DISPERSIVE APPROACH TO THREE-PARTICLE SYSTEMS

ANDREW JACKURA

INDIANA UNIVERSITY
JOINT PHYSICS ANALYSIS CENTER (JPAC)

INT WORKSHOP INT-18-70W
MULTI-HADRON SYSTEMS FROM LATTICE QCD

FEBRUARY 5-9, 2018
Outline

• Hadron Spectroscopy, and Phenomenology
• Review of $2 \to 2$ Reactions
• $3 \to 3$ Scattering Phenomenology
• Opportunities and Future Directions
Hadron Spectroscopy, and Phenomenology

\[ \text{Mass [GeV]} \]

\( J^P \)

\( I = \frac{1}{2} \) RPP

\( I = \frac{3}{2} \) RPP

\( I = \frac{1}{2} \) WIO8 SE L+P

\( I = \frac{3}{2} \) WIO8 SE L+P

\( N^* \)

\( \Delta^* \)

A. Jackura, Indiana University (ajackura@indiana.edu)
Hadron Spectroscopy

Constituent quark model has been successful in classifying the hadron spectrum, and gives guidance to the QCD substructure.

Search for exotics (non-quark model) is goal of many experiments (e.g. GlueX), and many new states have been discovered (XYZP’s).

- **Mesons**
  - $\eta, \eta'$

- **Baryons**
  - $\pi^+, \pi^0, \pi^-(u\bar{d})$
  - $K^0(d\bar{s}), K^+ (u\bar{s})$

- **Hybrids**
  - $\pi^+(ud), \pi^0(\frac{u\bar{u}+d\bar{d}}{\sqrt{2}})$

- **Glueballs**
  - $\eta, \eta'$

- **Mesonic-Molecules**
  - $K^0(s\bar{u}), K^+(s\bar{d})$

- **Tetraquarks**
  - $K^-(s\bar{u})$

- **Pentaquarks**
  - $\eta, \eta'$

The diagram shows the spin-parity $J^P$ for different states, with $S = \pm 1$ and $I_3 = \pm \frac{1}{2}$. The states are categorized by their $S$ and $I_3$ quantum numbers.
Hadron Spectroscopy

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Mesonic-Molecules

Hybrids

Glueballs

Tetraquarks

Pentaquarks

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<tr>
<th>Mass [GeV]</th>
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RPP2016

Godfrey & Isgur (1985)
Why 3-body Physics?

Advancements in theory and experiment require revisiting 3-body hadron scattering.

Lattice QCD has been computing scattering amplitudes - Requires 3-body formalism for continuing amplitudes to complex energies to investigate higher mass resonances.

New High-precision, high-statistics data collected on many 3-body meson systems - COMPASS, GlueX, …

New (and old) mysteries in the light-hadron sector, e.g., $a_1(1420)$

\[ a_1(1420) \rightarrow \pi^- \pi^- \pi^+ \]
Why 3-body Physics?

In heavy quarkonia, have discovered many non-quark model states (XYZs)

Many of these are found in 3-body decays, near thresholds - could 3-body effects contribute to the nature of these states?

\[ X(3872)/Z_c(3900) \to D\bar{D}\pi \]

\[ B^\pm \to K^\pm \pi^+\pi^- J/\psi \]

\[ X(3872) \]

\[ (u\bar{u}c\bar{c}) \]

Why 3-body Physics?

In heavy quarkonia, have discovered many non-quark model states (XYZs)

Many of these are found in 3-body decays, near thresholds - could 3-body effects contribute to the nature of these states?

Virtual pion-exchange cut

Real pion-exchange cut

Scattering Phenomenology

Model independent methods such as $S$-matrix theory provide constraints for reaction amplitudes: **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

**Data (Experimental/Lattice)**

![Diagram of scattering process](image)

$$\gamma p \rightarrow \pi^+ \pi^- p$$

- **Beam**
- **Target**
- **Exchange “Force”**

**Graph**

- **Intensity** vs. **$m_{\pi^+\pi^-}$ [GeV]**
Scattering Phenomenology

Model independent methods such as $S$-matrix theory provide constraints for reaction amplitudes: **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

### Amplitude Analysis

Amplitude Model

\[
t_\ell(s) = \frac{\mathcal{N}}{c_1 + c_2 s - \frac{s}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\rho(s')}{s'(s'-s)}}
\]

Functional form fixed from $S$-matrix constrains
Scattering Phenomenology

Model independent methods such as \( S \)-matrix theory provide constraints for reaction amplitudes: \textbf{Unitarity}, \textbf{Analyticity}, \textbf{Crossing}, and \textbf{Poincaré Symmetry}

\[
t_\ell(s) \rightarrow t_{\ell}^{\text{II}}(s) = \frac{t_\ell(s)}{1 + 2i\rho(s)t_\ell(s)}
\]
Scattering Phenomenology

Model independent methods such as $S$-matrix theory provide constraints for reaction amplitudes: **Unitarity**, **Analyticity**, **Crossing**, and **Poincaré Symmetry**

**Resonance Parameters**

$m_R = 769 \pm 1 \text{ MeV}$

$\Gamma_R = 149 \pm 2 \text{ MeV}$
Scattering Phenomenology

Model independent methods such as $S$-matrix theory provide constraints for reaction amplitudes: Unitarity, Analyticity, Crossing, and Poincaré Symmetry.

Imposing these constraints gives us general forms of reaction amplitudes, but there are still some degrees of freedom. These degrees of freedom are constrained by experiment, lattice QCD, models, …

$$m_R = 769 \pm 1 \text{ MeV}$$
$$\Gamma_R = 149 \pm 2 \text{ MeV}$$
Review of $2 \rightarrow 2$ Reactions
2→2 Elastic Scattering

Consider the elastic scattering of the 2→2 system \( ab \rightarrow ab \), where \( a \) and \( b \) are distinguishable particles

\[
\langle \{p'\} | T | \{p\} \rangle = (2\pi)^4 \delta^{(4)}(P' - P) \mathcal{F}(\{p', p\})
\]

---

- **Initial State**
- **Final State**
- **Rxn Flow**
- **2→2 Amplitude**
- **Construct unitarity constraints**
- **Partial Wave Expansion**
- **Dispersion Relations**
- **Parameterizations**
- **Relevant Kinematic Variables**
2→2 Elastic Scattering

Unitarity constrains the amplitude by fixing the imaginary part

Elastic Unitarity Relation \( (s < s_{inelas}) \)

\[
\text{Im } \mathcal{F}(\{p', p\}) = \rho_2(s) \int d\Omega_{p''} \mathcal{F}^*(\{p'', p'\}) \mathcal{F}(\{p'', p\}) \Theta(s - s_{th})
\]

Can reduce the unitarity relation by Partial Wave Expansion

\[
\mathcal{F}(\{p', p\}) = \sum_{\ell=0}^{\infty} \left( \frac{2\ell + 1}{4\pi} \right) f_{\ell}(s) P_{\ell}(\hat{p}' \cdot \hat{p})
\]

\( \hat{p}' \cdot \hat{p} = \cos \theta \)
2→2 Elastic Scattering

Unitarity constrains the amplitude by fixing the imaginary part

Elastic Unitarity Relation ( \( s < s_{inelas} \))

\[
\text{Im } \mathcal{F}(\{p', p\}) = \rho_2(s) \int d\Omega_{p''} \mathcal{F}^*(\{p'', p'\}) \mathcal{F}(\{p'', p\}) \Theta(s - s_{th})
\]

Partial wave unitarity relation is now algebraic

\[
\text{Im } f_\ell(s) = \rho_2(s) |f_\ell(s)|^2 \Theta(s - s_{th})
\]
2→2 Elastic Scattering

Dispersive representation for partial wave amplitudes

\[ f_\ell(s) = \frac{1}{\pi} \int_{-\infty}^{s_L} ds' \frac{\text{Im} f_\ell(s')}{{s'} - s} + \frac{1}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\rho_2(s')|f_\ell(s')|^2}{{s'} - s} \]

Nonlinear constraint for the amplitude \( f_i(s) \)

Left-hand cut physics comes from crossing

\[ \int_{-1}^{1} dz_s P_\ell(z_s) \sim \log(g(s)) \]

some function of kinematics
2→2 Elastic Scattering

Dispersive representation for partial wave amplitudes

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Nonlinear constraint for the amplitude \( f_i(s) \)

Left-hand cut physics comes from

\[
\int_{-1}^{1} dz_s P_\ell(z_s) \sim \log(g(s))
\]

Partial wave amplitudes have more complicated analytic structures - Careful in defining dispersive representations

some function of kinematics
2→2 Elastic Scattering

Can linearize the system via N-over-D method

\[ f_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)} \]

\[ N_\ell(s) = \frac{1}{\pi} \int_{-\infty}^{s_L} ds' \text{Im} \frac{f_\ell(s')}{s' - s} \]

\[ D_\ell(s) = D_\ell^{(0)}(s) - \frac{s}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\rho_2(s')N_\ell(s')}{s'(s' - s)} \]

Related to the K-matrix

\[ f_\ell^{-1}(s) = K_\ell^{-1}(s) - \frac{s}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\rho_2(s')}{s'(s' - s)} \]

Function not constrained by unitarity: CDD poles, polynomials, …
2→2 Elastic Scattering

Can linearize the system via N-over-D method

\[ f_\ell(s) = \frac{N_\ell(s)}{D_\ell(s)} \]

\[
N_\ell(s) = \frac{1}{\pi} \int_{-\infty}^{s_{L}} ds' \frac{D_\ell(s') \text{Im} f_\ell(s')}{s' - s}
\]

\[
D_\ell(s) = D^{(0)}_\ell(s) - \frac{s}{\pi} \int_{s_{th}}^\infty ds' \frac{\rho_2(s')N_\ell(s')}{s'(s' - s)}
\]

There is freedom in the function, not constrained by general principles - Must be determined by specific theory

Parameterize our Ignorance

Related to the K-matrix

\[ f^{-1}_\ell(s) = K^{-1}_\ell(s) - \frac{s}{\pi} \int_{s_{th}}^\infty ds' \frac{\rho_2(s')}{s'(s' - s)} \]

Function not constrained by unitarity: CDD poles, polynomials, ...
3→3 Scattering Phenomenology

\[ \pi^+ \rightarrow \pi^+ + \pi^- \]

\[ \pi^- \rightarrow \pi^- + \pi^0 \]

\[ \pi^0 \rightarrow \pi^0 + \pi^0 \]
3→3 Elastic Scattering

Consider the elastic scattering of 3-distinguishable particles $123 \rightarrow 123$

One approximation that is motivated by experimental analyses is the Isobar Model: Two particles resonant (called an Isobar) and the interact with third particle (called the Spectator)

C. Adolph et al. [COMPASS], Phys. Rev. D 95, no. 3, 032004 (2017)
Consider the elastic scattering of 3-distinguishable particles $\rightarrow 123$.

One approximation that is motivated by experimental analyses is the Isobar Model. Two particles resonant (called an Isobar) and the interact with third particle (called the Spectator).

\[ m_{3\pi} - 1318 \text{ MeV/c}^2 < 100 \text{ MeV/c}^2 \]
\[ m_{3\pi} - 1672 \text{ MeV/c}^2 < 100 \text{ MeV/c}^2 \]

C. Adolph et al. [COMPASS], Phys. Rev. D 95, no. 3, 032004 (2017)
Consider the elastic scattering of 3-distinguishable particles $1\rightarrow 2\to 3$.

One approximation that is motivated by experimental analyses is the Isobar Model. Two particles resonant (called an Isobar) and the interact with third particle (called the Spectator).

- Construct unitarity constraints
- Partial Wave Expansion
- Dispersion Relations
- Parameterizations

Multiple variables = Multiple discontinuities

Projecting above/below cuts may lead to different relations

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Consider the elastic scattering of 3-distinguishable particles $123 \rightarrow 123$. An approximation motivated by experimental analyses is the Isobar Model. Two particles resonant (called an Isobar) and the third interacts with a third particle (called the spectator).

- **Multiple variables = Multiple discontinuities**
- **Projecting above/below cuts may lead to different relations**
- **Singularities along paths of contours**

**Partial Wave Expansion**

**Dispersion Relations**

**Parameterizations**

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*C. Adolph et al. [COMPASS]*,
*Phys. Rev. D 95, no. 3, 032004 (2017)*
3→3 Elastic Scattering

Consider the elastic scattering of the 3→3 system 123→123, where 1, 2, and 3 are distinguishable particles

The S-matrix is decomposed as

\[
\langle \{p\}' \vert S \vert \{p\} \rangle = \langle \{p\}' \vert \{p\} \rangle + i \sum_j \delta(p_j' - p_j)(2\pi)^4 \delta(4)(Q_j' - Q_j) \mathcal{F}_j(\{p\}', \{p\})
\]

\[
+ i(2\pi)^4 \delta(4)(P' - P) A(\{p\}', \{p\})
\]

\[
= \sum_j \mathcal{F}_j(\{p\}', \{p\}) + A(\{p\}', \{p\})
\]
3→3 Elastic Scattering

3→3 amplitudes depend on 8 independent variables. One representation is

\[ A\{p', p\} = \sum_{J} \sum_{\lambda, \lambda'} \left( \frac{2J + 1}{8\pi^2} \right) A^J_{\lambda'\lambda}(\{E\}) D^{(J)}_{\lambda'\lambda}(R) \]

\[ \mathcal{R}_{jk} = (\varphi_j, \gamma_{jk}, \varphi'_k) \]

Invariant energies
Euler angles

\[ \{E\} = \{\sigma'_1, \sigma'_2, s, \sigma_1, \sigma_2\} \]

\[ s + t_{jk} + u_{jk} = \sigma_j + \sigma'_k + m^2_{j} + m^2_{k} \]

\[ \sum_{j=1}^{3} \sigma_j = s + \sum_{j=1}^{3} m^2_j \]

\[ \sum_{k=1}^{3} \sigma'_k = s + \sum_{k=1}^{3} m^2_k \]
Unitarity Relations

Disconnected 2→2 Unitarity Relation

\[ 2 \text{Im } F_j(\{p', p\}_j) = \rho_2(\sigma_j) \int d\Omega''_j \ F^*_j(\{p'', p'\}_j) F_j(\{p'', p\}_j) \]

Connected 3→3 Unitarity Relation

\[ 2 \text{Im } A(\{p', p\}) = \int \tilde{d}p'_1 \tilde{d}p'_2 \tilde{d}p'_3 \ (2\pi)^4 \delta^{(4)}(P'' - P) A^*(\{p'', p'\}) A(\{p'', p\}) \]
\[ + \sum_k \rho_2(\sigma'_k) \int d\Omega''_k \ F^*_k(\{p'', p'\}_k) A(\{p'', p\}) \Theta(\sigma'_k - \sigma_{th}(k)) \]
\[ + \sum_j \rho_2(\sigma_j) \int d\Omega''_j \ A^*(\{p'', p'\}) F(\{p'', p\}_j) \Theta(\sigma_j - \sigma_{th}(j)) \]
\[ + \sum_{j,k \neq k} 2\pi \delta(u_{jk} - m^2_{(jk)}) \ F^*_k(\{p'', p'\}_k) F_j(\{p'', p\}_j) \]
Unitarity Relations

Disconnected 2→2 Unitarity Relation

\[ 2 \text{Im} \]

Connected 3→3 Unitarity Relation

\[ 2 \text{Im} \]

\[ \sum_{k} \]

\[ \sum_{j,k} \]

\[ j \neq k \]
The Isobar Model

Assume that the amplitude can be expanded into *Isobar Amplitudes*

\[
\mathcal{A}(\{\mathbf{p}', \mathbf{p}\}) = \sum_{j,k} \mathcal{A}_{k,j}(\{\mathbf{p}', \mathbf{p}\}_{kj})
\]

Two particles interact before interacting with spectator

Sum over all allowed isobars
The Isobar Model

Assume that the amplitude can be expanded into *Isobar Amplitudes*

\[ A(\{p', p\}) = \sum_{j,k} A_{kj}(\{p', p\}_{kj}) \]

Two particles interact before interacting with spectator

\[ A_{kj} \rightarrow \sum_{s_j, s'_k} \sum_{\lambda_j, \lambda'_k} A_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) Y^*_{s_k}(\Omega_k) Y_{s_j}(\Omega_j) \]

Model involves only finite number of isobars

Sum over all allowed isobars
Isobar Model Unitarity Relations

Factorizes the sub-energy rescattering

\[
\hat{A}_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) = \frac{1}{D_k(\sigma'_k)} \hat{A}_{kj}(\sigma'_k, s, t_{jk}, \sigma_j) \frac{1}{D_j(\sigma_j)}
\]

2→2 Rescattering

Still sub-energy dependence

\[
f_j(\sigma_j) = N_j(\sigma_j)/D_j(\sigma_j)
\]
Isobar Model Unitarity Relations

\[ 2\text{Im} \]
Isobar Model Unitarity Relations

\[ 2 \text{Im} \]

\[ \sum_n \]

\[ \sum_{n \neq r} \]

\[ \sum_{n \neq k} \]

\[ \sum_{r \neq j} \]

\[ (j \neq k) \]
Isobar Model Unitarity Relations

\[ 2\text{Im} \int \sum_{n}^{\xi_k} \sum_{r \neq j}^{\xi_j} + \sum_{r \neq j}^{\xi_k} + \sum_{n \neq r}^{\xi_j} + \sum_{n \neq k}^{\xi_j} + \sum_{r \neq j}^{\xi_k} + \sum_{n \neq k}^{\xi_j} + \sum_{n \neq k}^{\xi_j} \]

\( (j \neq k) \)
Analytic Structure

We can split the imaginary part into discontinuities across all variables

Need to be careful on which direction we approach the real axis from the complex planes

\[ 2i \text{ Im } \tilde{A}_{kj}(\sigma'_k, s, t_{jk}, u_{jk}, \sigma_j) = \Delta_{\sigma'_k} \tilde{A}_{kj}(s+, t_{jk}+, u_{jk}+, \sigma_j+) \]
\[ + \Delta_s \tilde{A}_{kj}(\sigma'_k-, t_{jk}+, u_{jk}+, \sigma_j+) \]
\[ + \Delta_{t_{jk}} \tilde{A}_{kj}(\sigma'_k-, s-, u_{jk}+, \sigma_j+) \]
\[ + \Delta_{u_{jk}} \tilde{A}_{kj}(\sigma'_k-, s-, t_{jk}-, \sigma_j+) \]
\[ + \Delta_{\sigma_j} \tilde{A}_{kj}(\sigma'_k-, s-, t_{jk}-, u_{jk}-) \]

\[ x_\pm = x \pm i\epsilon \]
Analytic Structure

We can split the imaginary part into discontinuities across all variables.

Need to be careful on which direction we approach the real axis from the complex planes.

For \( j \neq k \), have to worry about singularities in \( u_{jk} \) from One Particle Exchange (OPE).

\[
x_{\pm} = x \pm i\epsilon
\]
Analytic Structure

\[
\Delta_s = i \sum_n \xi_i^{n} \xi_j^{n \neq r} + i \sum_{n,r} \xi_i^{n} \xi_j^{n \neq r}
\]

\[
\Delta_{\sigma'_k} = i \sum_{n \neq k} \xi_i^{n} \xi_j^{n}
\]

\[
\Delta_{\sigma_j} = i \sum_{r \neq j} \xi_i^{r} \xi_j^{r}
\]

\[
\Delta_{u_{jk}} = i \sum_{r \neq k} \xi_i^{r} \xi_j^{r}
\]  

\[
(j \neq k)
\]
$$\Delta_s = i \sum_n \xi_i' p' \xi_i p$$

$$\Delta_{\sigma'_k} = i \sum_{n \neq k} \xi_i' p' \xi_i p$$

$$\Delta_{\sigma_j} = i \sum_{r \neq j} \xi_i' p' \xi_i p$$

$$\Delta_{u_{jk}} = i \sum_{r \neq j} \xi_i' p' \xi_i p$$

Will turn this into an s-channel cut via Partial Wave Projection

$$u_{jk} = u_{jk} (\sigma_k', s, z_{jk}, \sigma_j)$$
We now want to consider partial wave projections of the amplitude

To simplify the expressions, let’s consider the case for $J = 0$, and spin-0 isobars

$$C_{kj}(\sigma'_k, s, \sigma_j) = \int_{-1}^{+1} dz_{jk} \hat{A}_{kj}(\sigma'_k, s, t_{jk}(s, z_{jk}), \sigma_j)$$

We can proceed to project out the discontinuities

**Note**: The off-diagonal ($j \neq k$) amplitudes have a subtlety because of the OPE amplitude
Partial wave projection of the OPE term gives an extra cut in the complex $s$-plane

$$\int_{-1}^{+1} dz_{jk} \delta(u_{jk}(s, z_{jk}) - m^2_{jk})$$

$$\sim \frac{2s}{\lambda^{1/2}(s, \sigma_j, m^2_j) \lambda^{1/2}(s, \sigma'_k, m^2_k)} \Theta(s - s^{(+)} \Theta(s^{(-)} - s)$$

Non-zero in Dalitz region

Exchange Mass

A. Jackura, Indiana University (ajackura@indiana.edu)
Want partial wave projection of

\[ \Delta_s \hat{A}_{kj}(\sigma_{k^{' -}}, s^+, u_{jk^+}, \sigma_{j^+}) + \Delta_{u_{jk}} \hat{A}_{kj}(\sigma_{k^{' -}}, s^-, u_{jk^+}, \sigma_{j^+}) \]

\[ u_{jk^+} = u_{jk} + i\epsilon \]
\[ u_{jk^-} = u_{jk} - i\epsilon \]

\[ s + t_{jk} + u_{jk} = \sigma_j + \sigma'_{k^-} + m_j^2 + m_k^2 \]
\[ s \pm i\epsilon \implies u_{jk} \mp i\epsilon \]

\[ \Delta_s \hat{A}_{kj}(\sigma_{k^{' -}}, s^+, u_{jk^+}, \sigma_{j^+}) = \hat{A}_{kj}(\sigma_{k^{' -}}, s^+, u_{jk^+}, \sigma_j) - \hat{A}_{kj}(\sigma_{k^{' -}}, s^-, u_{jk^+}, \sigma_j) \]

\[ \Delta_{u_{jk}} \hat{A}_{kj}(\sigma_{k^{' -}}, s^-, u_{jk^+}, \sigma_{j^+}) = \hat{A}_{kj}(\sigma_{k^{' -}}, s^-, u_{jk^+}, \sigma_j) - \hat{A}_{kj}(\sigma_{k^{' -}}, s^-, u_{jk^-}, \sigma_j) \]
One-Particle-Exchange

Want partial wave projection of

\[ \Delta_s \hat{A}_{k,j}(\sigma_{k_-}, s_+, u_{jk+}, \sigma_{j+}) + \Delta_{u_{jk}} \hat{A}_{k,j}(\sigma_{k_-}, s_-, u_{jk+}, \sigma_{j+}) \]

\[ u_{jk+} = u_{jk} + i \epsilon \]

\[ u_{jk-} = u_{jk} - i \epsilon \]

\[ \Delta_s \hat{A}_{k,j}(\sigma_{k_-}, s_+, u_{jk+}, \sigma_{j+}) = \hat{A}_{k,j}(\sigma_{k_-}, s_+, u_{jk+}, \sigma_{j+}) - \hat{A}_{k,j}(\sigma_{k_-}, s_-, u_{jk+}, \sigma_{j+}) \]

\[ \Delta_{u_{jk}} \hat{A}_{k,j}(\sigma_{k_-}, s_-, u_{jk+}, \sigma_{j+}) = \hat{A}_{k,j}(\sigma_{k_-}, s_-, u_{jk+}, \sigma_{j+}) - \hat{A}_{k,j}(\sigma_{k_-}, s_-, u_{jk-}, \sigma_{j+}) \]
One-Particle-Exchange

Want partial wave projection of

\[ \Delta_s \hat{A}_{kj}(\sigma_{k-}, s_+, u_{jk+}, \sigma_{j+}) + \Delta_u u_{jk} \hat{A}_{kj}(\sigma_{k-}, s-, u_{jk+}, \sigma_{j+}) \]

\[
\begin{align*}
    u_{jk+} &= u_{jk} + i\epsilon \\
    u_{jk-} &= u_{jk} - i\epsilon \\
    B(s_+) &= \Delta_s \hat{A}_{kj}(\sigma_{k-}, s_+, u_{jk+}, \sigma_{j+}) = \hat{A}_{kj}(\sigma_{k-}, s_+, u_{jk+}, \sigma_{j+}) - \hat{A}_{kj}(\sigma_{k-}, s_-, u_{jk+}, \sigma_{j+})
\end{align*}
\]

\[
\begin{align*}
B(s_-) &= \Delta_{u_{jk}} \hat{A}_{kj}(\sigma_{k-}, s-, u_{jk+}, \sigma_{j+}) = \hat{A}_{kj}(\sigma_{k-}, s-, u_{jk+}, \sigma_{j+}) - \hat{A}_{kj}(\sigma_{k-}, s-, u_{jk-}, \sigma_{j+})
\end{align*}
\]
One-Particle-Exchange

\[
\int_{-1}^{1} d\xi_{jk} \left[ \Delta_s \hat{A}_{kj}(\sigma_{k-}, s+, u_{jk+}, \sigma_{j+}) + \Delta_{ujk} \hat{A}_{kj}(\sigma_{k'}, s-, u_{jk+}, \sigma_{j+}) \right] \\
= B(s+) - B(s-) - (A(s) - B(s))
\]
One-Particle-Exchange

\[
\int_{-1}^{1} \, dz_{jk} \left[ \Delta_s \hat{A}_{k,j}(\sigma'_{k-}, s+, u_{jk+}, \sigma_{j+}) + \Delta_{u_{jk}} \hat{A}_{k,j}(\sigma'_{k-}, s-, u_{jk+}, \sigma_{j+}) \right] = B(s+) - B(s-) - (A(s-) - B(s-))
\]

leads to discontinuity across \(s\)

\[
\Delta_s C_{k,j}(\sigma'_{k-}, s+, \sigma_{j+}) = \Delta_s [\text{Boxes}] - \Delta [\text{OPE}]
\]
Triangle Diagrams

Kinematics may require deformation of dispersive contours

$$\Delta_{\sigma_1} \mathcal{C}_{31}(\sigma_{3-}'', s-, \sigma_{1+}) = i \rho_2(\sigma_{1+}) N_1(\sigma_{1+}) \int d\sigma'' D_3^{-1}(\sigma_{3''}) \mathcal{C}_{33}(\sigma_{3-}, s-, \sigma_{3-}')$$

Fix $s$, $\sigma_{3}'$, investigate contour in $\sigma_1$

$$\mathcal{C}_{31}(\sigma_{3-}'', s-, \sigma_{1+}) = \frac{1}{\pi} \int_{\sigma_{th}^{(1)}}^{(\sqrt{s-} - m_1)^2} d\sigma \frac{1}{\hat{\sigma} - \sigma_{1+}} \rho_2(\hat{\sigma}) N_1(\hat{\sigma}) b(\hat{\sigma}, s-, \sigma_{3-}')$$

Must deform contour

M. T. Grisaru
Phys. Rev. 146, 1098 (1966)
Opportunities and Future Directions
Quasi-2-Body Approximation

As a first approximation, we consider that the isobars are "quasi-stable" ⇒ Effective 2→2 system, with isobar decay correction in intermediate state

\[ \Delta_s = i \sum_n \xi_n \xi_n' + i \sum_{n \neq r} \xi_n \xi_n' \]

Effects of other terms can be estimated for cases where resonance is far from isobar-spectator threshold
Quasi-2-Body Approximation

As a first approximation, we consider that the isobars are “quasi-stable” ⇒ Effective 2→2 system, with isobar decay correction in intermediate state

\[ \Delta_s \]

\[ \sum_{r} r \]

\[ \sum_{n} n \]

\[ \sum_{n,r} n,r \]

\[ \sum_{r} r \]

\[ \sum_{n} n \]

Effects of other terms can be estimated for cases where resonance is far from isobar-spectator threshold

A. Jackura, Indiana University (ajackura@indiana.edu)
Quasi-2-Body Approximation

Assume that entire sub-energy dependence is purely isobar amplitude

\[ \xi'_k \xi_j \approx \xi'_k \xi'_k \xi_j \xi_j \]

\[ \hat{A}^J_{k,j}(\sigma'_k, s, \sigma'_j) \approx \hat{A}^J_{k,j}(s) \]

\[ \frac{1}{2i} \Delta_s \xi'_k \xi_j = \frac{1}{2} \xi'_k \xi''''_n \xi''_n \xi''_n \xi_j \]

\[
\text{Im } \hat{A}^J_{k,j}(s) = \sum_n \int_{\sigma''_n}^{(\sqrt{s} - m_n)^2} d\sigma''_n \rho_2(s, \sigma''_n, m^2_n) \text{Im } D_n^{-1}(\sigma''_n) \hat{A}^J_{k,n}(s) \hat{A}^J_{n,j}(s)
\]

\[ \equiv \sum_n \tilde{\rho}_n(s) \hat{A}^J_{k,n}(s) \hat{A}^J_{n,j}(s) \quad \text{Quasi - 2→2 Unitarity} \]
Quasi-2-Body Approximation

Have turned 3-body system into quasi-2→2 coupled-channel system

$$\text{Im } \hat{A}_{k,j}^J(s) = \sum_n \tilde{\rho}_n(s) \hat{A}_{kn}^J(s) \hat{A}_{nj}^J(s)$$

Can parameterize with N/D method

Isobar decay effects encoded into quasi-2-body phase space

$$\tilde{\rho}_n(s) = \int_{\sigma_{th}^{(n)}} (\sqrt{s} - m_n)^2 d\sigma'' \rho_2(s, \sigma''_n, m_n^2) \text{Im } D_n^{-1}(\sigma''_n)$$

Introduces additional cuts in s-plane (Woolly cuts)

$$\text{Im } s$$

3π

Re s

3-Body Resonance Pole
3\(\pi\) at COMPASS

COMPASS has largest dataset for 3\(\pi\) resonance production

JPAC in collaboration with COMPASS, developing analytic model to extract resonance poles for partial wave intensities

Interested in \(J^{PC} = 2^+, 1^{++}\) to investigate \(\pi_2\)- and \(a_1\)-systems, and non-resonant production mechanisms (e.g. Deck)

Implement quasi-2-body unitarity - High-energy process (190 GeV \(\pi\)-beam), can assume factorization of nuclear recoil
$f_2\pi \ S$
$(\pi\pi)_S\pi \ D$
$\rho\pi \ F$
$\rho\pi \ P$
$f_2\pi \ D$

M. Mikhasenko, AJ [JPAC], In Preparation

Simultaneous fit for 11 $t'$-bins
Tau-decay

The resonance $a_1(1260)$ pole position can be tested with the quasi-two-body model.
X(3872)

X(3872) is the most well-known XYZ state - Still controversial on the nature of the state (mesonic molecule, tetraquark, …)

Primary decay mode: $X(3872) \rightarrow \bar{D}D\pi$

Investigate effects of single pion exchange - Unitarization may result in pole

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$AJ \ et \ al. \ [JPAC], \ In \ Preparation$

Diagram:

- Virtual pion-exchange cut
- Real pion-exchange cut

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Outlook and Future Directions

Unitarity and Analyticity give consistent constraints on reaction amplitudes

$3 \rightarrow 3$ relations involve functions taken at different points in the complex planes - difficult to find an ‘easy’ parameterization

Work on-going to investigate the analytic structure, and derive a set of relations one could use for various parameterizations

Certain approximations (narrow-width, etc.) may potentially lead to some parameterizations that can be used in analyses