

Toward an AdS/cold atom correspondence

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Plan

- Unitarity regime
- Schrödinger symmetry: nonrelativistic conformal invariance
- Geometric realization of Schrödinger symmetry
- Green's functions in vacuum
- Many-body physics
- Conclusion

Collaborators: Y. Nishida, M. Rangamani, S. Ross, E. Thompson

Refs.: [Y. Nishida, DTS, 0706.3746 \(PRD\)](#)
[Balasubramanian and McGreevy, 0804.4053 \(PRL 2008\)](#)
[DTS, 0804.3972 \(PRD 2008\)](#)
[Rangamani, Ross, DTS, Thompson, arXiv:0811.2049](#)

Viewpoint

Glimmers of a connection between string theory and atomic physics

By **Shamit Kachru**

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Results from string theory, generalizing the anti-de Sitter/conformal field theory correspondence, may offer a fresh set of mathematical tools for understanding some kinds of phase transitions that occur in cold atomic systems.

Subject Areas: **Atomic and Molecular Physics, String theory**

A Viewpoint on:

Toward an AdS/cold atoms correspondence: A geometric realization of the Schrödinger symmetry

D. T. Son

Phys. Rev. D **78**, 046003 (2008) – Published August 4, 2008

Gravity Duals for Nonrelativistic Conformal Field Theories

Koushik Balasubramanian and John McGreevy

Phys. Rev. Lett. **101**, 061601 (2008) – Published August 4, 2008

String theory is well known as a possible route to unification of gravity and quantum mechanics. But its historical roots lie in another direction: strings were originally proposed as a way to describe the strong force that governs quarks—called quantum chromodynam-

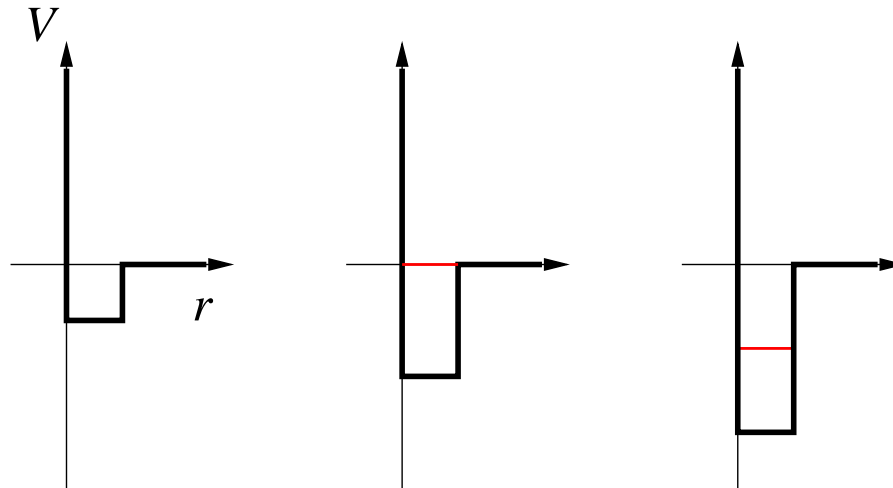
involving “fundamental” vs “effective” strings, have dovetailed nicely: it has turned out that certain special solutions of fundamental string theory provide effective string descriptions of normal field theories, similar in many ways to QCD.

Unitarity regime

Consider a BCS-BEC crossover.

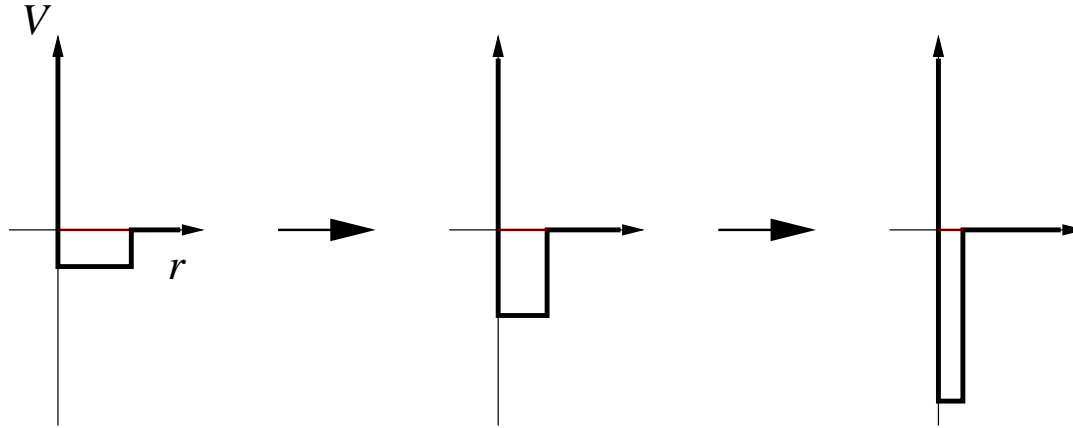
Assume a square-well potential, with fixed but small range r_0 .

- $V_0 < 1/mr_0^2$: no bound state
- $V_0 = 1/mr_0^2$: one bound state appears, at first with zero energy
- $V_0 > 1/mr_0^2$: at least one bound state



Unitarity regime (II)

Unitarity regime: take $r_0 \rightarrow 0$, keeping one bound state at zero energy.



In this limit: no intrinsic scale associated with the potential

In the language of scattering theory: infinite scattering length $a \rightarrow \infty$

s -wave scattering cross section saturates unitarity

Boundary condition interpretation

Unitarity: taking Hamiltonian to be free:

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m}$$

but imposing nontrivial boundary condition on the wavefunction:

$$\Psi(\underbrace{\mathbf{x}_1, \mathbf{x}_2, \dots}_{\text{spin-up}}, \underbrace{\mathbf{y}_1, \mathbf{y}_2, \dots}_{\text{spin=down}})$$

When $|\mathbf{x}_i - \mathbf{y}_j| \rightarrow 0$:

$$\Psi \rightarrow \frac{C}{|\mathbf{x}_i - \mathbf{y}_j|} + 0 \times |\mathbf{x}_i - \mathbf{y}_j|^0 + O(|\mathbf{x}_i - \mathbf{y}_j|)$$

Free gas corresponds to

$$\Psi \rightarrow \frac{0}{|\mathbf{x}_i - \mathbf{y}_j|} + C + O(|\mathbf{x}_i - \mathbf{y}_j|)$$

Field theory interpretation

Consider the following model

Sachdev, Nikolic; Nishida, DTS

$$S = \int dt d^d x \left(i\psi^\dagger \partial_t \psi - \frac{1}{2m} |\nabla \psi|^2 - c_0 \psi_\uparrow^\dagger \psi_\downarrow^\dagger \psi_\downarrow \psi_\uparrow \right)$$

Dimensional analysis:

$$[t] = -2, \quad [x] = -1, \quad [\psi] = \frac{d}{2}, \quad [c_0] = 2 - d$$

Contact interaction is irrelevant at $d > 2$

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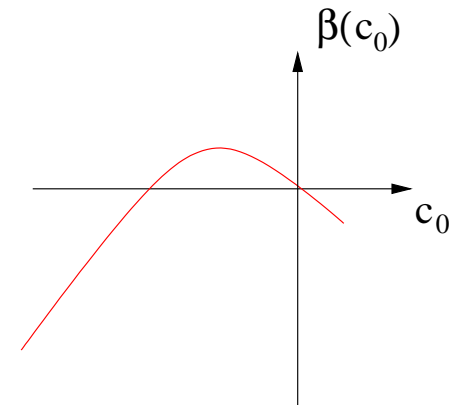
Contact interaction is irrelevant at $d > 2$

RG equation in $d = 2 + \epsilon$:

$$\frac{\partial c_0}{\partial s} = -\epsilon c_0 - \frac{c_0^2}{2\pi}$$

Two fixed points:

- $c_0 = 0$: trivial, noninteracting
- $c_0 = -2\pi\epsilon$: unitarity regime



Field theory in $d = 4 - \epsilon$ dimensions

Sachdev, Nikolic; Nishida, DTS; Nussinov and Nussinov

$$S = \int dt d^d x \left(i\psi^\dagger \partial_t \psi - \frac{1}{2m} |\nabla \psi|^2 - g\phi\psi_\uparrow^\dagger \psi_\downarrow^\dagger - g\phi^* \psi_\downarrow \psi_\uparrow + i\phi^* \partial_t \phi - \frac{1}{4m} |\nabla \phi|^2 + C\phi^* \phi \right)$$

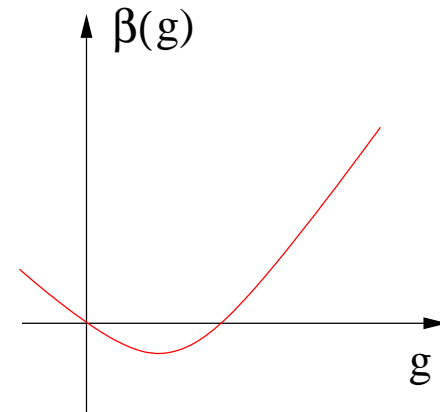
C finely tuned to criticality

Dimensions: $[g] = \frac{1}{2}(4 - d) = \frac{1}{2}\epsilon$

RG equation for g :

$$\frac{\partial g}{\partial \ln \mu} = -\frac{\epsilon}{2}g + \frac{g^3}{16\pi^2}$$

Fixed point at $g^2 = 8\pi^2\epsilon$



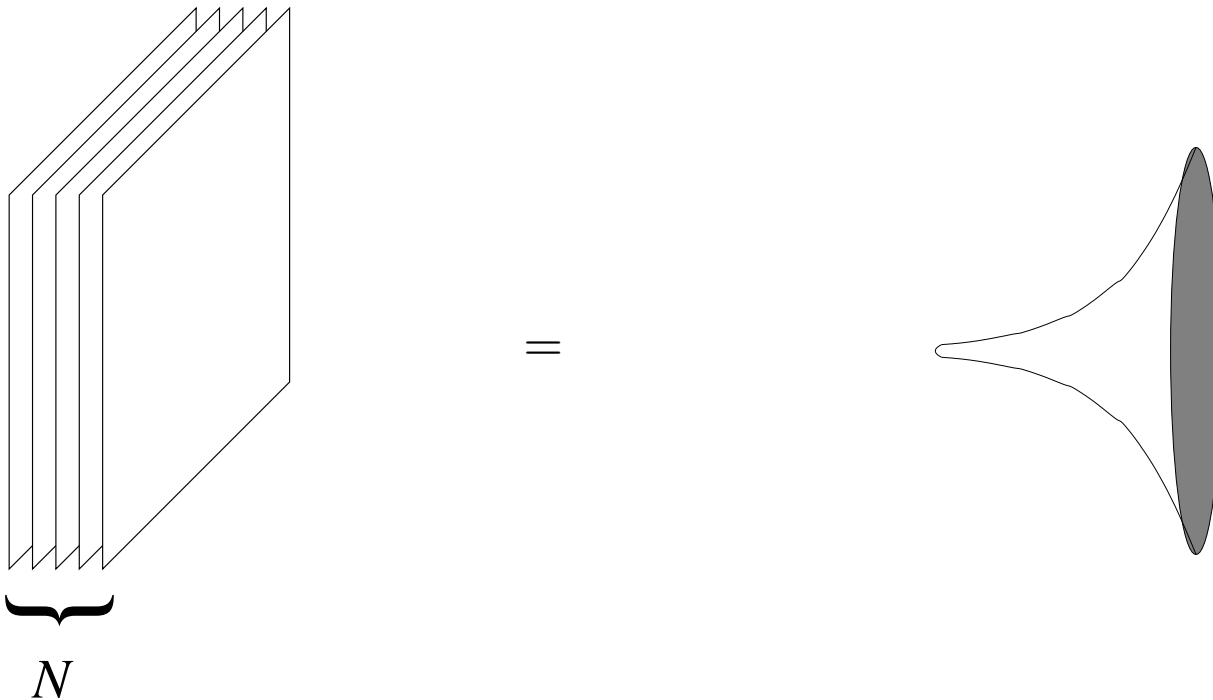
AdS/CFT

Maldacena : stack of N D3-branes in type IIB string theory can be described in two different pictures:

As a quantum field theory describing fluctuations of the branes: $\mathcal{N} = 4$ super-Yang-Mills theory

As string theory on a curved spacetime called $\text{AdS}_5 \times \text{S}^5$

$$ds^2 = \frac{r^2}{R^2} (-dt^2 + d\mathbf{x}^2) + \frac{R^2}{r^2} dr^2 + R^2 d\Omega_5^2$$



Applied string theory

AdS/CFT \rightarrow solution of $\mathcal{N} = 4$ SYM theory (and its cousins) at large N , large 't Hooft coupling λ limit

- Free energy at strong coupling

$$P(T)|_{\lambda \rightarrow \infty} = \frac{3}{4} P(T)_{\lambda \rightarrow 0}$$

- Shear viscosity: computable without kinetic theory

$$\frac{\eta}{s} = \frac{\hbar}{4\pi}$$

- Suggested modification of relativistic hydrodynamics in the presence of triangle anomalies:

$$j^\mu = \rho u^\mu + \text{diffusion} + \xi \epsilon^{\mu\nu\alpha\beta} u_\nu \partial_\alpha u_\beta$$

Chiral separation in rotating relativistic fluids

Analogy with AdS/CFT

Bertsch parameter:

$$\epsilon(n) = \xi \epsilon_{\text{free}}(n)$$

Current estimate: $\xi \approx 0.4$

Similar to pressure in $\mathcal{N} = 4$ SYM theory

$$P(T)|_{\lambda \rightarrow \infty} = \frac{3}{4} P(T)|_{\lambda \rightarrow 0}$$

Small viscosity at $T \gtrsim T_c$ Thomas, Schäfer

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Is there an useful AdS/CFT-type duality for unitarity Fermi gas?

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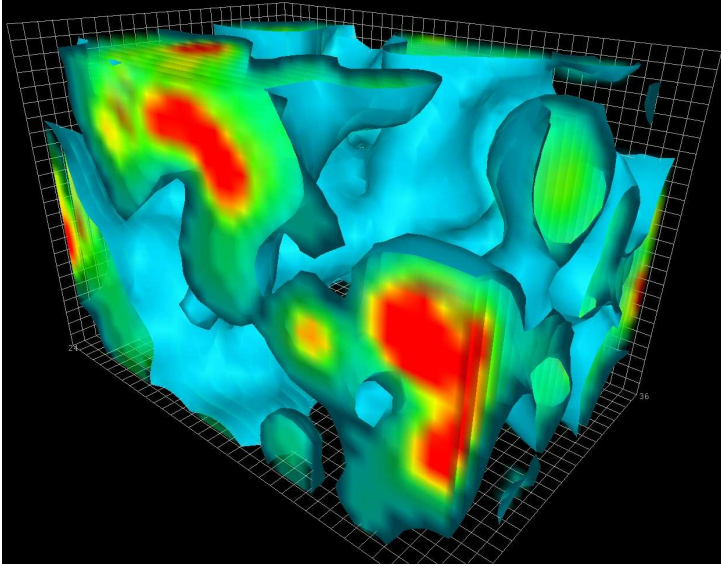
Is there an useful AdS/CFT-type duality for unitarity Fermi gas?

As in $\mathcal{N} = 4$ SYM, perhaps we should start with the vacuum (zero temperature and density)

- More symmetries
- T and μ can (hopefully) be accommodated later (μ can be treated as a spurion field)

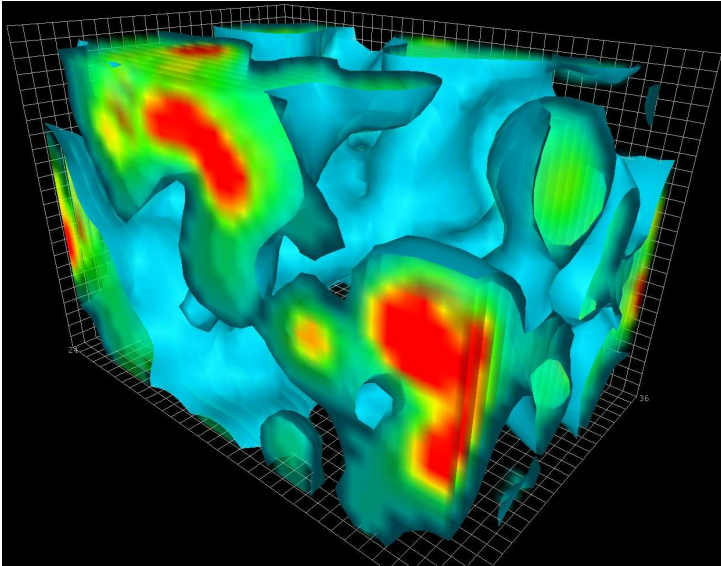
Vacuum

Relativistic vacuum



Vacuum

Relativistic vacuum

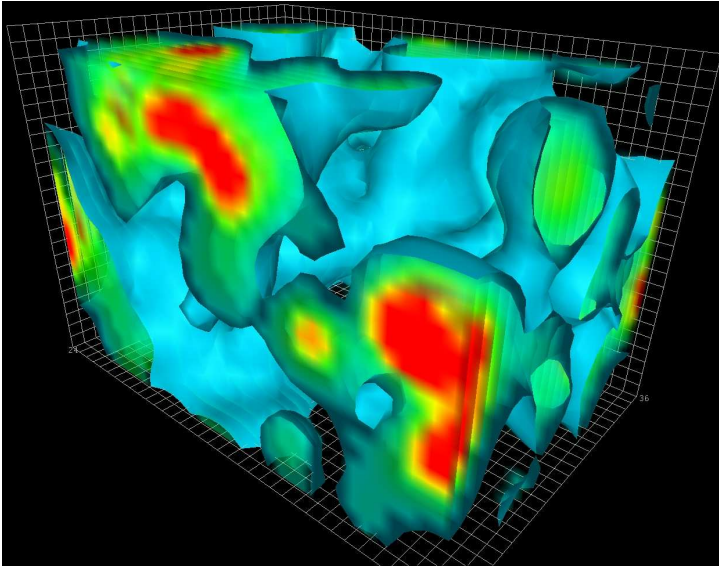


Nonrelativistic vacuum



Vacuum

Relativistic vacuum



Nonrelativistic vacuum



Nonrelativistic vacuum is much simpler (no particle-hole pair creation)

Still: nontrivial conformal dimensions and correlation functions.

AdS/CFT correspondence

$\mathcal{N} = 4$ super-Yang-Mills theory \Leftrightarrow type IIB string theory on $\text{AdS}_5 \times \text{S}^5$.

First evidence: matching of symmetries $\text{SO}(4, 2) \times \text{SO}(6)$.

- $\text{SO}(4, 2)$: conformal symmetry of 4-dim theories (CFT_4), isometry of AdS_5
- $\text{SO}(6)$: $\sim \text{SU}(4)$ is the R-symmetry of $\mathcal{N} = 4$ SYM, isometry of S^5 .

Conformal algebra: $P^\mu, M^{\mu\nu}, K^\mu, D$

$$[D, P^\mu] = -iP^\mu, \quad [D, K^\mu] = iK^\mu$$

$$[P^\mu, K^\nu] = -2i(g^{\mu\nu} D + M^{\mu\nu})$$

Schrödinger algebra

Nonrelativistic field theory is invariant under:

- Phase rotation M : $\psi \rightarrow \psi e^{i\alpha}$
- Time and space translations, H and P^i
- Rotations M^{ij}
- Galilean boosts K^i
- Dilatation D : $\mathbf{x} \rightarrow \lambda \mathbf{x}$, $t \rightarrow \lambda^2 t$
- Conformal transformation C :

$$\mathbf{x} \rightarrow \frac{\mathbf{x}}{1 - \lambda t}, \quad t \rightarrow \frac{t}{1 - \lambda t}, \quad \psi \rightarrow \dots$$

$$[D, P^i] = -iP^i, \quad [D, K^i] = iK^i, \quad [P^i, K^j] = -\delta^{ij} M$$

$$[D, H] = -2iH, \quad [D, C] = 2iC, \quad [H, C] = iD$$

D, H, C form a $\text{SO}(2,1)$

Chemical potential $\mu\psi^\dagger\psi$ breaks the symmetry.

Consequences of Schrödinger symmetry

Long-distance behavior at $T = 0$ parametrized by ξ

First corrections to hydrodynamics parametrized by c_1, c_2 DTS, Wingate

$$\text{phonon dispersion: } \omega(q) = c_s q \left[1 - \pi^2 \sqrt{2\xi} \left(c_1 + \frac{3}{2} c_2 \right) \frac{q^2}{k_F^2} \right]$$

Static response functions:

$$\chi(q) = -\frac{mk_F}{\pi^2 \xi} \left[1 + 2\pi^2 \sqrt{2\xi} \left(c_1 - \frac{9}{2} c_2 \right) \frac{q^2}{k_F^2} \right]$$

$$\chi^T(q) = -9c_2 \sqrt{\frac{\xi}{2}} v_F q^2$$

Total energy of N particles in a harmonic potential:

$$E = \frac{\sqrt{\xi}}{4} \bar{\omega} (3N)^{4/3} - \sqrt{2} \pi^2 \xi \left(c_1 - \frac{9}{2} c_2 \right) \frac{\omega_1^2 + \omega_2^2 + \omega_3^2}{\bar{\omega}} (3N)^{2/3}$$

$$\bar{\omega} = (\omega_1 \omega_2 \omega_3)^{1/3}$$

Generators

$$M = \int d\mathbf{x} n(\mathbf{x}), \quad P_i = \int d\mathbf{x} j_i(\mathbf{x})$$

$$K_i = \int d\mathbf{x} x_i n(\mathbf{x}), \quad C = \frac{1}{2} \int d\mathbf{x} x^2 n(x), \quad D = - \int d\mathbf{x} x_i j_i(\mathbf{x})$$

$H \rightarrow H + \omega^2 C$: putting the system in an external potential $V(\mathbf{x}) = \frac{1}{2}\omega^2 x^2$.

Operator \mathcal{O} has dimension Δ if $[D, \mathcal{O}(0)] = -i\Delta\mathcal{O}(0)$

$[D, K_i] = iK_i$: K_i lowers dimension by 1

$[D, C] = 2iC$: C lowers dimension by 2

Define primary operators: $[K_i, \mathcal{O}(0)] = [C, \mathcal{O}(0)] = 0$

Examples of primary operator for free fermions:

$\psi_\uparrow\psi_\downarrow, \psi_\uparrow\partial_i\psi_\downarrow - \partial_i\psi_\uparrow\psi_\downarrow$, but not $\partial_i(\psi_\uparrow\psi_\downarrow)$

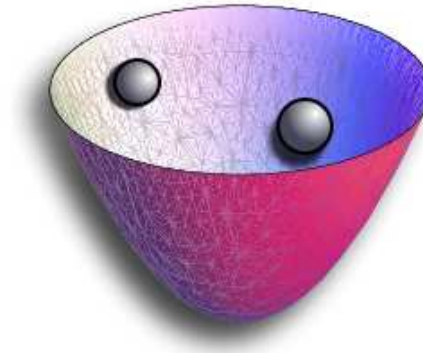
Operator-state correspondence

Nishida, DTS

Primary operator with dimension Δ \iff eigenstate in harmonic potential with energy $\Delta \times \hbar\omega$

$\psi_{\uparrow}\psi_{\downarrow}$

\iff



cf. Werner, Castin; Tan

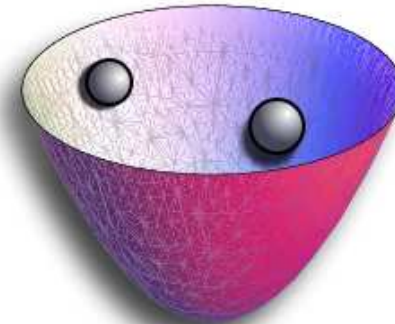
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cf. Werner, Castin; Tan

Proof: let $\omega = 1$, $H_{\text{osc}} = H + C$; Define $|\Psi_{\mathcal{O}}\rangle = e^{-H} \mathcal{O}^{\dagger}(0)|0\rangle$

From Schrödinger algebra one finds $e^H H_{\text{osc}} e^{-H} = C + iD$

from $[C, \mathcal{O}^{\dagger}(0)] = 0$, $[D, \mathcal{O}^{\dagger}(0)] = -i\Delta_{\mathcal{O}}$, and $C|0\rangle = D|0\rangle = 0$:

$$H_{\text{osc}}|\Psi_{\mathcal{O}}\rangle = \Delta_{\mathcal{O}}|\Psi_{\mathcal{O}}\rangle$$

Example

Two particles in harmonic potential: ground state with unitarity boundary condition can be found exactly

$$\Psi(\mathbf{x}, \mathbf{y}) = \frac{e^{-(x^2+y^2)/2}}{|\mathbf{x} - \mathbf{y}|}, \quad E_0 = 2\hbar\omega$$

→ Dimension of operator $O_2 = \psi_\uparrow\psi_\downarrow$ is 2 (naively 3)

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Can be see explicitly: two-particle wavefunctions behave as

$$\Psi(\mathbf{x}, \mathbf{y}) \sim \frac{1}{|\mathbf{x} - \mathbf{y}|}, \quad x \rightarrow y$$

the operator O_2 has to be regularized as

$$O_2(\mathbf{x}) = \lim_{\mathbf{x} \rightarrow \mathbf{y}} |\mathbf{x} - \mathbf{y}| \psi_\uparrow(\mathbf{x}) \psi_\downarrow(\mathbf{y})$$

so that $\langle 0 | O_2 | \Psi(\mathbf{x}, \mathbf{y}) \rangle$ is finite.

Lowest 3-body operators: $\Delta_{l=1} = 4.27272\dots$, $\Delta_{l=0} = 4.66622\dots$

Relation to Shina Tan's contact

Shina Tan has derived several exact relations, for example between total energy and occupation number:

$$E = \sum_{\sigma} \int \frac{d\mathbf{p}}{(2\pi)^3} \frac{p^2}{2m} \left(n_{\mathbf{p},\sigma} - \frac{C}{p^4} \right) + \frac{V}{4\pi am} C$$

The connection between C and operator O_2 is

$$C = \langle O_2^{\dagger}(\mathbf{x}) O_2(\mathbf{x}) \rangle$$

Embedding the Schrödinger algebra

- $Sch(d)$: is the symmetry of the Schrödinger equation

$$i \frac{\partial \psi}{\partial t} + \frac{\nabla^2}{2m} \psi = 0$$

- CFT_{d+2} : is the symmetry of the Klein-Gordon equation

$$\partial_\mu^2 \phi = 0, \quad \mu = 0, 1, \dots, d+1$$

In light-cone coordinates $x^\pm = x^0 \pm x^{d+1}$ the Klein-Gordon equation becomes

$$-2 \frac{\partial}{\partial x^+} \frac{\partial}{\partial x^-} \phi + \partial_i \partial_i \phi = 0, \quad i = 1, \dots, d$$

Restricting ϕ to $\phi = e^{-imx^-} \psi(x^+, \mathbf{x})$: Klein-Gordon eq. \Rightarrow Schrödinger eq.:

$$2im \frac{\partial}{\partial x^+} \psi + \nabla^2 \psi = 0, \quad \nabla^2 = \sum_{i=1}^d \partial_i^2$$

This means $Sch(d) \subset CFT_{d+2}$

Embedding (2)

$Sch(d)$ is the subgroup of CFT_{d+2} containing group elements which does not change the ansatz

$$\phi = e^{imx^-} \psi(x^+, x^i)$$

Algebra: $sch(d)$ is the subalgebra of the conformal algebra, containing elements that commute with P^+

$$[P^+, O] = 0$$

One can identify the Schrödinger generators:

$$M = P^+, \quad H = P^-, \quad K^i = M^{i-},$$

$$D_{\text{nonrel}} = D_{\text{rel}} + M^{+-}, \quad C = \frac{1}{2}K^+$$

Nonrelativistic dilatation

$$\begin{array}{ccccccc} & & D_{\text{rel}} & & M^{+-} & & \\ & & & & & & \\ x^+ & \rightarrow & \lambda x^+ & \rightarrow & \lambda^2 x^+ & & \\ x^- & \rightarrow & \lambda x^- & \rightarrow & x^- & & \\ x^i & \rightarrow & \lambda x^i & \rightarrow & \lambda x^i & & \end{array}$$

Geometric realization of Schrödinger algebra

Start from AdS_{d+3} space:

$$ds^2 = \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2}$$

Invariant under the whole conformal group, in particular with respect to relativistic scaling

$$x^\mu \rightarrow \lambda x^\mu, \quad z \rightarrow \lambda z$$

and boost along the x^{d+1} direction:

$$x^+ \rightarrow \tilde{\lambda} x^+, \quad x^- \rightarrow \tilde{\lambda}^{-1} x^-$$

Break the symmetry down to $\text{Sch}(d)$:

$$ds^2 = \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2} - \frac{2(dx^+)^2}{z^4}$$

The additional term is invariant only under a combination of relativistic dilation and boost:

$$x^+ \rightarrow \lambda^2 x^+, \quad x^- \rightarrow x^-, \quad x^i \rightarrow \lambda x^i, \quad z \rightarrow \lambda z$$

Model

$$ds^2 = \frac{-2dx^+ dx^- + dx^i dx^i + dz^2}{z^2} - \frac{2(dx^+)^2}{z^4}$$

Is there a model where this is a solution to the Einstein equation?

The additional term gives rise to a change in $R_{++} \sim z^{-4}$: we need matter that provides $T_{++} \sim z^{-4}$.

Can be provided by A_μ with $A_+ \sim 1/z^2$: has to be a massive gauge field.

$$S = \int d^{d+3}x \sqrt{-g} \left(\frac{1}{2}R - \Lambda - \frac{1}{4}F_{\mu\nu}^2 - \frac{m^2}{2}A_\mu^2 \right)$$

Two-point function

Following standard prescription

$$S = - \int d^{d+3}x \sqrt{-g} (g^{\mu\nu} \partial_\mu \phi^* \partial_\nu \phi + m_0^2 \phi^* \phi)$$

Consider $\phi \sim e^{iMx^-}$

$$S = \int d^{d+2}x dz \frac{1}{z^{d+3}} (2iMz^2 \phi^* \partial_t \phi - z^2 \partial_i \phi^* \partial_i \phi - m^2 \phi^* \phi), \quad m^2 = m_0^2 + 2M^2$$

$$\langle O(\tau, \mathbf{x}) O(0, 0) \rangle \sim \frac{\theta(\tau)}{\tau^\Delta} \exp\left(-\frac{Mx^2}{2\tau}\right)$$

where

$$\Delta = \frac{d+2}{2} + \nu, \quad \nu = \sqrt{m^2 + \frac{(d+2)^2}{4}}$$

Form dictated by Schrödinger symmetry.

Three-point function

$$\langle O_1(t_1, x_1) O_2(t_2, x_2) O_3^\dagger(0, 0) \rangle = \frac{\theta(t_1)\theta(t_2)}{t_1^{\Delta_{13,2}} t_2^{\Delta_{23,1}} (t_1 - t_2)^{\Delta_{12,3}}} \exp \left[-\frac{M_1 x_1^2}{2t_1} - \frac{M_2 x_2^2}{2t_2} \right] \Psi(y)$$

$\Delta_{ij,k} = \Delta_i + \Delta_j - \Delta_k$, where y is a Schrödinger invariant

$$y = \frac{x_1 t_2 - x_2 t_1}{(t_1 - t_2) t_1 t_2}$$

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Unitarity particles: $\Delta_1 = \Delta_2 = d/2$, $\Delta_3 = 2$

$$\Psi(y) \sim y^{1-d/2} \gamma \left(\frac{d}{2} - 1, y \right) \leftarrow \text{incomplete beta function}$$

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Holography: computing Witten diagram **Fuertes, Moroz**

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specializing for $M_1 = M_2 = 1$, $\Delta_1 = \Delta_2 = d/2$, $\Delta_3 = 2$:

Three-point function

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Holography: computing Witten diagram **Fuertes, Moroz**

$$\psi(y) \sim \int dv dv' e^{-iM_1 v - iM_2 v'} (v - v' + iy)^{-\Delta_{12,3/2}} (v')^{-\Delta_{23,1}} v^{-\Delta_{13,2}}$$

specializing for $M_1 = M_2 = 1$, $\Delta_1 = \Delta_2 = d/2$, $\Delta_3 = 2$:

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Klebanov-Witten's mnemonic

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Efimov effect \leftrightarrow violation of Breitenlohner-Freedman bound

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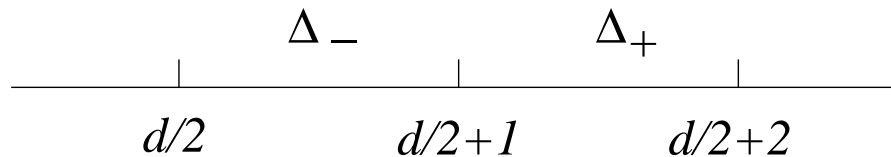
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- For a theory where some operator O has dimension $\Delta[O] = \Delta_+$,

$$d/2 + 1 < \Delta_+ < d/2 + 2$$

there exist another theory where O has dimension $\Delta_- = d + 2 - \Delta_+$,

$$d/2 < \Delta_- < d/2 + 1$$



Examples

Example 1:

● Free fermions: $[\psi_\uparrow\psi_\downarrow] = 3$

$$3\text{D: } \frac{d}{2} + 1 = 5/2$$

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Example 2: p -wave resonance is not universal

- $[\psi\nabla\psi] = 4$
- Outside the range $(5/2, 7/2)$.
- (if p -wave resonance could be made universal, the resulting theory would have operator of dimension 1, less than unitarity bound $3/2$)

Examples (continued)

Example 3: one-dimensional unitarity fermions [Nishida, Tan](#)

- In 1d: $[\psi] = 1/2$
- $[\psi_\uparrow\psi_\downarrow] = 1$, outside $(3/2, 5/2)$
- 4 types of fermions: $[\psi_1\psi_2\psi_3\psi_4] = 2$
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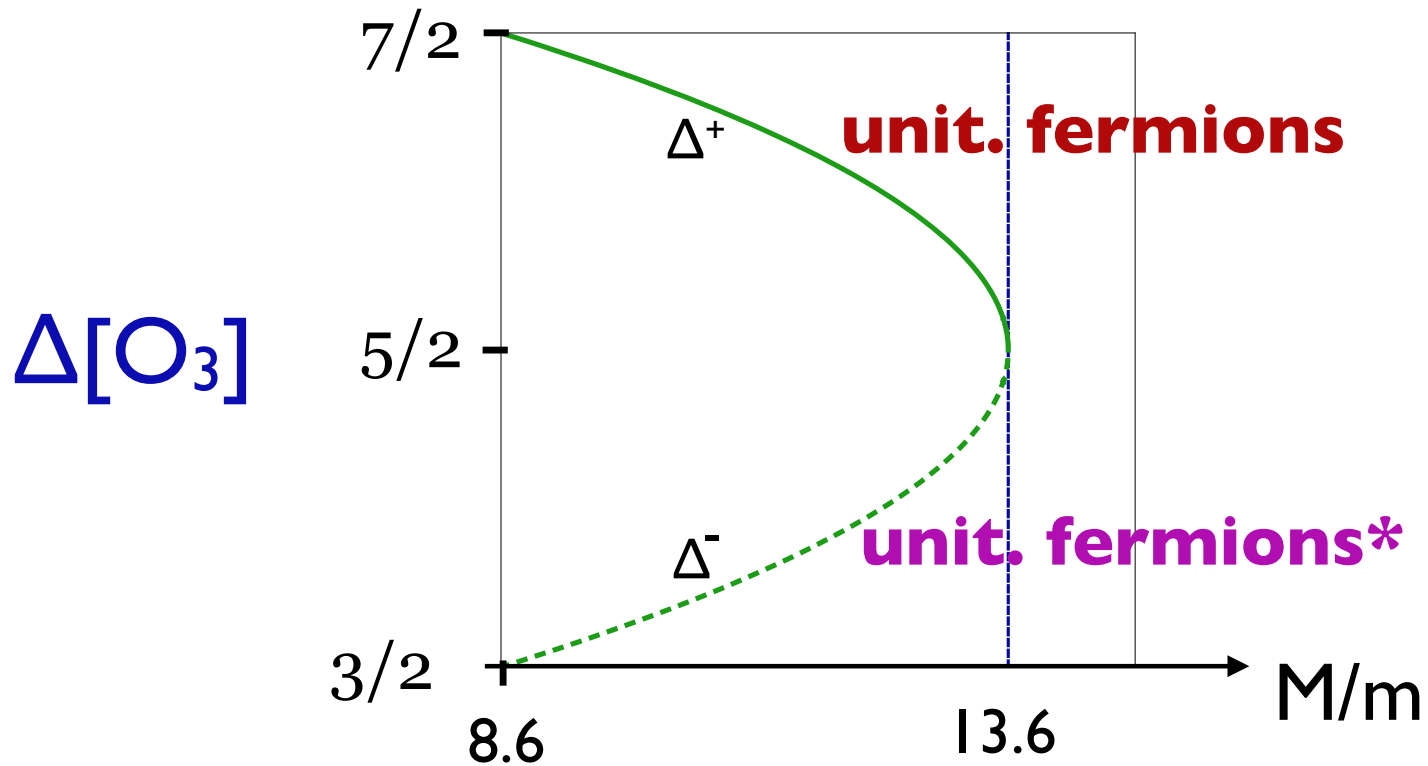
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Example 4: Unitarity fermions with different masses for 2 fermion types $M/m \neq 1$

- One three-body operator $\Psi\partial_i\Psi\psi$: dimension between $7/2$ and $5/2$ when M/m varies from 8.6 to 13.6
- There exists another scale-invariant theory with two and three-body resonances in this range of mass ratio [Nishida, DTS, Shina Tan](#)

Unitarity fermions*



Black holes

Schrödinger metric Can be realized in string theory ($d = 2$)

Herzog, Rangamani, Ross;

Maldacena, Martelli, Tachikawa;

Adam, Balasubramanian, McGreevy (2008)

$$S = \int d^5 x \sqrt{-g} \left(\frac{1}{2} R - \Lambda - \frac{1}{4} F_{\mu\nu}^2 - \frac{m^2}{2} A_\mu^2 \right) + \text{scalars}$$

Black-hole solutions also constructed: allow studying hydrodynamics

- Shear viscosity: $\eta/s = 1/(4\pi)$
- Thermal conductivity: Prandtl number = 1
- Thermodynamics $P(T, \mu) = \#T^4/\mu^2$: unrealistic

Why such equation of state?

- Dimensional reduction on light cone leaves an infinite number of fields with growing particle number (mass)
- They all contribute to the free energy **Barbon, Fuertes**

What we don't know

- Holographic renormalization
- Role of large N ? ($\text{Sp}(2N)$ model?)
- Hierarchical organization in NR field theories
 - to understand n -body sector we don't need to know solution to $n + 1$, $n + 2$ etc. body problem
 - In gravity: throwing away fields with mass $> n$ should be a consistent truncation
 - Not a feature of current string constructions of Schrödinger background
- Maybe one should get rid of the light-cone construction and start anew using a nonrelativistic gravity from the start (Newton-Cartan gravity?)

Speculations

More generally, AdS/CFT suggests that one can think about quantum field theories in a completely different language

- Perhaps one can reformulate unitarity Fermi gas in the language of local operators and their operator product expansions cf. Braaten, Platter
- Infinite number of operators, corresponding to short-range behavior of wavefunctions/ eigenstates in harmonic potential
- A Schwinger-Dyson-type hierarchy?
- Truncation to ψ and O_2 (+ additional approximations) \Rightarrow BCS mean-field theory?
- Keeping more fields \rightarrow more accurate approximations?

Conclusion

- Unitarity fermions have Schrödinger symmetry, a nonrelativistic conformal symmetry
- Universal properties, studied in experiments
- There is a metric with Schrödinger symmetry
- Starting point for dual phenomenology of unitarity fermions?
- Deeper connection between few- and many-body physics?