Manifest power-counting, the UV properties of $N=8$ supergravity and the origin of simplicity in $N=4$ SYM theory

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Based on work with Z. Bern, J.J. Carrasco, L. Dixon, H. Johansson and Q. Jin
The master plan:

1. $\mathcal{N}=8$ supergravity through 4 loops
   - why forms of amplitudes w/ manifest powercounting?
   - a curious relation between sYM and Sugra UV residues
   - 4-loops with manifest power counting
   - UV properties and more unexpected relations

2. Just how magical is nonplanar $\mathcal{N}=4$ sYM theory?
   - a close relative of $\mathcal{N}=4$ sYM theory though 2 loops
     with planar integrability & some tree-level numerator relations
   - how to calculate; find some all-loop prop’s of 2-trace terms
   - explore 1 & 2-loop nonplanar amplitudes

3. Comments and conclusions
Why chase presentations of amplitudes with manifest powercounting?

- usually simpler than other presentations
- (much) easier to analyze
- expose the UV properties of the theory:
  \[ N=8 \text{ supergravity: } D_c = 4 + \frac{6}{L} \text{ though 4 loops} \]
- (sufficiently many) symmetries are linearly realized
- may expose unexpected relations between theories and unexpected symmetries

Should they exist?

- not clear; perhaps akin to freedom of gauge choice
- depends on the symmetries at work (linear vs. nonlinear)
- confusing when several symmetries have similar consequences
An example:

- 2-loop sYM vs. 2-loop supergravity \( (d = 7) \)

\[
\mathcal{A}_4^{(2)} \bigg|_{\text{pole}}^{SU(N_c)} = -g^6 \mathcal{K} \left( N_c^2 \right) + 12 \left( \right)
\]

\[
\times \left( s (\text{Tr}_{1324} + \text{Tr}_{1423}) + t (\text{Tr}_{1243} + \text{Tr}_{1342}) + u (\text{Tr}_{1234} + \text{Tr}_{1432}) \right) + \ldots
\]

\[
\mathcal{M}_4^{(2)} \bigg|_{\text{pole}} = -2 \left( \frac{\kappa}{2} \right)^6 stu(s^2 + t^2 + u^2) M_4^{\text{tree}} \left( \right)
\]

\[
= -2 \left( \frac{\kappa}{2} \right)^6 stu(s^2 + t^2 + u^2) M_4^{\text{tree}} \left( \frac{\pi}{12(4\pi)^7} \right) + \ldots
\]
An example:

- 2-loop sYM vs. 2-loop supergravity \((d = 7)\)

\[
\mathcal{A}^{(2)}_{4}\bigg|_{\text{pole}^{SU(N_c)}} = -g^6 K \left( N_c^2 \begin{array}{c}
\text{Diag.1} \\
\text{Diag.2}
\end{array} + 12 \left( \begin{array}{c}
\text{Diag.3} \\
\text{Diag.4}
\end{array} \right) \right) \\
\times \left( s \left( \text{Tr}_{1324} + \text{Tr}_{1423} \right) + t \left( \text{Tr}_{1243} + \text{Tr}_{1342} \right) + u \left( \text{Tr}_{1234} + \text{Tr}_{1432} \right) \right) + \ldots
\]

\[
\mathcal{M}^{(2)}_{4}\bigg|_{\text{pole}^{SU(N_c)}} = -2 \left( \frac{\kappa}{2} \right)^6 stu (s^2 + t^2 + u^2) M_4^{\text{tree}} \frac{\pi}{12(4\pi)^7 \epsilon}
\]

- 3-loop sYM vs. 3-loop supergravity \((d = 6)\)

\[
\mathcal{A}^{(3)}_{4}\bigg|_{\text{pole}^{SU(N_c)}} = 2 g^8 K \left( N_c^3 \begin{array}{c}
\text{Diag.1} \\
\text{Diag.2} \\
\text{Diag.3}
\end{array} + 12 N_c \left( \begin{array}{c}
\text{Diag.4} \\
\text{Diag.5}
\end{array} + 3 \begin{array}{c}
\text{Diag.6} \\
\text{Diag.7}
\end{array} \right) \right) \\
\times \left( s \left( \text{Tr}_{1324} + \text{Tr}_{1423} \right) + t \left( \text{Tr}_{1243} + \text{Tr}_{1342} \right) + u \left( \text{Tr}_{1234} + \text{Tr}_{1432} \right) \right)
\]

\[
\mathcal{M}^{(3)}_{4}\bigg|_{\text{pole}^{SU(N_c)}} = - \left( \frac{\kappa}{2} \right)^8 (stu)^2 M_4^{\text{tree}} \left( 10 \left( \begin{array}{c}
\text{Diag.1} \\
\text{Diag.2} \\
\text{Diag.3}
\end{array} \right) + 3 \left( \begin{array}{c}
\text{Diag.4} \\
\text{Diag.5}
\end{array} \right) \right)
\]

\[
= - \left( \frac{\kappa}{2} \right)^8 (stu)^2 M_4^{\text{tree}} \left( 10 \times \frac{\zeta(3)}{2(4\pi)^9 \epsilon} \right)
\]
An example:

- **2-loop sYM vs. 2-loop supergravity** \((d = 7)\)
  \[
  \mathcal{A}_{4}^{(2)} \bigg|_{\text{pole}}^{SU(N_c)} = - g^6 \mathcal{K} \left( N_c^2 \begin{array}{c} \circ \end{array} + 12 \begin{array}{c} \circ \end{array} \right) \times \left( s (\text{Tr}_{1324} + \text{Tr}_{1423}) + t (\text{Tr}_{1243} + \text{Tr}_{1342}) + u (\text{Tr}_{1234} + \text{Tr}_{1432}) \right) + \ldots
  \]
  \[
  \mathcal{M}_{4}^{(2)} \bigg|_{\text{pole}} = - 2 \left( \frac{\kappa}{2} \right)^6 \cdot stu(s^2 + t^2 + u^2) \cdot \frac{\pi}{12(4\pi)^7 \epsilon} 
  \]

- **3-loop sYM vs. 3-loop supergravity** \((d = 6)\)
  \[
  \mathcal{A}_{4}^{(3)} \bigg|_{\text{pole}}^{SU(N_c)} = 2 g^8 \mathcal{K} \left( N_c^3 \begin{array}{c} \bigcirc \end{array} + 12 N_c \left( \begin{array}{c} \bigcirc \end{array} + 3 \begin{array}{c} \bigcirc \end{array} \right) \right) \times \left( s (\text{Tr}_{1324} + \text{Tr}_{1423}) + t (\text{Tr}_{1243} + \text{Tr}_{1342}) + u (\text{Tr}_{1234} + \text{Tr}_{1432}) \right)
  \]
  \[
  \mathcal{M}_{4}^{(3)} \bigg|_{\text{pole}} = - \left( \frac{\kappa}{2} \right)^8 (stu)^2 M_4^{\text{tree}} \left( 10 \times \frac{\zeta(3)}{2(4\pi)^9 \epsilon} \right)
  \]

Does this continue?
Supergravity amplitudes with manifest power counting

• 1-loop sYM vs. 1-loop supergravity

\[ A_{4}^{1\text{loop}}(1,2,3,4) = s_{12}s_{23}A_{4}^{\text{tree}}(1,2,3,4) \]

\[ M_{4}^{1\text{loop}}(1,2,3,4) = \left[s_{12}s_{23}A_{4}^{\text{tree}}(1,2,3,4)\right]^{2} \]

\[ \begin{bmatrix}
      2 & 3 & 4 \\
      1 & 2 & 3 \\
      1 & 3 & 2 \\
      2 & 3 & 4
    \end{bmatrix}
\]

Green, Schwarz

• 2-loop sYM vs. 2-loop supergravity

\[ \frac{i^{2}s_{12}s_{23}}{c_{1}s_{12} + c_{2}s_{12} + \text{perm's}} \]

Bern, Rozowski, Yan

Bern, Dixon, Dunbar, Perelstein, Rozowski

\[ \left[s_{12}s_{23}\right]^{2} \]

\[ \begin{bmatrix}
      2 & 3 \\
      1 & 2 \\
      1 & 3 \\
      2 & 3
    \end{bmatrix}
\]

\[ \begin{bmatrix}
      2 & 3 \\
      1 & 2 \\
      1 & 3 \\
      2 & 3
    \end{bmatrix}
\]

\[ \frac{s_{12}^{2}}{s_{12}^{2}} + \text{perm's} \]

Bern, Dixon, Dunbar, Perelstein, Rozowski

• At 3 loops: use color/kinematics + squaring relation
• Color-kinematics duality provides a continuation of such squaring relations

\begin{align*}
\text{(a)-(d)} & : s^2 \\
\text{(e)-(g)} & : \frac{s(-\tau_{35} + \tau_{45} + t) - t(\tau_{25} + \tau_{45}) + u(\tau_{25} + \tau_{35}) - s^2}{3} \\
\text{(h)} & : \frac{s(2\tau_{15} - \tau_{16} + 2\tau_{26} - \tau_{27} + 2\tau_{35} + \tau_{36} + \tau_{37} - u)}{3} + t(\tau_{16} + \tau_{26} - \tau_{37} + 2\tau_{36} - 2\tau_{15} - 2\tau_{27} - 2\tau_{35} - 3\tau_{17}) + s^2 \\
\text{(i)} & : \frac{s(-\tau_{25} - \tau_{26} - \tau_{35} + \tau_{36} + \tau_{45} + 2t)}{3} + t(\tau_{26} + \tau_{35} + 2\tau_{36} + 2\tau_{45} + 3\tau_{46}) + u(\tau_{25} + s^2) \\
\text{(j)-(l)} & : \frac{s(t-u)}{3}
\end{align*}

• It has the same cuts as previously-derived expressions
\[(s(\tau_{26} + \tau_{36}) + t(\tau_{15} + \tau_{25}) + st)^2 \]
\[+ (s^2(\tau_{26} + \tau_{36}) - t^2(\tau_{15} + \tau_{25}))(\tau_{17} + \tau_{28} + \tau_{39} + \tau_{4,10})\]
\[+ s^2(\tau_{17}\tau_{28} + \tau_{39}\tau_{4,10}) + t^2(\tau_{28}\tau_{39} + \tau_{17}\tau_{4,10})\]
\[+ u^2(\tau_{17}\tau_{39} + \tau_{28}\tau_{4,10})\]

\[(s\tau_{45} - t\tau_{46})^2 - \tau_{27}(s^2\tau_{45} + t^2\tau_{46})\]
\[\]
The 4-loop plan:
1. Construct 4-loop $\mathcal{N}=4$ sYM 4-point amplitude in BCJ form
   see J.J. Carrasco’s talk
2. Square numerators $\rightarrow$ candidate $\mathcal{N}=8$ amplitude
   (address subtleties - certain sYM contributions “square” to zero: $\frac{0}{0}$ vs. $\frac{0^2}{0}$)
3. Check that it is indeed the $\mathcal{N}=8$ amplitude (it works out)
4. Analyze the result; extract UV divergences; etc

Some features: Bern, Carrasco, Dixon, Johansson, RR (to appear)
1. Expressed in terms of 82 integrals (85 in sYM; 3 “nilpotent” contrib’s)
   19 different numerators (up to signs)
2. Manifest power counting; finite in $d = 5$ by inspection
3. Similar to the 3-loop amplitude it contains 3-point sub-amplitudes
Three classes of terms; examples (a factor of $stuM_{4}^{(0)}$ is stripped off)

13-propagator integrals:

$$N_{12}^{sYM} = \frac{1}{2} s(\tau_{16} - \tau_{26} - \tau_{35} + \tau_{45} + 2\tau_{56} + 2t) - 2(4\tau_{16}\tau_{25} + 4\tau_{15}\tau_{26} + \tau_{45}(\tau_{36} - 3\tau_{46}) + \tau_{35}(\tau_{46} - 3\tau_{36})))$$

$$N_{12}^{sugra} = \left[\frac{1}{2} s(\tau_{16} - \tau_{26} - \tau_{35} + \tau_{45} + 2\tau_{56} + 2t) - 2(4\tau_{16}\tau_{25} + 4\tau_{15}\tau_{26} + \tau_{45}(\tau_{36} - 3\tau_{46}) + \tau_{35}(\tau_{46} - 3\tau_{36})))\right]^2$$

12-propagator integrals:

$$N_{66}^{sYM} = s(4t(\tau_{35} - 2\tau_{36}) + 2u(\tau_{35} + 3\tau_{45} - 4\tau_{46}) - s(6u + \tau_{15} - 6t + 5\tau_{25} - 8\tau_{26}))$$

$$N_{66}^{sugra} = \left[s(4t(\tau_{35} - 2\tau_{36}) + 2u(\tau_{35} + 3\tau_{45} - 4\tau_{46}) - s(6u + \tau_{15} - 6t + 5\tau_{25} - 8\tau_{26}))\right]^2$$

11-propagator integrals:

$$N_{80}^{sYM} = 16s^2(u - t)$$

$$N_{80}^{sugra} = \left[16s^2(u - t)\right]^2$$

• Note mechanism for manifest powercounting

• in sYM: only 11-propagator integrals are divergent in $d=11/2$
In supergravity, most integrals are divergent in $d=11/2$
- leaves open the possibility of additional magic

Extract UV divergences in $d=11/2$ (in general in dimensions with log divergences)
- expand at small external momenta
- use Lorentz-invariance to reorganize tensor integrals

2-tensors
$$l_i^{\mu_i} l_j^{\mu_j} \mapsto \frac{1}{d} \eta^{\mu_i \mu_j} l_i \cdot l_j$$

4-tensors
$$l_i^{\mu_i} l_j^{\mu_j} l_k^{\mu_k} l_l^{\mu_l} \mapsto \frac{1}{(d-1)d(d+2)} (A \eta^{\mu_i \mu_j} \eta^{\mu_k \mu_l} + B \eta^{\mu_i \mu_k} \eta^{\mu_j \mu_l} + C \eta^{\mu_i \mu_l} \eta^{\mu_j \mu_k})$$

$$A = (d + 1) l_i \cdot l_j l_k \cdot l_l - l_i \cdot l_k l_j \cdot l_l - l_i \cdot l_l l_j \cdot l_k$$
$$B = -l_i \cdot l_j l_k \cdot l_l + (d + 1) l_i \cdot l_k l_j \cdot l_l - l_i \cdot l_l l_j \cdot l_k$$
$$C = -l_i \cdot l_j l_k \cdot l_l - l_i \cdot l_k l_j \cdot l_l + (d + 1) l_i \cdot l_l l_j \cdot l_k$$
In supergravity, most integrals are divergent in $d=11/2$
  - leaves open the possibility of additional magic

Extract UV divergences in $d=11/2$ (in general in dimension with $1^{st}$ log divergence)
  - expand at small external momenta
  - use Lorentz-invariance to reorganize tensor integrals
  - do the permutation sum

. external momentum dependence factorizes as

$$M_4^{(4)} \leftrightarrow stuM_4^{(0)}(s^2 + t^2 + u^2)^2$$
(vacuum integrals)

The examples:

$$I_{12}^{\text{sugra}} \leftrightarrow 4 (s^2 + t^2 + u^2)^2 \left( \frac{d^3 - 19d^2 + 146d - 96}{(d - 1)d(d + 2)} + \frac{17d - 25}{(d - 1)d(d + 2)} \right)$$

$$I_{66}^{\text{sugra}} \leftrightarrow \frac{16}{d} (s^2 + t^2 + u^2)^2 (17 - 64)$$

$$I_{80}^{\text{sugra}} \leftrightarrow 1024 (s^2 + t^2 + u^2)^2$$

$$\tau_{ab} = 2k_a \cdot k_b$$
In supergravity, most integrals are divergent in \(d=11/2\)
- leaves open the possibility of additional magic

Extract UV divergences in \(d=11/2\) (in general in dimensions with log divergences)
- expand at small external momenta
- use Lorentz-invariance to reorganize tensor integrals
- do the permutation sum
  - external momentum dependence factorizes as
    \[
    M_4^{(4)} \leftrightarrow stuM_4^{(0)}(s^2 + t^2 + u^2)^2
    \]
  - some statistics:
    - 69 vacuum graph topologies
    - 32 scalar integrals
    - 24 2-tensor integrals \((\tau_{ab} \text{ numerator})\)
    - 13 4-tensor integrals \((\tau^2_{ab} \text{ numerator})\)
  - Reducible (though Laporta algorithm) to only 3 vacuum scalar integrals
So the leading UV pole in $d=11/2$ is

$$
\mathcal{M}_4^{(4)}\bigg|_{\text{pole}} = -\frac{23}{8} \left( \frac{\kappa}{2} \right)^{10} stu(s^2 + t^2 + u^2)^2 \mathcal{M}_{\text{tree}}^4
$$

$$
-256 + \frac{2025}{8}
$$

11-propagator integrals; same as in sYM

12- and 13-propagator integrals

and it evaluates to

$$
\mathcal{M}_4^{(4)}\bigg|_{\text{pole}} = -\frac{23}{8} \left( \frac{\kappa}{2} \right)^{10} stu(s^2 + t^2 + u^2)^2 \mathcal{M}_{\text{tree}}^4
\times \left( \frac{30208}{2625} \Gamma^4\left(\frac{3}{4}\right) - \frac{1024}{525} \Gamma^3\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right) + \frac{4}{21} \Gamma\left(\frac{3}{4}\right) \right)
$$

$$
-6.198399226750(2)
$$
So the leading UV pole in $d=11/2$ is

$$
\mathcal{M}_4^{(4)}|_{\text{pole}} = -\frac{23}{8} \left(\frac{\kappa}{2}\right)^{10} stu(s^2 + t^2 + u^2)^2 M_4^{\text{tree}} (\text{same as in sYM})
$$

$$
-256 + \frac{2025}{8} \quad \text{12- and 13-propagator integrals}
$$

$$
\quad \text{11-propagator integrals; same as in sYM}
$$

As for comparison with the single-trace subleading color sYM

$$
\mathcal{A}_4^{(4)}|_{\text{pole}}^{SU(N_c)} = -6 g^{10} \mathcal{K} N_c^2 \left( N_c^2 + 2 \right) \times \left( s (\text{Tr}_{1324} + \text{Tr}_{1423}) + t (\text{Tr}_{1243} + \text{Tr}_{1342}) + u (\text{Tr}_{1234} + \text{Tr}_{1432}) \right)
$$

It seems unlikely that relation is a coincidence; its origin and implications however are not clear; may continue at higher loops
Supergravity in the UV: the status

• Explicit calculations: $D_c = 4 + \frac{6}{L}$ for $L=2,3,4$

• Arguments based on fixed-order calculations for all-loop cancellations of certain dangerous terms

• String dualities reduced to $d=4$ suggest finite eff. act.
  - Certain kind of string-inspired superspace
  - susy Ward identities + duality symmetry

• Power counting: $L=7, d=4$ divergence shows up in higher dimensions at lower loops: $L=5$ in $d=5-1/5$. A 5-loop calculation is needed to settle this issue

Bern, Dixon, RR

Green, Russo, Vanhove

Bjornson, Green;
Beisert, Elvang,
Freedman, Kiermaier,
Morales, Stieberger

in progress: Bern, Carrasco, Dixon, Johansson, RR
A curious feature of $\mathcal{N}=8$ supergravity

Lift in different duality frames:
- 2- vs. 3-form vector field strength

$d=4$ $\mathcal{N}=4$ multiplet

$d=5$ sYM

$d=6$ $(1,1)$ sYM multiplet

$d=6$ $(2,0)$ multiplet

$PSU(2, 2|4)$

$OSp(8^*|4)$

Conjectured absence of quantum violation of scale invariance

Douglas
A curious feature of $\mathcal{N}=8$ supergravity

$\mathcal{N}=4$ multiplet $PSU(2, 2|4)$

$d=4$

$Lip$ in different duality frames: 2- or 3-form vector field strength

$d=6 (1,1)$ sYM multiplet $OSp(8^*|4)$

$d=6 (2,0)$ multiplet

conjectured absence of quantum violation of scale invariance

$d=6 (4,0)$ multiplet $OSp(8^*|8)$

$d=6 (2,2)$ Sugra multiplet

$Lif$ in different duality frames

$d=5$

$sYM$

$d=5$

$d=5$ multiplet $OSp(8^*|4)$

$d=5$

$d=5$

$Lip$ in different duality frames: 2- or 3-form vector field strength

$d=6 (1,1)$ sYM multiplet $OSp(8^*|4)$

$d=6 (2,0)$ multiplet

conjectured absence of quantum violation of scale invariance

$d=6 (4,0)$ multiplet $OSp(8^*|8)$

$d=6 (2,2)$ Sugra multiplet

$Lif$ in different duality frames
Irreducible, positive energy representation of $OSp(8^* | 8)$

<table>
<thead>
<tr>
<th>6D Field</th>
<th>$SU^*(4)_D$</th>
<th>$USp(8)$</th>
<th>4D Decomposition</th>
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<tr>
<td>$\phi^{[ABCD]}$</td>
<td>(0,0,0)</td>
<td>42</td>
<td>$\phi^{[ABCD]}$</td>
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<tr>
<td>$\lambda^{[ABC]}_{\alpha}$</td>
<td>(1,0,0)</td>
<td>48</td>
<td>$\lambda^{[ABC]}<em>{\alpha} \oplus \lambda^{[ABC]}</em>{\alpha}$</td>
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<tr>
<td>$h^{[AB]}_{\alpha\beta}$</td>
<td>(2,0,0)</td>
<td>27</td>
<td>$h^{[AB]}<em>{\alpha\beta} \oplus h^{[AB]}</em>{\alpha\beta} \oplus \partial_{\alpha\beta} \phi^{[AB]}$</td>
</tr>
<tr>
<td>$\psi^A_{(\alpha\beta\gamma)}$</td>
<td>(3,0,0)</td>
<td>8</td>
<td>$\psi^A_{(\alpha\beta\gamma)} \oplus \psi^A_{(\alpha\beta\gamma)} \oplus \partial_{\gamma}(\alpha \lambda^A_{\beta}) \oplus \partial_{\alpha}(\beta \lambda^A_{\gamma})$</td>
</tr>
<tr>
<td>$R_{(\alpha\beta\gamma\delta)}$</td>
<td>(4,0,0)</td>
<td>1</td>
<td>$R_{(\alpha\beta\gamma\delta)} \oplus R_{(\alpha\beta\gamma\delta)} \oplus \partial_{\delta}(\gamma h^0_{\alpha\beta}) \oplus \partial_{\delta}(\gamma h^0_{\alpha\beta}) \oplus \partial_{(\alpha}(\gamma \partial_{\delta)}h^0_{\beta})$</td>
</tr>
</tbody>
</table>
Comments

• Discussed advantages of manifest power-counting presentations of scattering amplitudes

• extracted the pole in $d=11/2$; confirmed critical dimension pattern

• result has unexpected features
  - transcendental part of residue is the same as the residue of the $\mathcal{N}=4$ $1/N^2$-suppressed single-trace terms

• pointed out a curious kinematic similarity between the lift of the $\mathcal{N}=4$ vector multiplet to $d=6$ and that of the $\mathcal{N}=8$ multiplet and the potential importance of the duality frame.

Interesting to explore the existence of an interacting theory

see Y.-t Huang’s talk
What makes the amplitudes of $\sqrt{\mathcal{N}}$ sYM “simple”?
Non-planar amplitudes are simpler than what they could have been and, to some extent, related to their planar counterparts:

**U(1) decoupling:**
- 1-loop sub-leading color i.t.o. leading color
  - combination of box integrals
  - parts of 2-loop 2-trace related to leading color
    - Bern, Rozowsky, Yan; Bern, de Freitas, Dixon

- combination of

Higher loops:
- 3 & 4 loops: 2-trace better in UV than rest

\[ D_c = 4 + \frac{6}{L} \quad \text{vs} \quad D_c = 4 + \frac{8}{L} \]

**Color-kinematic duality:**
- (potential) all-order relations between l. and sub-l. color
- simple and structured expressions

To have a glimpse at the origin of some of these properties...
Analyze QFT-s which share most of the properties of $\mathcal{N} = 4$ sYM

→ Deform it in a controlled way

1. orbifolds

$$\varphi_i^I = R^I J g^{-1} \varphi_i^J g \quad R \in SU(4) \quad g \in SU(4) \subset SU(N)$$

Inheritance principle: Bershadsky, Johansen

Bershadsky, Kakushadze, Vafa

2. the $h$ deformation

$$W = \text{Tr}[\Phi_1[\Phi_2, \Phi_3]] \longrightarrow f(h, N)(\text{Tr}[\Phi_1[\Phi_2, \Phi_3]] + h(\text{Tr}[\Phi_1^3] + \text{Tr}[\Phi_2^3] + \text{Tr}[\Phi_3^3]))$$

Leigh, Strassler

3. the $\beta$ deformation

$$W = \text{Tr}[\Phi_1[\Phi_2, \Phi_3]] \longrightarrow f(\beta, N)\text{Tr}[\Phi_1(e^{i\beta} \Phi_2 \Phi_3 - e^{-i\beta} \Phi_3 \Phi_2)]$$

Leigh, Strassler

<table>
<thead>
<tr>
<th></th>
<th>super-conf.</th>
<th>dual super-conf.</th>
<th>planar integrable</th>
<th>Amp/W.L.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>yes; $N=2, 1, 0$</td>
<td>yes; inherited</td>
<td>yes</td>
<td>quite likely</td>
</tr>
<tr>
<td>2</td>
<td>yes; $N=1$</td>
<td>not (well) known</td>
<td>sometimes</td>
<td>not clear</td>
</tr>
<tr>
<td>3</td>
<td>yes; $N=1, 0$</td>
<td>yes</td>
<td>yes</td>
<td>yes</td>
</tr>
</tbody>
</table>
The supersymmetric $\beta$-deformed $\mathcal{N} = 4$ super-Yang-Mills theory

- the same field content as $\mathcal{N} = 4$ sYM
- real $\beta$: the same planar properties except for supersymmetry
- a pattern for the deformation: Lunin, Maldacena

\[ \hat{\beta}_{ij} = -\hat{\beta}_{ji}, \quad \hat{\beta}_{12} = \hat{\beta}_{23} = \hat{\beta}_{31} = \beta \]

<table>
<thead>
<tr>
<th>$J_{12}$</th>
<th>$J_{34}$</th>
<th>$J_{56}$</th>
<th>$\phi^{14}$</th>
<th>$\phi^{24}$</th>
<th>$\phi^{34}$</th>
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<th>$\psi^2$</th>
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Some consequences:

- most non-commutative results survive; planar amplitudes are inherited in dimensional regularization Filk (space-time noncommutativity); Khoze; ...
- vector U(1) factors decouple; chiral superfield U(1) factors are coupled
- both $f_{abc}$ and $d_{abc}$ couplings
• non-vanishing tree-level double-trace amplitudes
\[ L_{2\text{tr}} = \frac{1}{2N} |f(\beta, N)|^2 \epsilon_{ijk} \epsilon^{ilm} \text{Tr}[[\phi^j, \phi^k]_\beta] \text{Tr}[[\bar{\phi}_l, \bar{\phi}_m]_\beta] \]

=> crucial for finiteness; also \(|f(\beta, N)|^2 = \frac{g_{YM}^2}{1 - \frac{4}{N^2} \sin^2 \beta}\)

With the same planar properties, differences are creeping in at subleading color level in dimensional regularization

• Single-trace amplitudes:
\[ A^{(0)} = \sum_{\rho \in S_n/Z_n} \text{Tr}[T^{a_{\rho(1)}} \ldots T^{a_{\rho(n)}}] A^{(0)}(k_{\rho(1)} \ldots k_{\rho(n)}) \]
\[ A^{(0)}(k_1 \ldots k_n) \rightarrow e^{i\Theta(1, \ldots, n)} A^{(0)}(k_1 \ldots k_n); \quad \Theta(1, \ldots, n) = \sum_{1 \leq i < j \leq n} q_i \cdot \hat{\beta} \cdot q_j \]

- Account for the \(O(1/N^2)\) deformation of the coefficient of the superpotential

Here: focus on double-trace terms; ignore \(O(1/N^2)\) corrections
• Modified color-kinematic-like (BCJ) duality:

\[
\mathcal{A}_4^{\beta,(0)} (1g^+, 2\phi^{23}, 3f^{134}, 4f^{124}) = \frac{n_{12}}{s_{12}} f^{12a} f_\beta^{34} a + \frac{n_{23}}{s_{23}} f_\beta^{23a} f^{14} a + \frac{n_{13}}{s_{13}} f^{31a} f_\beta^{24} a
\]

\[
\mathcal{A}_4^{\beta,(0)} (1\phi^{23}, 2\phi^{14}, 3\phi^{13}, 4\phi^{24}) = \frac{n_{12}}{s_{12}} f^{12a} f^{34} a + \frac{n_{23}}{s_{23}} f^{23a} f^{14} a + \frac{n_{13}}{s_{13}} f_\beta^{31a} f_\beta^{24} a
\]

\[
f_{\beta}^{abc} = \text{Tr}[T^a [T^b, T^c]]_{\beta} = e^{i\Phi(a,b,c)} \text{Tr}[T^a T^b T^c] - e^{i\Phi(a,c,b)} \text{Tr}[T^a T^c T^b]
\]

Numerator factors -- same as in \(\mathcal{N}=4\) sYM:

\[n_{12} + n_{23} + n_{13} = 0\]

Color factors – different; generically no Jacobi identity:

\[f^{[12}_a f^{3]4a} = 0\quad f^{[12}_a d^{3]4a} = 0\]

but no Jacobi-like identity involving only d-structure constants
Some explicit examples 4-point loop amplitudes: 1 loop

- Construct using generalized unitarity
  - use color-dressed cuts
  - supersums: use pictorial rules
    - supersums dressed with the extra phase factors (structure hints at hidden susy)
  - focus on 3 terms: \( \text{Tr}[T^{a_1} T^{a_i}] \text{Tr}[T^{a_j} T^{a_k}] \quad i, j, k = 2, 3, 4 \)

- Classify following the number of vector multiplets
  - 4 vector multiplets: same as in \( \mathcal{N} = 4 \) sYM
  - 3 vector multiplets + 1 chiral multiplet: vanish identically
  - 2 vector multiplets + 2 chiral multiplets: \( A(1_{g^+}, 2_{g^-}, 3_\phi^{34}, 4_\phi^{12}) \)
Some explicit examples 4-point loop amplitudes: 1 loop

- Classify following the number of vector multiplets
  - 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
  - 3 vector multiplets + 1 chiral multiplet: vanish identically
  - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$

$$A_{1234}^{(1)\beta}(\mathcal{N} = 4) = A_{1234}^{(1)\beta}(\mathcal{N} = 4) - 8 \sin^2 \beta (\text{Tr}_{13} \text{Tr}_{24} + \text{Tr}_{14} \text{Tr}_{23}) A_{1234}^{(1)\text{extra}}$$

$$A_{1234}^{(1)\text{extra}} = \frac{\langle 23 \rangle^2}{\langle 13 \rangle^2} \left[ - s_{12} s_{23} \left( \begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \right) + s_{12} \left( \begin{array}{c} 2 \\ 1 \\ 3 \\ 4 \end{array} \right) + s_{23} \left( \begin{array}{c} 3 \\ 2 \\ 4 \\ 1 \end{array} \right) ight]$$

- IR finite
- in $d=4$, expressible in terms of the $d=6$ box integral
- UV divergent in 6 dimensions; standard expectation for a conformal $\mathcal{N} = 1$ theory
Some explicit examples 4-point loop amplitudes: 1 loop

- Classify following the number of vector multiplets

  - 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
  - 3 vector multiplets + 1 chiral multiplet: vanish identically
  - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$
  - 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$

\[
A^{(1)2tr}(1234) = \cos \beta A^{(1)2tr}_{\mathcal{N}=4} (1234) (\text{Tr}_{12} \text{Tr}_{34} + \text{Tr}_{13} \text{Tr}_{24} + \text{Tr}_{14} \text{Tr}_{23})
\]

\[
A^{(1)2tr}_{\mathcal{N}=4} = -2 s_{12} s_{23} \frac{[23][34]}{[12][13]} \left( \begin{array}{c}
\begin{array}{cccc}
2 & 3 & 4 & 1 \\
1 & 4 & 3 & 2
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{cccc}
2 & 3 & 1 & 4 \\
3 & 1 & 4 & 2
\end{array}
\end{array} + \begin{array}{c}
\begin{array}{cccc}
2 & 1 & 4 & 3 \\
4 & 3 & 2 & 1
\end{array}
\end{array} \right)
\]
Some explicit examples 4-point loop amplitudes: 1 loop

• Classify following the number of vector multiplets

• 4 vector multiplets: same as in \( \mathcal{N} = 4 \text{ sYM} \)
• 3 vector multiplets + 1 chiral multiplet: vanish identically
• 2 vector multiplets + 2 chiral multiplets: \( A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}}) \)
• 1 vector multiplet + 3 chiral multiplets: \( A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2}) \)
• 4 chiral multiplets: \( A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^1}, 4_{\psi^1}), A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi^1}, 4_{\psi^2}) \)

\[
A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^1}, 4_{\psi^1})^{(1)\beta}_4 = A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^1}, 4_{\psi^1})^{(1)\mathcal{N}=4}_4 - 8 \sin^2 \beta \text{Tr}_{12} \text{Tr}_{34} A(1234)^{(1)\text{extra}}
\]

\[
\frac{A(1234)^{(1)\text{extra}}}{\cos^2 \beta} = \frac{\langle 34 \rangle}{\langle 12 \rangle} \left[ s_{13}s_{14} - s_{13}(\begin{array}{c}
\begin{array}{cc}
1 & 4 \\
3 & 2 \\
\end{array}
\end{array}) - s_{14}(\begin{array}{c}
\begin{array}{cc}
1 & 2 \\
3 & 4 \\
\end{array}
\end{array}) + \right]
\]

\[
= - \frac{\langle 34 \rangle}{\langle 12 \rangle} \frac{G[l, 1, 4, 2]}{s_{13}s_{14}} \left[ \begin{array}{c}
\begin{array}{cc}
1 & 4 \\
3 & 2 \\
\end{array}
\end{array}\right]
\]
Some explicit examples 4-point loop amplitudes: 1 loop

- Classify following the number of vector multiplets

  - 4 vector multiplets: same as in $\mathcal{N} = 4$ sYM
  - 3 vector multiplets + 1 chiral multiplet: vanish identically
  - 2 vector multiplets + 2 chiral multiplets: $A(1_{g^+}, 2_{g^-}, 3_{\phi^{34}}, 4_{\phi^{12}})$
  - 1 vector multiplet + 3 chiral multiplets: $A(1_{g^-}, 2_{\phi^{34}}, 3_{\psi^1}, 4_{\psi^2})$
  - 4 chiral multiplets: $A(1_{\psi^1}, 2_{\psi^1}, 3_{\psi^1}, 4_{\psi_2}), A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi^1}, 4_{\psi_2})$

\[
A(1_{\psi^1}, 2_{\psi^2}, 3_{\psi^1}, 4_{\psi_2})^{(1)\beta} = A_4^{(1)\mathcal{N} = 4} - 8 \sin^2 \beta \operatorname{Tr}_{12} \operatorname{Tr}_{34} A(1234)_{12;34}^{(1)\text{extra}} - 8 \sin^2 \beta \operatorname{Tr}_{14} \operatorname{Tr}_{23} A(1234)_{14;23}^{(1)\text{extra}}
\]

\[
A(1234)_{12;34}^{(1)\text{extra}} = \cos^2 \beta \frac{\langle 34 \rangle}{\langle 12 \rangle} \left( \frac{1}{2} s_{12}s_{13} - s_{12} \left( \frac{1}{2} s_{12}s_{13} - s_{12}s_{14} - s_{12}s_{13} \right) \right)
\]

\[
A(1234)_{14;23}^{(1)\text{extra}} = \frac{\langle 34 \rangle}{\langle 12 \rangle} \left[ \cos^2 \beta \frac{G[l, 1, 2, 4]}{2s_{12}s_{13}} \left( \frac{1}{2} s_{12}s_{13} - s_{12}s_{14} - s_{12}s_{13} \right) \right]
\]
Some comments:

• results consistent with expected structure of IR divergences
  -- most corrections are in fact IR-finite; consistent with structure of IR div’s
  -- only small changes in the soft anomalous dimension matrix

• no real improvement over a finite “garden variety” $\mathcal{N} = 1$ theory
  -- except perhaps absence of incomplete cancellations (of bubbles)

• some details are as if there were more than $\mathcal{N} = 1$ susy
  -- supersums are perfect squares, characteristic to $\mathcal{N} = 2$
  -- persists at higher loops

• no immediate manifestation of modified color-kinematics relations
More explicit examples 4-point loop amplitudes: 2 loops

**Same classification:**

- 2 vector multiplets + 2 chiral multiplets: \( A(1_{g+}, 2_{g-}, 3_{\phi^{34}}, 4_{\phi^{12}}) \)

\[
A(1234)_{4, 2}^{(2)\beta} = A(1234)_{4, 2}^{(2)N=4} - 8 \sin^2 \beta \text{Tr}_{13} \text{Tr}_{24} A_{13;24}^{(2)\text{extra}} - 8 \sin^2 \beta \text{Tr}_{14} \text{Tr}_{23} A_{14;23}^{(2)\text{extra}}
\]

\( \alpha \)

\[ C \quad U \]  
Symmetries of \( A_{14;23}^{(2)\text{extra}} \)

- \( C : (1 \leftrightarrow 2, 3 \leftrightarrow 4) \)
- \( U : (1 \leftrightarrow 4, 2 \leftrightarrow 3) \)

\( U' \) \quad \( A_{14;23}^{(2)\text{extra}} = \sum \alpha_i I_i \)  
\quad + (1 + C) \sum \beta_i J_i 
\quad + (1 + U) \sum \gamma_i K_i 
\quad + (1 + U)(1 + C) \sum \delta_i L_i
\begin{align*}
\alpha_1 &= \tau_{1,4} \left( \tau_{1,8}^2 + \tau_{2,5}^2 + \tau_{4,7}^2 + \tau_{3,6}^2 + \tau_{1,2}(\tau_{1,8} + \tau_{2,5} + \tau_{4,7} + \tau_{3,6} - 2\tau_{1,4}) - \tau_{1,8}\tau_{2,5} - \tau_{4,7}\tau_{3,6} \right) \\
\alpha_2 &= \tau_{1,4} \left( \tau_{1,8}^2 + \tau_{2,6}^2 + \tau_{4,7}^2 + \tau_{3,5}^2 + \tau_{1,2}(\tau_{1,8} + \tau_{2,6} + \tau_{4,7} + \tau_{3,5}) + \tau_{1,8}\tau_{2,6} + \tau_{4,7}\tau_{3,5} \right) \\
\alpha_3 &= -4\tau_{1,3}\tau_{1,4} \\
\beta_1 &= 2\tau_{1,4} \left( \tau_{1,8}^2 + \tau_{4,7}^2 + \tau_{1,3}(\tau_{1,8} + \tau_{4,7}) \right) \\
\beta_2 &= -2\tau_{1,4}^2 \\
\beta_3 &= 2\tau_{1,4}^2 \\
\beta_4 &= 2\tau_{1,2} \\
\beta_5 &= (\tau_{1,4} - 2\tau_{1,2}) \\
\gamma_1 &= \tau_{1,2} \left( \tau_{1,5}^2 + \tau_{2,6}^2 + \tau_{1,2}(\tau_{1,5} + \tau_{2,6}) \right) \\
\gamma_2 &= 2\tau_{1,2}^2 \\
\gamma_3 &= 2\tau_{1,2} \\
\gamma_4 &= (\tau_{1,3} - \tau_{1,2}) \\
\delta_1 &= \tau_{1,4}^2 \\
\delta_2 &= -\tau_{1,4}\tau_{2,5} + \frac{1}{2}\tau_{12}(3\tau_{1,3} + \tau_{2,5} - 2\tau_{3,5} + 2\tau_{2,7})
\end{align*}
- The other trace structure $A_{13;24}^{(2)\text{extra}}$: similar structure with $A_{14;23}^{(2)\text{extra}}$ with a few twists
  - planar double-boxes are absent
  - additional symmetries:
    - $C : (1 \leftrightarrow 2, 3 \leftrightarrow 4)$
    - $U : (1 \leftrightarrow 3, 2 \leftrightarrow 4)$
    - $E : (1 \leftrightarrow 3)$

$$A_{13;24}^{(2)\text{extra}} = (1 + C) \sum_i \beta_i J'_i + (1 + U) \sum_i \gamma_i K'_i$$

$$+ (1 + U)(1 + C) \sum_i \delta_i L'_i + (1 + U)(1 + C)(1 + E) \sum_i \epsilon_i M'_i$$
More comments

• Despite extensive planar similarity with $\mathcal{N} = 4$ sYM and tree-level numerator relations, the $\beta$-deformed theory does not show much simplicity at the non-planar level.

• $\beta$-deformed theory seems to exhibit all the features of a typical finite $\mathcal{N} = 1$ theory.

• Calculations suggest that details of the theory are crucial for transferring planar info to non-planar level.

• Disentangle effects of reduced supersymmetry and $d_{abc}$?
  ($d_{abc}$ is also an obstruction for constructing a gravity theory via KLT relations.)
Recap

• constructed one for $\mathcal{N}=8$
• extracted the pole in $d=11/2$; divergence is indeed present there
• result has unexpected features
  - transcendental part of residue is the same as the residue of the $\mathcal{N}=4$ $1/N^2$-suppressed single-trace terms

• a 5-loop calculation will test these observations as well as the 7-loop UV behavior of $\mathcal{N}=8$ supergravity in $d=4$

• pointed out a curious kinematic similarity between the lift of the $\mathcal{N}=4$ vector multiplet to $d=6$ and that of the $\mathcal{N}=8$ multiplet and the potential importance of the duality frame

• In theories with several multiplets, the details of the interactions between multiplets are crucial for transferring planar simplicity to non-planar level via color/kinematics duality

• a better picture: disentangle effects of $d_{abc}$
Extra slides
Expect similar features at 4 loops

The plan:
1. Construct 4-loop N=4 sYM 4-point amplitude in BCJ form
2. Square numerators $\rightarrow$ candidate N=8 amplitude (address subtleties)
3. Check that it is indeed the N=8 amplitude (it works out)
4. Analyze the result; extract UV divergences; etc

To anticipate:
1. extensive cancellations between individual diagrams despite apparent perfect square integrands (Minkowski signature is important)
2. approx. 69 vac. int’s irreducible through momentum conservation
3. 3 master integrals
\[ V_1 = \frac{1}{(4\pi)^{11}} \epsilon \left[ \frac{512}{5} \Gamma^4\left(\frac{3}{4}\right) - \frac{2048}{105} \Gamma^3\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right) \right] + \mathcal{O}(1) \]

\[ V_2 = \frac{1}{(4\pi)^{11}} \epsilon \left[ -\frac{4352}{105} \Gamma^4\left(\frac{3}{4}\right) + \frac{832}{105} \Gamma^3\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right) \right] + \mathcal{O}(1) \]

\[ V_8 = \frac{1}{(4\pi)^{11}} \frac{4}{21} \frac{1}{\Gamma\left(\frac{3}{4}\right)} \frac{1}{\epsilon} \left[ -\frac{5248}{125} \Gamma^5\left(\frac{3}{4}\right) + \frac{224}{25} \Gamma^4\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{2}\right) \Gamma\left(\frac{1}{4}\right) + 2 \text{NO}_m \right] + \mathcal{O}(1) \]
Nonplanar amplitudes of $\mathcal{N} = 4$ super-Yang-Mills theory are not directly constrained properties of planar amplitudes:

- dual super-conformal invariance
- integrableness of planar dilatation operator
- amplitudes / Wilson loops relation
- amplitudes / correlation function relation
\[ \beta_1 = 2\tau_{1,3} \left( \tau_{2,5}^2 + \tau_{4,6}^2 - \tau_{1,3}(\tau_{2,5} + \tau_{4,6}) \right) \]

\[ \beta_2 = -2\tau_{1,2} \]

\[ \beta_3 = 2\tau_{1,4} - 2\tau_{1,3} \]

\[ \gamma_1 = \tau_{1,2} \left( \tau_{4,5}^2 + \tau_{3,6}^2 + \tau_{1,3}(\tau_{4,5} + \tau_{3,6}) \right) \]

\[ \gamma_2 = \tau_{1,2}^2 \]

\[ \gamma_3 = 2\tau_{1,3} \]

\[ \delta_1 = \frac{1}{2} \tau_{1,4}(2\tau_{1,7} + \tau_{2,7} + \tau_{1,4}) + \frac{1}{2} \tau_{1,2}(4\tau_{1,4} - 3\tau_{1,2} + 2\tau_{4,7} + \tau_{2,5}) \]

\[ \epsilon_1 = 2\tau_{1,3}\tau_{2,5} \]