Part I

QGP instabilities under the influence of collisions

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Outline

• Kinetic instabilities in the anisotropic QGP

• **Model** for the inclusion of collisions among the hard particles

• Effect of the collisions on the stable modes

• Effect on instabilities
Motivation

• Fast equilibration of QGP seen experimentally
• QCD-Plasma instabilities are one means to explain this: Exponential growth of the collective modes

Strong fields influence particle distributions  
→ possible explanation for fast equilibration seen in experiments

• Collisions have so far been neglected in all perturbative calculations
• However with $\alpha_s \approx 0.3$, as at RHIC energies, they are present and not negligible in general.  
  Do they reduce the growth of the instabilities? Maybe eliminate them totally?
Anisotropy and instabilities

- Due to expansion the parton distribution functions become locally \textit{anisotropic} in momentum space for times $\tau > \langle p_T \rangle^{-1}$.

$$\langle p_T \rangle \sim Q_s$$

$$\langle p_L \rangle \sim 1/\tau$$

- Squeeze isotropic distr. in one direction:

$$f(p) = N(\xi) f_{\text{iso}} \left( p^2 + \xi (p \cdot \hat{n})^2 \right)$$

- In a system with anisotropic momentum distribution \textit{kinetic instabilities} can occur.
Vlasov equation

- Linearized transport equations: \((i \in \{g, q, \bar{q}\})\)
  
  \[
  \left[ V \cdot D_X, \delta n^i(p, X) \right] + g\theta_i V_\mu F^{\mu\nu}(X) \partial^{(p)}_\nu n^i(p) = 0
  \]

  \[
  D_X = \partial_X + igA(X)
  \]

  \[
  F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu - ig[A^\mu, A^\nu]
  \]

  \[
  \theta_g = \theta_q = 1 \text{ and } \theta_{\bar{q}} = -1
  \]

- Induced current per color channel:

  \[
  j^{i\mu}_{\text{ind}} = g \int_p V^\mu \left\{ 2N_c \delta f^g_d(p, X) + N_f [\delta f^q_d(p, X) - \delta f^{\bar{q}}_d(p, X)] \right\}
  \]

  at leading order in the coupling constant
Solution for collisionless case

• Neglect terms of subleading order in \( g \):
  All color channels decouple

\[
V \cdot \partial_X \delta f^i_a(p, X) + g \theta_i V_{\mu} F^\mu\nu_a(X) \partial_\nu^{(p)} f^i(p) = 0
\]

• Solution for the induced current:

\[
J^\mu_{\text{ind}a}(K) = g^2 \int_p V^\mu \partial_\nu^{(p)} f(p) \left( g_{\gamma\beta} - \frac{V_{\gamma} K_\beta}{K \cdot V + i\epsilon} \right) A_\alpha^\gamma(K)
\]

with so far arbitrary

\[
f(p) = 2N_c f^g(p) + N_f \left[ f^q(p) + f^{\bar{q}}(p) \right]
\]
Inclusion of collisions

- Expect reduction of instability growth when collisions are included – want to estimate the effect quantitatively

- Add Bhatnagar-Gross-Krook (BGK) - type collision term:

\[ V \cdot \partial_X \delta f^i_a(p, X) + g \theta_i V \mu F^\mu\nu_a(X) \partial_\nu f^{(p)}(p) = C^i_a(p, X) \]

with \( C^i_a(p, X) = -\nu \left[ f^i_a(p, X) - \frac{N^i_a(X)}{N_{eq}^i} f_{eq}^i(|p|) \right] \)

with collision rate \( \nu \).

- BGK is improved relaxation time approximation:
  Conserves particle number locally
Analytical solution

- Solution for $\delta f^i_\alpha(p, X)$ yields the induced current
- Self energy is obtained from ind. current via:

$$\Pi^{\mu\nu}_{ab}(K) = \frac{\delta J^{\mu}_{\text{ind}_a}(K)}{\delta A^b_{\nu}(K)}$$

$$\Pi^{\mu\nu}_{ab}(K) = \delta_{ab} g^2 \int_p V^\mu \nabla^\nu f(p) M^{\mu\nu}(K, V) D^{-1}(K, v, \nu)$$

$$+ \delta_{ab} g^2 (i\nu) \int \frac{d\Omega}{4\pi} V^\mu D^{-1}(K, v, \nu) \int_p \partial^V f(p') M^{\mu\nu}(K, V') D^{-1}(K, v', \nu) W^{-1}(K, v)$$

$$M^{\mu\nu}(K, V) := g^{\nu\beta}(\omega - k \cdot v) - V^\nu K^\beta$$

$$D(K, v, \nu) := \omega + i\nu - k \cdot v$$

$$W(K, \nu) := 1 - i\nu \int \frac{d\Omega}{4\pi} D^{-1}(K, v, \nu)$$

Not just the replacement of $\omega \rightarrow \omega + i\nu$
Dispersion relations

\[ J_{\text{ind}}^\mu(K) = \Pi^{\mu\nu}(K)A_\nu(K) \]

in Maxwell's equation \( iK_\mu F^{\mu\nu}(K) = J_{\text{ind}}^\nu + J_{\text{ext}}^\nu \)
leads to

\[
\left[ K^2g^{\mu\nu} - K_\mu K^\nu + \Pi^{\mu\nu}(K) \right] A_\mu(K) = J_{\text{ext}}^\nu(K)
\]

temporal axial gauge \( A_0 = 0 \):

\[
\left[ (k^2 - \omega^2)\delta^{ij} - k^i k^j + \Pi^{ij}(K) \right] E^j(K) = i\omega J_{\text{ext}}^i(K)
\]

\[ [\Delta^{-1}(K)]^{ij} \]

Poles of the propagator \( \Delta(K) \rightarrow \text{dispersion relations} \)
Tensor basis

- Decompose self energy using tensor basis for anisotropic system:

\[ \Pi^{ij} = \alpha A^{ij} + \beta B^{ij} + \gamma C^{ij} + \delta D^{ij} \]

\[ A^{ij} = \delta^{ij} - k^i k^j / k^2 \]
\[ B^{ij} = k^i k^j / k^2 \]
\[ C^{ij} = \bar{n}^i \bar{n}^j / \bar{n}^2 \]
\[ D^{ij} = k^i \bar{n}^j + k^j \bar{n}^i \]

- In this basis the inverse propagator can be written as

\[ \Delta^{-1}(K) = (k^2 - \omega^2 + \alpha) A + (\beta - \omega^2) B + \gamma C + \delta D \]

- The propagator reads

\[ \Delta(K) = \Delta_A (A - C) + \Delta_G \left[ (k^2 - \omega^2 + \alpha + \gamma) B + (\beta - \omega^2) C - \delta D \right] \]

\[ \Delta_A^{-1}(K) = k^2 - \omega^2 + \alpha \]
\[ \Delta_G^{-1}(K) = (k^2 - \omega^2 + \alpha + \gamma)(\beta - \omega^2) - k^2 \bar{n}^2 \delta^2 \]

Direction with maximal instability

- Choose $\mathbf{k}$ and $\mathbf{\hat{n}}$ to be parallel.

- $\Rightarrow$ Only $\alpha$ and $\beta$ contributions which correspond to transverse and longitudinal modes, respectively.

- Find analytic solutions for both structure functions

- $\alpha$ - mode:

$$
\alpha(\omega, k, \xi, \nu) = \frac{m^2_D}{4} \frac{\sqrt{1 + \xi}}{k(1 + \xi Z^2)^2} \left\{ \left( k(z^2 - 1) - i\nu(1 + \xi z^2) - (\xi - 1)(\xi(1 + \xi z^2) - i\nu) \right) \ln \left[ \frac{z + 1}{z - 1} \right] \right. \\
\left. - \frac{i}{\sqrt{\xi}} \left[ z\nu(1 + (3 + z^2(1 - \xi))\xi) + ik(1 - \xi + z^2(1 + \xi(2z(\xi - 1) + \xi))) \right] \right\} \arctan \sqrt{\xi}
$$

$$
z = (\omega + i\nu)/k
$$
Solutions in $\omega/k$ - plane

- Poles for $\alpha$- and $\beta$- mode:
  \[
  k^2 - \omega^2 + \alpha = 0, \\
  \beta - \omega^2 = 0
  \]

No collisions and $k \parallel \hat{n}$

\[\Rightarrow\]

Anisotropic poles at positive $\xi$. 

\[\text{collisions and } k \parallel \hat{n}\]
Stable modes

• Transverse modes

• Longitudinal modes
β-mode: Riemann sheets

- Extend log to lower Riemann sheets:

$$\ln\left(\frac{z + 1}{z - 1}\right) = \ln\left(\left|\frac{z + 1}{z - 1}\right|\right) + i \left[\arg\left(\frac{z + 1}{z - 1}\right) + 2\pi N\right]$$
Unstable Modes

- Purely imaginary solution $\omega(k) = i\Gamma(k)$
- Growth rate $\Gamma(k)$ decreases with increasing $\nu$

Instabilities cease to exist above a critical collision rate.

For $\xi \to \infty$: $\nu_{\text{crit}} \approx 0.6 \, m_D$
Estimate for $\nu$

$$\nu = \nu_{\text{hard}} \sim \alpha_s^2 \log \alpha_s^{-1}$$

**Equilibrium** estimate (for pure glue):

$$\nu = 5.2 \alpha_s^2 T \log(0.25 \alpha_s^{-1})$$

negative for large $\alpha_s$

missing full calculation – let’s play games:

$$\nu \sim 5.2 \alpha_s^2 T \log(0.25 \alpha_s^{-1} + 1)$$


$$\alpha_s \approx 0.2 - 0.4 \Rightarrow \nu \approx 0.1 - 0.2 m_D$$
Summary of part I

• BGK-type model for the inclusion of collisions

• Modification of the stable modes

• Collisional damping of the unstable modes

• Existence of critical collision rate even in the extremely anisotropic limit

• For realistic $\alpha_s \approx 0.3 \Rightarrow \nu \approx 0.1 - 0.2 \, m_D$
Part II

Fermionic collective modes of an anisotropic QGP

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Overview of part II

- Wish to explore the fermionic collective modes
- Do not expect unstable modes within the HL approximation (no fermionic bound states)
- Proof explicitly that they do not exist
- For the anisotropic plasma, the fermionic self energy shows strong angular dependence
- Calculate self energies in the Keldysh picture. We are then able to calculate photon production from an anisotropic QGP
Quark hard loop self energy


\[ \Sigma(K) = \frac{C_F}{4} g^2 \int \frac{d^3p}{(2\pi)^3} \frac{f(p)}{|p|} \frac{P \cdot \gamma}{P \cdot K} \]

\[ k_0 \sim k \sim gp_{\text{hard}} \quad \text{and} \quad p_0 \sim p \sim p_{\text{hard}} \]

\[ f(p) = 2(n(p) + \bar{n}(p)) + 4n_g(p) \]

- Again squeeze isotropic distr. in one direction:

\[ f(p) = N(\xi) f_{\text{iso}} \left( p^2 + \xi(p \cdot \hat{n})^2 \right) \]
Directional dependence of the quark self energy

• The self energy has the simple structure:

\[ \Sigma(K) = \gamma^0 \Sigma_0 + \gamma \cdot \Sigma \]
Fermionic collective modes

Collective modes are determined by solving

$$\det S^{-1} = 0$$

with $$iS^{-1}(P) = \gamma^\mu p_\mu - \Sigma \equiv \gamma^\mu A_\mu$$

where we defined

$$A(K) = (k_0 - \Sigma_0, k - \Sigma)$$

Using

$$\det(\gamma^\mu A_\mu) = (A^\mu A_\mu)^2$$ and defining $$A_s^2 = A \cdot A$$

we can rewrite the equation to be solved to

$$A_0 = \pm A_s$$
Nyquist analysis

- We show for 2 special cases that there are no more collective modes than the 4 also present in the isotropic plasma.

- For \( \mathbf{k} \parallel \hat{\mathbf{n}} \): \( A_0 = \pm A_s \Rightarrow \omega - \Sigma_0 = \pm (k - \Sigma_z) \)

\[
\frac{1}{2\pi i} \int_C dz \frac{f'(z)}{f(z)} = N - P
\]

\( z = \omega / k \)

\( f(\omega, k, \xi) = \omega - \Sigma_0(\omega, k, \xi) \pm [k - \Sigma_z(\omega, k, \xi)] \)
Nyquist analysis II

- Yields $N=4: 2$ for positive and $2$ for negative $\omega$

- Same procedure for $\xi \to \infty$ and arbitrary angle: result: $N=4$

- For arbitrary angle and finite $\xi$ we look for additional solutions numerically and find none.

- Unstable fermionic modes do not exist within the HL approximation
Towards photon production

\[
q + \overline{q} \rightarrow g + \gamma \quad \text{annihilation} \quad q(\overline{q}) + g \rightarrow q(\overline{q}) + \gamma \quad \text{Compton scattering}
\]

\[
E \frac{dR}{d^3q} = \frac{i}{2(2\pi)^3} \Pi_{12}^{\mu}(Q)
\]

Soft part:

\[
i \Pi_{12}^{\mu} \sim \cdots
\]

\[
S^*_{12/21}(P)|_{HL} = S^*_{ret}(P)|_{HL} \Sigma_{12/21}(P) S^*_{adv}(P)|_{HL}
\]
We calculate the off diagonal HL self energies explicitly and find

\[ S_{12/21}^*(P) |_{HL} = S_{\text{ret}}^*(P) |_{HL} \Sigma_{12/21}(P) S_{\text{adv}}^*(P) |_{HL} \]

where \( \Sigma \) is the retarded self energy discussed before.

So we only need the retarded self energy to calculate the soft part of the non equilibrium photon production rate.

The result is ultraviolet divergent - the hard part will be infrared divergent – we expect the full result to be independent of the momentum cutoff as in the equilibrium calculation.
Summary of part II

- Showed numerically and analytically for 2 special cases that there are no unstable fermionic collective modes in the HL framework

- Calculated the off-diagonal parts of the self energy needed for off equilibrium photon production

- Found that $\Sigma_{12} = -\Sigma_{21} = -i \text{Im}\Sigma$

- Set the stage for the calculation of photon production from an anisotropic quark gluon plasma

- Outlook: Calculation of soft part is done – need to calculate the hard part and combine to get the full result