A_3^{(u)}(0) | p \rangle + \mathcal{O}(e^{-\delta t})
\]
Proton axial charge

- Extract matrix element through linear response of correlators to the background field
- Form ratios to cancel leading time-dependence

\[
R_p(t) = \frac{\left( C_{\lambda_u,\lambda_d=0}^{(p)}(t) - C_{\lambda_u=0;\lambda_d}^{(p)}(t) \right) |_{\mathcal{O}(\lambda)}}{C_{\lambda_u=0;\lambda_d=0}^{(p)}(t)}
\]

At late times:

\[
R_p(t + 1) - R_p(t) \xrightarrow{t \to \infty} \frac{g_A}{Z_A}
\]

- Matrix element revealed through “effective matrix elt. plot”

![Graph showing the variation of \( g_A/Z_A \) with time (t/a). The graph includes data points and a fitted curve indicating a constant fit to the plateau region.]
Simplest semileptonic weak decay of a nuclear system

Gamow-Teller (axial current) contribution to decays of nuclei not well-known from theory

Understand multi-body contributions to \( \langle GT \rangle \) better predictions for decay rates of larger nuclei

Calculate

\[ g_A \langle GT \rangle = \langle ^3\text{He}| \overline{q} \gamma_k \gamma_5 \tau^- q | ^3\text{H} \rangle \]
Tritium $\beta$-decay

\[
\frac{(1 + \delta_R)f_V}{K/G^2_V} t_{1/2} = \frac{1}{\langle F \rangle^2 + f_A/f_V g_A^2 \langle GT \rangle^2}
\]

known from theory or expt.

- Form ratios of compound correlators to cancel leading time-dependence:

\[
\frac{R_{3H}(t)}{R_p(t)} \xrightarrow{t \to \infty} \frac{g_A(3H)}{g_A} = \langle GT \rangle
\]

- Ground state ME revealed through "effective ME plot"

constant fits to plateau region

smeared-smeared smeared-point
Proton-proton fusion

- Stars emit heat/light from conversion of H to He
- Sun + cooler stars: proton-proton fusion chain reaction

Related to:
- Neutrino breakup reaction (SNO)
- Muon capture reaction (MuSun)

We calculate \( \langle d; 3|A_3^3|pp\rangle \)

\[ L_{1,A}, \ell_{1,A}, \bar{L}_{1,A}, \ldots \]

\[ pp \rightarrow de^+\nu \] cross-section

\[ p + p \rightarrow ^2H + e^+ + \nu_e \]
\[ ^2H + p \rightarrow ^3\text{He} \]
\[ ^3\text{He} + ^3\text{He} \rightarrow ^4\text{He} + 2p \]
Proton-proton fusion

- Form ratios of compound correlators to cancel leading time-dependence

\[ R_{3S_1, 1S_0}(t) = \frac{C_{\lambda_u, \lambda_d=0}^{(3S_1, 1S_0)}(t) - C_{\lambda_u=0, \lambda_d=0}^{(3S_1, 1S_0)}(t)}{\sqrt{C_{\lambda_u=0, \lambda_d=0}^{(3S_1, 3S_1)}(t) C_{\lambda_u=0, \lambda_d=0}^{(1S_0, 1S_0)}(t)}} \]

- Fit a constant to the ‘effective matrix element plot’ at late times

\[ R_{3S_1, 1S_0}(t + 1) - R_{3S_1, 1S_0}(t) \xrightarrow{t\to\infty} \langle 3S_1; J_z = 0 | A_3^3 | 1S_0; I_z = 0 \rangle / Z_A \]

\[ = \frac{\langle d; 3 | A_3^3 | pp \rangle}{Z_A} \]

\[ \frac{\langle d; 3 | A_3^3 | pp \rangle}{Z_A} \]
Proton-proton fusion

Want to relate lattice QCD ME to
- LECs of EFTs
- pp-fusion cross section

Finite-volume quantisation condition: relate $\langle d; 3|A_3^3|pp\rangle$ to scale-indep. LECs
- Pionless EFT: $\bar{L}_{1,A}$
- Dibaryon formalism: $\bar{\ell}_{1,A}$

Define a new related quantity, $L_{1,A}^{sd-2b}$, which should have mild pion-mass dependence (remove effective range terms in $\bar{L}_{1,A}$)

Extrapolate $L_{1,A}^{sd-2b}$ to the physical point

- Prediction for $\bar{L}_{1,A}, \bar{\ell}_{1,A}$ at the physical point
- Prediction for physical cross-section

Finite-volume quantisation

- Axial field splits degeneracy of the nucleon doublet
- $^3S_1$ and $^1S_0$ channels mix
- Construct 2x2 inverse scattering amplitude matrix in background field

\[
\begin{pmatrix}
\text{continuum integrals from bubble diagrams} & \text{discrete sums} \\
\text{Det} = 0 & \text{poles of scattering amplitude} \leftrightarrow \text{eigenenergies}
\end{pmatrix}
\]

Finite-volume quantisation

- Det of inverse scattering matrix = 0 \iff eigenenergies are solutions of

\[
[p \cot \delta^3 S_1 + \delta G_0^V (p; L)] [p \cot \delta^1 S_0 + \delta G_0^V (p; L)] = [W_3 g_A M_{L_{1, A}} - W_3 g_A G_1^V (p; L)]^2
\]

from effective range expansion

finite-volume sums

weak coupling

two-body LEC

- Matrix element related to LEC

\[
|\delta E^{3 S_1 - 1 S_0}| / W_3 = |\langle 3 S_1 | A_3^3 | 1 S_0 \rangle| = Z_d^2 (4 g_A \gamma L_{1, A} + 2 g_A)
\]

- Define combination that characterises two-nucleon contribution

Expect mild pion-mass dependence \iff can extrapolate

experience from \( np \to d\gamma \)

\[
L_{1, A}^{sd-2b} = (\langle d; 3 | A_3^3 | pp \rangle - 2 g_A) / 2
\]

\[
Z_d = 1 / \sqrt{1 - \rho \gamma}
\]

Proton-proton fusion

\[ \frac{L_{1,A}^{sd-2b}}{Z_A} = -0.0107(12)(49) \]

Extrapolate,
predict physical

cross-section
Proton-proton fusion

Low-energy cross section for $pp \rightarrow de^+\nu$ dictated by the matrix element

$$|\langle d; j | A_k^- | pp \rangle| \equiv g_A C_\eta \sqrt{\frac{32\pi}{\gamma^3}} \Lambda(p) \delta_{jk}$$

Relate $\Lambda(0)$ to extrapolated LEC using EFT

$$\Lambda(0) = \frac{1}{\sqrt{1 - \gamma \rho}} \left\{ e^\chi - \gamma a_{pp} [1 - \chi e^\chi \Gamma(0, \chi)] + \frac{1}{2} \frac{\gamma^2 a_{pp} \sqrt{r_1 \rho}}{g_A} \right\} - \frac{1}{2g_A} \gamma a_{pp} \sqrt{1 - \gamma \rho} L^{sd-2b}_{1,A}$$

N$^2$LO $\not\!$ EFT with effective range contributions resummed using the dibaryon approach

---

Physical cross-section dictated by

\[ \Lambda(0) = 2.6585(6)(72)(25) \]

Can also extract

\[ L_{1,A} = 3.9(0.1)(1.0)(0.3)(0.9) \text{ fm}^3 \]

renormalisation scale \( \mu = m_\pi \)

higher-order \( \not\) EFT corrections (power-counting)
Fusion cross section dictated by

$$\Lambda(0) = 2.6585(6)(72)(25)$$


Relevant counter-term in EFT

$$L_{1,A} = 3.9(0.1)(1.0)(0.3)(0.9) \text{ fm}^3$$

Higher-order insertions

- Can access terms with more current insertions from same calculations
- Recall: background field correlation function

\[ C_{\lambda_{u};\lambda_{d}}(t) = \left( \begin{align*}
+ \lambda^3 \\
\end{align*} \right) \]

Linear response gives axial matrix element

Implicit sum over current insertion times
Higher-order insertions

- Can access terms with more current insertions from same calculations
- Recall: background field correlation function

\[ C_{\lambda_u;\lambda_d}(t) = \left( \begin{array}{c} + \lambda^3 \\ + \lambda^2 \\ + \lambda \\ + \lambda \end{array} \right) \]

Quadratic response from two insertions on different quark lines

Implicit sum over current insertion times
Double $\beta$-decay

- Certain nuclei allow observable $\beta\beta$ decay

\[ T_{1/2}^{2\nu\beta\beta} \geq 10^{19} \text{ y} \]

- If neutrinos are massive Majorana fermions $0\nu\beta\beta$ decay is possible

\[ T_{1/2}^{0\nu\beta\beta} > 10^{25} \text{ y} \]

Calculate two-current nuclear matrix elements dictate half-life
Second order weak interactions

Background axial field to second order

\[ M_{GT}^{2\nu} = 6 \int d^4x d^4y \langle pp |T [J^+_3(x)J^+_3(y)] |nn\rangle \]

many technical complications

Non-negligible deviation from long distance deuteron intermediate state contribution

PROPAGATING DEUTERON

LONG-DISTANCE PIECE

SHORT-DISTANCE PIECE

Non-negligible deviation from long distance deuteron intermediate state contribution

\[ M_{GT}^{2\nu} = -\left| \frac{M_{pp \rightarrow d}}{E_{pp} - E_d} \right|^2 + \beta_{A}^{(I=2)} \]

Quenching of \( g_A \) in nuclei is insufficient!

TBD: connect to EFT for larger systems
Gluon structure of nuclei

How does the gluon structure of a nucleon change in a nucleus?

European Muon Collaboration (1983): “EMC effect”
Modification of per-nucleon cross section of nucleons bound in nuclei
Gluon analogue?

Ratio of structure function $F_2$ per nucleon for iron and deuterium

$$F_2(x, Q^2) = \sum_{q=u,d,s\ldots} x e_q^2 \left[ q(x, Q^2) + \bar{q}(x, Q^2) \right]$$

Number density of partons of flavour $q$
Look for **nuclear (EMC) effects** in the first moments of the spin-independent gluon structure function.

Doubly challenging

- Nuclear matrix element
- Gluon observable (suffer from poor signal-to-noise)

**Deuteron gluon momentum fraction**

Ratio $\propto$ matrix element for $0 \ll \tau \ll t$

![Graph showing the ratio of three-point to two-point functions against operator insertion time $\tau$.]
Matrix elements of the **Spin-independent gluon operator** in nucleon and light nuclei

- Present statistics: can’t distinguish from no-EMC effect scenario
- Small additional uncertainty from mixing with quark operators

\[ \langle x \rangle_{\pi} \sim 450 \text{ MeV} \]

\[ \langle x \rangle_{\pi} \sim 800 \text{ MeV} \]
**Gluonic Transversity**

**Double helicity flip structure function $\Delta(x,Q^2)$**


- **Hadrons:** Gluonic Transversity (parton model interpretation)

\[
\Delta(x, Q^2) = -\frac{\alpha_s(Q^2)}{2\pi} \operatorname{Tr} Q^2 \ x^2 \int_x^1 \frac{dy}{y^3} \left[ g_{\hat{x}}(y, Q^2) - g_{\hat{y}}(x, Q^2) \right]
\]

$g_{\hat{x},\hat{y}}(y, Q^2)$: probability of finding a gluon with momentum fraction $y$ linearly polarised in $\hat{x}$, $\hat{y}$ direction

- **Nuclei:** Exotic Glue

\[
\langle p|\mathcal{O}|p \rangle = 0
\]

\[
\langle N, Z|\mathcal{O}|N, Z \rangle \neq 0
\]

gluons not associated with individual nucleons in nucleus
Gluonic Transversity

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**Nuclei:** Exotic Glue

gluons not associated with individual nucleons in nucleus

$$\langle p|\mathcal{O}|p \rangle = 0$$

$$\langle N, Z|\mathcal{O}|N, Z \rangle \neq 0$$
Non-nucleonic glue in deuteron

First moment of gluon transversity distribution in the deuteron, $m_\pi \sim 800$ MeV

- First evidence for non-nucleonic gluon contributions to nuclear structure
- Hypothesis of no signal ruled out to better than one part in $10^7$
- Magnitude relative to momentum fraction as expected from large-$N_c$

PRD96 094512 (2017)

\[
\text{Ratio } \propto \text{matrix element for } 0 \ll \tau \ll t
\]

Ratio of 3pt and 2pt functions

\[
\frac{C_3(t,\tau)}{A^2}\frac{C_2(t)}{A^2},
\]

\[
\begin{align*}
    t &= 2 \\
    t &= 3 \\
    t &= 4 \\
    t &= 5 \\
    t &= 6
\end{align*}
\]
Scalar & tensor nuclear MEs

- Axial, scalar, tensor charges of light nuclei A<4, at unphysical value of the quark masses m_π ~800 MeV

- Complete flavour-decomposition including strange quarks

Scalar

- Possible DM interaction is through scalar exchange
- Direct detection depends on nuclear matrix element

Tensor

- Quark electric dipole moment (EDM) contributions to the EDMs of light nuclei
- Input for searches for nuclear EDMs as evidence for BSM CP violation

arXiv:1712.03221
Strange matrix elements

- Complete flavour-decomposition including strange quarks
- Disconnected contributions estimated stochastically

[Arjun Gambhir, LLNL & LBNL]
Scalar & tensor nuclear MEs

- Naive expectation determined by baryon#, isospin, spin
- O(10%) nuclear effects in the scalar charges
- Nuclear modifications scale with magnitude of corresponding charge (i.e., baryon# for scalar, spin for tensor, axial)

arXiv:1712.03221
Nuclear MEs from LQCD

Nuclear matrix elements important to experimental programs e.g.,

- Neutrino breakup reaction (SNO)
- Muon capture reaction (MuSun)
- Double-beta decay
- Electron-Ion Collider
- Nuclear electric dipole moments
- Dark matter direct detection

Current state-of-the-art: significant systematics but phenomenologically interesting at current precision
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