Degenerate fermionic matter at N^{3}LO

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T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Degenerate fermionic matter at N^{3}LO: Quantum Electrodynamics,* [2204.11893], T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Soft photon propagation in a hot and dense medium to next-to-leading order,* [2204.11279]

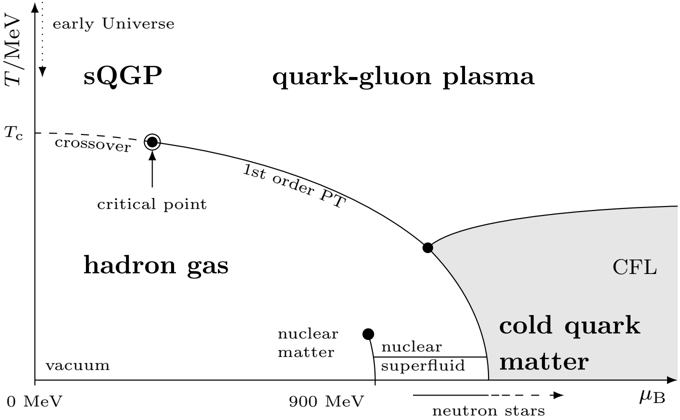
Motivation

Does deconfined matter exist inside neutron stars?

High-T: quark-gluon plasma in early universe or heavy-ion collision

High-μ: cold quark matter conjectured in neutron stars (NS)

Pressure (p) encodes bulk thermodynamics.



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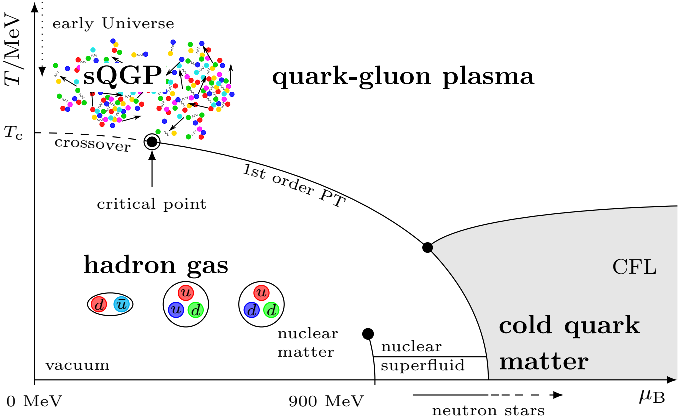
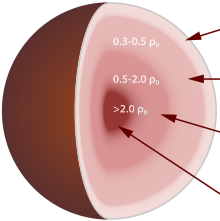


Dense strongly interacting matter

Thermodynamic properties of cold ($T = 0$) and dense ($\mu_B \neq 0$) matter.

Application: Neutron-star (NS) matter equation of state (EoS).

Link between micro and macro from GR and EoS:[1] $\varepsilon(p) \Rightarrow M(R)$.



[1] F. Özel, D. Psaltis, T. Guver, G. Baym, C. Heinke, and S. Guillot, *The Dense Matter Equation of State from Neutron Star Radius and Mass Measurements*, Astrophys. J. **820** (2016) 28 [1505.05155]

figure by P. Haensel, A. Y. Potekhin, and D. G. Yakovlev, *Neutron stars 1: Equation of state and structure*, vol. 326. Springer, New York, USA, 2007

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TOV equations:

$$\frac{\mathrm{d}M(r)}{\mathrm{d}r} = 4\pi r^2 \varepsilon(r) \; ,$$

$$\frac{\mathrm{d}p(r)}{\mathrm{d}r} = -\frac{G\varepsilon(r)M(r)}{r^2} \frac{\left[1 + \frac{p(r)}{\varepsilon(r)}\right]\left[1 + \frac{4\pi r^3 p(r)}{M(r)}\right]}{\left[1 - \frac{2GM(r)}{r}\right]} \; .$$

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figure by E. Annala, T. Gorda, E. Katerini, A. Kurkela, J. Nättilä, V. Paschalidis, and A. Vuorinen, *Multimessenger Constraints for Ultradense Matter,* Phys. Rev. X **12** (2022) 011058 [2105.05132]

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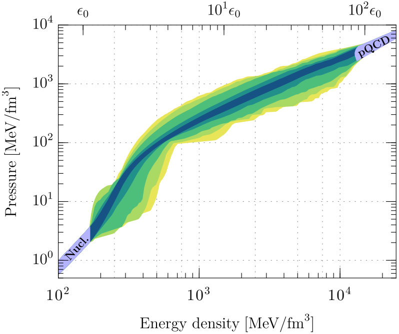

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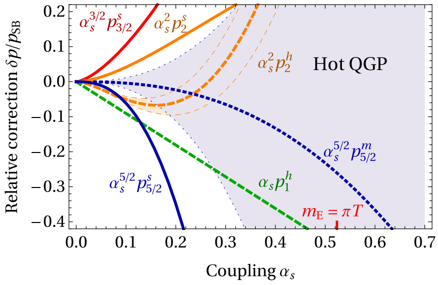
At finite μ pQCD is the only reliable first-principles method

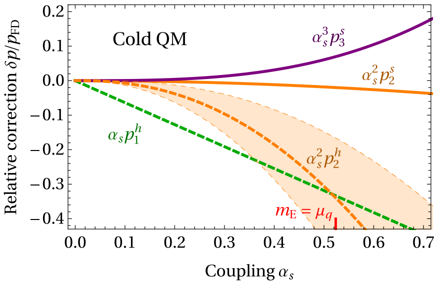
Finite T

- pQCD at high T
- Lattice QCD applicable
- Pressure well understood in broad T range

Finite μ

- pQCD at high μ
- Sign problem $\Rightarrow$ lattice QCD not applicable





figures by T. Gorda, A. Kurkela, R. Paatelainen, S. Säppi, and A. Vuorinen, *Soft Interactions in Cold Quark Matter*, Phys. Rev. Lett. **127** (2021) 162003 [2103.05658]

Cold and dense field theory

Framework for calculating dense pQCD pressure

❶ Generate Feynman diagrams from partition function.

$$p(\mu) \sim \ln \mathcal{Z} = \ln \int \mathcal{D}\bar{\psi}\psi\bar{c}cA\, e^{-S_{\rm QCD}}$$

Imaginary-time formalism[3] for thermodynamics and static quantities.

❷ Evaluate master integrals in $D = 4 - 2\epsilon$ dimensions. UV-finite physical quantity.

IR divergences cancel non-trivially – needs work.

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[3] I. Ghisoiu, T. Gorda, A. Kurkela, P. Romatschke, M. Säppi, and A. Vuorinen, *On high-order perturbative calculations at finite density,* Nucl. Phys. B **915** (2017) 102 [1609.04339]

IR divergences handled via effective field theory (EFT)

Finite T

3 scales:

- πT Hard. Full-theory
- gT Soft. Dimensionally reduced EFT for chormo-electric fields.
- g^2T Ultrasoft. Non-perturbative lattice EFT for chromo-magnetic fields.

Finite μ

2 scales:

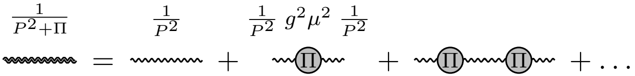
- μ Hard. Full-theory
- $g\mu$ Soft. Hard thermal loop (HTL) EFT for gluon fields.

No thermal excitation of gluons at $T = 0$. No Linde IR problem.[4]

[4] A. Linde, *Infrared problem in the thermodynamics of the Yang-Mills gas*, Phys. Lett. B **96** (1980) 289

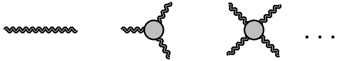
Hard thermal loop (HTL) resummation for soft gluons

All-loop resummation via HTL effective theory[5]



HTL gluon self-energy Π has LO $\sim g^2\mu^2$ contribution. HTL self-energy is the dominant contribution to a self-energy for *soft external momenta* $(P \sim g\mu \ll \mu)$.

Soft gluon propagators with $P \sim g\mu$ must be resummed.
HTL vertex functions must be resummed:



Loop and coupling expansion do not align $\Rightarrow$
Perturbative series contains non-analytic terms as $\ln g$, $\ln^2 g$.

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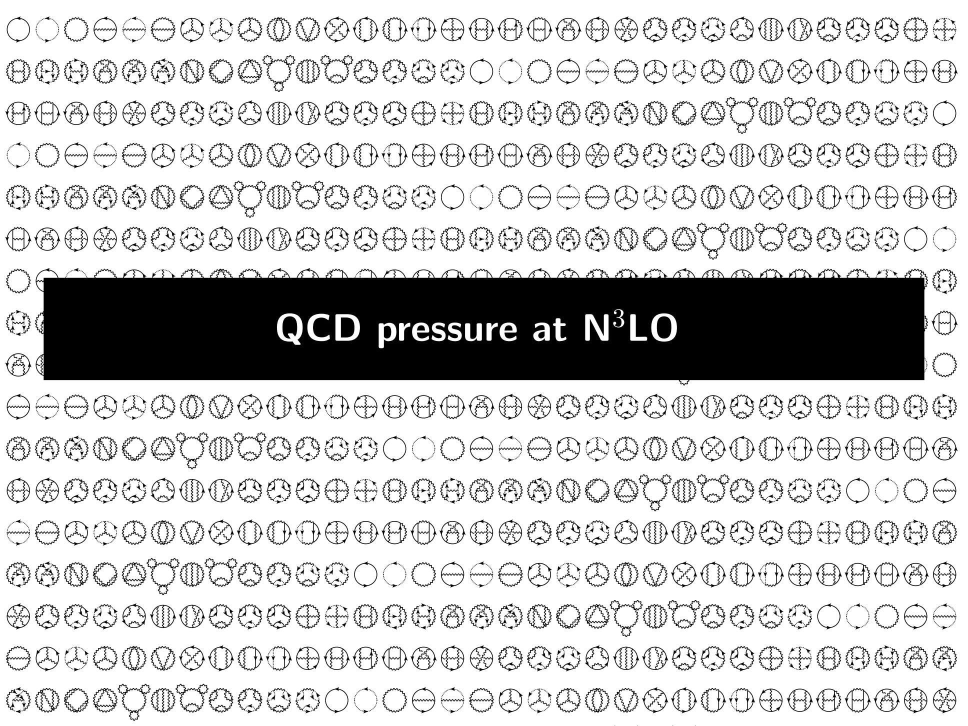
[5] E. Braaten and R. D. Pisarski, *Soft amplitudes in hot gauge theories: A general analysis,* Nucl. Phys. B **337** (1990) 569



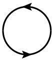
QCD pressure at N^{3}LO

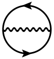
Structure of 3-loop pressure (N^2LO) at finite μ

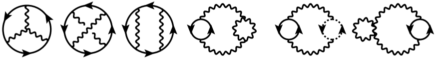
N^2LO result know for long time[6]

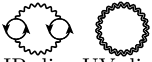
$$p(\mu) = a_0 + a_1 g^2 + a_{2,1} g^4 \ln g + a_{2,0} g^4 + \mathcal{O}(g^6)\ .$$

 = soft gluon,  = hard gluon,  = fermion

$a_0 =$  (free Fermi pressure)

$a_1 =$ 

$a_{2,0} \supset$  (IR safe)

$a_{2,0}, a_{2,1} \supset$ 

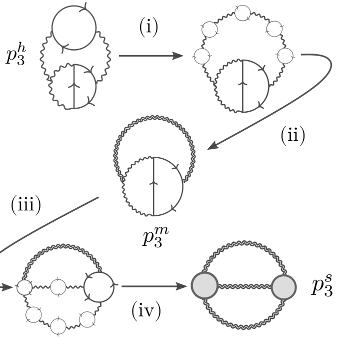
IR div. UV div.

$1/\epsilon$ cancel and finite $\ln g$ contribution from lifted divergence $\epsilon \times 1/\epsilon$.

[6] B. A. Freedman and L. D. McLerran, *Fermions and gauge vector mesons at finite temperature and density. I. Formal techniques,* Phys. Rev. D **16** (1977) 1130

HTL resummation at 4-loop level[7] (N^3LO)

(i) Hard sector (start). One soft gluon in IR divergent hard diagram. Dressed with HTL self-energies.
(ii) Mixed sector: HTL self-energies (resummed propagator)
(iii) Two soft gluons. More HTL self-energy and vertex insertions.
(iv) Soft sector: Fully HTL-resummed diagrams (2-loop)



[7] T. Gorda, A. Kurkela, R. Paatelainen, S. Säppi, and A. Vuorinen, *Soft Interactions in Cold Quark Matter*, Phys. Rev. Lett. **127** (2021) 162003 [2103.05658]

State-of-the-art pQCD pressure

Contains: soft contributions[8] at $\mathcal{O}(g^6)$.

Missing: mixed and hard contributions

[8] T. Gorda, A. Kurkela, R. Paatelainen, S. Säppi, and A. Vuorinen, *Soft Interactions in Cold Quark Matter*, Phys. Rev. Lett. **127** (2021) 162003 [2103.05658]

Quantum electrodynamics (vanilla QCD)

Remainder of the talk: QED.

N^{3}LO contribiton (e^6) to QED pressure with soft/mixed/hard organisation

$$\alpha^3 p_3 = \alpha^3 (p_3^s + p_3^m + p_3^h) \, ,$$

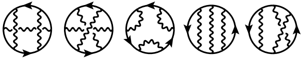
differs from N^{2}LO and contains $\ln^2 \alpha$ term

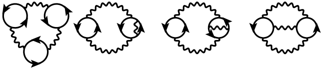
$$\alpha^3 p_3 = \alpha^3 \big(a_0 + a_1 \ln \alpha + a_2 \ln^2 \alpha \big) \, .$$

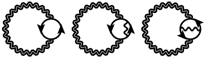
Simplification: In QED photons do not self-interact, HTL vertex functions vanish *viz.* no fully soft parts $p_3^s = 0 \Rightarrow a_2 = 0$.

The result is given by **mixed** p_3^m and **hard** p_3^h contributions.

Contributions to N^{3}LO QED pressure

$\alpha_e^3 p_3^h =$  (IR safe)

$\alpha_e^3 p_3^h =$  (IR div.)

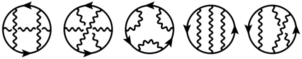
$\alpha_e^3 p_3^m =$  (UV div.)

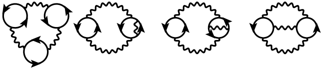
Divergences cancel in $\alpha_e^3 p_3^m + \alpha_e^3 p_3^{h,\text{IR div}}$ and yield $\alpha_e^3 \ln \alpha_e$ coefficient.

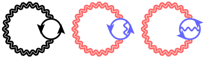
Diagrams $\alpha_e^3 p_3^m$ contain 2-loop self-energy insertions with soft external photons.

Next step: extend HTL photon self-energy from LO $\rightarrow$ NLO.

Contributions to N^{3}LO QED pressure

$\alpha_e^3 p_3^h =$  (IR safe)

$\alpha_e^3 p_3^h =$  (IR div.)

$\alpha_e^3 p_3^m =$  (UV div.)

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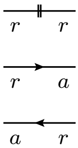
Intermezzo: NLO photon self-energy

(Non)-equilibrium thermodynamics: Real-time formalism

$\rho(\beta) = e^{-\beta\mathcal{H}} \to \mathcal{U}(t) = e^{-i\mathcal{H}t}$. Relating density operator to time evolution corresponds to path integral over complex time contour.[9]

Contour-ordered propagators and self-energies are 2×2 matrices ($\mathbf{D}$). Elements have manifest causality properties in r/a basis:



$$= D^{rr}(P)$$
$$= D^{\mathrm{R}}(P)$$
$$= D^{\mathrm{A}}(P)$$

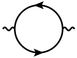
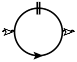
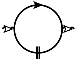
$$\mathbf{D} = \begin{pmatrix} \langle\phi_r\phi_r\rangle & \langle\phi_r\phi_a\rangle \\ \langle\phi_a\phi_r\rangle & \langle\phi_a\phi_a\rangle \end{pmatrix} = \begin{pmatrix} D^{rr} & D^{\mathrm{R}} \\ D^{\mathrm{A}} & 0 \end{pmatrix}$$

Framework tailored for calculating n-point functions:

- ▷ Minkowskian signature. Analytic continuation not required.
- ▷ μ-dependence is manifest in propagators by distribution functions $D^{rr}(P) \supset n_{\mathrm{B/F}}(p^0)$; cf. Euclidean $P^\alpha = (p^0 - i\mu, \mathbf{p})$.

[9] J. Ghiglieri, A. Kurkela, M. Strickland, and A. Vuorinen, *Perturbative thermal QCD: Formalism and applications*, Phys. Rep. **880** (2020) 1 [2002.10188]

HTL photon self-energy at LO[10]

$\Pi^{\mathrm{LO}}_{\mu\nu} \sim$  $=$  $+$ 

External momentum K is soft, $K \sim e\mu \ll \mu$

HTL limit: leading term in $\frac{K}{\mu}$ expansion.

Broken Lorentz symmetry: $\Pi_{\mu\nu}$ splits into transverse (T) and longitudinal (L) components

$$\Pi_{\mu\nu} = \mathbb{P}^{\mathrm{T}}_{\mu\nu}\Pi_{\mathrm{T}} + \mathbb{P}^{\mathrm{L}}_{\mu\nu}\Pi_{\mathrm{L}}$$

$$\Pi^{\mathrm{LO}}_{\mathrm{T}} = \frac{e^2}{2}\left(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\right)\left[\frac{k_0^2}{k^2} + \left(1 - \frac{k_0^2}{k^2}\right)\frac{k^0}{2k}\log\frac{k^0 + k + i\eta}{k^0 - k + i\eta}\right],$$

$$\Pi^{\mathrm{LO}}_{\mathrm{L}} = e^2\left(\frac{T^2}{3} + \frac{\mu^2}{\pi^2}\right)\left(1 - \frac{k_0^2}{k^2}\right)\left[1 - \frac{k^0}{2k}\log\frac{k^0 + k + i\eta}{k^0 - k + i\eta}\right].$$

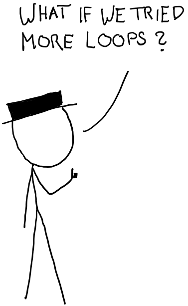
[10] e.g. J. I. Kapusta and C. Gale, *Finite-Temperature Field Theory*. Cambridge University Press, Cambridge, Jan, 2006

HTL photon self-energy at higher orders

Expansion in small momentum K and coupling e

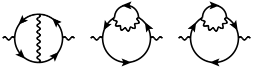
$$\Pi(K^2) = \sum_{n=0}^{\infty} [K^2]^n \sum_{\ell=1}^{\infty} e^{2\ell} \Pi_\ell^{(n)}(0)$$

$$\begin{aligned} \text{LO} : n+\ell = 1 \quad & n=0, \ell=1 \quad \mathcal{O}(e^2\mu^2) \\ \text{NLO} : n+\ell = 2 \quad & n=1, \ell=1 \quad \mathcal{O}(e^4\mu^2) \quad \text{(1-loop, power correction)} \\ & n=0, \ell=2 \quad \mathcal{O}(e^4\mu^2) \quad \text{(2-loop, difficult)} \end{aligned}$$



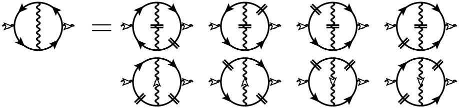
Computing the 2-loop photon self-energy

Need HTL limit (soft external line) of three 2-loop diagrams

$$\Pi^{\text{2loop}}_{\mu\nu} =$$ 

recently finished computation[11] including finite μ.

In real-time formalism have new analytic structure and lots of contributions:



Automation possible!

[11] K. Seppännen, *HTL Self-energies in Hot and Dense QCD,* Master's thesis, Helsinki U., 2021 T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Soft photon propagation in a hot and dense medium to next-to-leading order,* [2204.11279]

Results

2-loop part[12] of NLO results are $\mathcal{O}(e^4\mu^2)$

$$\Pi_{\mathrm{T}}^{\mathrm{NLO}} = -\frac{e^4\mu^2}{8\pi^4}\frac{k^0}{2k}\log\frac{k^0+k+i\eta}{k^0-k+i\eta}\,,$$

$$\Pi_{\mathrm{L}}^{\mathrm{NLO}} = -\frac{e^4\mu^2}{8\pi^2}\left\{1+2\left(1-\frac{k_0^2}{k^2}\right)\left[1-\frac{k^0}{2k}\log\frac{k^0+k+i\eta}{k^0-k+i\eta}\right]^2\right\}.$$

Including $\mathcal{O}(\epsilon)$ terms to lift $1/\epsilon$ pressure contributions.

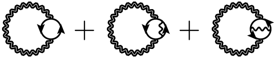
- UV finite since renormalisation $Z_e = Z_\psi$ in QED
- Gauge-independent

[12] T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Soft photon propagation in a hot and dense medium to next-to-leading order,* [2204.11279]

Calculating the N^{3}LO QED pressure[13]

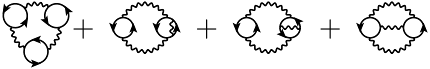
Mixed diagrams: NLO self-energy × HTL-resummed propagator (done)

$$\alpha_e^3 p_3^m =$$



Hard IR-sensitive 4-loop diagrams

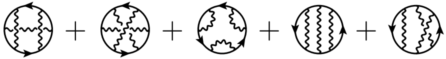
$$\alpha_e^3 p_3^{h,\mathrm{IR\ div}} =$$



contain divergences (cancel with mixed diagrams), explicit $\bar{\Lambda}$ logarithms (done), N_f^3 contribution (done).

Hard IR-safe 4-loop diagrams

$$\alpha_e^3 p_3^{h,\mathrm{IR\ safe}} =$$



almost computed to N^{3}LO and contain explicit $\bar{\Lambda}$ logarithms (done) and a pure number suppressed by N_f.

[13] T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Degenerate fermionic matter at N^{3}LO: Quantum Electrodynamics*, [2204.11893]

Results: N^{3}LO correction to QED pressure

Update[14] 45-year-old result[15] ($\delta \simeq -0.8563832$)

$$\frac{p_3}{p_{\rm LO}} = N_{\rm f}^2\left(\frac{\alpha_e}{\pi}\right)^3\left[a_{3,1}\ln^2\left(N_{\rm f}\frac{\alpha_e}{\pi}\right) + a_{3,2}\ln\left(N_{\rm f}\frac{\alpha_e}{\pi}\right) + a_{3,3}\ln\left(N_{\rm f}\frac{\alpha_e}{\pi}\right)\ln\frac{\bar\Lambda}{2\mu} + a_{3,4}\ln^2\frac{\bar\Lambda}{2\mu} + a_{3,5}\ln\frac{\bar\Lambda}{2\mu} + a_{3,6}\right]$$

$a_{3,1}$	0
$a_{3,2}$	$-\frac{5}{4} + \frac{33}{2}N_{\rm f}^{-1} + \frac{1}{48}\left(7 - 60N_{\rm f}^{-1}\right)\pi^2$
$a_{3,3}$	2
$a_{3,4}$	$-\frac{2}{3}$
$a_{3,5}$	$-\frac{79}{9} + \frac{2}{3}\pi^2 + \frac{2}{3}(13 - 8\ln 2)\ln 2 + \delta - \frac{31}{4}N_{\rm f}^{-1}$
$a_{3,6}$	$1.02270(2) + \left(2.70082 + \frac{1}{2}c_{0,1}\right)N_{\rm f}^{-1} + \frac{1}{2}c_{0,2}N_{\rm f}^{-2}$

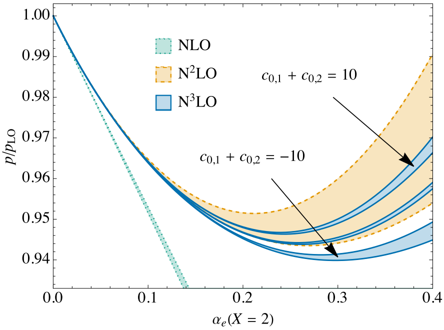
[14] T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Degenerate fermionic matter at N^{3}LO: Quantum Electrodynamics,* [2204.11893]

[15] B. A. Freedman and L. D. McLerran, *Fermions and gauge vector mesons at finite temperature and density. I. Formal techniques,* Phys. Rev. D **16** (1977) 1130

Cold and dense QED pressure up to N^{3}LO

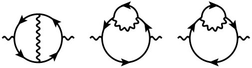
For physical QED, $N_\mathrm{f} = 1$. Renormalisation scale $\bar{\Lambda} = X\mu$, varied $X \in (2^0, \ldots, 2^2)$.

Dramatic decrease in $\bar{\Lambda}$-dependence.



Next step: Extend to QCD

1. Generalise NLO photon self-energy to QCD
   - ▷ Fully soft 1-loop diagrams
   - ▷ Gauge independent(?)

$\Pi_{\mu\nu}^{\text{2loop}} =$ 

2. Generalise N^3LO QED pressure to QCD
   - ▷ Finite soft sector[16]
   - ▷ Mixed and hard sectors

3. Accuracy of dense pQCD pressure improved

[16] T. Gorda, A. Kurkela, R. Paatelainen, S. Säppi, and A. Vuorinen, *Soft Interactions in Cold Quark Matter*, Phys. Rev. Lett. **127** (2021) 162003 [2103.05658]

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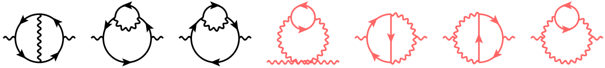

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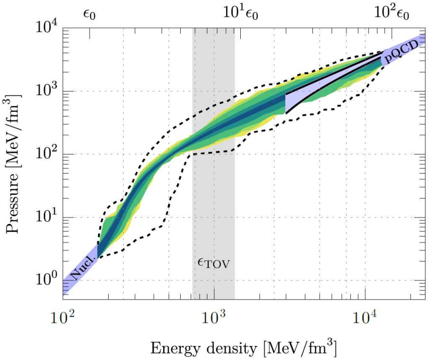
Conclusions

- ☆ Finite-μ QCD pressure pushed to similar accuracy as known from finite-T[17]
- ☆ pQCD result constrains[18] neutron-star matter EoS



[17] T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Degenerate fermionic matter at N^{3}LO: Quantum Electrodynamics,* [2204.11893], T. Gorda, A. Kurkela, J. Österman, R. Paatelainen, S. Säppi, P. Schicho, K. Seppänen, and A. Vuorinen, *Soft photon propagation in a hot and dense medium to next-to-leading order,* [2204.11279]

[18] E. Annala, T. Gorda, E. Katerini, A. Kurkela, J. Nättilä, V. Paschalidis, and A. Vuorinen, *Multimessenger Constraints for Ultradense Matter,* Phys. Rev. X **12** (2022) 011058 [2105.05132]