# Tail States and Collective Fluctuations in Wilson Dirac Spectra

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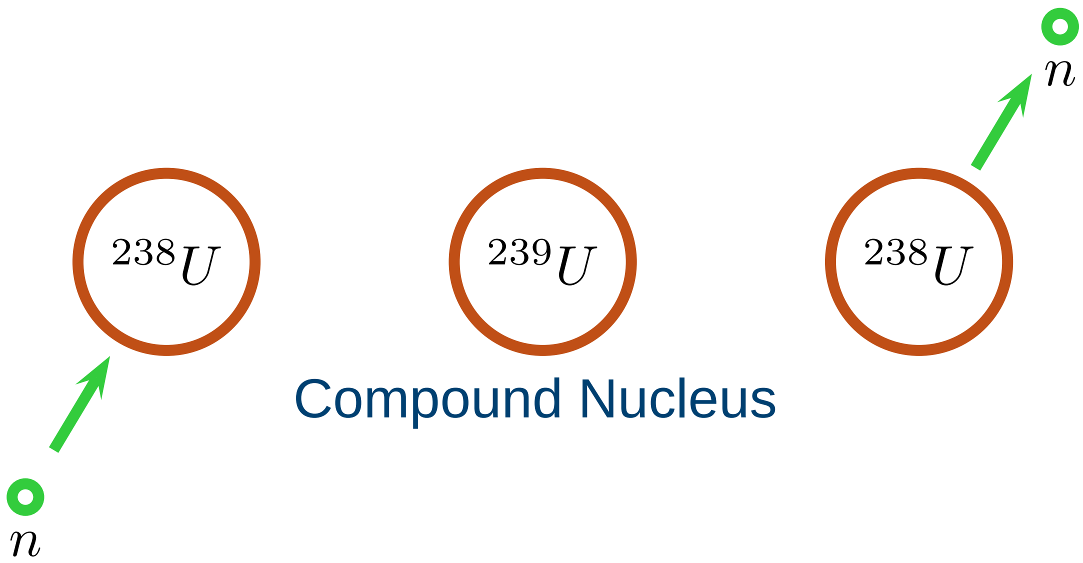
# I. Introduction

Compound Nucleus

$S$ -matrix Fluctuations

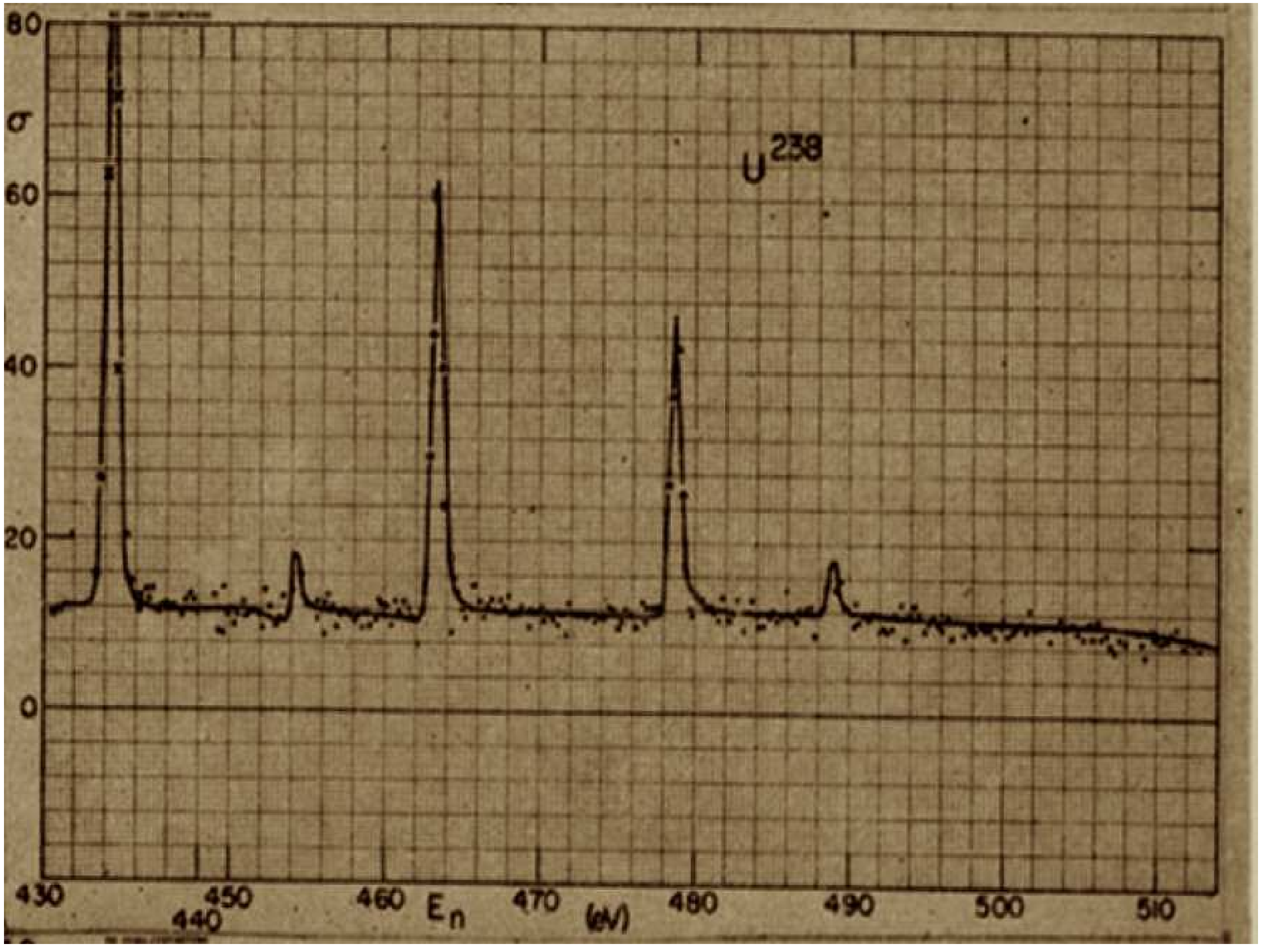
Universal Conductance Fluctuations

# Compound Nucleus




The energy of the incoming neutron gets redistributed rapidly over the nucleons in the nucleus and it a long time before the energy is concentrated in a single neutron so that it can decay.

# Compound Nucleus Scattering




Total cross section versus energy (in $eV$).

Garg-Rainwater-Petersen-Havens,1964

# $S$-matrix Fluctuations

- Distribution of the spacings of resonances with the same angular momentum and parity is given by Random Matrix Theory

$$P(S) \approx \frac{\pi}{2} \frac{S}{\langle S \rangle} e^{-\frac{\pi}{4} \frac{S^2}{\langle S \rangle^2}}.$$

- Stochastic $S$ -matrix is given by

$$S_{ab} = \delta_{ab} - 2\pi i w_{a\mu} D^{-1}_{\mu\nu} w_{\nu b}, \qquad D_{\mu\nu} = E\delta_{\mu\nu} + H^{RMT}_{\mu\nu} + \pi i \sum_a w_{\mu a} w_{a\nu}.$$

Weidenmüller

- $S_{ab} = \langle S_{ab} \rangle + S^{fl}_{ab}$
- The cross section $\langle |S_{ab}|^2 \rangle$ may be much larger than $|\langle S_{ab} \rangle|^2$ .

# $S$-matrix Fluctuations

$$\langle S_{ab}^{(\mathrm{CN\ fl})}(E_1)(S_{cd}^{(\mathrm{CN\ fl})}(E_2))^*\rangle = \prod_{i=1}^{2}\int_0^{+\infty} \mathrm{d}\lambda_i \int_0^1 \mathrm{d}\lambda$$
$$\times \frac{1}{8}\ \mu(\lambda_1,\lambda_2,\lambda)\ \exp\left\{-\frac{i\pi\varepsilon}{d}(\lambda_1+\lambda_2+2\lambda)\right\}$$
$$\times \prod_e \frac{(1-T_e\lambda)}{(1+T_e\lambda_1)^{1/2}(1+T_e\lambda_2)^{1/2}}$$
$$\times J_{abcd}(\lambda_1,\lambda_2,\lambda)\ . \qquad (75)$$

The factor $\mu(\lambda_1,\lambda_2,\lambda)$ is an integration measure and given by

$$\mu(\lambda_1,\lambda_2,\lambda) = \frac{(1-\lambda)\lambda|\lambda_1-\lambda_2|}{\prod_{i=1}^{2}[((1+\lambda_i)\lambda_i)^{1/2}(\lambda+\lambda_i)^2]}\ , \qquad (76)$$

while

$$J_{abcd}(\lambda_1,\lambda_2,\lambda) = (\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc})T_aT_b$$
$$\times\left(\sum_{i=1}^{2}\frac{\lambda_i(1+\lambda_i)}{(1+T_a\lambda_i)(1+T_b\lambda_i)}+\frac{2\lambda(1-\lambda)}{(1-T_a\lambda)(1-T_b\lambda)}\right)$$

+ one more term.

Verbaarschot-Weidenmüller-Zirnbauer-1985

See Review by Michell-Richter-Weidenmüller-2010.

- $S$ -matrix is unitary and has to be analytic and causal. Minimizing the information entropy determines the distribution, $P(S)$ , of the $S$ -matrix elements.

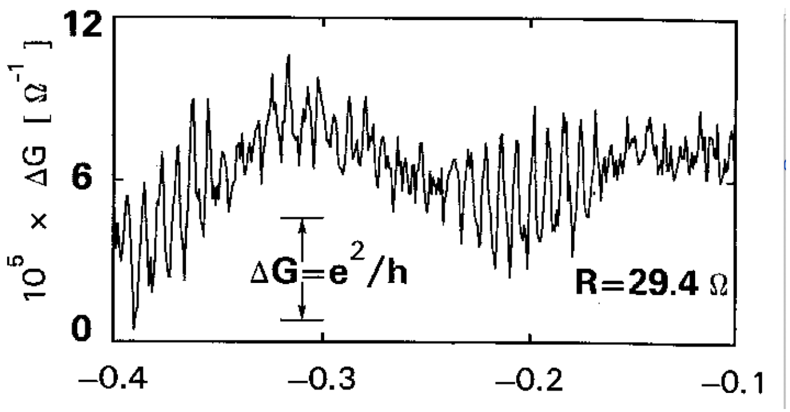
  Mello-Pereyra-Seligman-1985

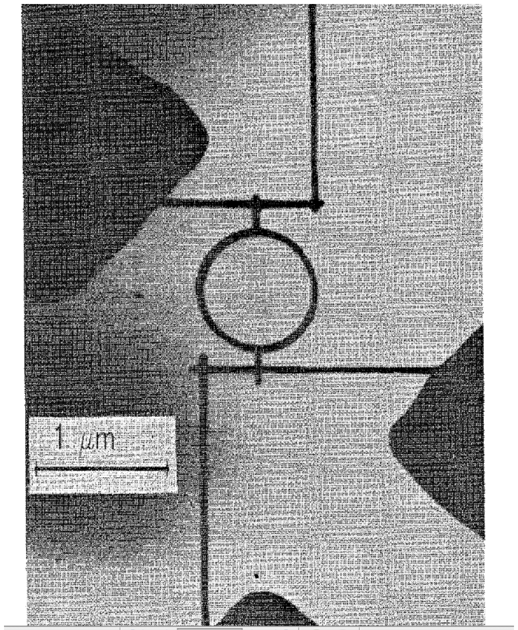
- It is an expression in terms of the transmission coefficients

$$T_a = 1 - |\langle S_{aa}\rangle|^2.$$

# Universal Conductance Fluctuations

Conductivity is measured in a small gold ring in a magnetic field




Washburn-Webb-1986

RMT prediction: var $\frac{g}{g_0} = \frac{1}{15}$

$\frac{2e^2}{\hbar}$

Altshuler-1985, Lee-Stone-1985

# II. Disordered Systems versus Lattice QCD

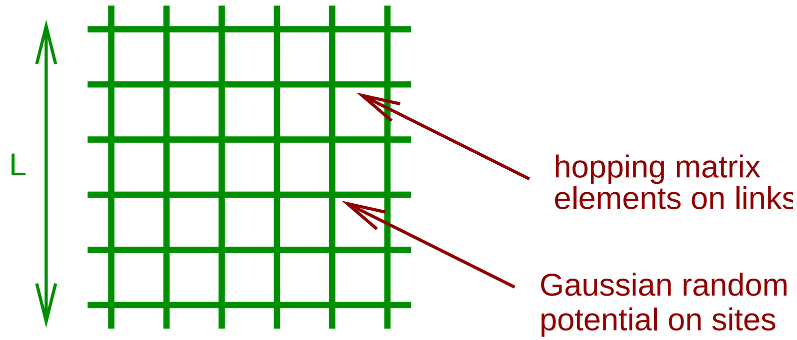
Comparison and Differences

Low Energy Effective Theory

Diffusive Motion and Thouless Energy

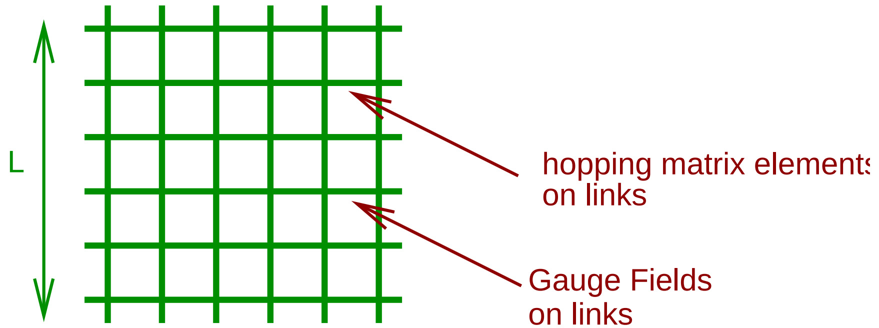
# Disordered System versus lattice QCD

## Disordered System




- Random potential + hopping matrix elements
- The lattice is physical and we have to add lattice effects to the continuum theory.

## Lattice QCD




- Random Gauge Potential
- The lattice is unphysical and we have to subtract discretization effects from lattice QCD calculations.

# Main Differences

| Disordered System | Lattice QCD |
|---|---|
| Hamiltonian | Euclidean Dirac operator |
| 2 or 3 dimensional | 4 dimensional |
| Quenched disorder | Dynamical quarks |
| Lattice is physical | Lattice is an artifact |
| Diffusion | Pion |
| Cooperon | Diquark |
| Time Reversal Symmetry | Broken Time Reversal Symmetry |

# Some More Differences

- The Dirac operator has a chiral symmetry, resulting in the block structure

$$D = \begin{pmatrix} 0 & id \\ -id^\dagger & 0 \end{pmatrix}$$

- Gauge fields have topology resulting in exact zero modes of the Dirac operator. The small quark masses they dominate the propagator

$$\frac{1}{D+m}$$

# Low Energy Effective Theory

### Disordered System

- Effective theory describing the diffusive motion of the electrons is a nonlinear $\sigma$ model

$$\mathcal{L} = \frac{1}{8} D \mathrm{Tr} \partial_k Q \partial_k Q - \frac{1}{4} i\omega \mathrm{Tr}[\Lambda Q]$$

$$Q \in U(2n)/U(n) \times U(n) \ .$$

- $D$ is the diffusion constant.

### QCD

- Long distance diffusion of quarks is described by the chiral Lagrangian of pions

$$\mathcal{L} = \frac{1}{4} F_\pi^2 \mathrm{Tr} \partial_\mu U \partial_\mu U^\dagger - \mathrm{Tr}[MU^\dagger +$$

$$U \in SU(N_f)$$

- For QCD in 3d we have that $U \in U(2N_f)/U(N_f) \times U(N_f)$ .
- The pion decay constant, $F_\pi^2$, is the diffusion constant.

# Diffusive Motion

$$\langle \Delta x^2 \rangle = D \Delta t.$$

Information on geometry is lost when $\Delta x = L$ . This corresponds to an energy scale

$$E_{\mathrm{Th}} = \frac{\hbar}{\Delta t} = \frac{\hbar D}{L^2}.$$

This is the Thouless energy.

**Disordered Systems**

Ergodic domain: $E \ll E_{Th}$
Zero dimensional limit

**Lattice QCD**

$\epsilon$-domain: $E \ll E_{Th}$
Microscopic limit

# Thouless Energy

- Eigenvalue spacing

$$\Delta\lambda = \frac{\Sigma}{V}.$$

- Thouless energy in units of the eigenvalues spacing

$$\frac{E_{th}}{\Delta\lambda} = \frac{\hbar D}{\Sigma L^{2-d}}.$$

- No separation of scales takes place for $d = 2$ . The reason is the absence of spontaneous symmetry breaking and Goldstone bosons.

# The $\epsilon$ Expansion

Expansion of the chiral Lagrangian with

$$p \sim \frac{1}{L}, \qquad m \sim \frac{1}{V}, \qquad \lambda = \frac{1}{V}, \qquad a \sim \frac{1}{\sqrt{V}}.$$

To leading order the partition function factorizes into a zero momentum part and a nonzero momentum part.

The zero momentum part is equivalent to a random matrix theory with the same global symmetries.

The thermodynamic limit with $mV$ , $\lambda V$ and $a^2 V$ fixed is known as the microscopic limit of QCD.

Although the physical quark mass is not in the microscopic domain, eigenvalues of the Dirac operator are in the microscopic domain.

# Remarks

- It came as a surprise that the behavior of the smallest Dirac eigenvalues of a complicated nonlinear field theory can be obtained analytically.
- Powerful RMT techniques can be used to evaluate many observables.
- chiral Random Matrix Theory has become a standard tool in lattice QCD.
- Because of these success it has also been used as a successful (phenomenological) model, in particular for QCD at nonzero chemical potential and the sign problem.

# III. Wilson Dirac Operator

QCD Partition Function

Chiral Condensate

Dirac Operator

Banks-Casher Relation

Wilson Dirac Operator

# QCD Partition Function

The QCD partition at temperature $1/\beta$ and chemical potential $\mu$ is given by

$$Z_{\rm QCD} \quad = \quad \sum_k e^{-\beta(E_k-\mu)},$$

where the sum is over all states. At zero momentum the energy of the states is equal to the mass of the particles.

This partition function can be rewritten as a Euclidean quantum field theory

$$Z_{\rm QCD} \quad = \quad \langle \det(D + m + \mu\gamma_0)\rangle_{\rm YM}.$$

Dirac operator

quark mass matrix

imaginary vector potential

# Dirac Operator

Chiral Block Structure

$$D = \begin{pmatrix} 0 & id \\ -id^\dagger & 0 \end{pmatrix}.$$

Eigenvalues are zero or occur in pairs $\pm\lambda_k$ .

$$Z_{\rm QCD} = m^\nu \langle \prod_k (\lambda_k^2 + m^2) \rangle.$$

# Chiral Condensate

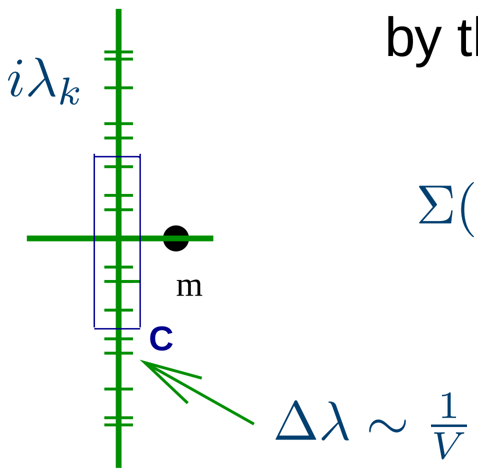
Order parameter for the chiral phase transition

$$\begin{aligned}\langle\frac{1}{V}\int d^4x\bar{\psi}\psi\rangle &= \frac{1}{V}\frac{d}{dm}\log Z(m)\\ &= \frac{1}{V}\sum_k\left\langle\frac{1}{m+i\lambda_k}\right\rangle.\end{aligned}$$

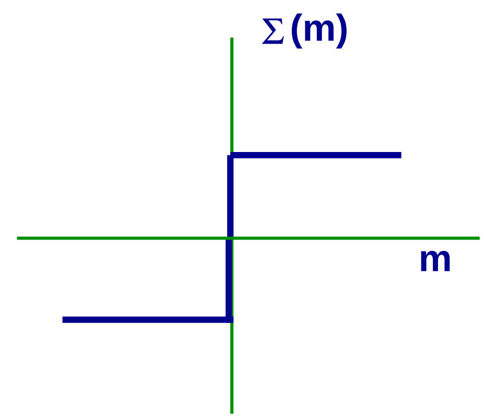
Experimentally, $\langle\bar{\psi}\psi\rangle = (-220MeV)^3$ .

# Banks-Casher Formula

The chiral condensate, $\Sigma(m)$ , is given
by the Banks-Casher formula




$$\begin{aligned} \Sigma(m) &= \lim_{V\to\infty} \left\langle \frac{1}{V} \sum_k \frac{1}{i\lambda_k + m} \right\rangle \\ &= \lim_{V\to\infty} \frac{\pi\rho(0)}{V} \quad \text{for } \; m \to 0 \end{aligned}$$




The chiral condensate is the discontinuity of the resolvent for $V \to \infty$ when $m$ crosses the imaginary axis. That is why we are interested in the behavior of the small eigenvalues.

The chiral condensate is the “electric field” in two dimensions at $m$ of charges at $i\lambda_k$ .

Eigenvalue spacing near zero: $\Delta\lambda = \frac{1}{\rho(0)} = \frac{\pi}{\Sigma V}.$

# Note on Determinantal Theories

$$Z(m) = \langle \det(D+m) \rangle$$

- Partition function remains the same if we multiply $D+m$ by an unimodular matrix.
- Chiral condensate: $\Sigma(m) = \frac{1}{V}\frac{d}{dm} log(Z(m)\Big|_{m=0}$ .
- Eigenvalues are interesting if $m$ is a multiple of the identity.
- For the Wilson Dirac operator we consider both $D_W + m$ and $D_5 \equiv \gamma_5(D+m)$ .
- $D_W$ is nonhermitian while $D_5^\dagger = D_5$ .

# Doubling Problem

- Because the Dirac operator in linear in the derivatives we have the dispersion relation

$$E = \sin pa.$$

  This results in an unwanted low energy mode.
- Way out (Wilson 1974) : Add $a\nabla^2$ to the Dirac operator so that the modes at $p = \pi/a$ are lifted.
- In agreement with the Nielsen-Ninomiya theorem this term violates the chiral symmetry of the Dirac operator.

# Wilson Dirac operator

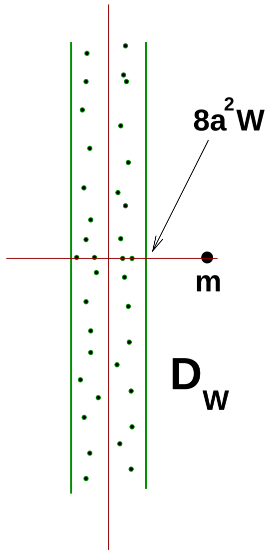
Wilson Dirac operator: $D_W = \frac{1}{2}\gamma_\mu(\nabla_\mu + \nabla_\mu^*) - \frac{1}{2}a\nabla_\mu\nabla_\mu^*.$

$$\{D_W, \gamma_5\} \neq 0$$

Block structure

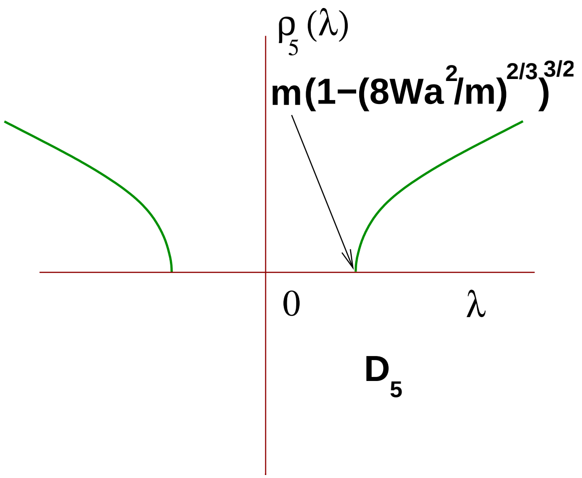
$$D_W = \begin{pmatrix} aA & id \\ id^\dagger & aB \end{pmatrix}$$

with $A^\dagger = A,\ B^\dagger = B.$




$$D_W = \gamma_5 D_W^\dagger \gamma_5$$

$$D_5 \equiv \gamma_5(D_W + m) = D_5^\dagger$$



# IV. Tail States

Bogolubov Hamiltonian

Tail States

Determinantal Theories

Numerical Spectra

# Bogolubov versus Wilson

**Bogolubov Hamiltonian**

$$H = \begin{pmatrix} S + i\alpha A & \Delta \\ \Delta & S - i\alpha A \end{pmatrix}.$$

Beloborodov-Narozhny-Aleiner-1999

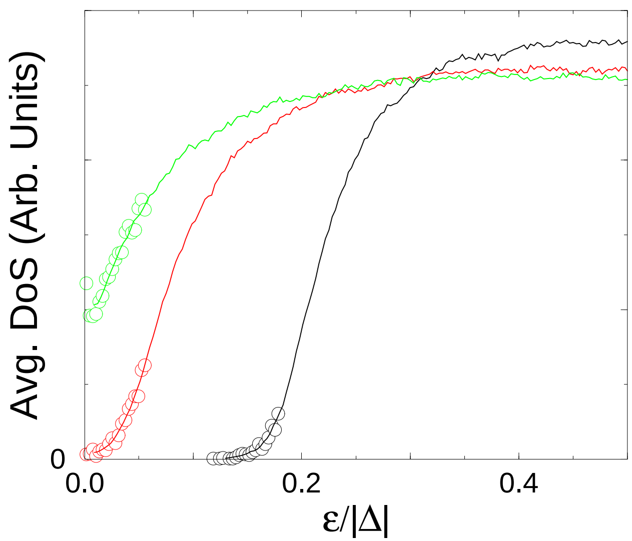
- The time reversal invariance breaking term, $i\alpha A$, results from magnetic impurities.
- Eigenvalues for $\alpha = 0$: $\lambda_k = \pm\sqrt{s_k^2 + \Delta^2}$

**Wilson Dirac Operator**

$$D_5 = \begin{pmatrix} aA + m & d \\ d^\dagger & -aA - m \end{pmatrix}$$

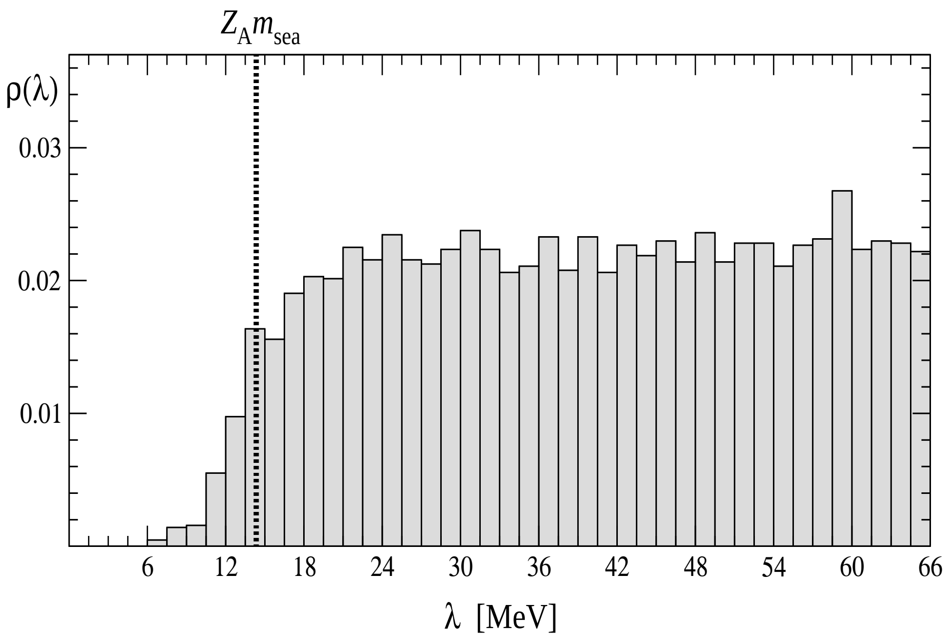
- The $aA$ term is due to the Wilson term.
- Eigenvalues for $a = 0$: $\lambda_k = \pm\sqrt{d_k^2 + m^2}$.

# Lattice Results for the Wilson Dirac Spectrum




Spectral density of a Bogolubov Hamiltonian in the presence of magnetic impurities.

Lamacraft-Simons-2001




Spectral density of $\gamma_5(D_W + m)$ on a $48 \times 24^3$ lattice. Lüscher-2007

- Dirac spectrum has a gap.
- A Gaussian tail intrudes inside the gap.

# Bogolubov versus Wilson

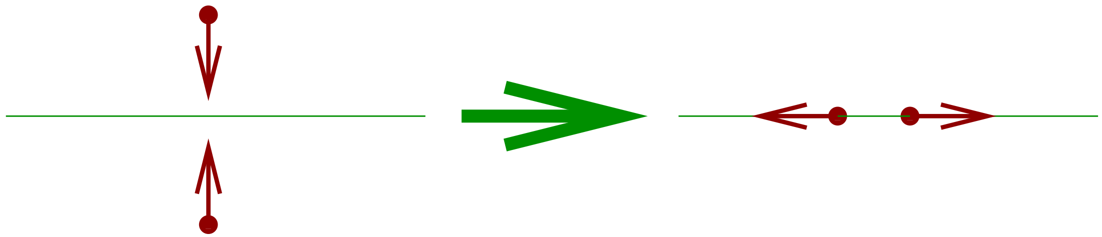
**Bogolubov Hamiltonian** **Wilson Dirac Operator**

$$\gamma_5 H = \begin{pmatrix} S + i\alpha A & \Delta \\ -\Delta & -S + i\alpha A \end{pmatrix} \qquad D_W = \gamma_5 D_5 = \begin{pmatrix} aA + m & d \\ -d^\dagger & aA + m \end{pmatrix}$$

- In both cases the operator is pseudo-Hermitian, $O^\dagger = \gamma_5 O \gamma_5$ .
- Eigenvalues occur in complex conjugate pairs.

# Pseudo Hermitian Operators

$$H^\dagger \neq H$$



$$H^\dagger = \gamma_5 H \gamma_5$$

- Two complex eigenvalues can collide and turn into a pair of real eigenvalues. The number of real eigenvalues does not change under small deformations of the operator.
- What is the distribution of the real pairs?
- What happens with the eigenvalues of the pseudo-hermitian Dirac operator when the gap closes?

# Tail States

For $|z-m|/a$ fixed for $a \to 0$ the spectral density inside the gap can also be obtained from a saddle point analysis:

$$\rho(z) \sim e^{-\Sigma^2 V(z-m)^2/16a^2 W_8} \qquad \text{for} \quad 0 < z \ll m.$$

The width parameter is given by Damgaard-Splittorff-JV-2010

$$\sigma^2 = \frac{8a^2 W_8}{V\Sigma^2}, \qquad \frac{\sigma}{\Delta\lambda} = \frac{\sqrt{8}}{\pi} a\sqrt{W_8 V}.$$

This is exactly the scaling behavior found by Del Debbio-et al-2006.

- Typical lattice parameters are $mV\Sigma = 6$ and $a\sqrt{W_8 V} = 0.2 - 0.5$ .
- For $mV\Sigma \gg 1$ and $a^2 W_8 V \gg 1$ the distribution of the smallest eigenvalue is given by the Tracy-Widom distribution .

  This behavior was also found for the Gorkov Hamiltonian with magnetic impurities. Lamacraft-Simons-2001, Vavilov-Brouwer-Ambegaokar-Beenakker-2000,Beloborodov-Narozhny-Aleiner-1999

# V. Collective Fluctuations of Dirac Spectra

Collective Fluctuation and Symmetries

Thouless Energy due to Collective Fluctuations

Wilson Chiral Lagrangian

Sign of Low Energy Constants

Collective Fluctuations and First Order Scenario

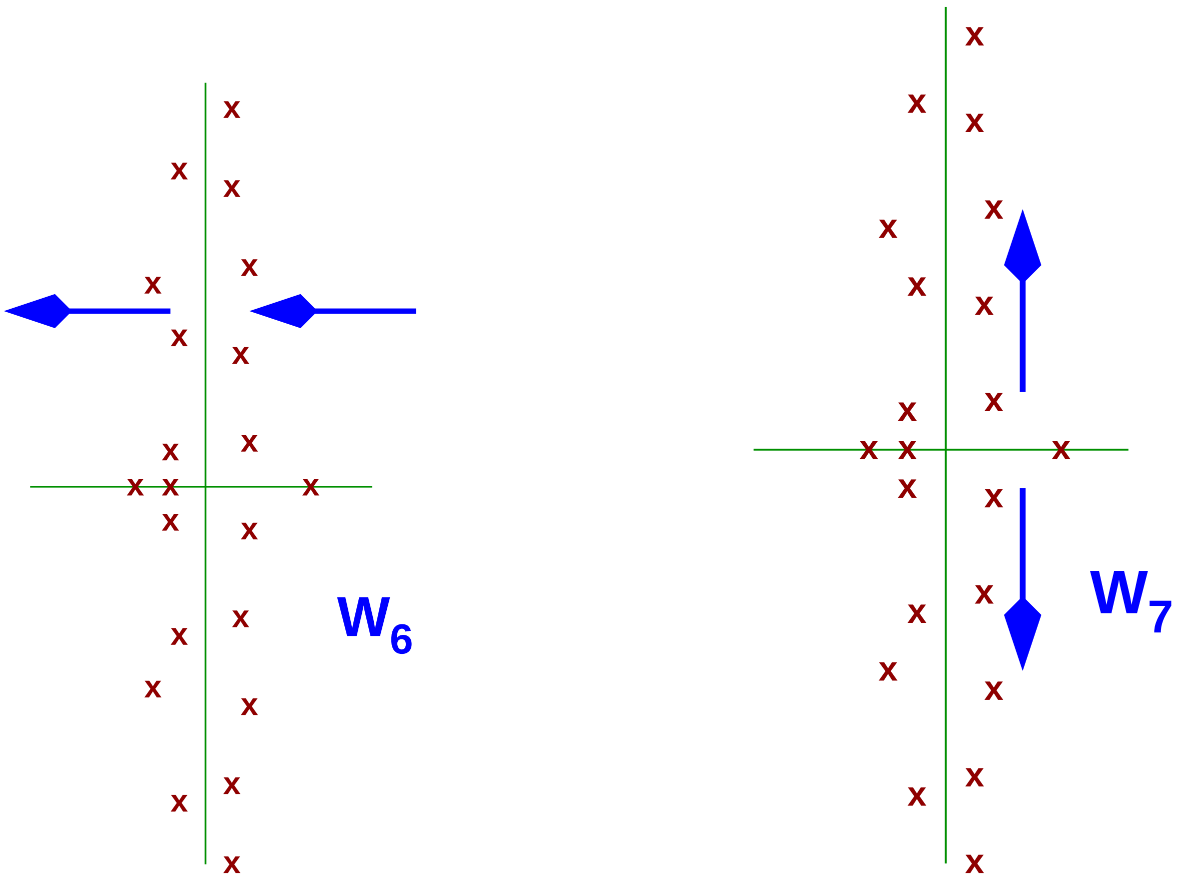
# Microscopic versus Collective Spectral Fluctuations

- Microscopic fluctuations occur on the scale of the average eigenvalue spacing.
- Collective fluctuations occur on the scale of many eigenvalue spacing.

# Thouless Energy due to Collective Fluctuations

- Spectral fluctuations on the scale of $\sqrt{V}$ give rise to the Thouless energy.
- Which modes of the gauge field configuration is responsible for these spectral fluctuations?
- The fluctuations number of quasi zero modes of the Dirac operator scales with $\sqrt{V}$ and induces fluctuation in the average spectral density of $\delta\rho/\langle\rho\rangle \sim 1/\sqrt{V}$ . This gives rise to a Thouless energy with the correct scaling behavior.

# Possible Collective Spectral Fluctuations




Collective fluctuations consistent with complex conjugation.

Is the positive definiteness of the Wilson term the source of these fluctuations? Kieburg-Splittorff-JV-2013

# Chiral Lagrangian

Chiral Lagrangian for Wilson Fermions in the microscopic domains

$$-\mathcal{L} \;=\; \frac{1}{2}mV\Sigma\mathrm{Tr}(U+U^\dagger)+$$
$$a^2VW_6[\mathrm{Tr}(U+U^\dagger)]^2+a^2VW_7[\mathrm{Tr}(U-U^\dagger)]^2-a^2VW_8\mathrm{Tr}(U^2+U^{-2}$$

Sharpe-Singleton-1998, Rupak-Shoresh-2002, Bär-Rupak-Shoresh-2004,Damgaard-Splittorff-JV- 2011

Nonlinear $\sigma$-model for Bogolubov Hamiltonian in the presence of magnetic impurities,

$$\mathcal{L}=-i\Delta V\mathrm{Tr}\sigma_2 Q-\alpha^2V\mathrm{Tr}Q^2, \qquad Q=\Sigma_3 Q_0 \quad \text{with} \quad Q_0\in U(2n)/U(n)\times U(n$$

Lamacraft-Simons-2001

Where are the trace squared terms?

# Trace Squared Terms

- Single trace terms can be linearized at the expense of a Gaussian integral and then can be added to the mass term.
- This random mass can interpreted in terms collective fluctuations of Dirac eigenvalues.
- Collective spectral fluctuations must be consistent with the symmetries of the QCD Dirac operator.

# Collective Eigenvalue Fluctuations and $W_6$

For $W_6 < 0$ we have

$$e^{-a^2 V W_6 \mathrm{Tr}^2(U+U^{-1})} \sim \int dy e^{-y^2/(16V|W_6|a^2) - \frac{1}{2} y \mathrm{Tr}(U+U^{-1})}$$

The partition function can be written as

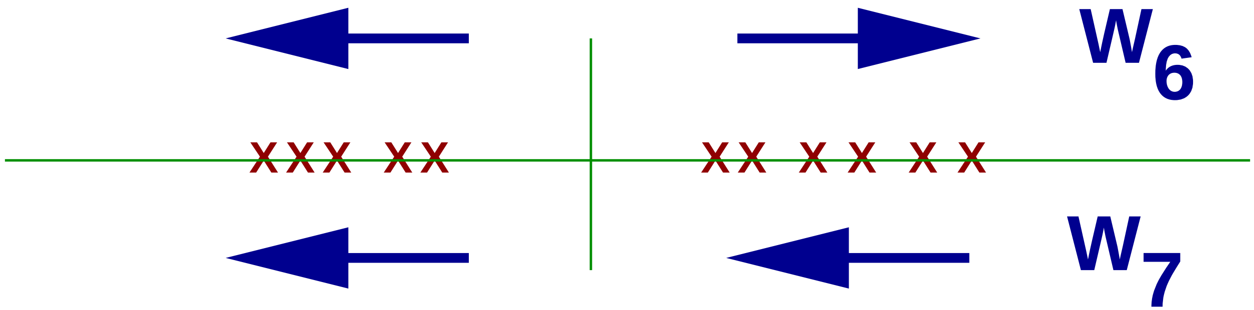
$$\begin{aligned} Z(m; W_6, W_8) &= \int dy e^{-y^2/16V|W_6|a^2} Z(m-y; W_6 = 0, W_8) \\ &= \int dy e^{-y^2/a6V|W_6|a^2} \langle \prod_k (m-y-\lambda_k) \rangle. \end{aligned}$$

A negative $W_6$ therefore corresponds to collective fluctuations of the strip of eigenvalues.

# Sign of $W_6$ , $W_7$ and $W_8$

- A positive $W_6$ would correspond to collective eigenvalues fluctuations in the imaginary direction which are forbidden because of the complex conjugation property. This implies that $W_6 < 0$ .
  Akemann-Damgaard-Splittorff-JV-2010, Kieburg-Splittorff-JV-2012
- A similar argument can be made for $W_7$ . This results in a random mass term $\gamma_5 y$ and the requirement that $W_7 < 0$ .
- The positivity of $W_8 - W_6 - W_7$ follows from pseudo-Hermiticity of the Wilson Dirac operator. Akemann-Damgaard-Splittorff-JV-2011, Hansen-Sharpe-2011

# Collective Fluctuations in Terms of $D_5$



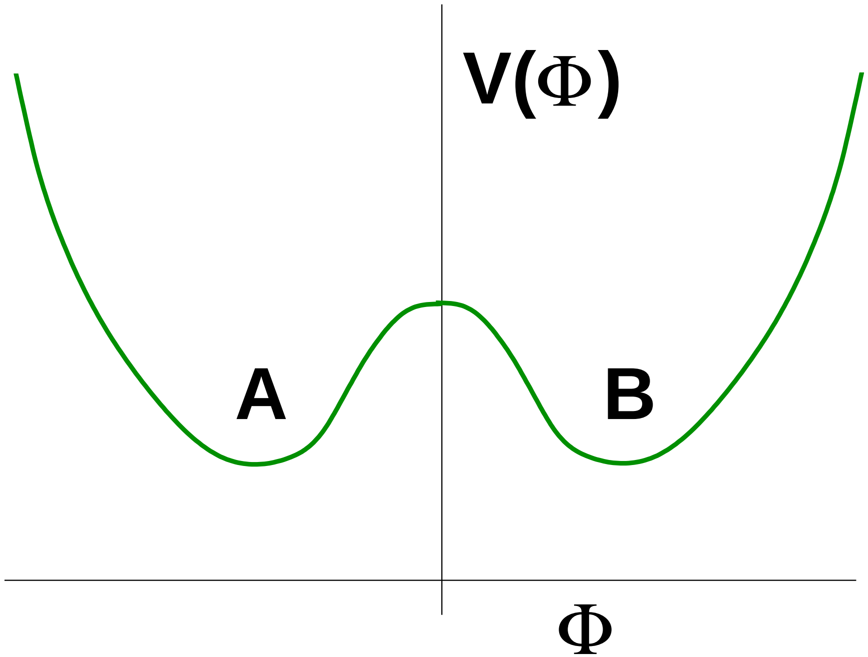

Spectrum of $D_5$

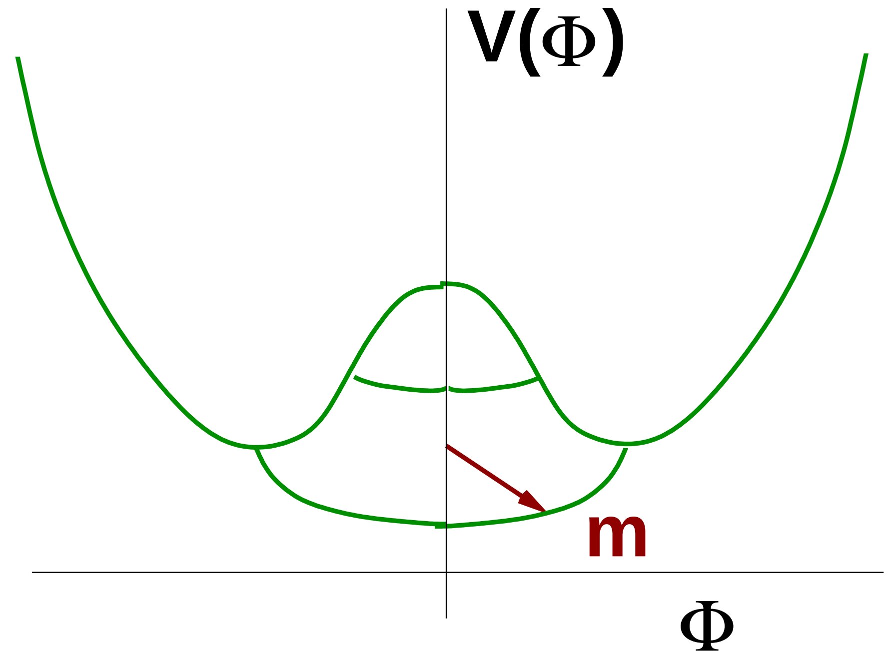
.

# Spectra and Phases

- A phase transition takes place when the gap closes. In QCD we enter the Aoki phase while for the Bogolubov Hamiltonian superconductivity ceases to exist.
- The transition to the Aoki phase is continuous.
- Sharpe and Singleton proposed another possibility that results in a first order phase transition.
- How can we understand the first order behavior in terms of collective fluctuations of Dirac eigenvalues?

# First Order Scenario




In the first order scenario the effective potential for the chiral condensate has only two minima. A nonzero quark mass slightly tilts the potential and the condensate jumps from one minimum the the other. Sharpe-Singleton

There is no flat direction and the pion mass does not vanish for vanishing quark mass.

Effective potential for the chiral condensate in continuum limit. The chiral condensate rotates the direction of the quark mass.

Because of the flat direction the pion mass vanishes for vanishing quark mass

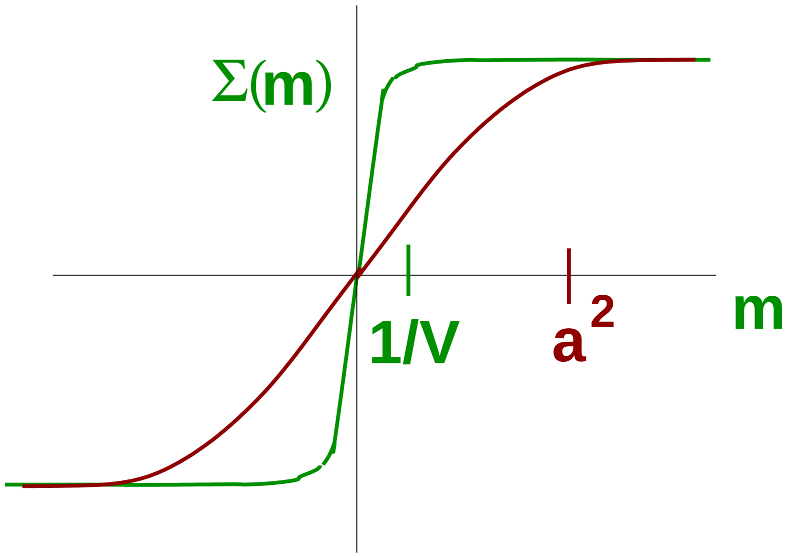
$$m_\pi \sim \sqrt{m}.$$

In the Aoki phase the direction of the chiral condensate is realigned by discretization effects.

# First Order Scenario and Dirac Spectra

- How can we understand a first order scenario in terms of Dirac spectra?
- Banks-Casher Relation

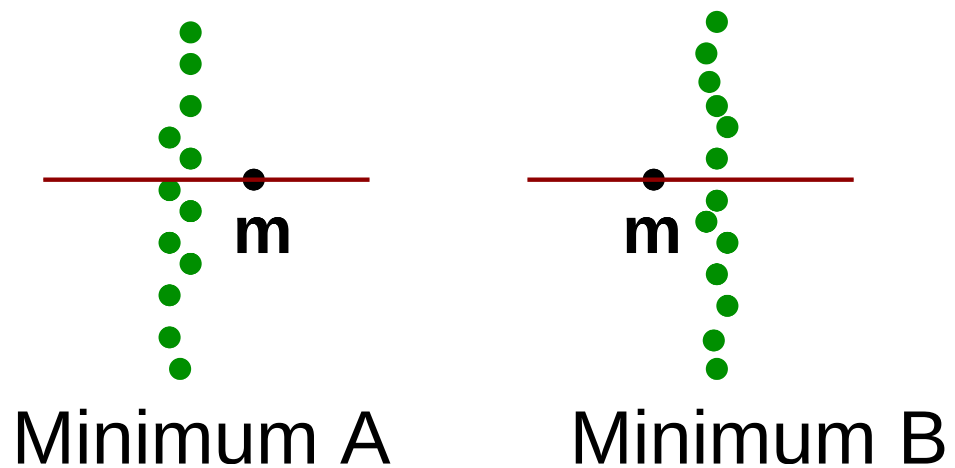
$$\Sigma(m) = \lim \frac{1}{V} \sum_k \frac{2m}{\lambda_k^2 + m^2}$$




Mass Dependence of the Chiral Condensate in a First Order Scenario (green) and for the Aoki phase (red).

# First Order Scenario and Dirac Spectra

- Because the Wilson Dirac operator in neither Hermitian nor anti-Hermitian its eigenvalues can move.
- Because of the fermion determinant they will be repelled from the quark mass.
- The finite jump of the Dirac spectrum results in a first order phase transition.



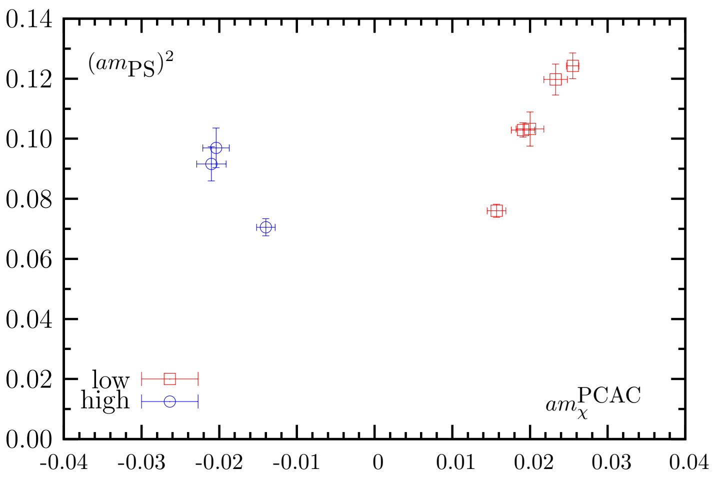

The fuzzy string of eigenvalues id repelled from the mass, $m$ , which results in a first order phase transition.

# Predictions

- Pion mass Sharpe-Singleton , Münster

$$m_\pi^2 = \frac{2|m|\Sigma - 16(W_8 + 2W_6)a^2}{F_\pi^2}.$$

When $W_8 + 2W_6 < 0$ we have a minimum pion mass. This has been observed in lattice simulations with twisted mass fermions. Jansen-etal-2005




The minimum pion mass is $O(a)$ .

- The first order scenario has only been observed for dynamical Wilson quarks, whereas the Aoki phase has been found both in the quenched case and in the case with dynamical Wilson quarks.

# Signs of Low-Energy Constants

- $W_8 > 0$ independent of the value of $W_6$ and $W_7$ .

  Akemann-Damgaard-Splittorf-JV-2010,Hansen-Sharpe-2011

- Positivity of the QCD partition function requires that $W_8 - W_6 - W_7 > 0$ .

- Interpretation in terms of eigenvalue fluctuations requires that $W_6 < 0$ , $W_7 < 0$ .

- For twisted mass Wilson fermions lattice simulations find that $m_0^{\rm PS} < m_\pm^{\rm PS}$ and chiral perturbation theory tells us

  Münster-2004, Sharpe-Wu-2004

$$m_0^{\rm PS\,2} - m_\pm^{\rm PS\,2} = \frac{16a^2(W_8 + 2W_6)}{F_\pi^2}.$$

# Sign of Low Energy Constants

A consistent picture emerges if

$$W_8 > 0, \quad W_6 < 0, \quad W_7 < 0,$$
$$W_8 + 2W_6 < 0.$$

Kieburg-Splittorff-JV-2012,Hansen-Sharpe-2011

# VI. Wilson Dirac Spectra

Random Matrix Theory for Wilson Dirac Spectra

Exact Results

Monte Carlo Simulations

Sign of Low Energy Constants

Collective Fluctuations and First Order Scenario

Observables in the small $a$ -limit

# Random Matrix Theory for the Wilson Dirac Operator

Since the chiral Lagrangian is determined uniquely by symmetries, in the microscopic domain it also can be obtained from a random matrix theory with the same symmetries. In the sector of index $\nu$ the random matrix partition function is given by

$$Z^{\nu}_{N_f} = \int dAdBdW \, \det{}^{N_f}(D_W + m + z\gamma_5) \, P(D_W),$$

with

$$D_W = \begin{pmatrix} aA & C + aD \\ -C^\dagger + aD^\dagger & aB \end{pmatrix}. \qquad \text{and} \quad A^\dagger = A, \qquad B^\dagger = B.$$

where $A$ is a square matrix of size $n \times n$, and $B$ is a square matrix of size $(n+\nu) \times (n+\nu)$. The matrices $C$ and $D$ are complex $n \times (n+\nu)$ matrices. Damgaard-Splittorff-JV-2010

# Comments

- The index of the $D_W$ defined by spectral flow is equal to $\nu$ .
- Universality: the properties of this random matrix theory do not depend on the details of $P(D_W)$ .
- In the microscopic domain, the Random Matrix Theory partition function reduces to the (twisted mass) chiral Lagrangian introduced before with $W_8 > 0$ and $W_6 = W_7 = 0$ .
- The presence of a $D$ term does not change the chiral Lagrangian. Such term simplifies random matrix calculations.
- The advantage of a random matrix formulations is that we can use powerful random matrix methods to calculate spectral observables. Kieburg-Zafeiropoulos-JV-2011, Akemann-Nagao-2011

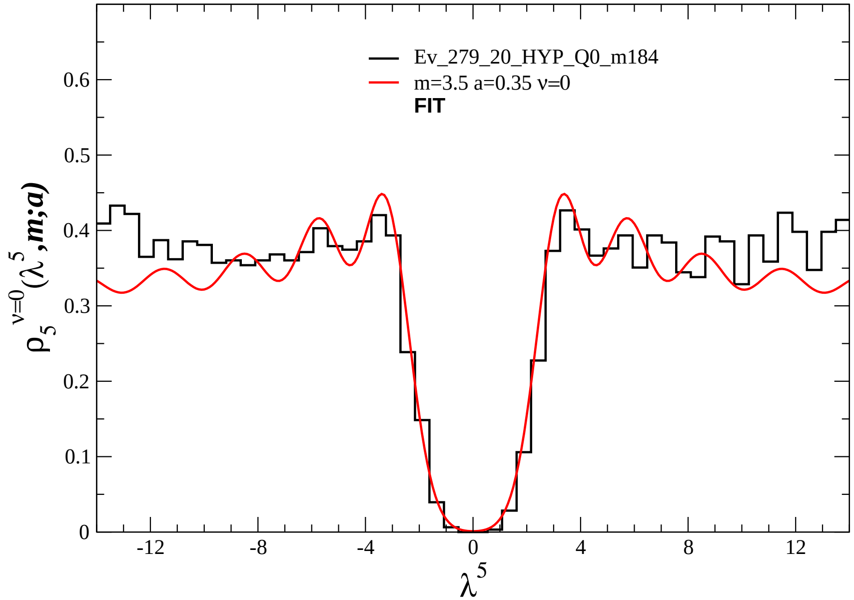
# Exact Results

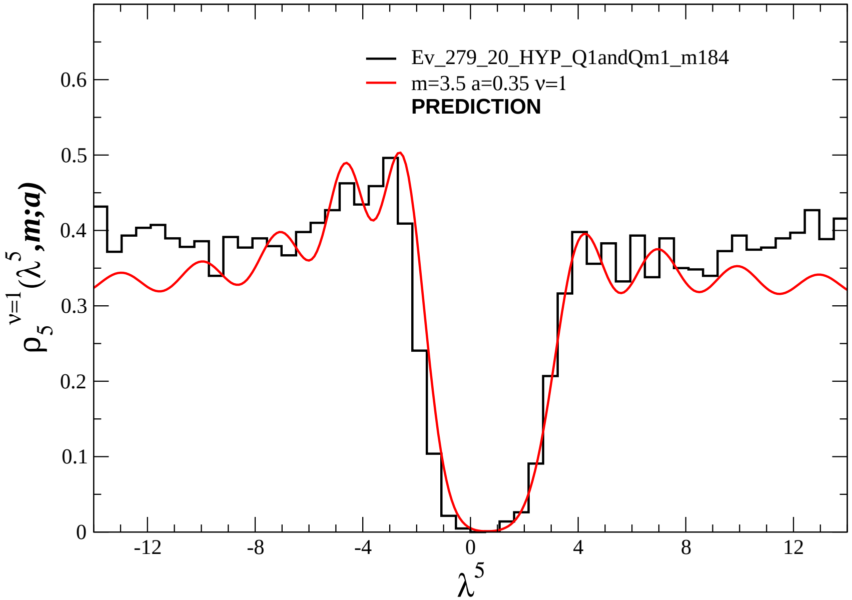
Exact results have been derived for four different observables:

- The eigenvalue density of $D_5$ .
  Damgaard-Splittorff-JV-2010,Akemann-Nagao-2011,Larssen-2012
- The density of the complex eigenvalues of $D_W$ .
  Kieburg-JV-Zafeiropoulos-2011
- The density of the real eigenvalues of $D_W$ .
  Kieburg-JV-Zafeiropoulos-2011
- The distribution of the chiralities over the real eigenvalues of $D_W$ .
  Damgaard-Splittorff-JV-2010,Splittorff-JV-2011
  Kieburg-JV-Zafeiropoulos-2011,Kieburg-Splittorff-JV-2012

Analytical results have been derived from a supersymmetic chiral Lagrangian as well as from Wilson Random Matrix Theory.

# Quenched Lattice Results at fixed $\nu$




The microscopic spectrum of $\gamma_5(D_W + m)$ fr $\nu = 0$ (left) and $\nu = 1$ (right).

Damgaard-Heller-Splittorff-2011

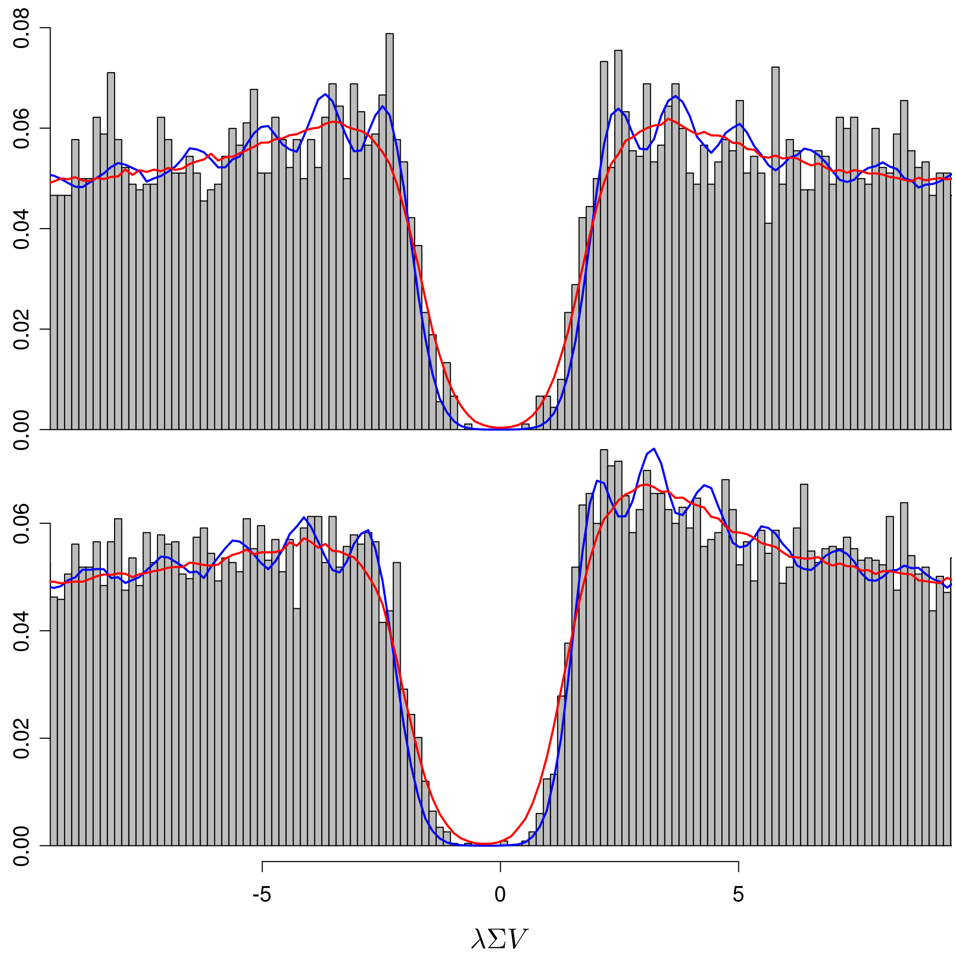
The red curve represents the analytical result for the resolvent. Splittorff-JV-2011

$$G^\nu(m,z;a) = \frac{1}{16a^2\pi}\int \frac{dsdt}{t-is} e^{-[(s+iz)^2+(t-z)^2]/16a^2} \frac{(m-is)^\nu}{(m-t)^\nu} \tilde{Z}^\nu_{1|1}(\sqrt{m^2+s^2}, \sqrt{m^2-t^2}; a$$

where

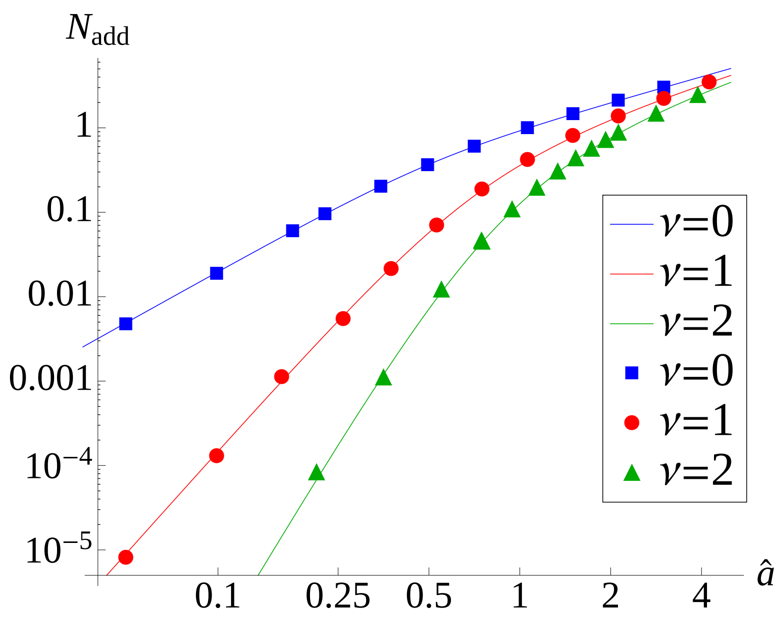
$$\tilde{Z}^\nu_{1|1}(x,y;a=0) = \frac{y^\nu}{x^\nu}[yK_{\nu+1}(y)I_\nu(x) + xK\nu(y)I_{\nu+1}(x)].$$

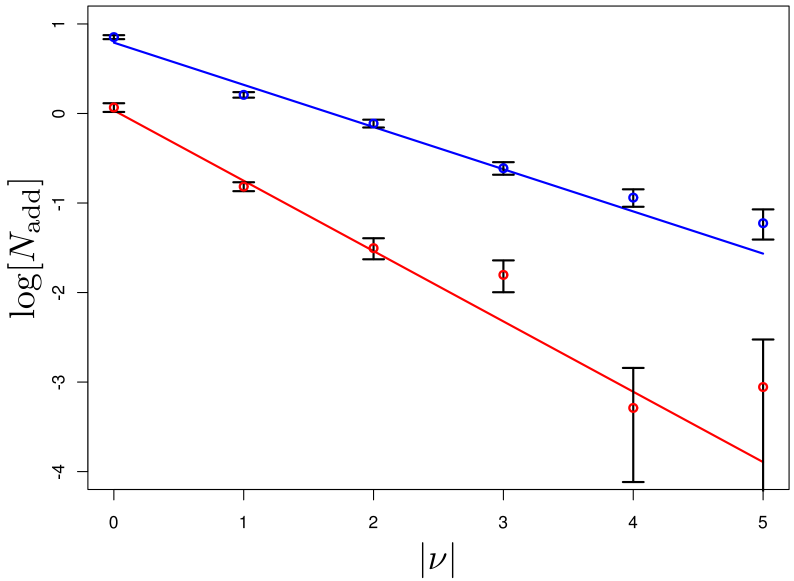
# More Quenched Lattice Dirac Spectra at fixed $\nu$




The spectrum of the Hermitian Wilson Dirac operator for $\nu = 0$ (top) and $\nu = 1$ (bottom). The blue curve is the WRMT result with $W_8 \neq 0$ and $W_6 = W_7 = 0$ while for the red curve they are nonzero. Deuzeman-Wenger-Wuilloud-2011

# Additional Real Modes




Additional number real modes of the Wilson Dirac operator. In the left figure we show the analytical result compared to random matrix simulations (Kieburg-JV-Zafeiropoulos-2011) and in the right figure we show lattice result of (Deuzeman-Wenger-Wuilloud-2011) for two different lattice spacings. The lattice data are consistent with a logarithmic $a$ -dependence.

# Observables in the small $a$ Limit

- The density of the projection of the eigenvalues of $D_W$ on the imaginary axis. According to the Banks-Casher formula we have

$$\Delta\lambda = \frac{\pi}{\Sigma V}.$$

- The average number of the additional real modes for $\nu = 0$ :

$$N_{\rm add}^{\nu=0} \quad \overset{a\ll 1}{=} \quad 2Va^2(W_8 - 2W_7).$$

- The width of the Gaussian shaped strip of complex eigenvalues:

$$\frac{\sigma^2}{(\Delta\lambda)^2} \quad \overset{a\ll 1}{=} \quad \frac{4}{\pi^2} a^2 V(W_8 - 2W_6),$$

# Observables in the small $a$ Limit

- The variance of the distribution of chirality over the real eigenvalues:

$$\langle (\frac{x}{\Delta\lambda})^2 \rangle_{\rho_\chi^\nu} \overset{a \ll 1}{=} \frac{8}{\pi^2} V a^2 (\nu W_8 - W_6 - W_7), \ \nu > 0.$$

There are linear dependencies between the relations. This results in the consistency condition

$$\frac{\langle \widetilde{x}^2 \rangle_{\rho_\chi}^{\nu=1}}{\Delta\lambda^2} = \frac{\sigma^2}{\Delta\lambda^2} + \frac{2}{\pi^2} N_{\rm add}^{\nu=0}.$$

Kieburg-JV-Zafeiropoulos

# VI. Conclusions

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- We have seen a close analogy between tail states due to magnetic impurities and tail stated due to nonzero lattice spacing.
- Collective spectral fluctuations limit the domain of validity of random matrix theory
- Collective spectral fluctuations drive the first order phase transition for Wilson fermions.
- We have constructed a random matrix theory that reproduces the chiral Lagrangian for the Wilson Dirac operator in the microscopic domain.

# Questions

- Which modes of the gauge field configuration are responsible for the Thouless energy?
- What is the effect of trace squared terms in disordered superconductors?
- What is the source of the collective spectral fluctuations of the Wilson Dirac operator?
- What are the effects of nonzero lattice spacing on disordered superrconductors?