STATISTICAL SCATTERING OF WAVES: FROM NUCLEI TO DISORDERED WAVEGUIDES.

An Overview of Past and New Results [a]

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INTRODUCTION

Complex scattering of waves

has captured the interest of physicists for a long time. For instance, the problem of

coherent multiple scattering of waves

has long been of great importance in physics, and particularly in optics.

The present talk fits in the general topic of:

"Statistical theory of complex wave-interference phenomena".

In particular, we shall study the

statistical fluctuations

of transmission and reflection of waves,

which are of considerable interest in

mesoscopic physics.

In the problems to be discussed here, complexity in wave scattering may derive from:

• the *chaotic* nature of the underlying classical dynamics.

Our discussion will find applications to:

a) Electronic transport in ballistic quantum dots,

b) Transport of electromagnetic waves, or other classical waves, (like elastic waves) through cavities,

in both cases with a chaotic classical dynamics.

• the quenched randomness of scattering potentials in a disordered medium.

Our discussion will find applications to:

a) Electronic transport in disordered quantum conductors,

b) Transport of electromagnetic waves, or other classical waves (like elastic waves) through waveguides.

Why do *statistics*?

The interference pattern resulting from

coherent multiple scattering

is so *complex*

(a change in some external parameter changes it completely),

that only a *statistical* treatment is feasible and meaningful.

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The most remarkable feature we shall encounter is that the results

will be expressible in terms of a few

relevant physical parameters,

the remaining details being just "scaffoldings".

This feature has often been be captured employing a

maximum-entropy approach.

PURPOSE OF THIS TALK

- Review past and recent work where ideas that originated from nuclear physics, like the nuclear optical model and the statistical theory of nuclear reactions, have been used to give a unified treatment of wave scattering (QM and classical) in classically chaotic cavities

  Discuss a number of unsolved problems

- Review past and recent work in the study of electronic transport in disordered conductors, and transport of classical waves in disordered waveguides.

  Discuss limitations of earlier approaches and achievements of new ones

  Discuss a number of unsolved problems

- Convince experimentalists to measure some of the quantities that we encounter!

ATOMIC NUCLEI AND MICROWAVE CAVITIES

• Historically, *nuclear physics*, a *complicated* many-body problem (dimensions $\approx$ fm $= 10^{-13}$ cm) has offered very good examples of *complex quantum-mechanical scattering*.

The *statistical theory of nuclear reactions* has been of great interest for many years, in those cases where, due to the presence of many resonances, the excitation function is very complicated as a function of energy and *its detailed structure is of little interest*. A *statistical treatment* is then called for.

Fluctuations in Nuclear Reactions, T. Ericson and T. Mayer-Kuckuk, Ann. Rev. Nucl. Sc, **16**, 183 (1966).

"Nuclear-reaction theory is equivalent to the theory of *waveguides* $\cdots$. We will concentrate on processes in which the incident wave goes through a highly complicated motion in the nucleus $\cdots$. We will picture the nucleus as a closed *cavity*, with reflecting, but *highly irregular walls*."

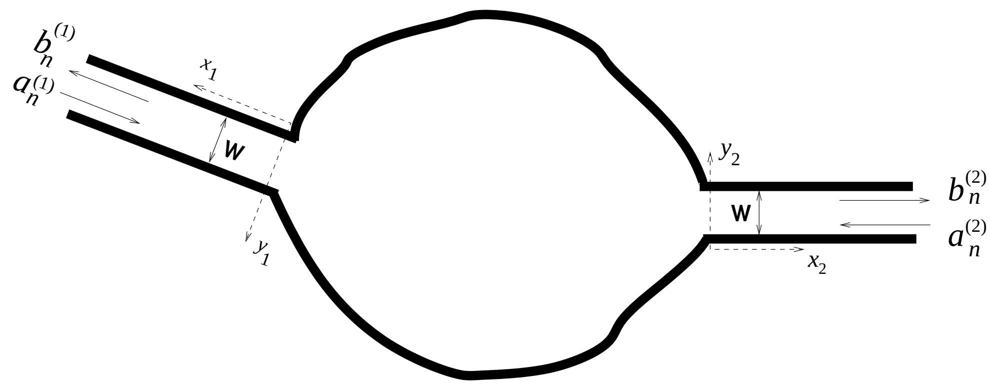
We now know that the "irregular walls" are not necessary and that a scheme like the *nuclear optical model* accounts for many features of *microwave propagation* (typical dimensions of 0.5 m) through *cavities* and of *quantum-mechanical scattering* in *simple one-particle systems* (typical dimensions of 1 μm), as long as the corresponding *classical dynamics is chaotic*.

Size of these systems spans $\approx$ 14 orders of magnitude

(It's really kL that matters)

A MICROWAVE CAVITY OR A BALLISTIC QUANTUM DOT

Cavity connected to the external world by two waveguides



A wave is sent in from one of the waveguides

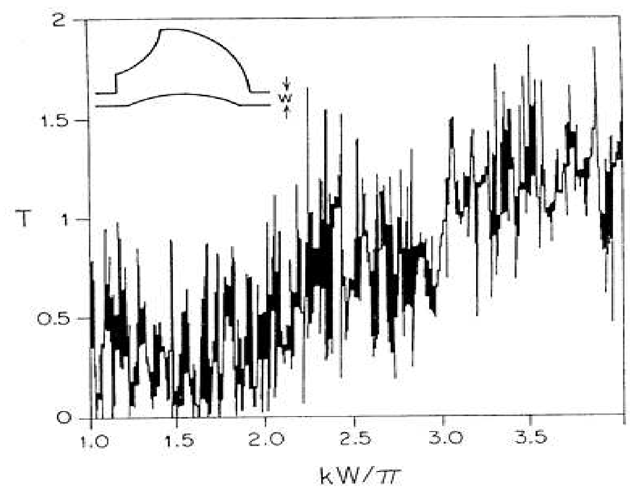
Result: a reflected wave and a transmitted wave

A point inside the cavity receives the contributions from many *multiply scattered waves* that have bounced from the inner surface

$$\{\text{outgoing amplitudes}\} = S\ \{\text{incoming amplitudes}\}, \quad S = \begin{pmatrix} r & t' \\ t & r' \end{pmatrix}$$

DOING STATISTICS WITH THE SCATTERING MATRIX

Numerical simulation (H. Baranger) of quantum scattering by a $2D$ cavity with a chaotic classical dynamics (Bunimovich stadium/4)



Changing E, $S(E)$ and hence $T(E)$ fluctuate in a very complex way

We are not interested in the detailed structure of $T(E)$.

Hence we are led to a *statistical description* of the problem

We shall do statistics directly on the S matrix

CONSTRUCTING THE RANDOM-S-MATRIX MODEL

• Assume in the scattering process only *two distinct time scales*:

i) *prompt* response from short paths

ii) *delayed* response from very long paths

Energy average of $S(E)$, $\bar{S}$, also known as the *"optical" S* matrix, is much smoother than $S(E)$: it describes short-time processes.

• As in *statistical mechanics*: think of an *ensemble of macroscopically identical cavities*, i.e., an *ensemble of S matrices*. Idealize $S(E)$, $\forall$ real E, as a *stationary random-matrix function* of E satisfying the condition of $\boxed{\textit{ergodicity}}$

energy averages $\Rightarrow$ ensemble averages   E.g., $\bar{S} = \langle S \rangle$

• Analyticity of $S \Rightarrow p(S)$ must be a *"reproducing kernel"*

$$\boxed{\langle f(S) \rangle = \int f(S) p_{\langle S \rangle}(S) d\mu(S) = f(\langle S \rangle)} \quad S = aS + bS^2 + \cdots$$

POISSON'S KERNEL

This is the probability density

$$\boxed{p_{\langle S\rangle}(S) = \frac{[\det(I-\langle S\rangle\langle S\rangle^\dagger)]^{(2\beta N+2-\beta)/2}}{|\det(I-S\langle S\rangle^\dagger)|^{2\beta N+2-\beta}}}$$

whose *information entropy* (Shannon)

$$\mathcal{S}[p] \equiv -\int p_{\langle S\rangle}(S)\ln p_{\langle S\rangle}(S)d\mu(S)$$

is greater than or equal to that of any other probability density satisfying the reproducibility requirement for the same optical $\langle S\rangle$.

System-specific *details other than the optical $\langle S\rangle$ are assumed to be irrelevant.*

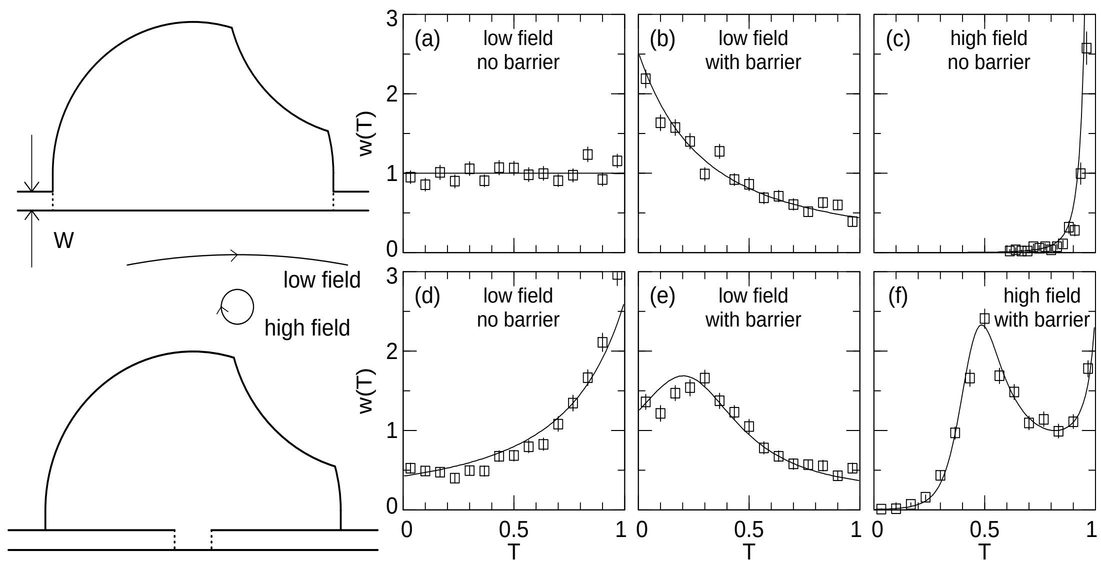
Hauser-Feshbach as a consequence of Poisson's kernel (W. Friedman and PAM)

Cf.: Hua, *Harmonic Analysis in the Complex Domain*

Jackson, *Classical Electrodynamics*

NUMERICAL SIMULATIONS vs THE MAXIMUM ENTROPY MODEL

□ num. sol. of Schrödinger's eqn.; ——— theoretical predictions



CONDUCTANCE DISTRIBUTION;

$N = 1, \beta = 2$ (H. U. Baranger and PAM, 1996)

COMMENTS

- Statistics collected over energy intervals ΔE containing ≈ 200 uncorrelated points

$\star$ $\langle S(E) \rangle \approx$ constant inside ΔE

$\star$ That such energy scales can be found gives evidence of two rather widely separated time scales in the problem.

- Sampling over an ensemble of 10 structures to improve statistics

$\star$ We are mainly averaging over energy. We rely on ergodicity to compare numerical distributions with theory

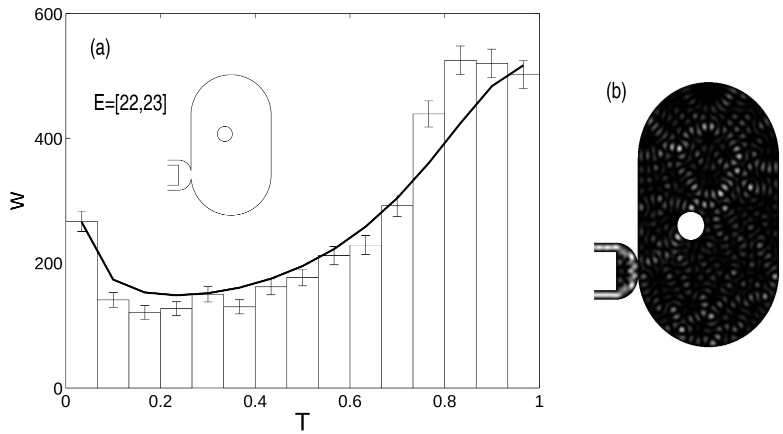
$\star$ Optical S matrix extracted directly from the numerical data and used as $\langle S \rangle$ in Poisson's kernel:

theoretical curves represent *parameter free* predictions.

$\star$ In Figs. 3(e,f) four subintervals –of 50 energies each– had to be used,

since each showed a slightly different optical S.

Systems with NO approximate validity of stationarity and ergodicity



CONDUCTANCE DISTRIBUTION: $N = 1, \beta = 1$ [a]

Small stadium ("whispering gallery mode"): short paths

Large stadium ("sea" of fine structure resonances): long paths

Agreement reasonable; but discrepancy found to be systematic

- Statistics collected over energy windows $\Delta E \sim 20$ points
  - $\star$ energy variation of $\langle S(E) \rangle$ inside ΔE was not negligible
  - $\star$ two not very widely separated time scales
- Sampling over an ensemble of 200 structures by moving obstacle

[a] Bulgakov, Gopar, PAM and Rotter, PRB **73**, 155302 (2006)

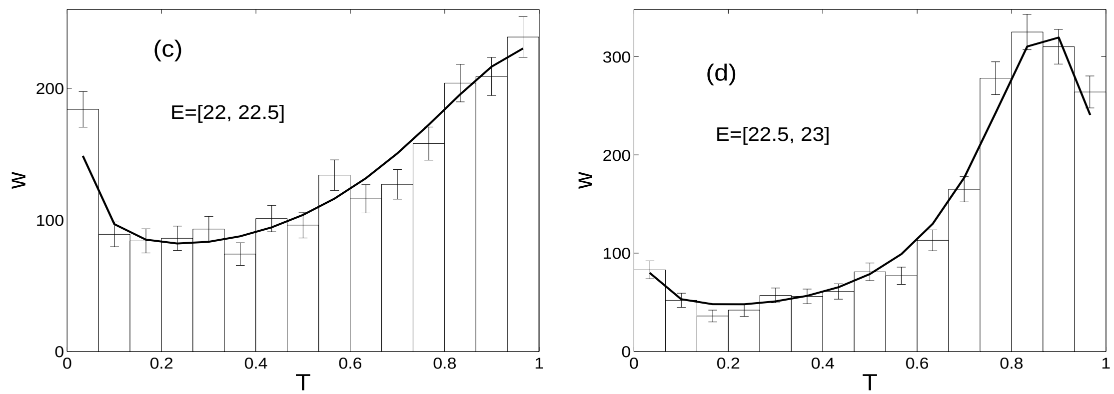
• Reduce ΔE to $\Delta E/2 \approx 10$ points only, so that:

$\Rightarrow$ Variation of $\langle S(E)\rangle$ in $\Delta E/2$ smaller: OK!

• But number of resonances inside $\Delta E/2$ reduced 50 %

• Ensemble consisting of 200 positions of the obstacle

• In contrast to the previous results, now we are "mainly" averaging over the ensemble of obstacles



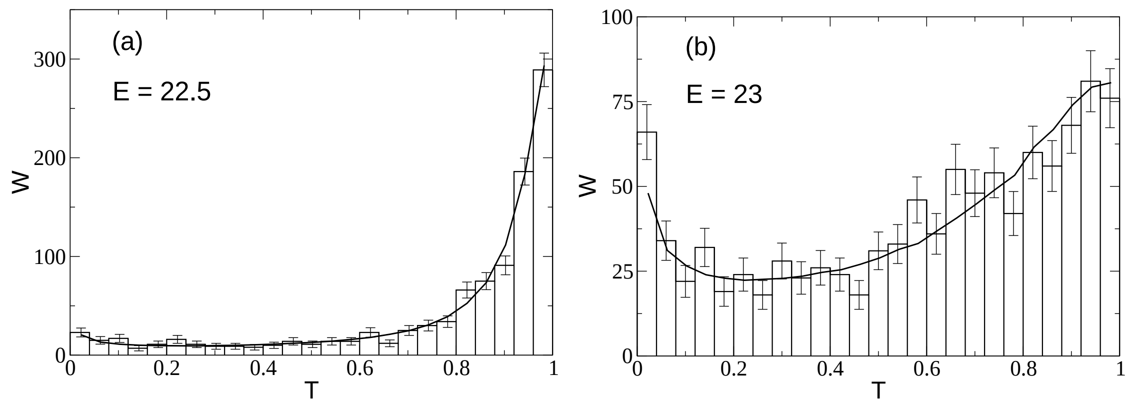
• Stationarity and ergodicity are ever less fulfilled!

Agreement much better! Surprising!

Results of a recent calculation are even more intriguing!

ΔE literally reduced to a point:

the distribution actually represents statistics collected across the ensemble at a fixed energy



Two samples collected at different fixed energies:

very good agreement with theory!

Surprising!

• Stationarity –and hence ergodicity– is an extreme idealization

• In realistic dynamical problems, stationarity is approximate: across ΔE the "local" optical $\langle S(E) \rangle$ should be approximately constant, while ΔE should contain many fine-structure resonances.

• We saw examples where this "compromise" is approximately fulfilled, and cases where it isn't.

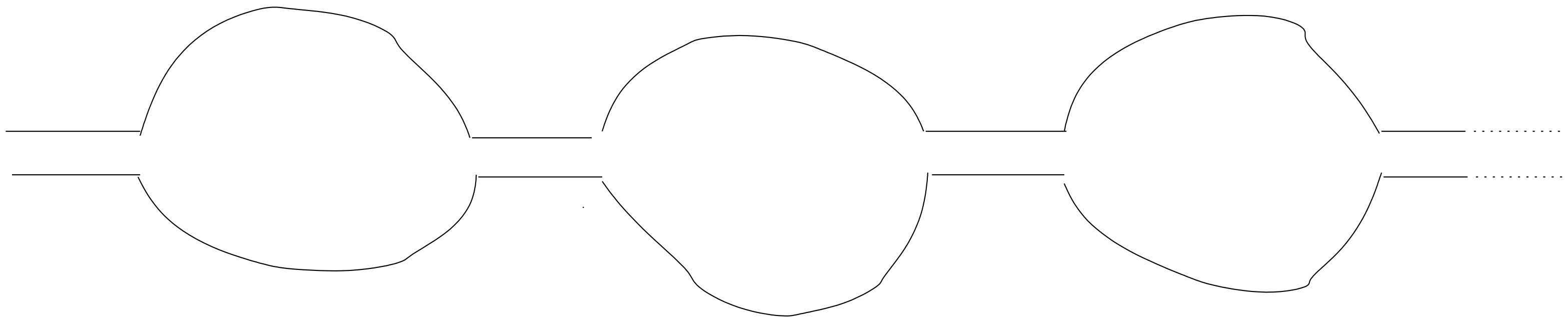
• Poisson's kernel was found to give a good description of the data taken across the ensemble for a fixed energy: Poisson's kernel seems to be valid beyond the situation where it was originally derived!

• Ingredient to construct Poisson's kernel: the reproducing property, defined for the ensemble at a *fixed energy*: we only know how to justify it through stationarity and ergodicity.

• It is as if the reproducing property were valid even without stationarity and ergodicity, which were originally used to derive it. We don't understand this!!

•• Comment on two-point functions!!

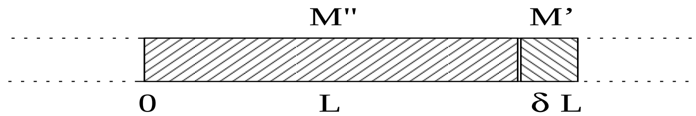
MODELLING A QUASI-1D DISORDERED WAVEGUIDE



Putting many ballistic cavities in series one could model a q1D disordered waveguide (Heidelberg).

Here we shall proceed in an alternative way

THE STATISTICAL PROBLEM



- To a waveguide of length L, with one configuration of disorder, assign a transfer matrix M''

$$M''\{\text{left amplitudes}\} = \{\text{right amplitudes}\}$$

- Assign a probability density $p_L(M'')$: what results is a *Random-Matrix Theory of transfer matrices*.

- Add a “Building Block”: $\delta L \ll L$, but *still containing many weak scatterers* (DWSL): M' and M'' statistically independent. Probability density: $p_{\delta L}(M')$.

- Combination: $M = M'M''$

$$p_{L+\delta L}(M) = \int p_L((M')^{-1}M)\, p_{\delta L}(M')\, dM'$$

Similar to Smoluchowski equation in Brownian motion theory

THE MAXIMUM-ENTROPY APPROACH and the Diffusion Equation

Expecting the *results to be largely independent of details of the BB,* the distribution $p_{\delta L}(M')$ for the BB was originally modelled by *maximizing the Shannon entropy* subject to the constraint that the mean-free-path ℓ be given. This led to a

$$\Downarrow$$

Diffusion equation in transfer-matrix space (Dorokhov+MPK) for the "evolution" with L of the probability distribution [a]:

$$\boxed{\frac{\partial w_{L/\ell}(\lambda)}{\partial L/\ell} = \frac{2}{N+1} \sum_{a=1}^{N} \frac{\partial}{\partial \lambda_a} \left[\lambda_a (1+\lambda_a) J(\lambda) \frac{\partial}{\partial \lambda_a} \frac{w_{L/\ell}(\lambda)}{J(\lambda)} \right]}$$

$$N, \quad g = \sum_{ab} T_{ab} = \sum_{a=1}^{N} \frac{1}{1+\lambda_a}, \quad J(\lambda) = \prod_{a<b} \left| \lambda_a - \lambda_b \right|, \quad w_0(\lambda) = \delta(\lambda)$$

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[a] PAM, P. Pereyra and N. Kumar, Ann. Phys. **181**, 290 (1988)

CENTRAL-LIMIT THEOREMS

• It was shown later [a] that *the maximum-entropy model selects the limiting distribution* [in the sense of the *dense-weak-scattering limit* (DWSL) = *large* density of *weak* scatterers, with a *fixed* mfp ℓ] within a class of models characterized by random phases ("isotropy" assumption) for the transfer matrices M_i of the *individual scattering units*.
Only one *relevant physical parameter* occurs: the mfp ℓ, the only property arising from individual scatterers that survives in the DWSL.

• Thus the *limiting distribution* described by the resulting diffusion equation (same as DMPK) can be interpreted as a *generalized central-limit theorem* (CLT).

• *A Maximum-Entropy model seems to work when there is a CLT "behind the scenes"*.

[a] PAM and B. Shapiro, PRB **37**, 5960 (1988)

• PAM and Tomsovic's model (MT) [b] generalized the above CLT relaxing the condition of "isotropy".

Scaling parameters = the mfp's ℓ_{ab} for the various processes $b \to a$.

Motivation: describe the physics in the transverse direction.

MT model is a possible candidate to study, for waveguides with surface disorder, the influence of the specific scattering properties of the various modes.

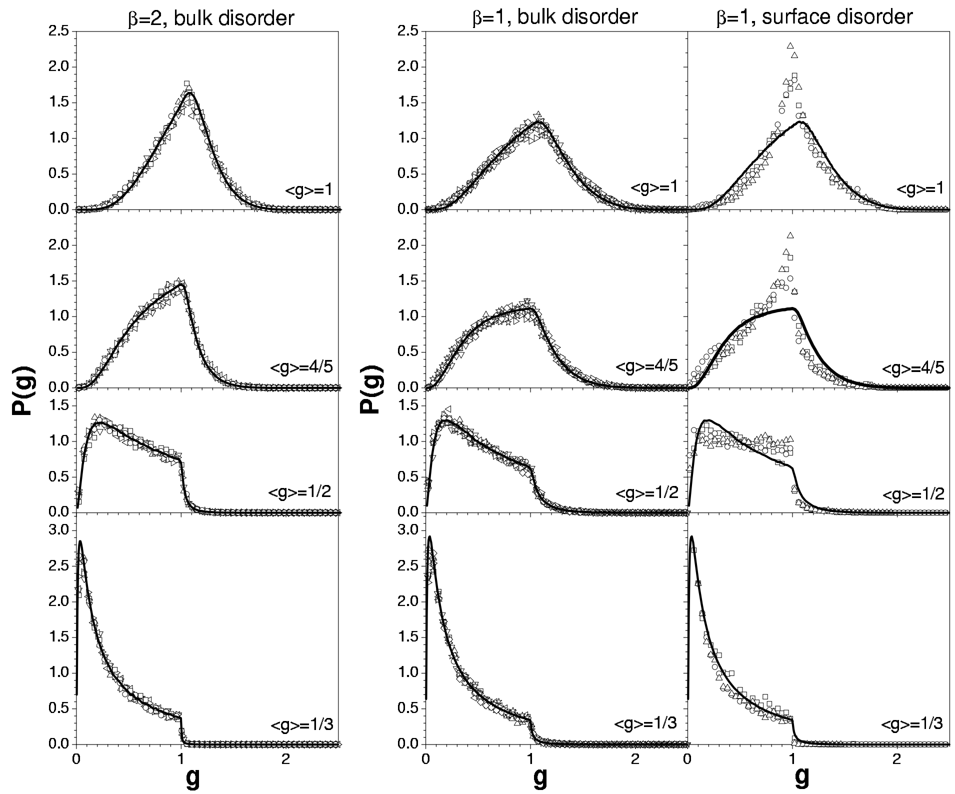
MT model reduces to DMPK when $\ell_{ab} = \ell$ (no isotropy assumption, which is at the level of the phases, is needed!).

No maximum-entropy ansatz known to describe this problem ...

[b] PAM and S. Tomsovic, PRB **46**, 15963 (1992)

NUMERICAL STUDIES IN THE CROSSOVER REGIME [a]

$\overline{\qquad\qquad}$ = DMPK; $\quad \circ, \diamond, \cdots =$ Anderson model



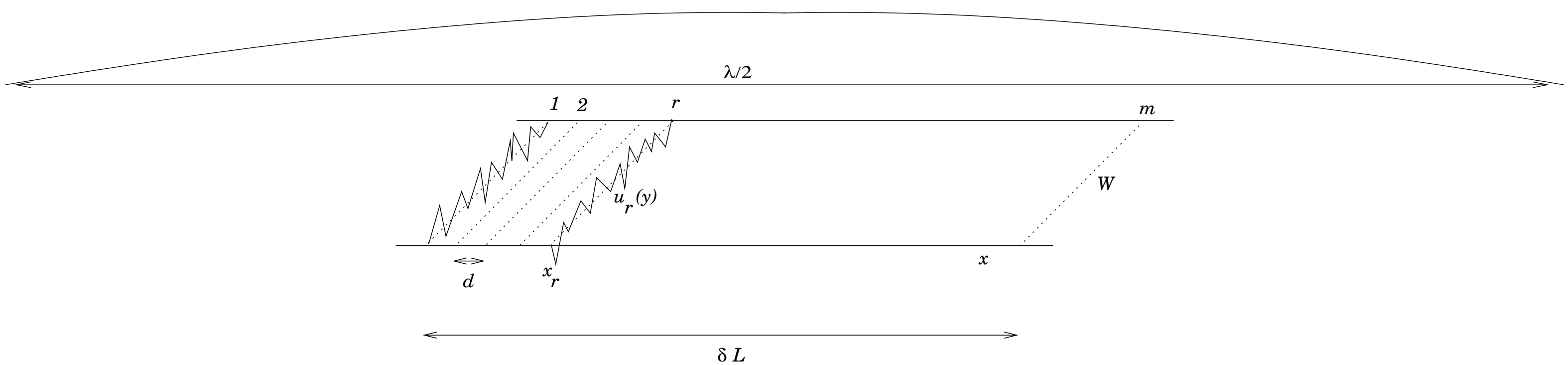
DMPK is a very good description of bulk disorder, but not of surface disorder!

[a]L. S. Froufe et al, PRL **89**, 246403 (2002)

POTENTIAL MODEL FOR THE DISORDERED WAVEGUIDE [a]

- Recently we studied a potential model to derive the statistical properties the BB and then the "evolution" with L of the expectation value of physical quantities.

Sequence of random δ-potential slices



BB contains, inside δL, $m \gg 1$ δ-slices

Regime: $$d \ll \delta L \ll \lambda, \ell$$

[a] L.S. Froufe, M. Yépez, PAM, J.J. Sáenz, Phys. Rev. B **75**, 031113 (2007)

M O T I V A T I O N

- We study wave-transport phenomena where the physics of the various modes is relevant, as for waveguides with surface disorder.
- The energy appears explicitly in the present model. In earlier models it was simply the energy at which ℓ was to be taken.
- Good description of amplitudes, i.e., involving phases, generally not described in previous models (some in MT).
- MT: short-wavelength approximation (SWLA), i.e, result to order $1/(k\ell)^0$, of the present model
- DMPK then emerges by taking all $\ell_{ab} = \ell$.

Given an "observable" $F(M)$, like t_{ab}, G, etc., one then finds, in the Dense-Weak-Scattering-Limit (DWSL), the Fokker-Planck equation:

$$\boxed{\frac{\partial\langle F(M)\rangle_L}{\partial L} = D^{jk,lm}_{ab,cd}(k,L)\left\langle M^{kn}_{be} M^{mp}_{df} \frac{\partial^2 F}{\partial M^{jn}_{ae}\partial M^{lp}_{cf}}\right\rangle_L}$$

"Diffusion coefficients" $D^{jk,lm}_{ab,cd}$ defined by

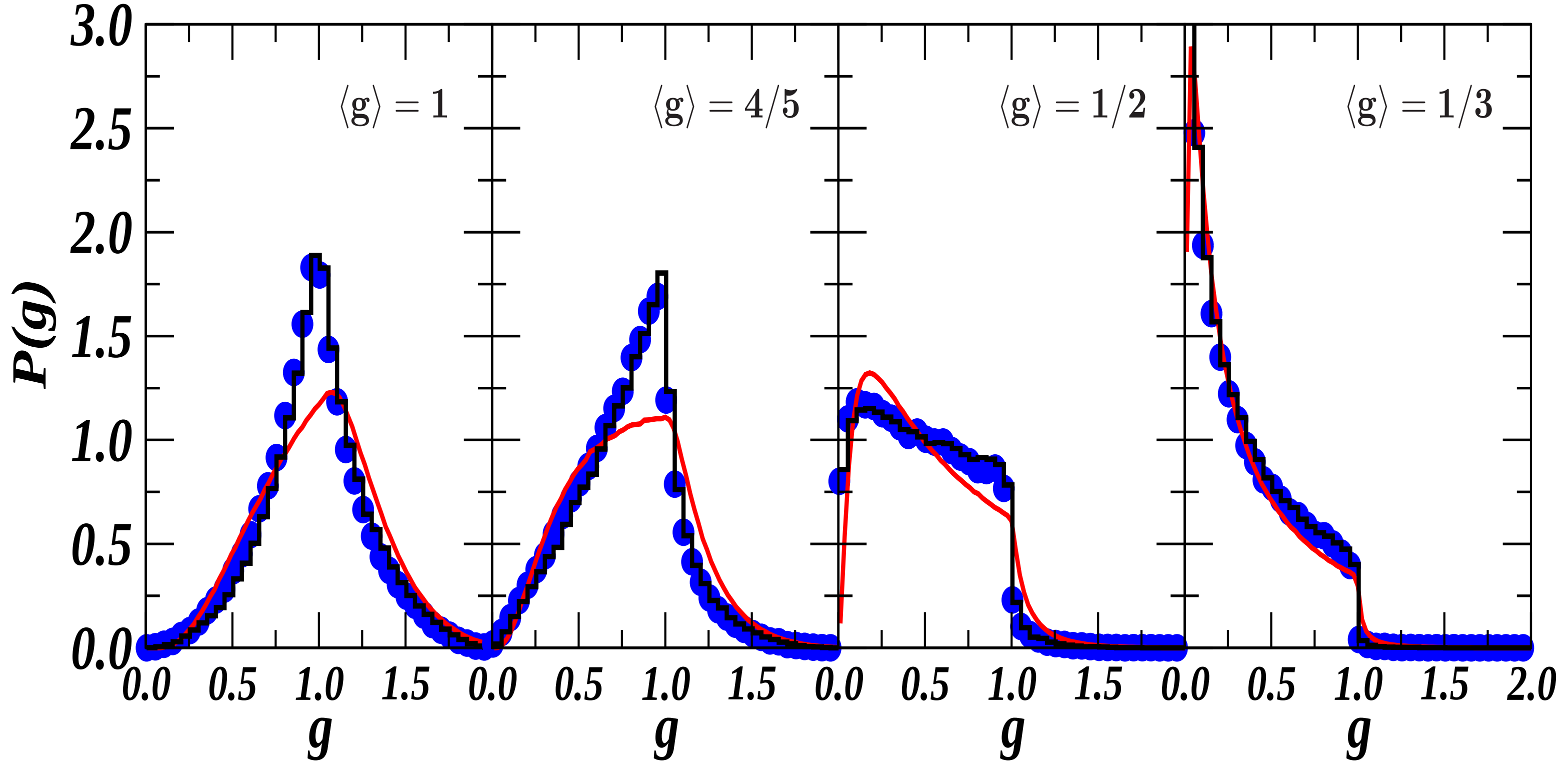
$$\left\langle \varepsilon^{jk}_{ab}\varepsilon^{lm}_{cd}\right\rangle_{L,\delta L} = 2D^{jk,lm}_{ab,cd}(k,L)\,\delta L + \cdots$$

are given in terms of the mfp's ℓ_{ab} and these in terms of the cumulants $\kappa_2(ab,cd)$ for the potential.

<u>Higher-order cumulants κ_4, etc., do not contribute in the DWSL !!</u>

For given $\{\ell_{ab}\}$ this is a universal result:
it is a <u>*generalized Central-Limit Theorem*</u> (CLT).

SURFACE DISORDER
CONDUCTANCE DISTRIBUTION

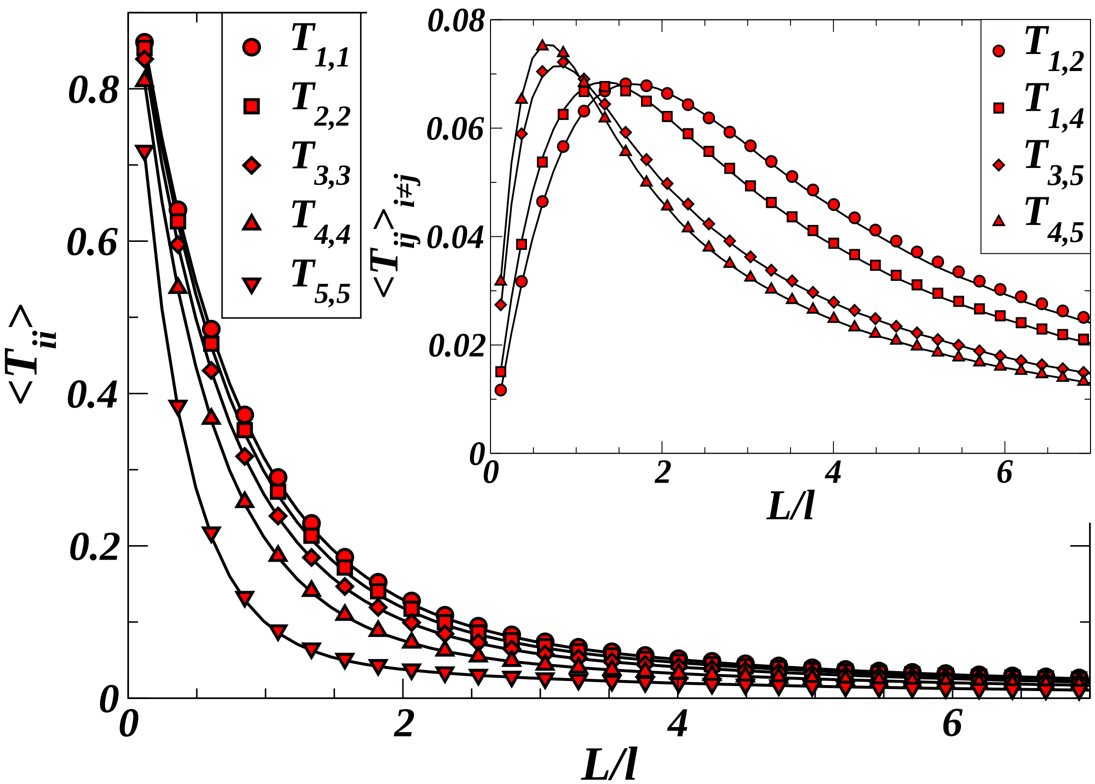


black: microscopic model for the potentials

blue: random walk in transfer-matrix space, in the SWLA

red: DMPK. (After L. Froufe's Thesis).

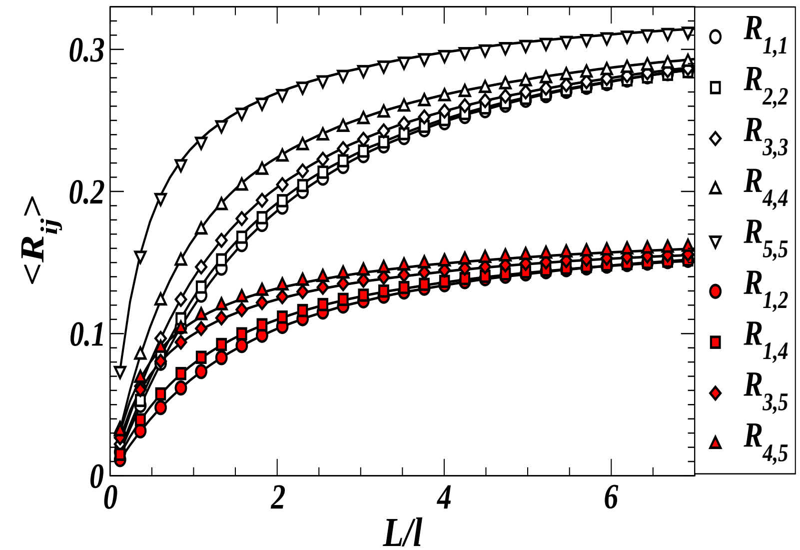
BULK DISORDER
AVERAGE TRANSMISSION COEFFICIENTS



Symbols: microscopic model for the potentials

Bold line: random walk in transfer-matrix space, in the *short-wavelength approximation (SWLA)* $[d \ll \lambda \ll \delta L \ll \ell]$ –essentially MT– with ℓ_{ab} calculated analytically. (After L. Froufe's Thesis).

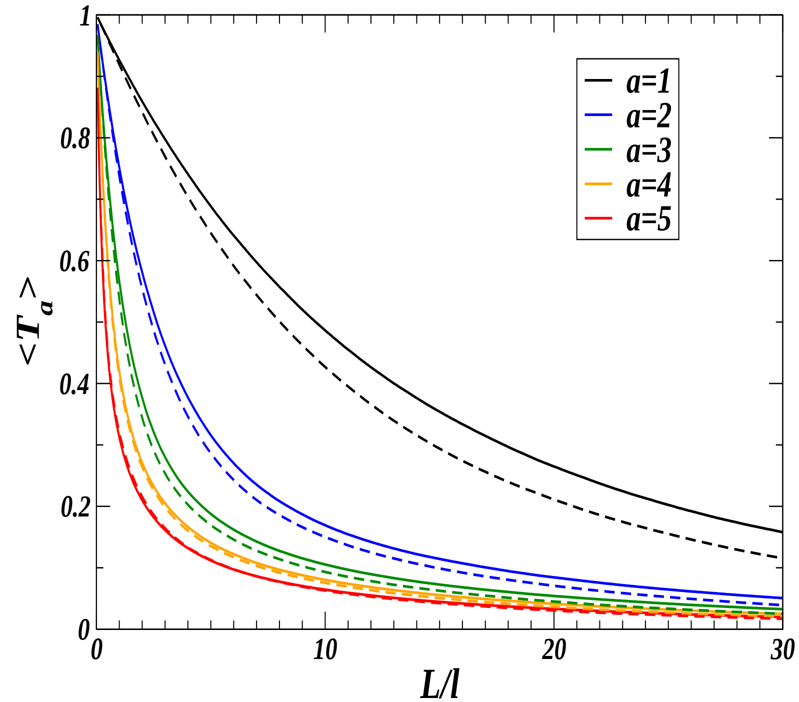
BULK DISORDER
AVERAGE REFLECTION COEFFICIENTS



Symbols: microscopic model for the potentials

Bold line: random walk in transfer-matrix space, in the *short-wavelength approximation (SWLA)*. (After L. Froufe's Thesis).

SURFACE DISORDER



———: microscopic model for the potentials

- - - -: random walk in transfer-matrix space, in the *short-wavelength approximation (SWLA)*, with ℓ_{ab} calculated analytically. (After L. Froufe's Thesis).

CONCLUSIONS

- A Maximum-Entropy model seems to work when there is a CLT “behind the scenes”.
- A CLT was found that gave support to the DMPK equation, originally obtained through a Maximum-Entropy Ansatz.
- A more general CLT was obtained by PAM and S. Tomsovic.
- We have shown a CLT for the model of a random distribution of δ-potential slices. Result depends on the mfp’s ℓ_{ab}.
- Results of “random walk in transfer-matrix space” in the SWLA compare very well with numerical simulations with bulk disorder and “reasonably” well in the case of surface disorder.
- Progress in the solution of the diffusion equation?
- A Maximum-Entropy Approach to this problem?
- The problem of evanescent modes?