ESTIMATING AND CHECKING TRUNCATION ERRORS IN EFFECTIVE FIELD THEORY

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How many would like to see Bayesian analysis:

How Bayesian analysis actually is:
Bayesian Flow

Diagnostics:
- Setup
  - specify priors
- Guidance
  - Evidence Ratios
  - Hyperparameter posteriors
- Parameter Estimation
  - Stability ($x_{max}$, $\bar{a}$)
- Validation
  - Cross-validation
  - Lepage Plots

Lepage Plots

- 1997 - How to Renormalize the Schrödinger equation
- "..mimic the real short-distance structure of the target and probe by simple short-distance structure..."
- Low-energy data will never contain sufficient information to tell the difference between this mimicry and reality

- structure as an expansion in a small parameter
- Lepage plot = Error Plot
- Goal to diagnose whether the expansion is "working"
Lepage Plots

- Approximate unknown high-energy potentials with smeared delta functions
- Impose an ultraviolet cutoff to remove less-understood physics, $\Lambda$
- Add correction terms which imitate short range physics – each one will bring an additional parameter

\[ |\Delta \delta_0(E)| \]

G.P. Lepage (1997)

- Look at $^1S_0$ phase shifts as corrections are added
- should see power law scaling
Lepage Plot as Error Plot

- Validation Stage
- Q: Can we convince ourselves that the next term indeed behaves as $x^{k+1}$?
  - Translated as expectation on residuals
  - not a new concept

- Is it distributed around zero?
- Is it normal?
Is there a way to account for errors from data and lower-order coefficients? Yes!

\[ g(x) = \left( \frac{1}{2} + \tan\left( \frac{\pi x}{2} \right) \right)^2 \]
Statistical Errors

- Both statistical and truncation errors are expressed through marginalization integrals

- **Truncation**: Retains information only of the posterior of $\bar{c}$

\[
pr(\Delta_k(x) | D, k) = \frac{1}{x^{k+1}} \int d\bar{c} \int \cdots \int dc_{k+2} \cdots dc_{\infty}
\]

\[
pr(c_{k+1} = \frac{1}{x^{k+1}} (\Delta_k(x) - \sum_{n=k+2}^{\infty} c_n x^n), c_{k+2}, \cdots c_{\infty} | \bar{c})pr(\bar{c} | D, k)
\]

- **Statistical**: Benefits from coefficient posteriors and correlation matrices
Statistical Errors

- Account for errors from data and lower-order coefficients

\[ c_0 = \mu_{c_0} \pm \sigma_{c_0} \]

\[ \vdots \]

\[ c_k = \mu_{c_k} \pm \sigma_{c_k} \]

- Anti-correlations expected from polynomial structure

\[ C = \left( G^T E^{-1} G + \frac{1}{c} I \right)^{-1} \]

\[ = R \Lambda R^{-1} = R \Lambda R^T \]

- \( R \) - orthogonal matrix of eigenvectors

- \( \Lambda \) - Diagonal matrix of inverse variances in SVD frame.

- Transfer between spaces:

\[ c_{EFT} = R^{-1} c_{SVD} \]

- Errors now independent:

\[ \sigma_f(x) = \sqrt{(RG)^T} \cdot \text{Diagonal} [\Lambda] \]
Statistical Errors

- check analytics with MCMC
Lepage Plots in Practice – Polynomial Residuals

linear behaviour, dominance of first-order truncation?

Residual

k+1 = 5, LO
k+1 = 5, NLO
k+1 = 5, NNLO

Theory
Breakdown
Scale?

Fluctuations about zero
z value
Evolution of Lepage Plots in $k$

missed $c_0$

Underfitting
Slopes vs Noise

\[ g(x) = \left( \frac{1}{2} + \tan\left( \frac{\pi x}{2} \right) \right)^2 \]
Scaling > Value

"...see clear evidence that the theory is improved by adding higher-order corrections, and that it is improved in just the manner predicted. This convinces me that the effective theory is working.." Lepage (1997)
LP as Diagnostic

- **Uniform Prior**

\[ \sigma_{M=4}(x) = (0.33 \pm 0.07) \]

\[ - (1.88 \pm 2.69)x + (44.65 \pm 32.6)x^2 \]

\[ - (181.9 \pm 149.79)x^3 + (263.61 \pm 228.5)x^4 \]

- **Gaussian Prior (\( \tilde{c} = 5 \))**

\[ \sigma_{M=4}(x) = (0.247 \pm 0.024) \]

\[ + (1.65 \pm 0.46)x + (2.98 \pm 2.38)x^2 \]

\[ + (0.38 \pm 4.4)x^3 - (0.02 \pm 4.9)x^4 \]

Can we quantify this objection to the results of the Uniform Prior?
Residual Scaling - Uniform Prior

\[ \left( \frac{1}{2} \tan \left( \frac{\pi x}{2} \right) \right)^2 \]

- \( k+1 = 2 \)
- \( k+1 = 3 \)
- \( k+1 = 4 \)
- \( k+1 = 5 \)
"if one did not know the underlying values of $a_0$ and $a_1$ one might be hard put to explain the extent to which the fit at order 2 is superior to that at order 3, or indeed, that at order 5."

Coincidentally (?), the fit at order 2 is the only order where we see the correct scaling.

Lepage plots as model selection?

Remember: this is still a toy-model-sample-size of 1
Model Selection

\[ \text{pr}(\bar{c}|k, D) = \frac{\text{pr}(D|\bar{c}, k)\text{pr}(\bar{c}|k)}{\text{pr}(D|k)} \]

- \( \text{pr}(\text{Pregnant}|\text{Woman}) \neq \text{pr}(\text{Woman}|\text{Pregnant}) \)
- Take uniform prior on \( k \)
- \( \text{pr}(D|k) \propto \text{pr}(k|D) \)

\[ \frac{\text{pr}(M_i|D)}{\text{pr}(M_8|D)} = \frac{\int_{0}^{100} \text{pr}(D|M_i,R)\text{pr}(M_i|R)\text{pr}(R)}{\int_{0}^{100} \text{pr}(D|M_8,R)\text{pr}(M_8|R)\text{pr}(R)} \]
Model Selection

Is different information than

\[
\frac{(1 + \tan(\frac{\pi x}{2}))^2}{2}
\]

number of coefficients

different information than

\[
\Pr[M+1|D] \quad \Pr[M+1 = 8|D]
\]

D1 \{R,0,100\}

- do Lepage plots offer a new window to the Bayesian analysis or a different perspective on an old one?

- Different Questions?
  - How big of a model is justified by the data?
  - Which model scales correctly?
  - May be too soon to tell..
Residual Scaling - Gaussian Prior

\[ \left( \frac{1}{2 + \tan\left(\frac{\pi \chi}{2}\right)} \right)^2 \]
The Diagnostic

- Slopes of first-order approximation obscured by statistical fluctuations.
  - *Seeing statistically significant* changes in slope at values of $x$ near the breakdown scale may be sufficient?
- To what extent may this inform model selection?
- When parameter estimation fails, slopes will be defined by residuals as e.g.
  \[ \delta c_0 + \delta c_1 x + \sum_{n=2}^{\infty} c_n x^n \]
  - Could this discrepancy be turned into a parameter estimation diagnostic?
- This has been a *quick* glance at a *single* toy problem...more for the future.
THEORISTS ANONYMOUS

- Admit that you have a problem: your theory has uncertainties
- Acknowledge the existence of a higher power
- Seek to understand its impact on your theory
- Make a searching and fearless inventory of errors
- Acknowledge your mistakes
- Make amends for those mistakes
- Help others who must deal with the same issues
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■ Help others who must deal with the same issues
  ■ Attend INT Bayesian Program
  ■ Thank You!
Concerns...continued

- Is the first term expansion good enough?
- Can we extrapolate the correlation matrix from the fit \( c_0, \ldots c_k \) to the marginalization for truncated terms?

\[
\sigma_\Sigma^2 = \sum_{j=0}^{k} \sigma_{c_j, SVD}^2 x^{2j} + \sum_{j=k+1}^{k+1+n} \bar{c}^2 x^{2j}
\]