Probing Neutron Star Interiors with Gravitational Waves

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Can you spot the Gravitational Wave?
Exciting Times for Gravitational Wave Astrophysics!

![Graph showing strain data from Hanford, Washington (H1) and Livingston, Louisiana (L1).]
From now on...

- Gravitational Waves and Data Analysis
- Neutron Star Compact Binaries
- Model Selection
- Results
Gravitational Waves

**Definition:** Wave-like perturbation of the gravitational field
\[ \Box h_{\mu\nu} = T_{\mu\nu} \]

**Generation:** Accelerating masses (changing quadrupole and higher multipole moments)
\[ h_{ij} \sim \frac{1}{R} \frac{d^2 Q_{ij}}{dt^2} \]

**Amplitude:** Small
\[ h \sim \frac{G m u^2}{c^4 R} \sim 10^{-22} \]

**Propagation:** Light speed, weakly interacting

**Spectrum:** Kepler 3rd Law:
\[ f \sim \sqrt{\frac{m}{r^{3/12}}} \sim \frac{1}{m} , \quad E_{rad} \sim \% m \]

Example: for GW150914,
\[ E_{GW} \sim 3 M_\odot \sim 10^3 E_{SN} \sim 0.6 E_{GRB} \]
Gravitational Wave Detectors

\[ h = \frac{\Delta L}{L} \Rightarrow \Delta L = 10^{-19} m \]
Gravitational Wave Data Analysis

\[ d = R[h'(x')] + n \]

- data
- detector response
- noise

Fit the data with a theoretical model for the GW signal

1) Get data
2) Select a model
3) Calculate the residual
4) Is this just noise?

\[ d \]
\[ h'(x') \]
\[ r = d - R[h'(x')] \]
\[ p(r) = p(n) = p(d|h'(x')) \]
Bayesian Probability Theory

Degree of belief interpretation of probability

Initial Understanding + New Observations = Updated Understanding

\[ p(\bar{x}) \quad p(d|\bar{x}) \quad p(\bar{x}|d) \]

Prior + Likelihood = Posterior

Bayes’ Theorem

\[ p(\bar{x}|d, M) = \frac{p(\bar{x}|M)p(d|\bar{x}, M)}{p(d|M)} \]

Evidence

\[ p(d|M) = \int p(\bar{x}|M)p(d|\bar{x}, M)\,d\bar{x} \]
Bayesian Model Selection

Probability of model $M$  

$$p(M|d) \sim p(M)p(d|M)$$

Odds ratio  

$$O_{ij} = \frac{p(M_i|d)}{p(M_j|d)}$$

$$= \frac{p(M_i)p(d|M_i)}{p(M_j)p(d|M_j)}$$

Bayes Factor  

(\text{the evidence from before})
Quasicircular Compact Binary Inspirals

NS/NS: [20 mins, 10,000 cycles, (10-a few k)Hz] ~400km

BH/NS: [(3-7) mins, (1,000-5,000) cycles, (10-merger)Hz] ~(500-700)km

BH/BH: [secs - mins, (100-700) cycles, (10-merger)Hz] ~(600-1000)km
\[ \bar{x} = (m_1, m_2, \bar{S}_1, \bar{S}_2, D_L, \theta_N, \phi_N, \theta_L, \phi_L, t_c, \phi_c) \]
Coalescing Neutron Stars and Nuclear Physics

Hotokezaka et al.
Neutron Star Inspirals

Tidal deformability

\[ Q_{ij} = -\lambda \varepsilon_{ij} \]
Tidal Deformability

\[ \vec{x} = (m_1, m_2, \vec{S}_1, \vec{S}_2, D_L, \theta_N, \phi_N, \theta_L, \phi_L, t_c, \phi_c) \]

\[ + \{ \lambda_i(m_i, \text{EoS}), Q_i(m_i, \text{EoS}) \} \]

= 19 parameters
Tidal Deformability

We can measure the tidal deformability* with a few bright sources

Read et al. (2009)
Del Pozzo et al. (2013)
Wade et al. (2014)
Agathos et al. (2015)
Lackey and Wade (2015)
# Equation of State

<table>
<thead>
<tr>
<th>EoS</th>
<th>Method/Model</th>
<th>Composition</th>
</tr>
</thead>
<tbody>
<tr>
<td>AP4</td>
<td>variational</td>
<td>n, p, e, μ</td>
</tr>
<tr>
<td>GCR</td>
<td>variational</td>
<td>n</td>
</tr>
<tr>
<td>SV</td>
<td>SHF</td>
<td>n, p, e, μ</td>
</tr>
<tr>
<td>SGI, SkI4</td>
<td>SHF</td>
<td>n, p, e, μ</td>
</tr>
<tr>
<td>DBHF(^{(2)})(A) MPa</td>
<td>BHF</td>
<td>n, p, e, μ</td>
</tr>
<tr>
<td>G4, GA-FSU2.1</td>
<td>RMF</td>
<td>n, p, e, μ</td>
</tr>
<tr>
<td>SGI-YBZ6-S(\Lambda)A3, SkI4-YBZ6-S(\Lambda)A3</td>
<td>SHF</td>
<td>n, p, e, μ, H</td>
</tr>
<tr>
<td>NiY5KK*</td>
<td>BHF</td>
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<tr>
<td>MPaH</td>
<td>BHF</td>
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<tr>
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</tr>
<tr>
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<td>SHF</td>
<td>n, p, e, μ, K</td>
</tr>
<tr>
<td>SV222</td>
<td>SHF</td>
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<tr>
<td>GA-FSU2.1-180</td>
<td>RMF</td>
<td>n, p, e, μ, K</td>
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<tr>
<td>ALF4, ALF5</td>
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<tr>
<td>GCR-ALF</td>
<td>variational</td>
<td>n, Q</td>
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<tr>
<td>SQM3</td>
<td>MIT bag</td>
<td>Q (u, d, s)</td>
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*Tidal Deformability

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</tr>
<tr>
<td>( \text{DBHF}^{(2)}(A) )</td>
<td>BHF</td>
<td>n, p, e, μ</td>
</tr>
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<td>MPa</td>
<td>BHF</td>
<td>n, p, e, μ</td>
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Can GWs Distinguish NS Binaries with Different Internal Composition?
Model Selection

We need to calculate the evidence and the odds ratio

\[ O_{ij} = \frac{p(M_i) \ p(d|M_i)}{p(M_j) \ p(d|M_j)} \]

How much we believe in each model before acquiring the data. Based on our previous experience, observational evidence, and theoretical understanding of the Universe.

Which of two competing model fits the data at hand better.
Bayes Factor

$$BF = \frac{p(d|M_i)}{p(d|M_j)}$$

When is the BF ‘large enough’?

<table>
<thead>
<tr>
<th>BF</th>
<th>Interpretation</th>
</tr>
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<tbody>
<tr>
<td>&lt;1</td>
<td>Negative</td>
</tr>
<tr>
<td>&lt;3</td>
<td>Barely worth mentioning</td>
</tr>
<tr>
<td>&lt;10</td>
<td>Strong</td>
</tr>
<tr>
<td>&lt;100</td>
<td>Very Strong</td>
</tr>
<tr>
<td>&gt;100</td>
<td>Desicive</td>
</tr>
</tbody>
</table>

Jeffreys scale of BF interpretation
The Evidence (or the ratio)

- Laplace Approximation
- Schwarz-Bayes Information Criterion
- Reversible Jump MCMC
- Thermodynamic Integration
- Nested Sampling
- Savage-Dickey Density Ratio
\[ \tilde{x} = (m_1, m_2, \tilde{S}_1, \tilde{S}_2, D_L, \theta_N, \phi_N, \theta_L, \phi_L, t_c, \phi_c) \]

+ \{\text{EoS}\}

15 continuous parameters, and 1 discrete
Reversible Jump Markov Chain Monte Carlo

Bayes Factor = \frac{\# \text{ of iterations in model 1}}{\# \text{ of iterations in model 2}}
Errors (with RJMCMC)

Bayes Factor = \frac{\text{# of iterations in model 1}}{\text{# of iterations in model 2}}

For well-mixed chains

\text{Var}(\text{BF}) = BF^2 \left( \frac{N_1 - N_{12}}{N_1 N_{12}} + \frac{N_2 - N_{21}}{N_2 N_{21}} \right)
Prior

\begin{align*}
\begin{array}{ll}
m_1 & \text{Uniform in [0.1, 3.2] } M_\odot \\
m_2 & \\
\vec{S}_2 & \text{Uniform in direction and magnitude in } [0, m_i^2] \\
\vec{S}_1 & \\
D_L & \text{Uniform in volume} \\
\theta_N & \text{Uniform in the sky} \\
\phi_N & \\
\theta_L & \text{Uniform in direction} \\
\phi_L & \\
e_x^D & \\
e_y^D & \\
e_z^D & \\
\end{array}
\end{align*}

\[ \vec{L}(\theta_L, \phi_L) \]
Likelihood: the Noise Model

\[ p(d|h) = p(d - R[h]) = p(n) \]

Correlated Gaussian noise

\[ p(n_1 \ldots n_N) = \frac{1}{\sqrt{\text{det}(2\pi C)}} e^{-\frac{1}{2} n_i C_{ij}^{-1} n_j} \]

Stationary noise

\[ C_{f_i f_j} \sim \delta_{ij} S(f_i) \]
Likelihood: the Noise Model

Easier to evaluate

\[ n_i C_{ij}^{-1} n_j = (n|n) \sim \int \frac{\tilde{n}(f)\tilde{n}^*(f)}{S(f)} df \]

Our noise model

\[ p(d|\tilde{x}) \sim e^{-\frac{(d-h(\tilde{x})|d-h(\tilde{x}))}{2}} \]
Noise related to $S(f)$
Building Models

\[ G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu} \]

\[ \frac{Gm_{\text{tot}}}{c^2 r_{12}} \ll 1 \]

\[ \frac{v}{c} \ll 1 \]

(Blanchet, LRR)
Models: Inspiral GW

\[ \vec{x} = (m_1, m_2, \vec{S}_1, \vec{S}_2, D_L, \theta_N, \phi_N, \theta_L, \phi_L, t_c, \phi_c) + \{\text{EoS}\} \]

- GW described by \( \vec{x} \), \( (m_1, m_2) \leq M_{\text{max}}(\text{EoS}) \)
- No GW, otherwise

Reasonably fast to evaluate
Proposal Distribution

**Prior**

Provides access to the entire prior volume

*(essential to pass the constant likelihood test)*

**Jiggle**

Search around the current position

\[ \vec{\theta} + \vec{\epsilon} \]
Fisher

Jump along the eigendirections (scaled by the eigenvalues) of the Fisher Information Matrix

$$F_{ij} = (h_i | h_j)$$

Langevin

jump along the likelihood gradient
Proposal Distribution

Differential Evolution

(technically it is not memoryless)

Model jumps

Pilot runs

Braak (2005)
Proposal Distribution

Sky jumps

Customized

LIGO
Finding the Highest Peak in Gallatin Range

Electric Peak: 3,343 m
MCMC
$p(d | \bar{x}) \rightarrow p(d | \bar{x})^{1/T}$
Exchanges

Wide exploration

Limited exploration

Good solutions

Kirkpatrick, Gelatt, Vecchi (1983)
Parallel Tempering
If an EoS with kaons fits the data better than an otherwise identical EoS without kaons, then we have detected kaons in a NS interior.
Bayes Factors

![Graph showing Bayes Factors vs SNR for different models: AP4 and SQM3 with various parameter sets. The graph illustrates the relationship between BF and SNR, highlighting the performance of each model.]
### Results

<table>
<thead>
<tr>
<th></th>
<th>Strange Quark Stars</th>
<th>Hybrid Quark Stars</th>
<th>Kaons</th>
<th>Hyperons</th>
</tr>
</thead>
<tbody>
<tr>
<td>aLIGO</td>
<td>Yes!</td>
<td>Maybe</td>
<td>Unlikely</td>
<td>Unlikely</td>
</tr>
<tr>
<td>SNR</td>
<td>20</td>
<td>30-40</td>
<td>50-60</td>
<td>50-60</td>
</tr>
<tr>
<td>mass</td>
<td>$(1.2, 1.5)M_\odot$</td>
<td>$1.4M_\odot$</td>
<td>$2M_\odot$</td>
<td>$2M_\odot$</td>
</tr>
</tbody>
</table>
Mass Matters

Directly measure low mass effects

Partial information on high mass effects
Errors

Systematic Errors: our models might be wrong
  General Relativity might be wrong
  Perturbative models not accurate enough
  Models not accurate astrophysically
  Unknown noise contribution
  Detector Calibration

Statistical Errors: finite signal strength
  Width of the Posterior
  Noise Realization
  Marginalization
Further work

Further meaningful comparisons
Inspiral phase: improve modeling
Merger phase: modeling

Efficient trans-model jumps
Exploration of disfavored models
Thermodynamic Integration
Merger phase: unmodeled search

Thank you!
It is possible for the wrong model to be preferred.

Or for the correct model to be preferred less and less as the signal strength increases.
Occam Penalty

A model that requires more parameters to fit the data is penalized

\[ \frac{\delta \theta}{\Delta \theta} \]

But what if it’s the denominator that changes between the various models?
Toy Model

We get N data from a signal \( d(f) = f \)

Two competing models

\[ h_1 = af \quad a \in (0, 2) \]

\[ h_2 = af^{1.5} \quad a \in (0, 2\kappa) \]

Likelihood

\[
L_i = \frac{1}{\sqrt{2\pi\sigma}} \exp \left\{ - \sum_{i=1}^{N} \frac{[d(f) - h_i(f)]^2}{2\sigma^2} \right\}
\]
Same Dimensionality, Different Prior Volume

\[ h_1 = af \quad a \in (0, 2) \]

\[ h_2 = af^{1.5} \quad a \in (0, 2\kappa) \]