March 2^{nd} 2023

The Stability-Causality Theorem

Lorenzo Gavassino

Based on:

Phys. Rev. X **12**, 041001 **(2022)**

VANDERBILT UNIVERSITY

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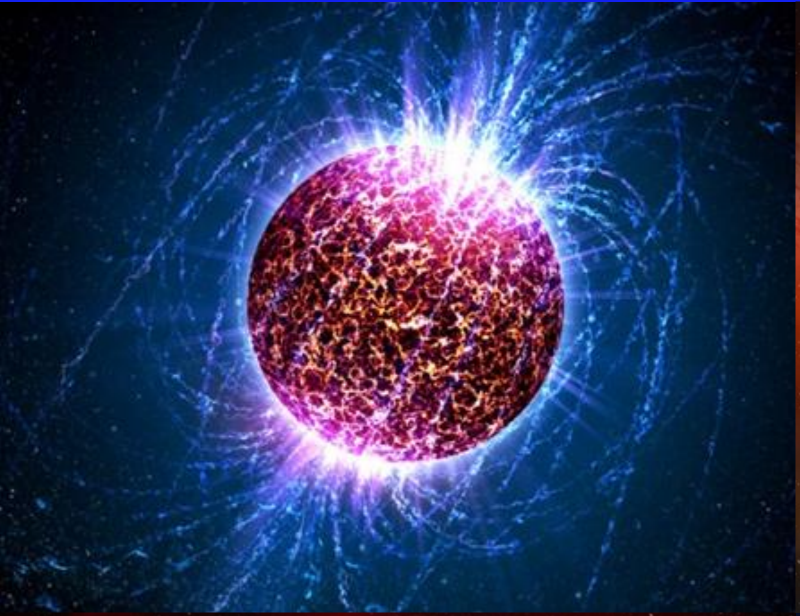
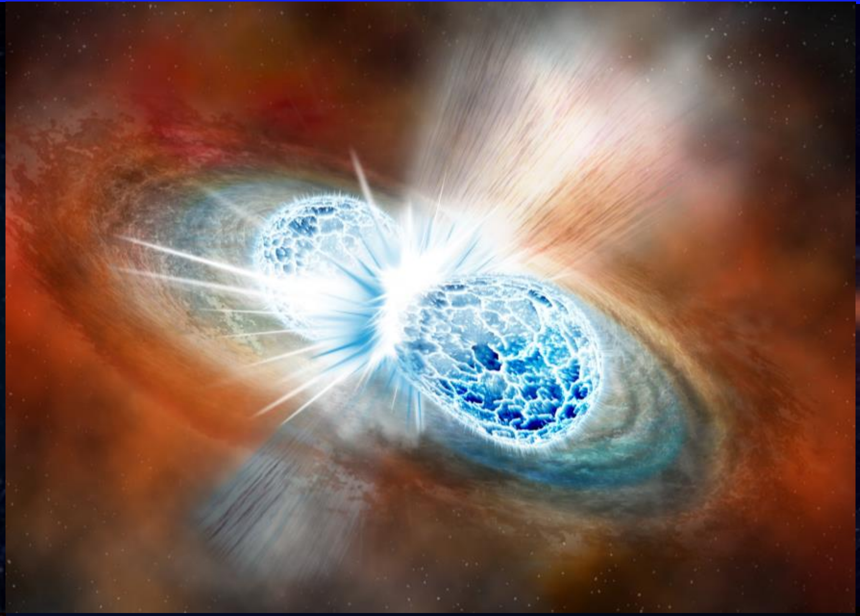
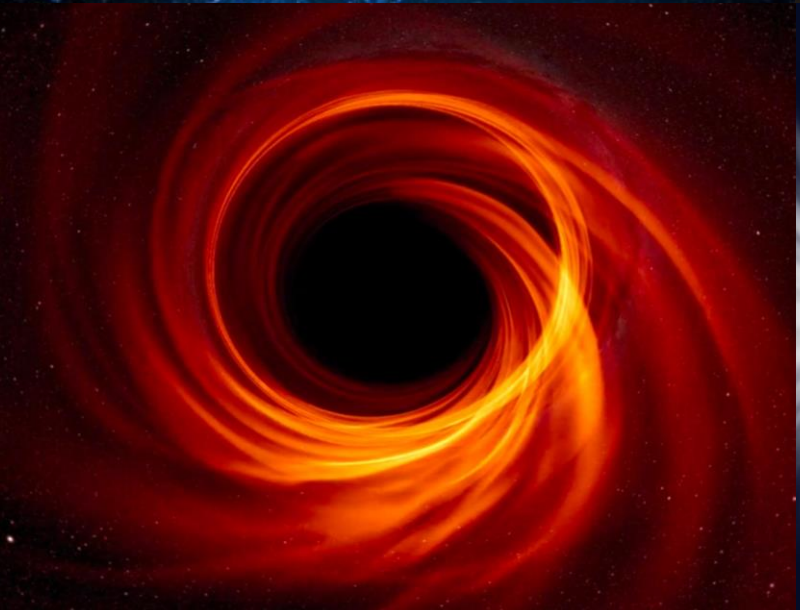
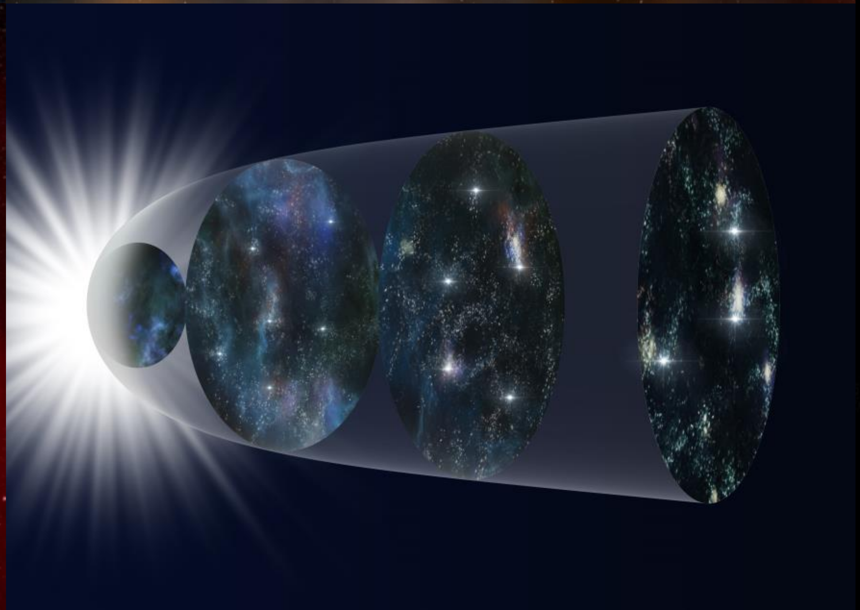
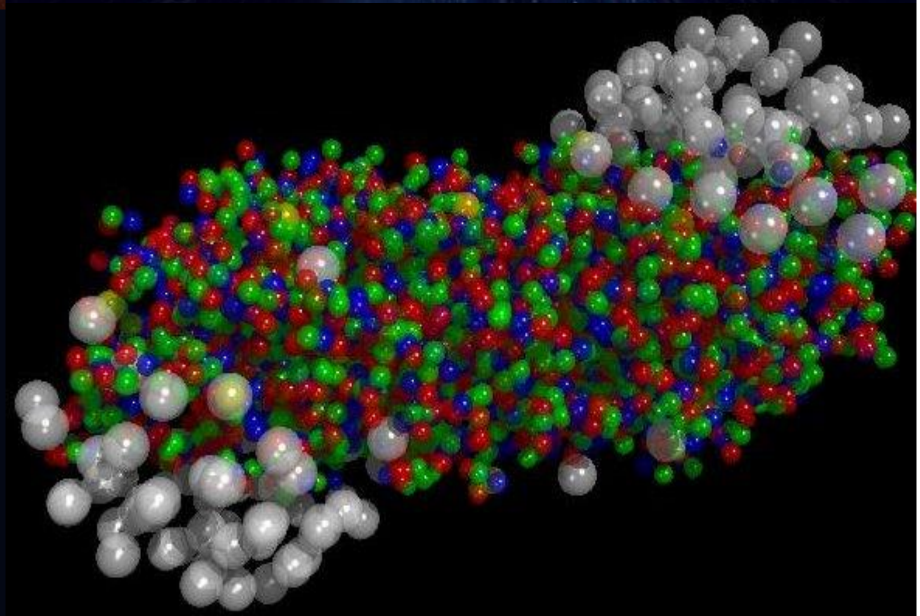
- Polish National Science Centre grant OPUS 2129/33/B/ST9/00942
- Vanderbilt Seeding Success Grant

Nicolaus Copernicus Astronomical Center of the Polish Academy of Sciences

We need relativistic hydro!

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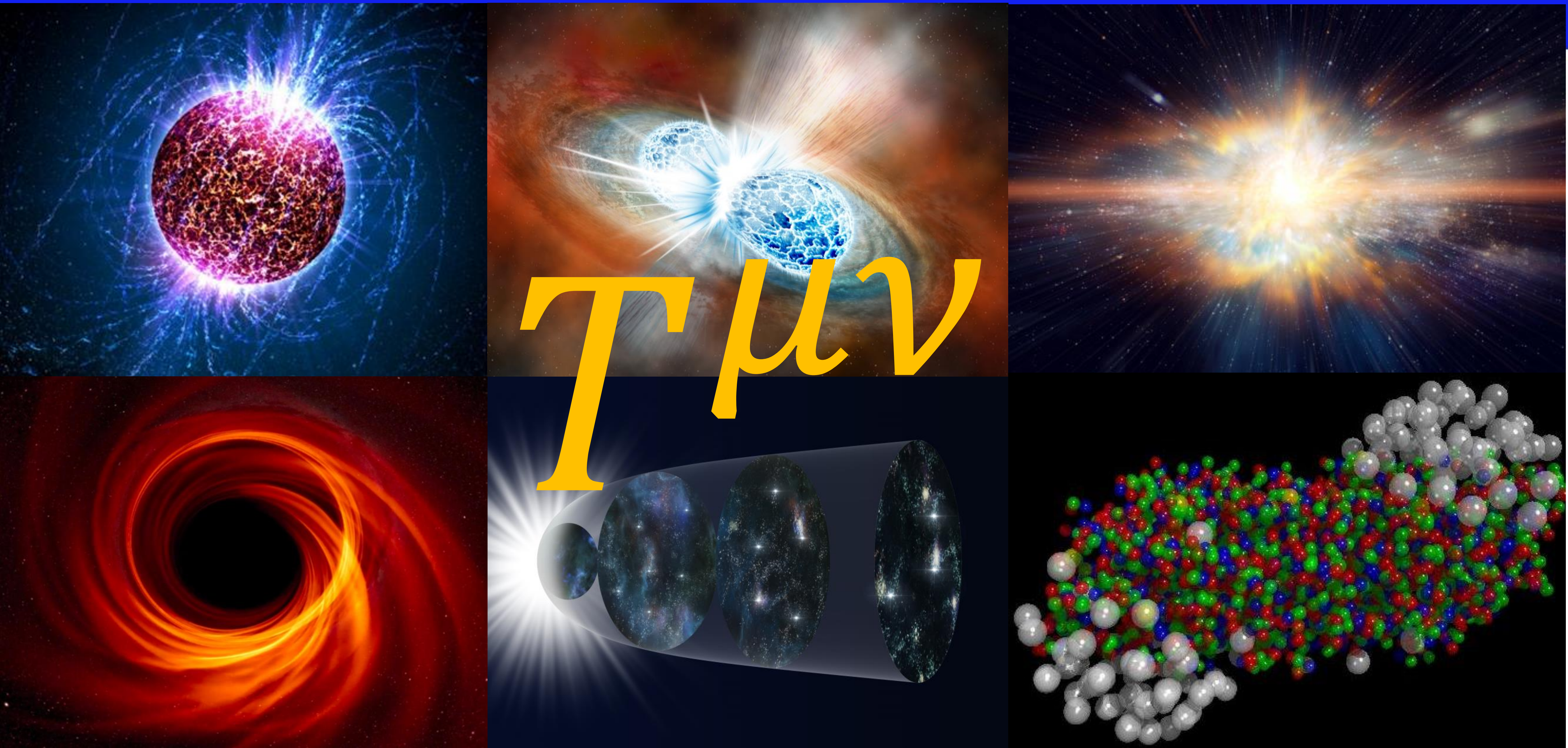

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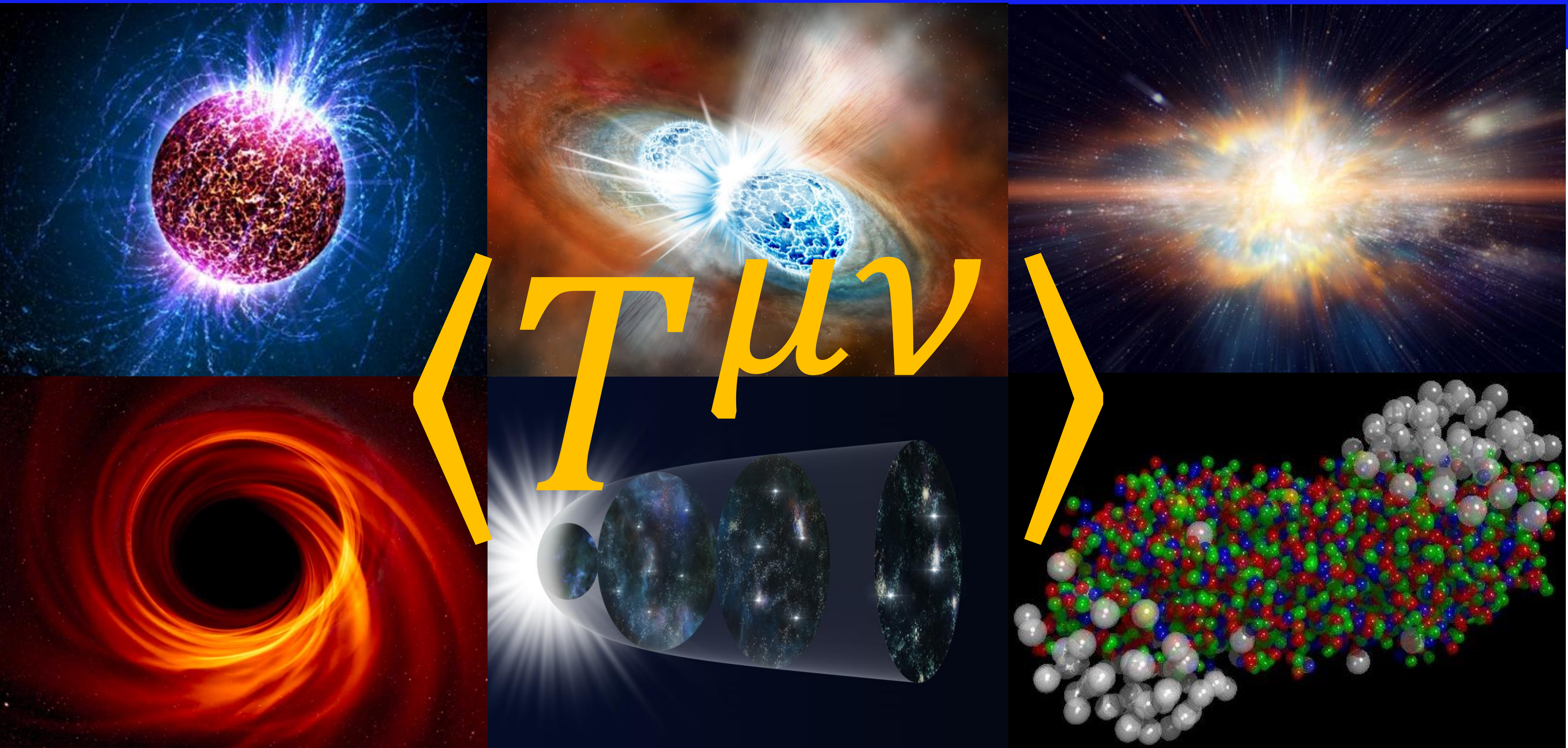
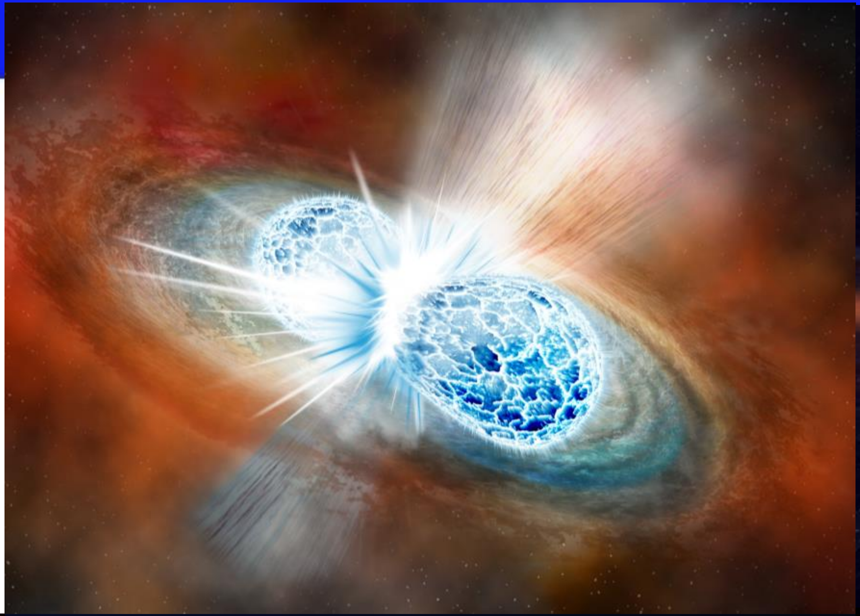
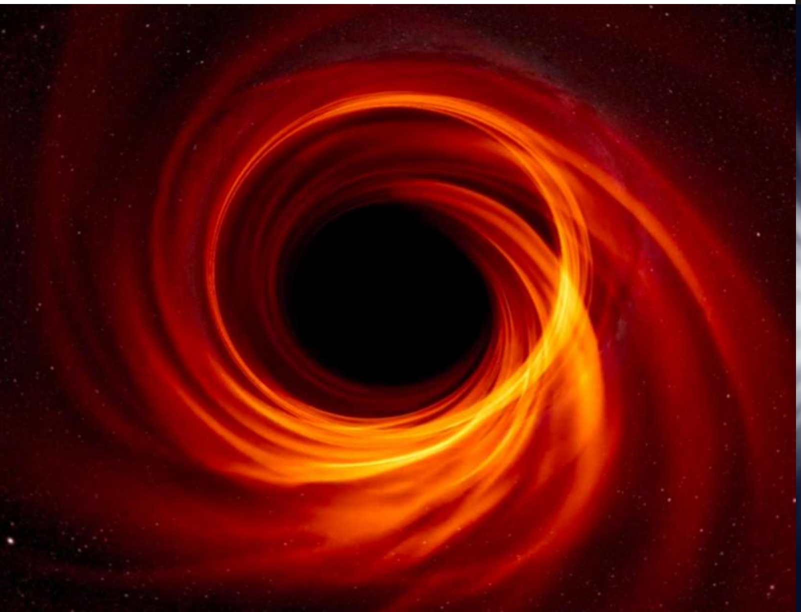
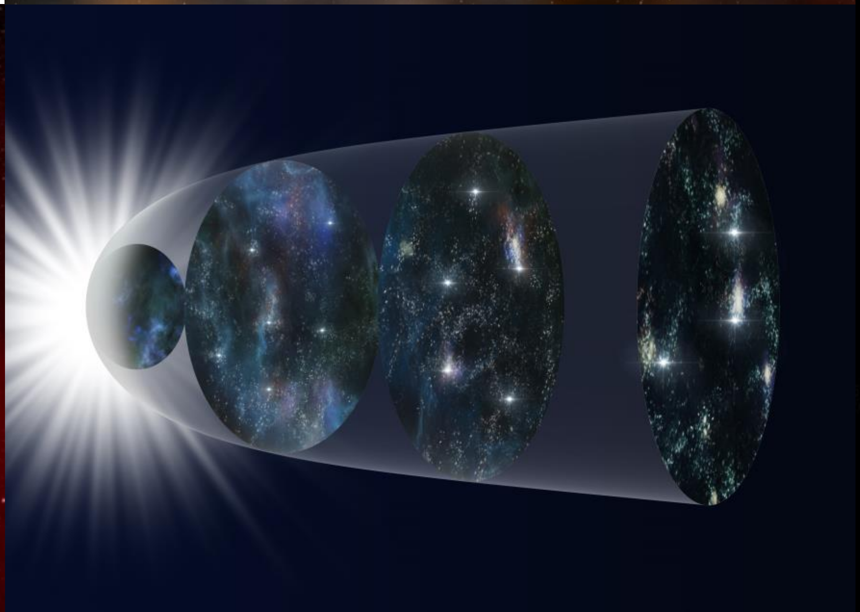
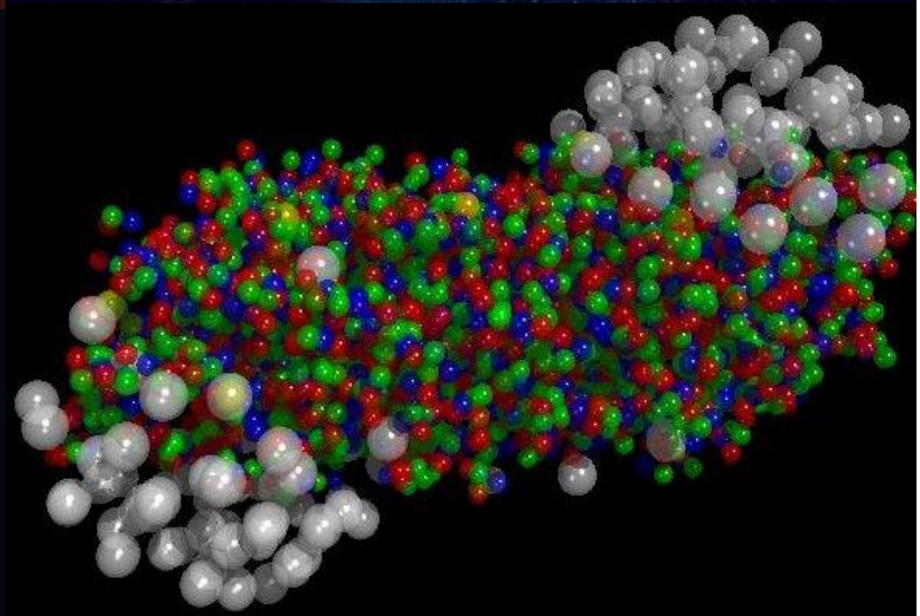


We need relativistic hydro!

- Superfluids
- Superconductors
- Solids
- Supersolids



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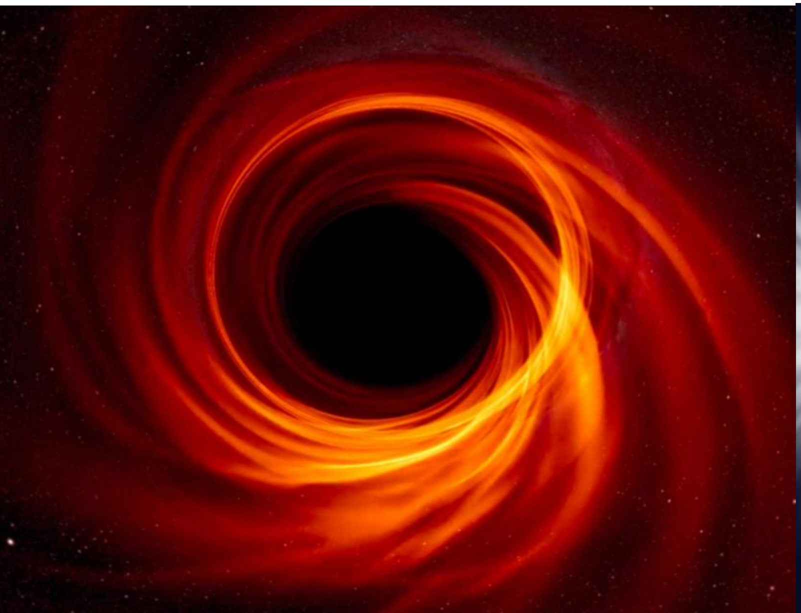
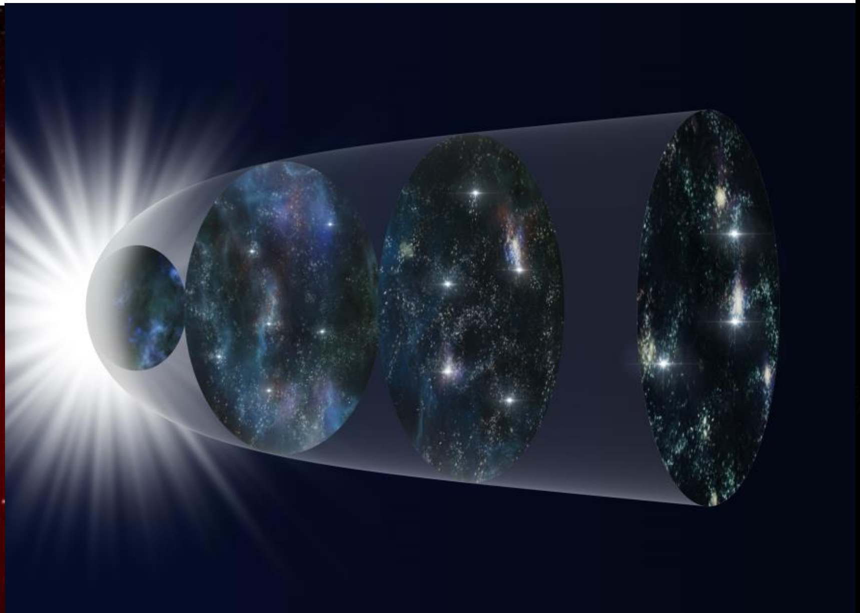
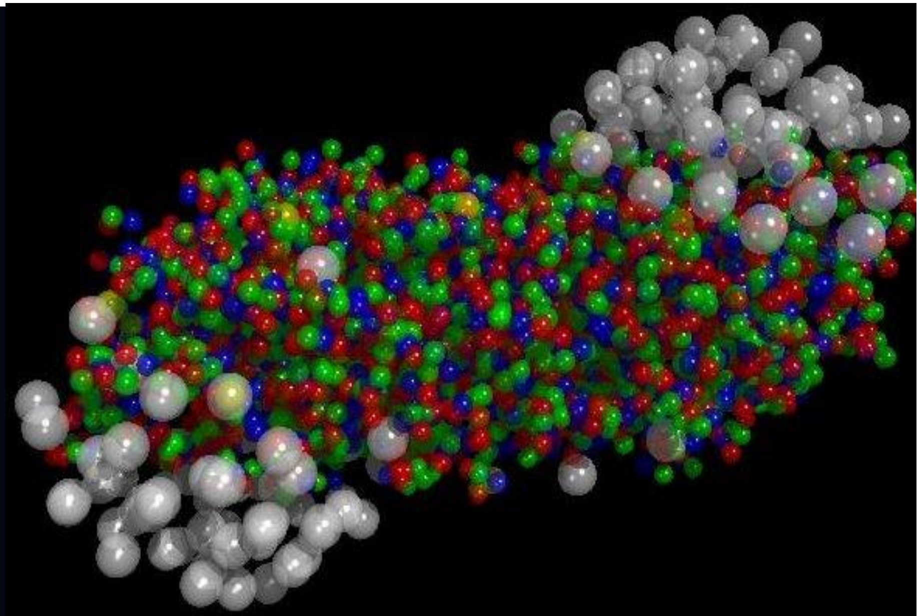
- Chemistry
- Coupling with neutrino radiation
- Phase transitions







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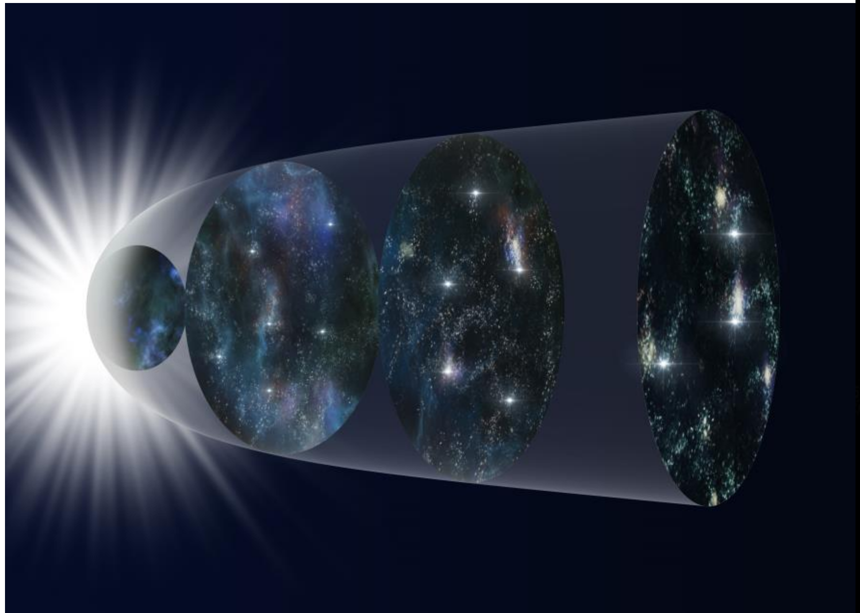
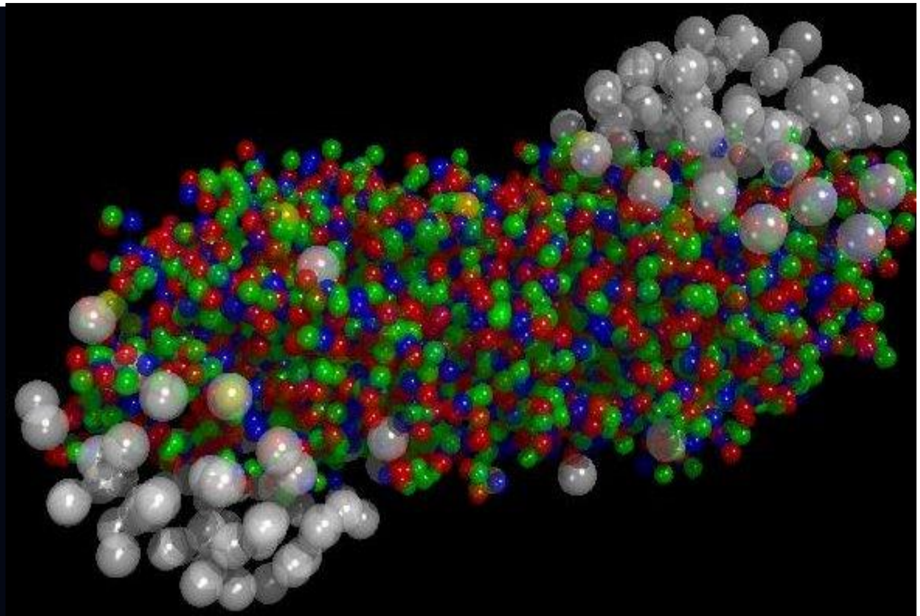
- MHD
- Magnetization
- Radiation hydro
- Plasmas
- Two temperatures





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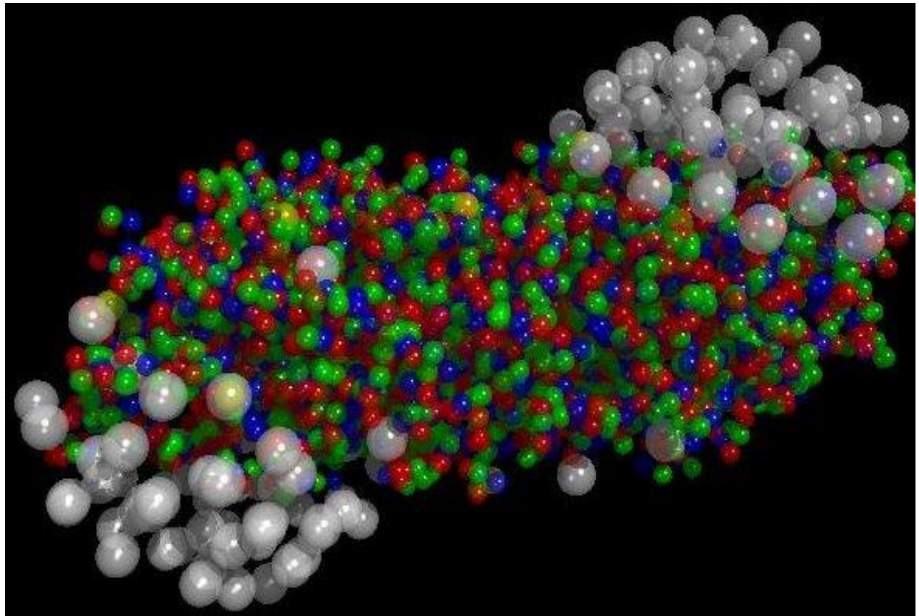
- Bulk viscosity far from equilibrium
- Multifluids



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- Shear viscosity
- Exotic degrees of freedom (holography)
- Spin hydro

We need relativistic DISSIPATIVE hydro!

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- Solids
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- Phase transitions

Dissipative

- MHD
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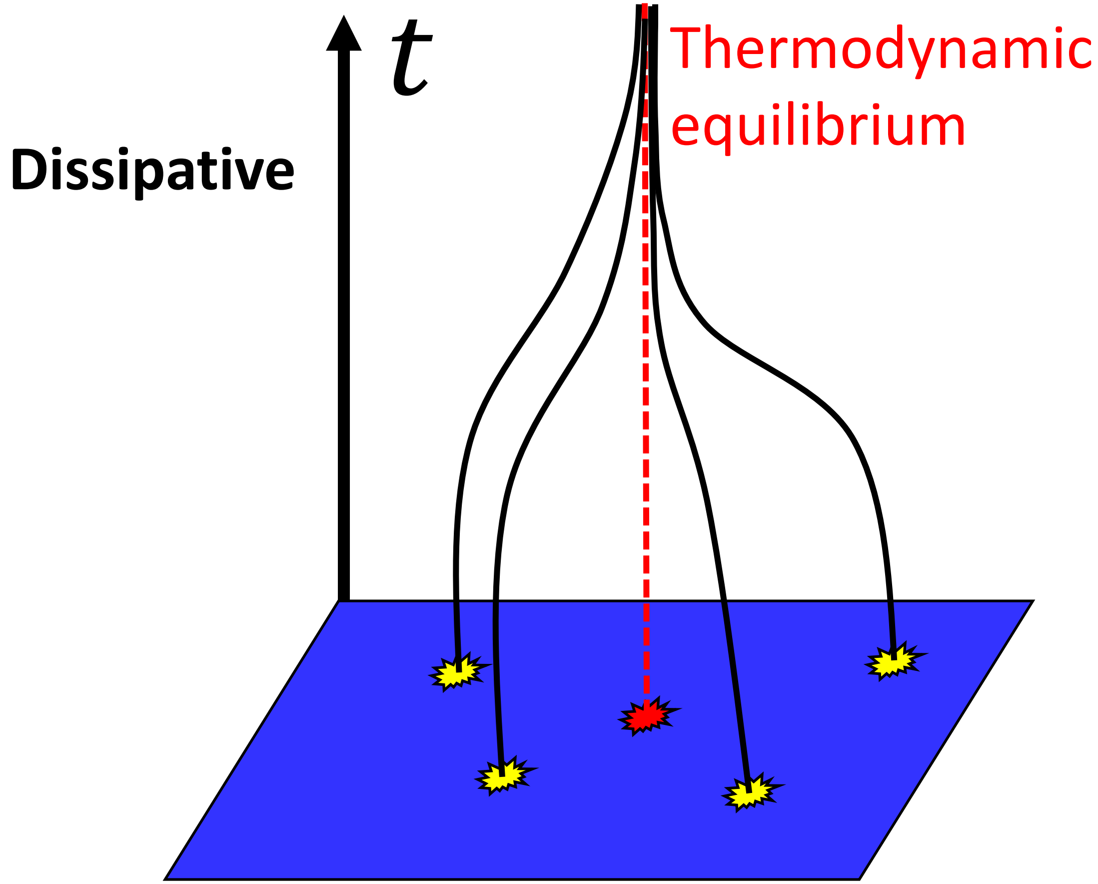
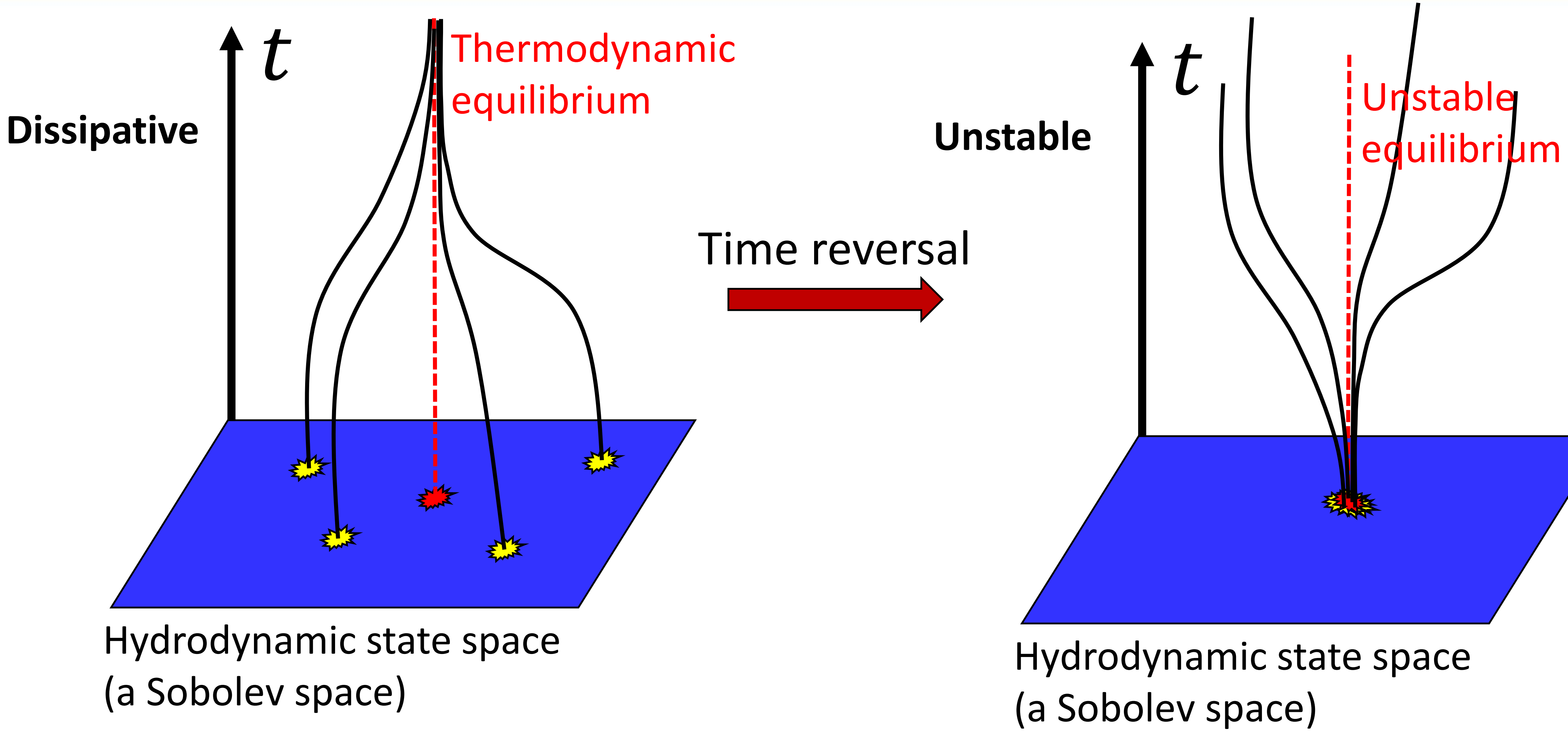
What does "dissipation" mean?

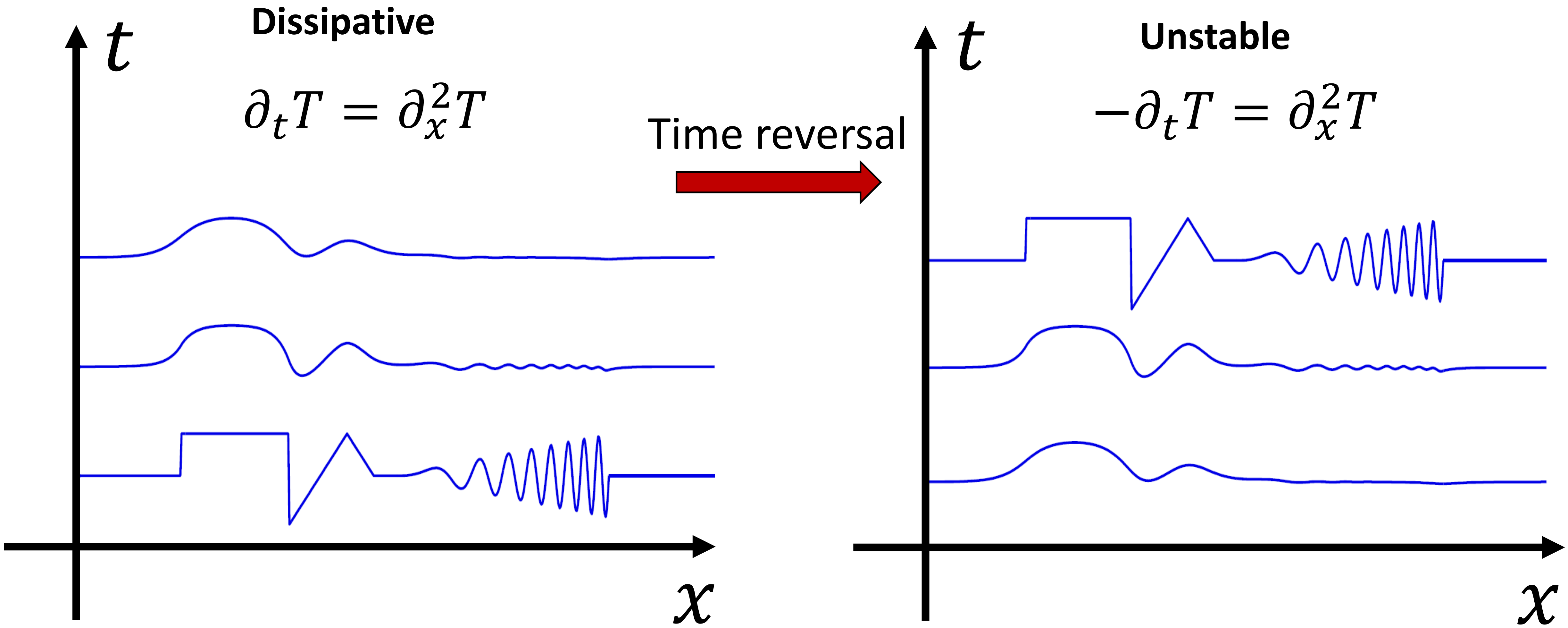


Hydrodynamic state space
(a Sobolev space)

Dissipation is irreversibility



Heat equation vs Backward heat equation



Goal: Implement this behaviour in relativity

We know that it is possible (Israel-Stewart, divergence-type, BDNK,…).

But for some reason it is hard!

There is something about dissipation that makes it “delicate”.

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We know that it is possible (Israel-Stewart, divergence-type, BDNK,…).
But for some reason it is hard!
There is something about dissipation that makes it “delicate”.

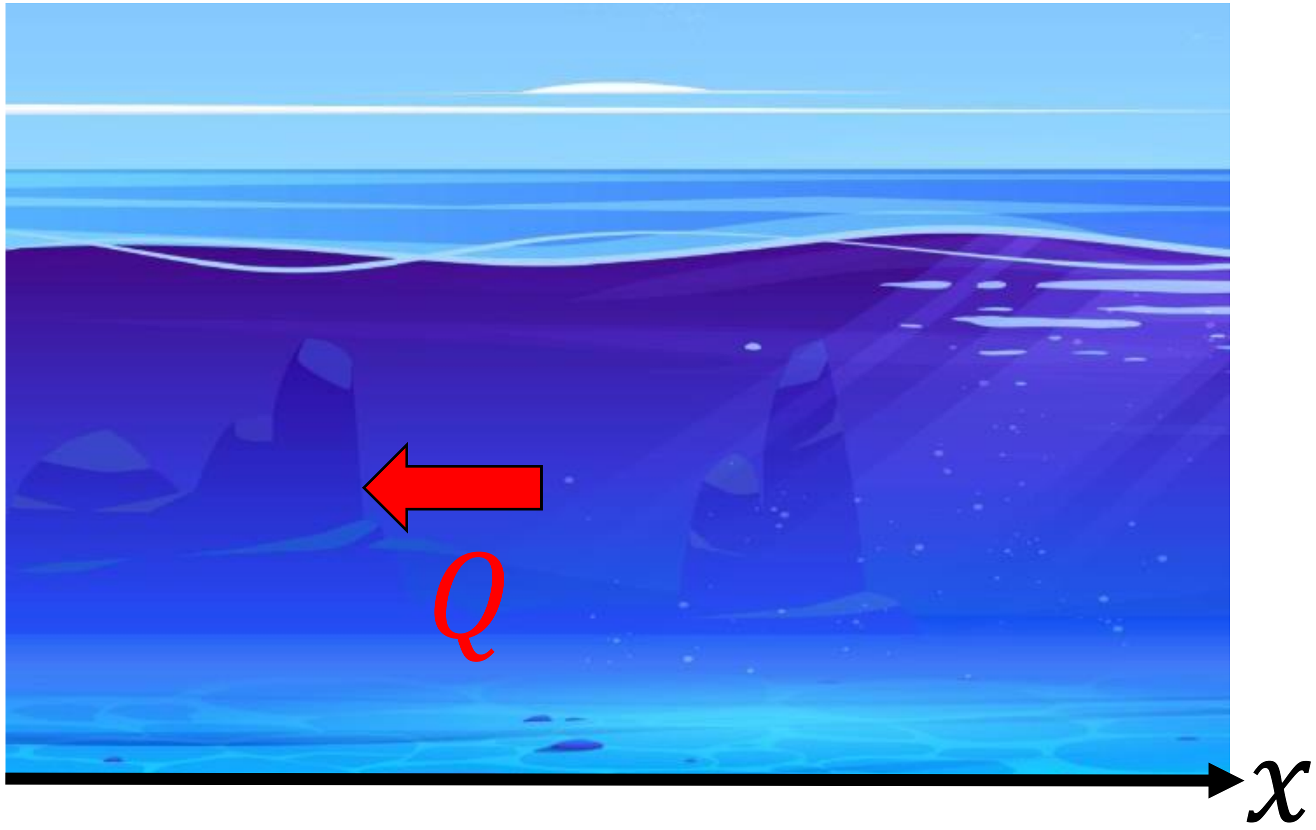
Example:

- Misner-Thorne-Wheeler (“Gravitation”): Exercise 22.7;
- Weinberg (“Gravitation and cosmology”): Pages 53-58;

They present the theory of Eckart, the first relativistic viscous theory (1940).

Eckart theory

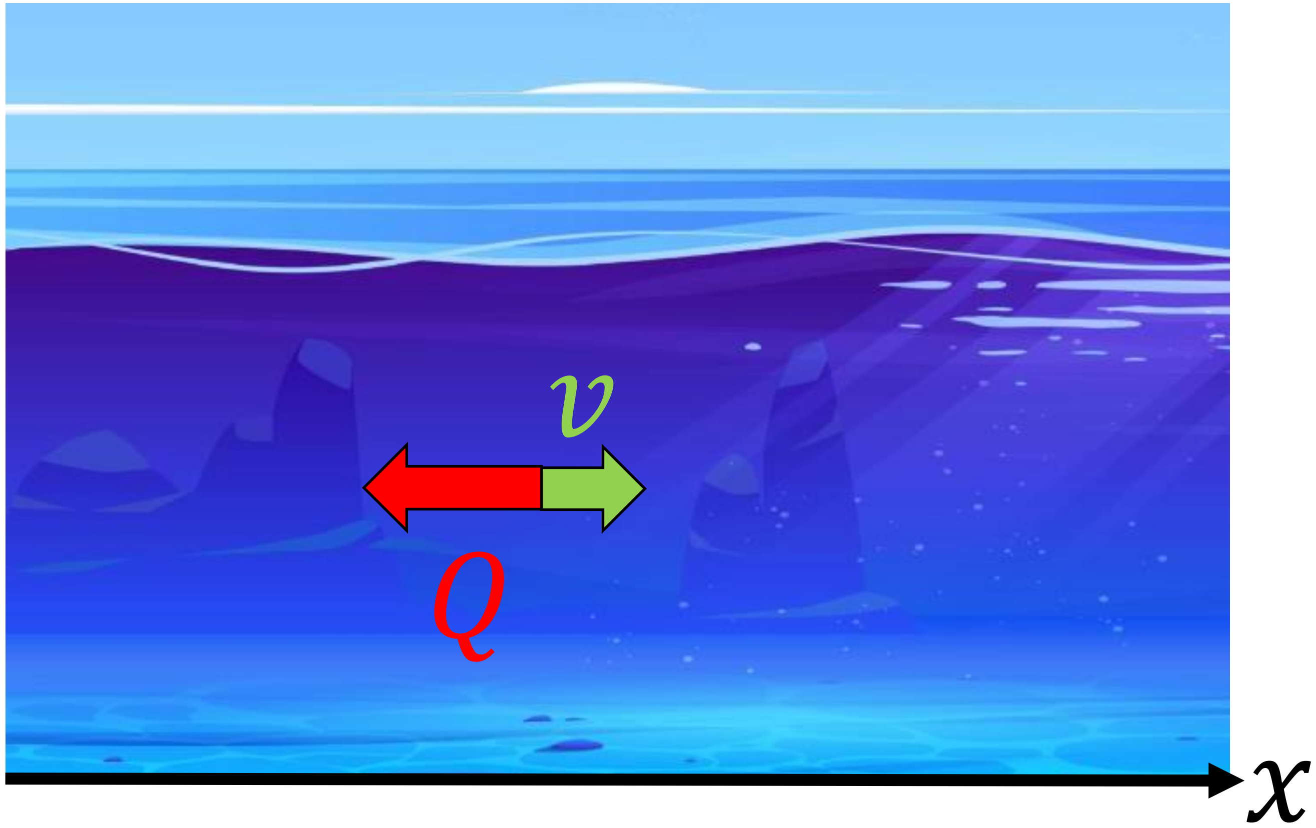
Let’s use it to describe the ocean…



Assume $p_x = 0$

Eckart theory

Recall that $E = mc^2$



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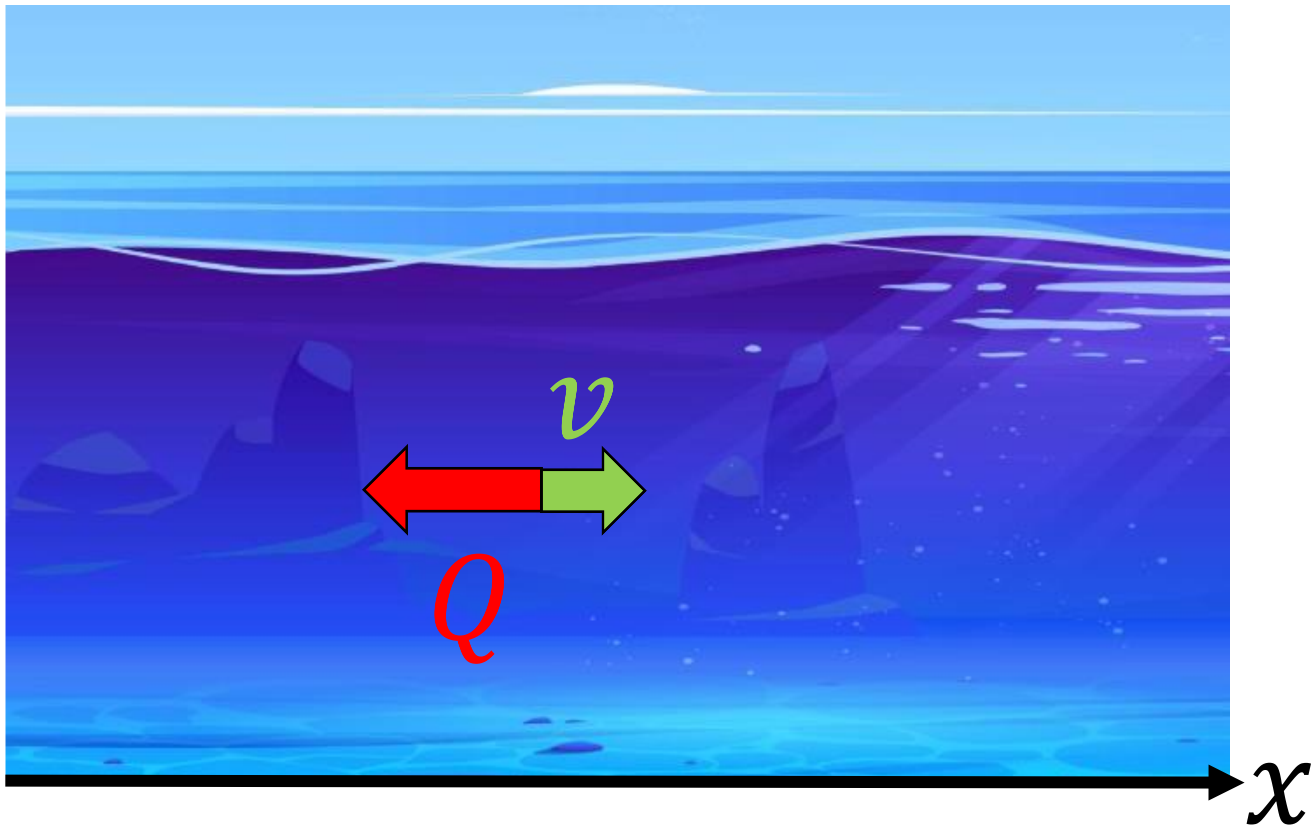
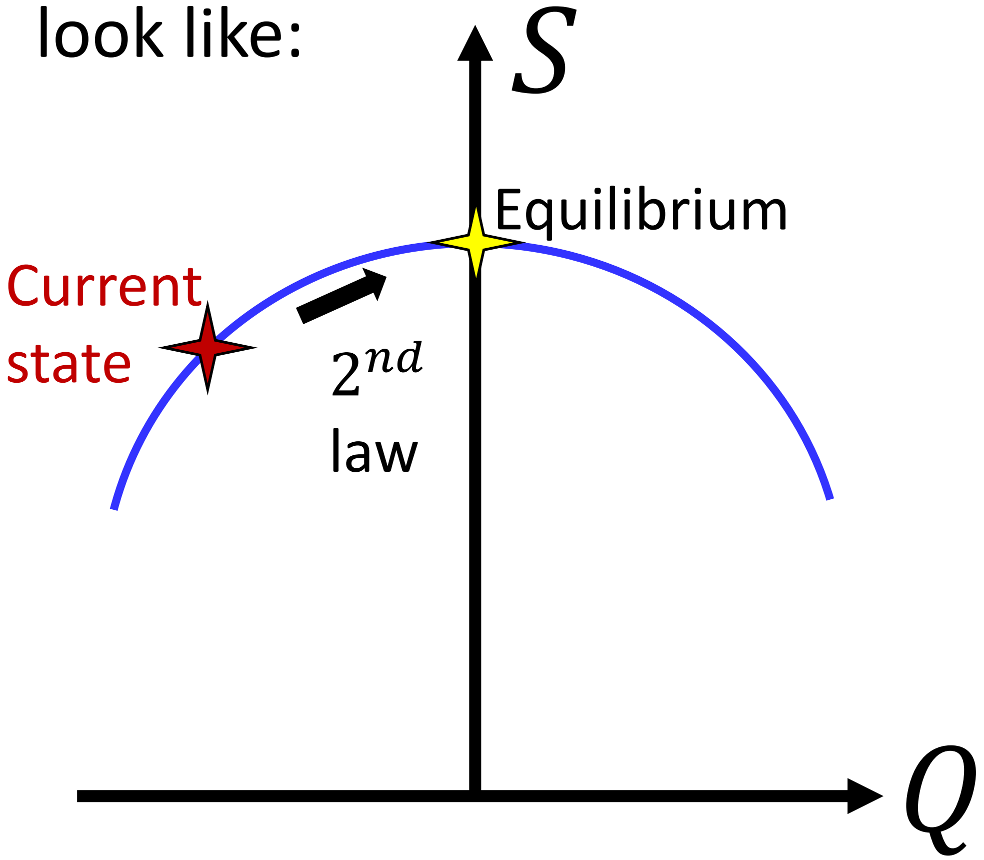


Assume $p_x = 0$

How the entropy should look like:



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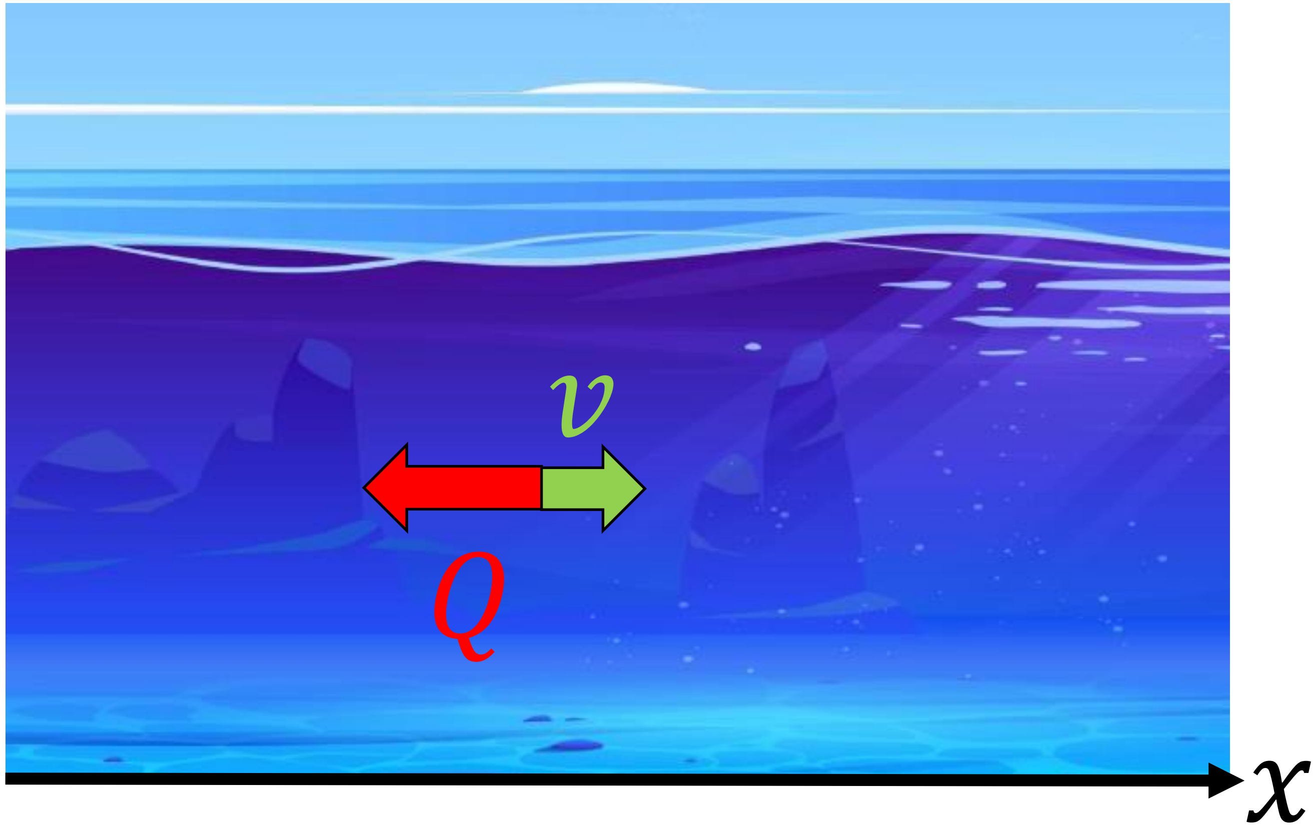
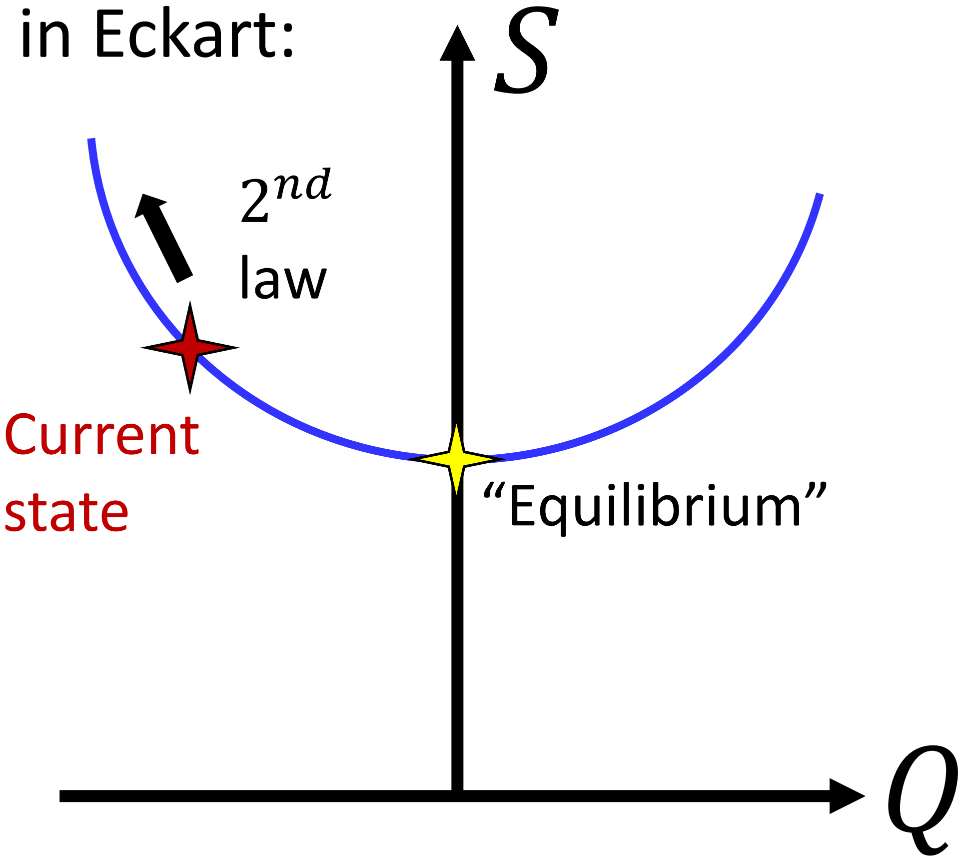


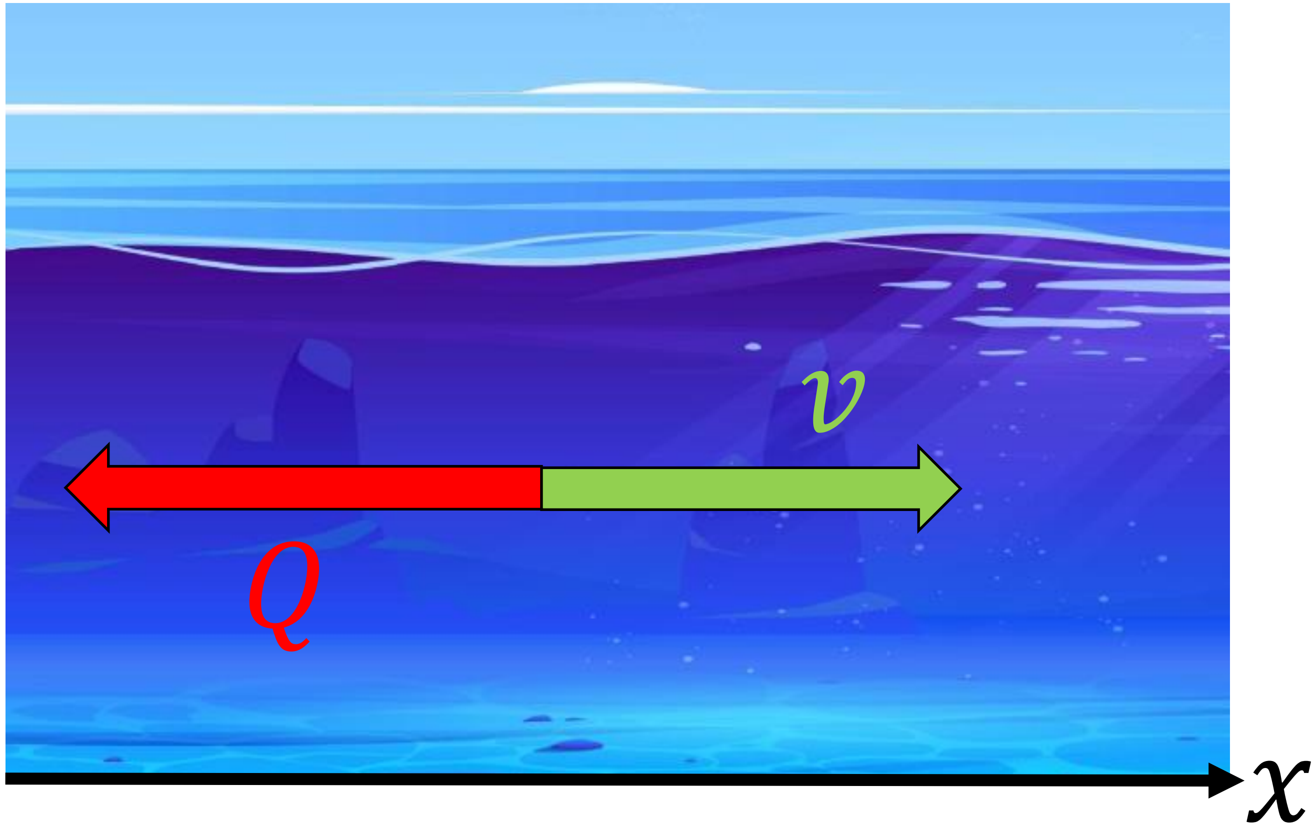
Assume $p_x = 0$

How it actually look like in Eckart:



Eckart theory

The ocean accelerates!



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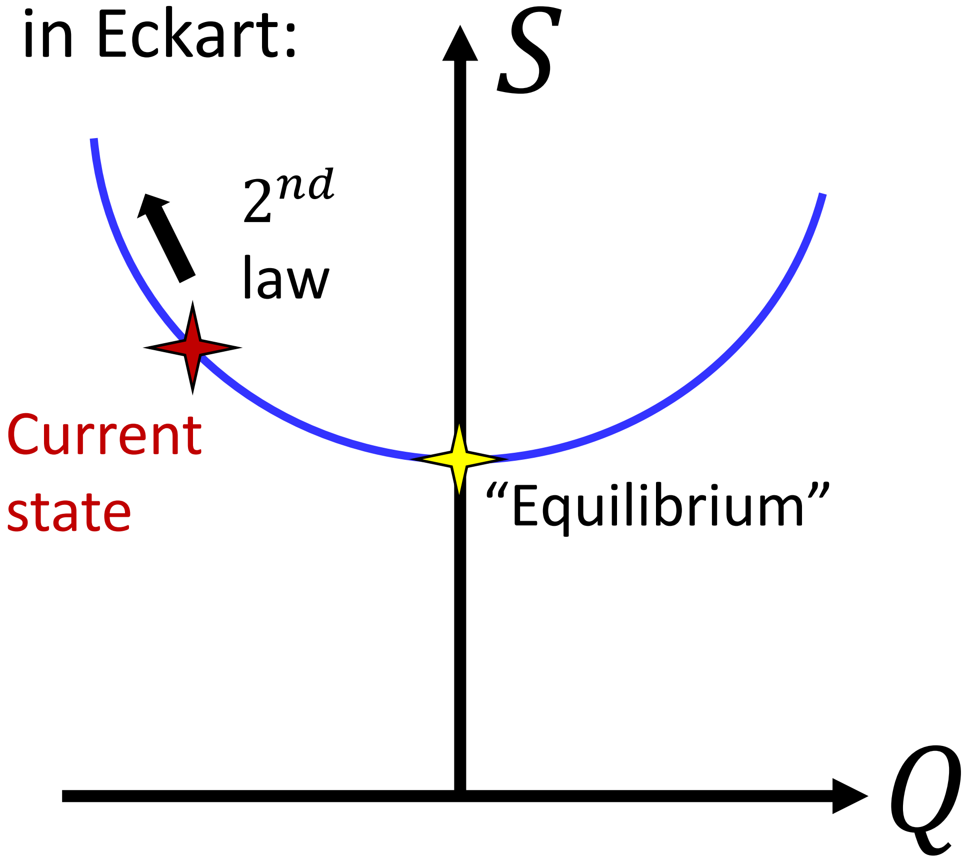

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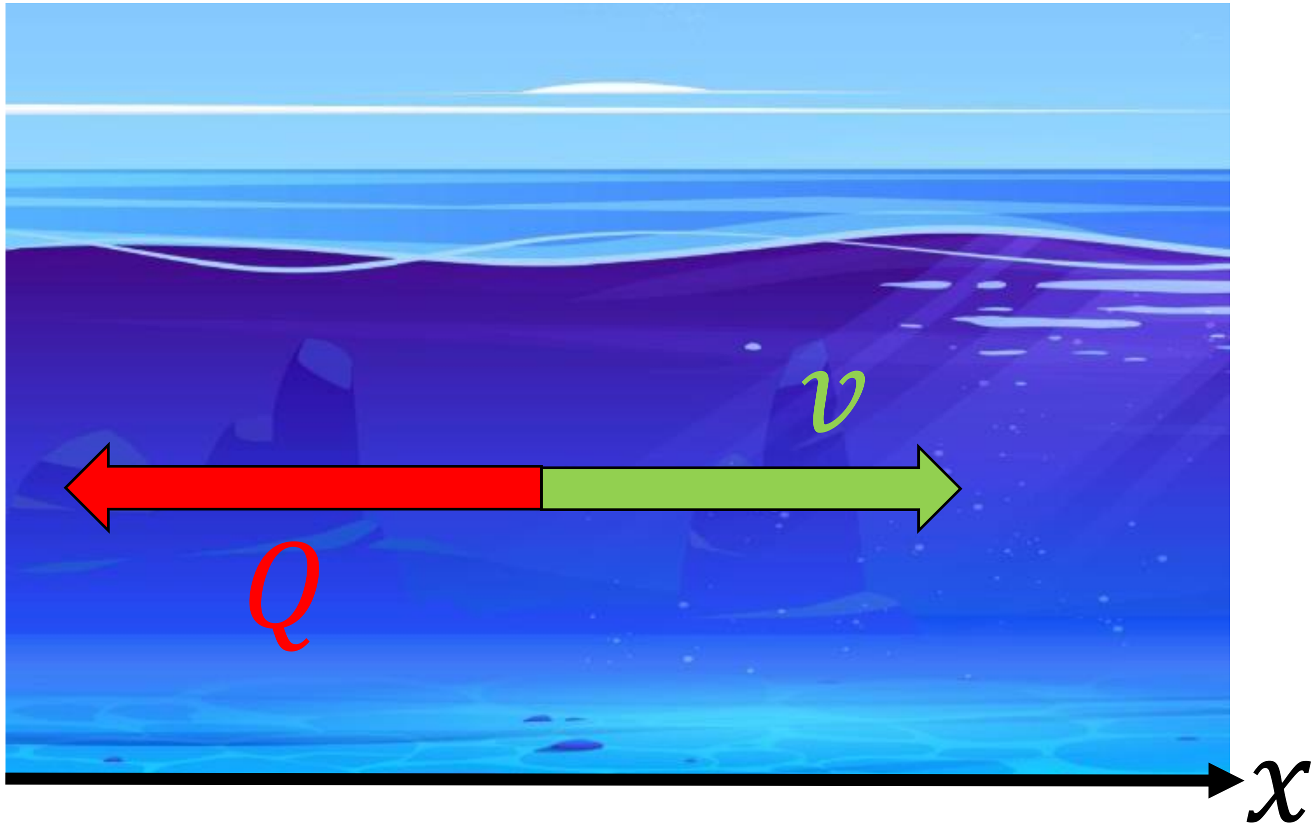
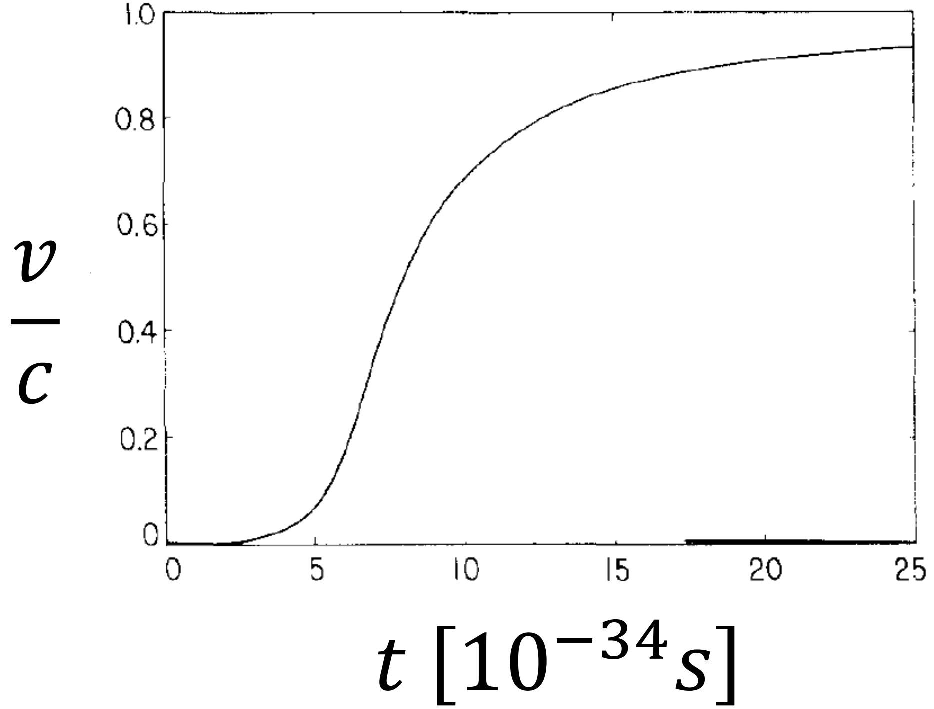




Assume $p_x = 0$

Diffusion equation

$$\partial_t T = \partial_x^2 T$$

It is first order in time. Thus, you need to fix $T(0,x)$.

Boost it!

$$\gamma(\partial_{t'} - v\partial_{x'})T = \gamma^2\left(\partial_{x'}^2 - 2v\partial_{x'}\partial_{t'} - v^2\partial_{t'}^2\right)T$$

It is second order in time. Thus, you need to fix $T(0,x'), \partial_{t'}T(0,x')$.

You have 'double' solutions.

$$T(t',x') = T_D(t',x') + T_U(t',x')$$

Dissipative

Looks like

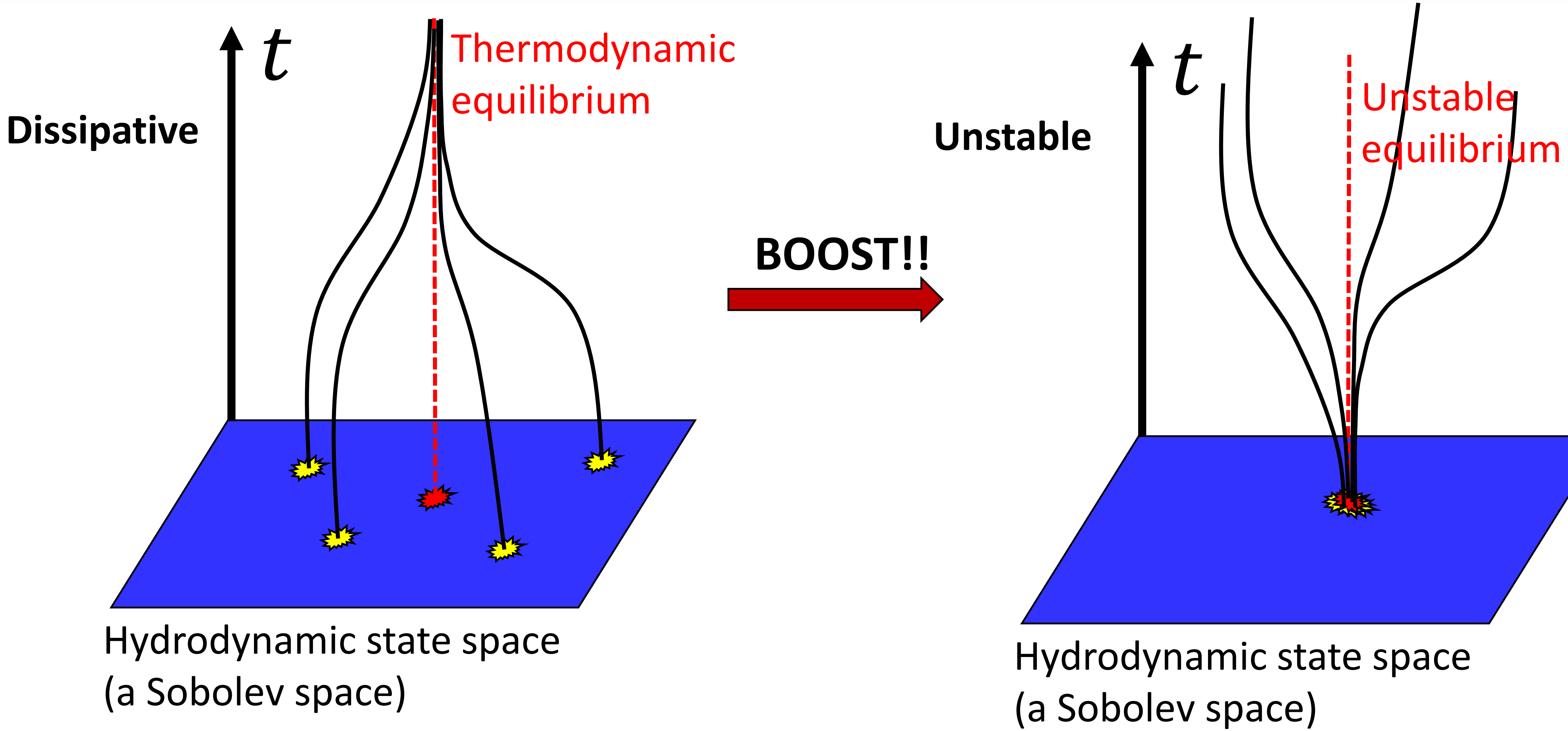
" $\partial_t T = \partial_x^2 T$ "

Unstable

Looks like

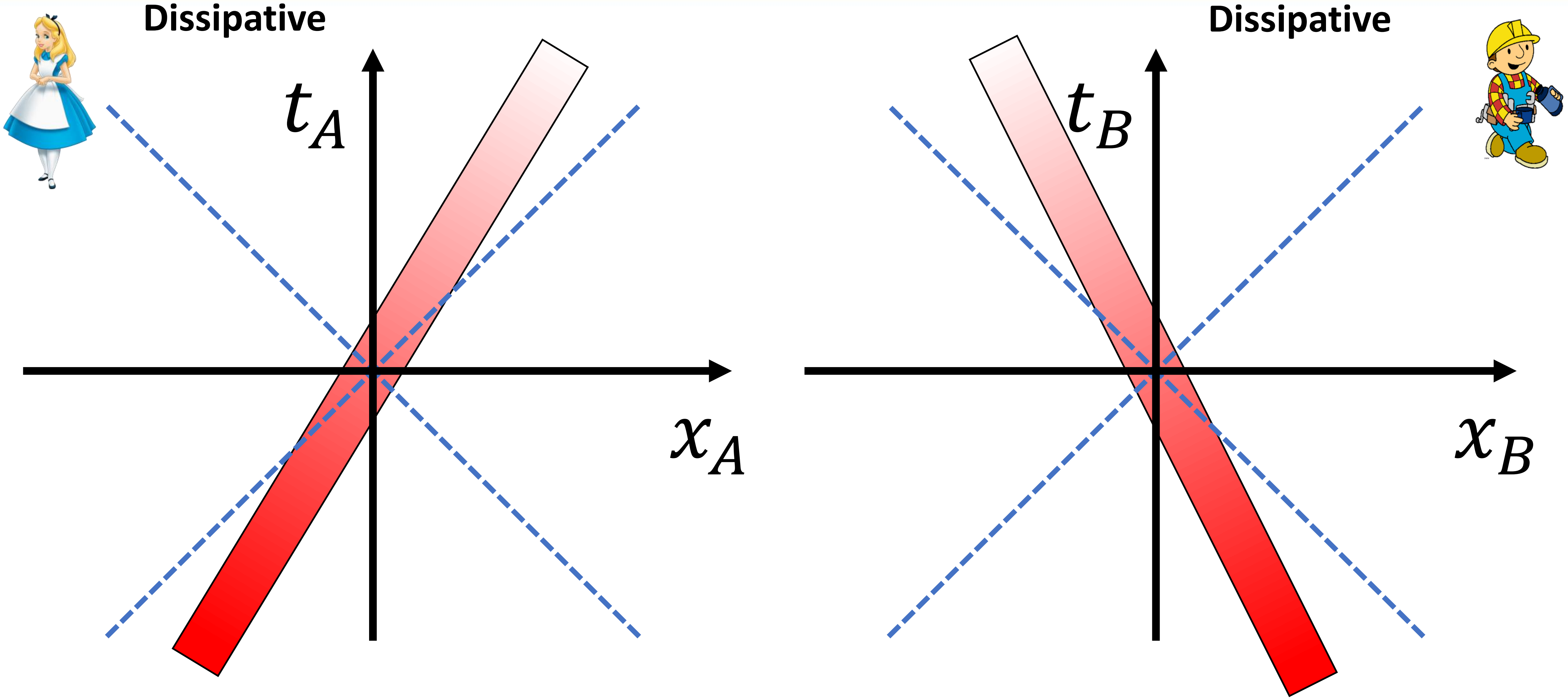
" $-\partial_t T = \partial_x^2 T$ "

$Apparently,\ \Lambda(Dissipative) = Unstable$

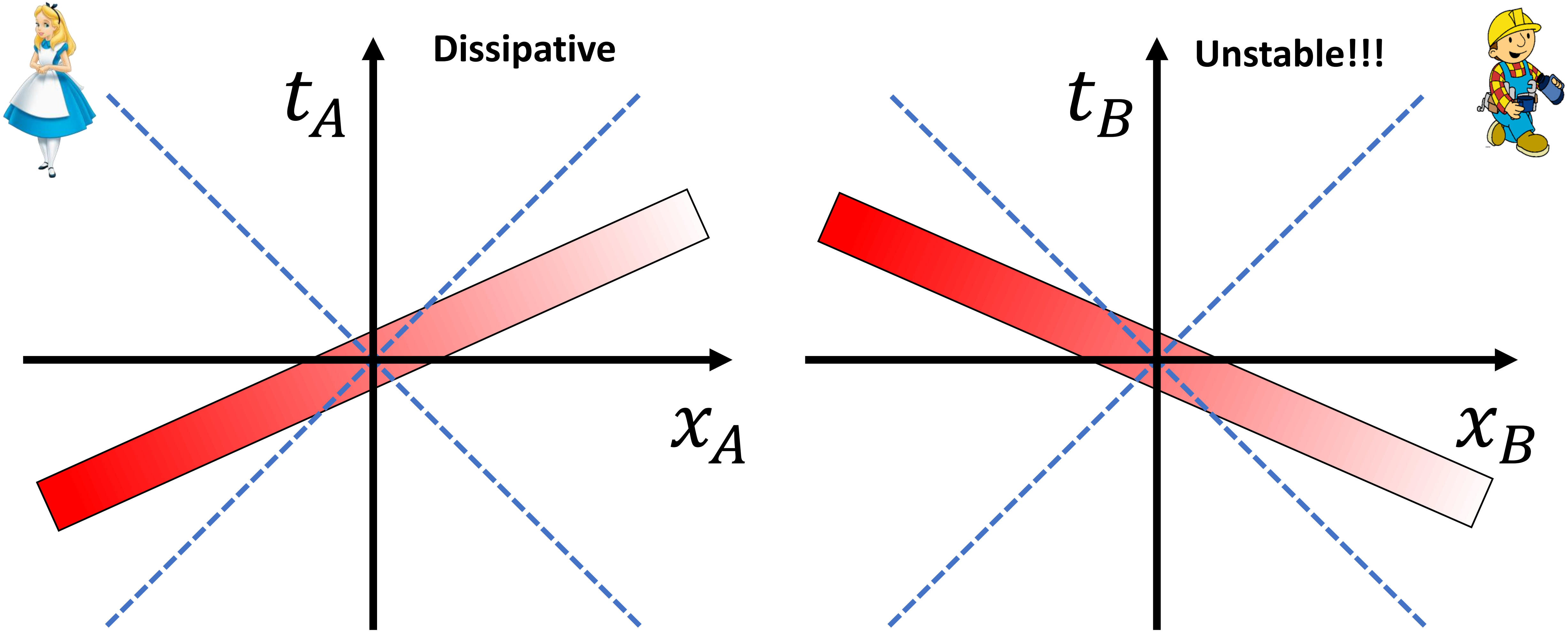


It is time to explain this!

Perturbations in the real world…



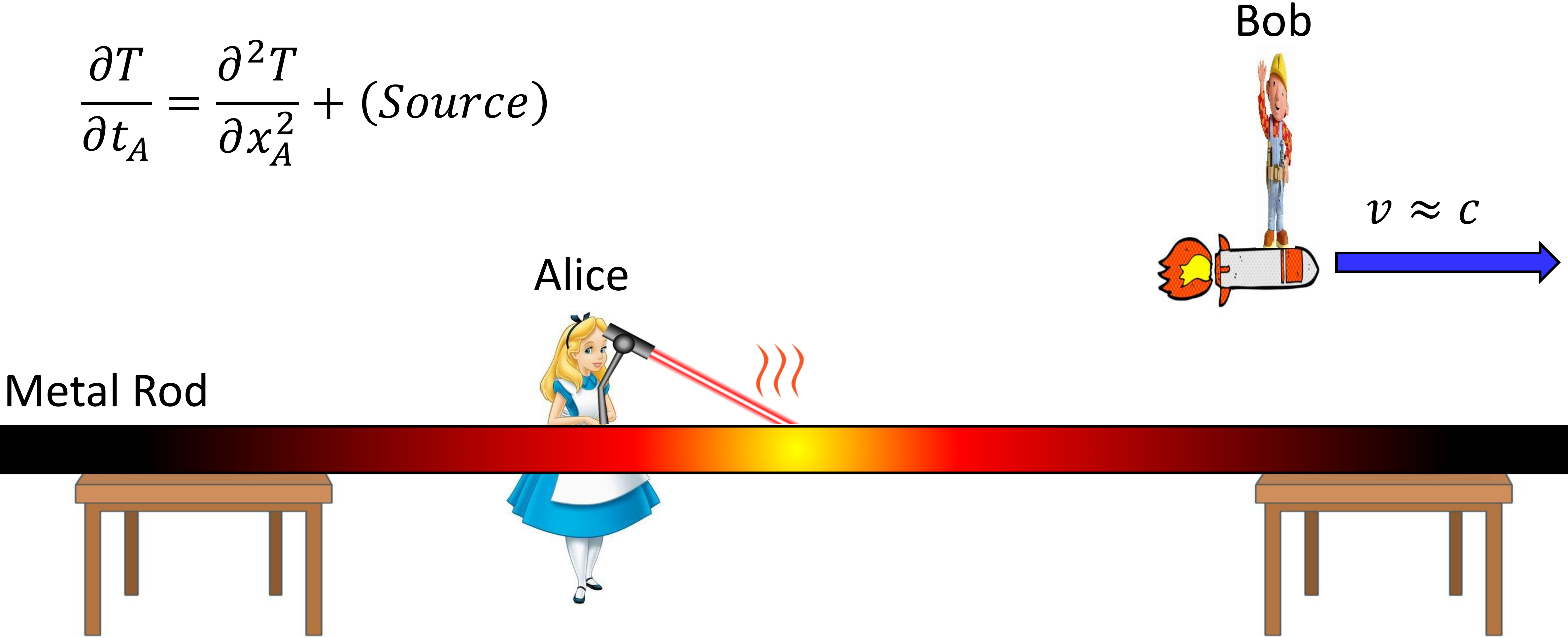
…But what if, for some reason, something propagates outside the lightcone?



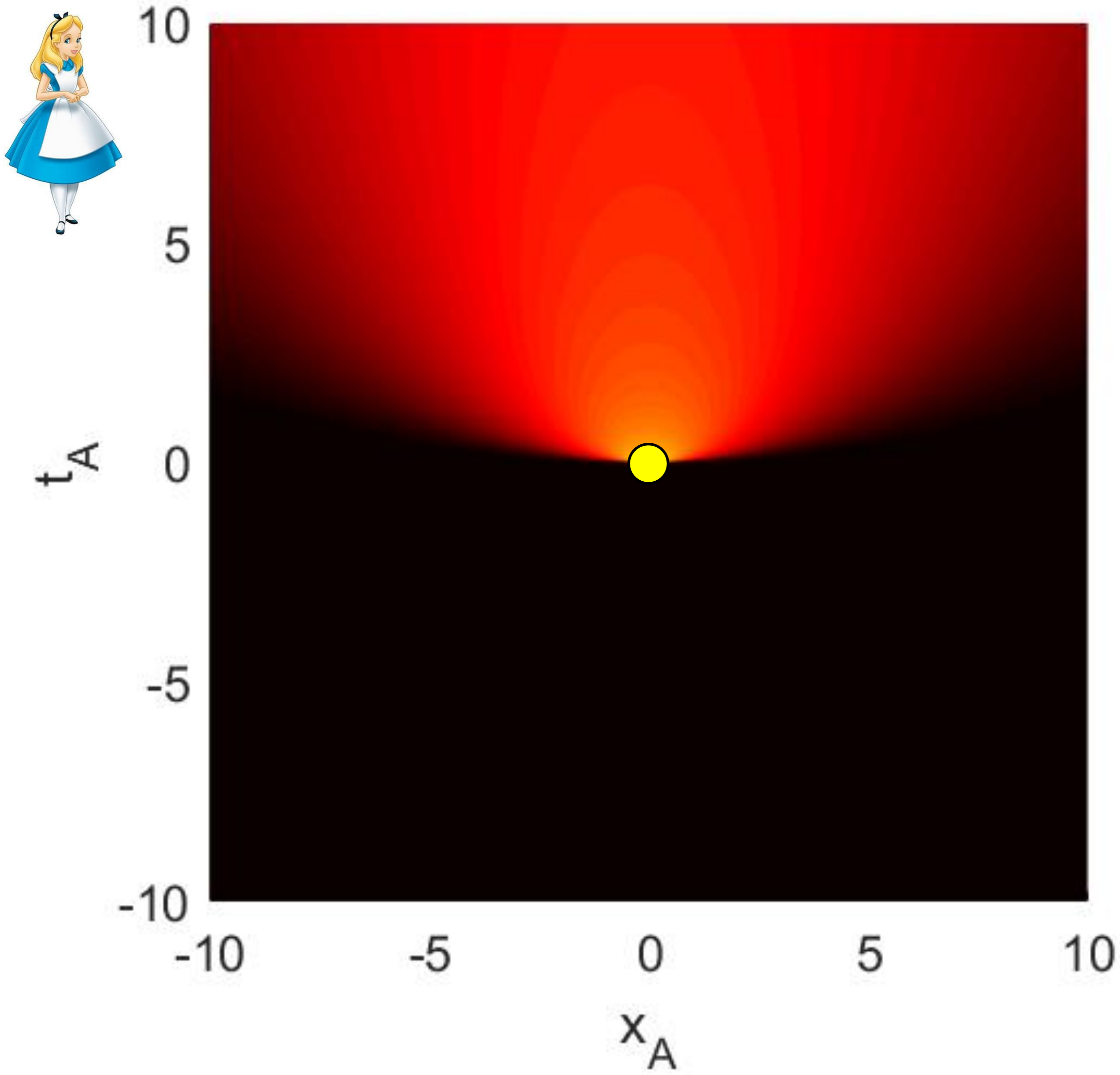
Dissipation cares about chronology. Causality violations reverse it in Bob's frame.

A concrete example

$$\frac{\partial T}{\partial t_A} = \frac{\partial^2 T}{\partial x_A^2} + (Source)$$



Minkowski Diagram

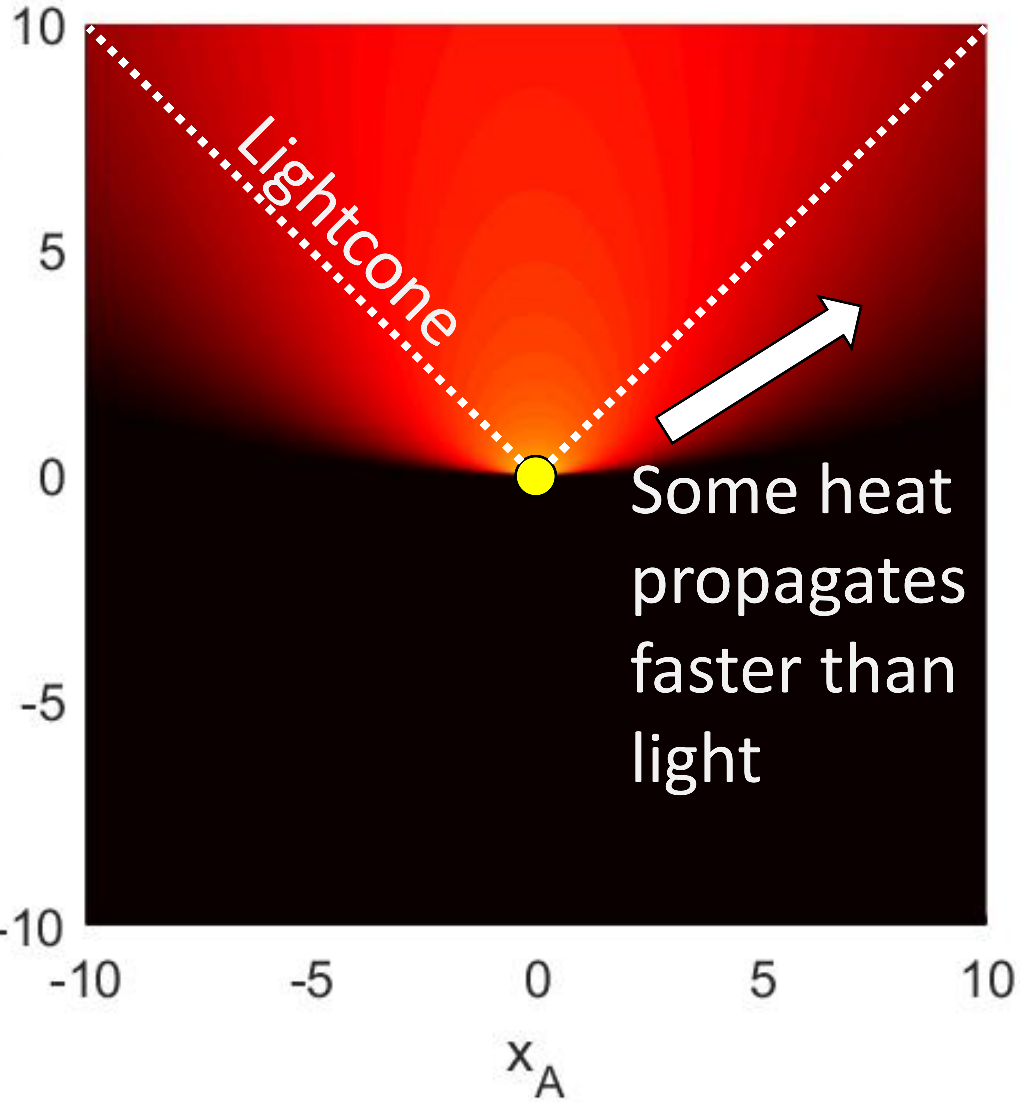


$$\partial_{t_A} T = \partial^2_{x_A} T + \delta(t)\delta(x)$$

Retarded Green function:

$$T(t_A, x_A) = \frac{\Theta(t_A)}{\sqrt{4\pi t_A}} \exp\left(-\frac{x_A^2}{4t_A}\right)$$

Superluminal propagation



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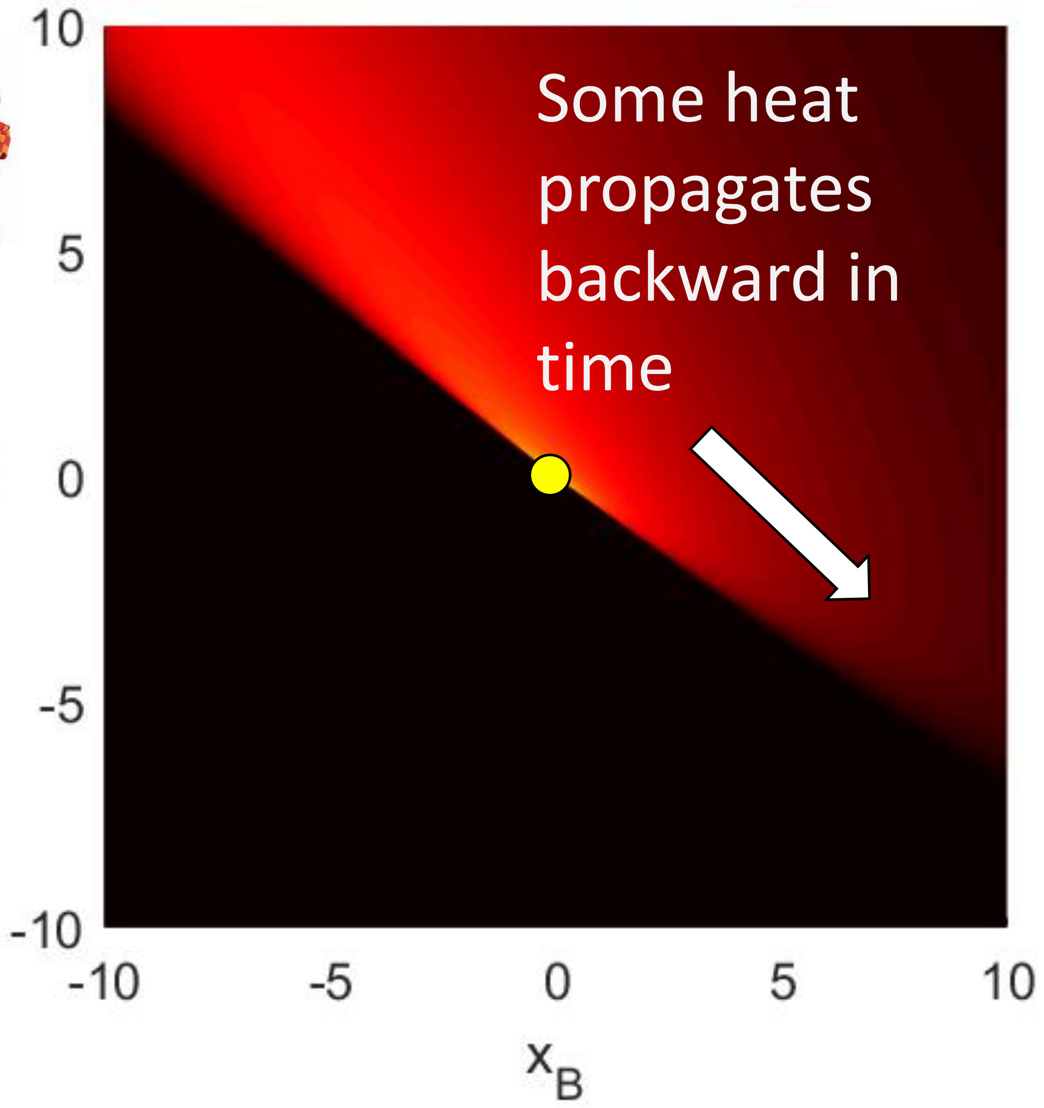
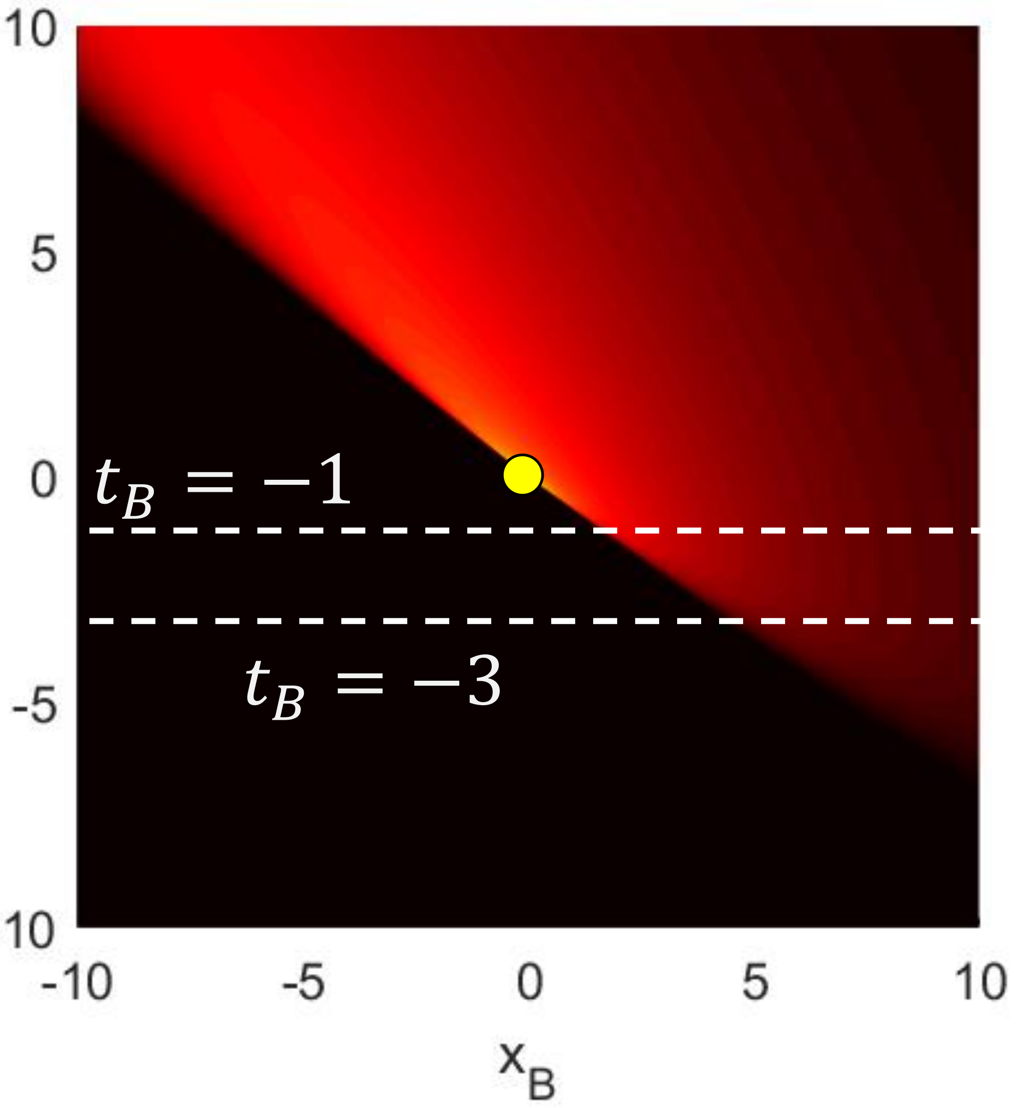
The effect precedes the cause

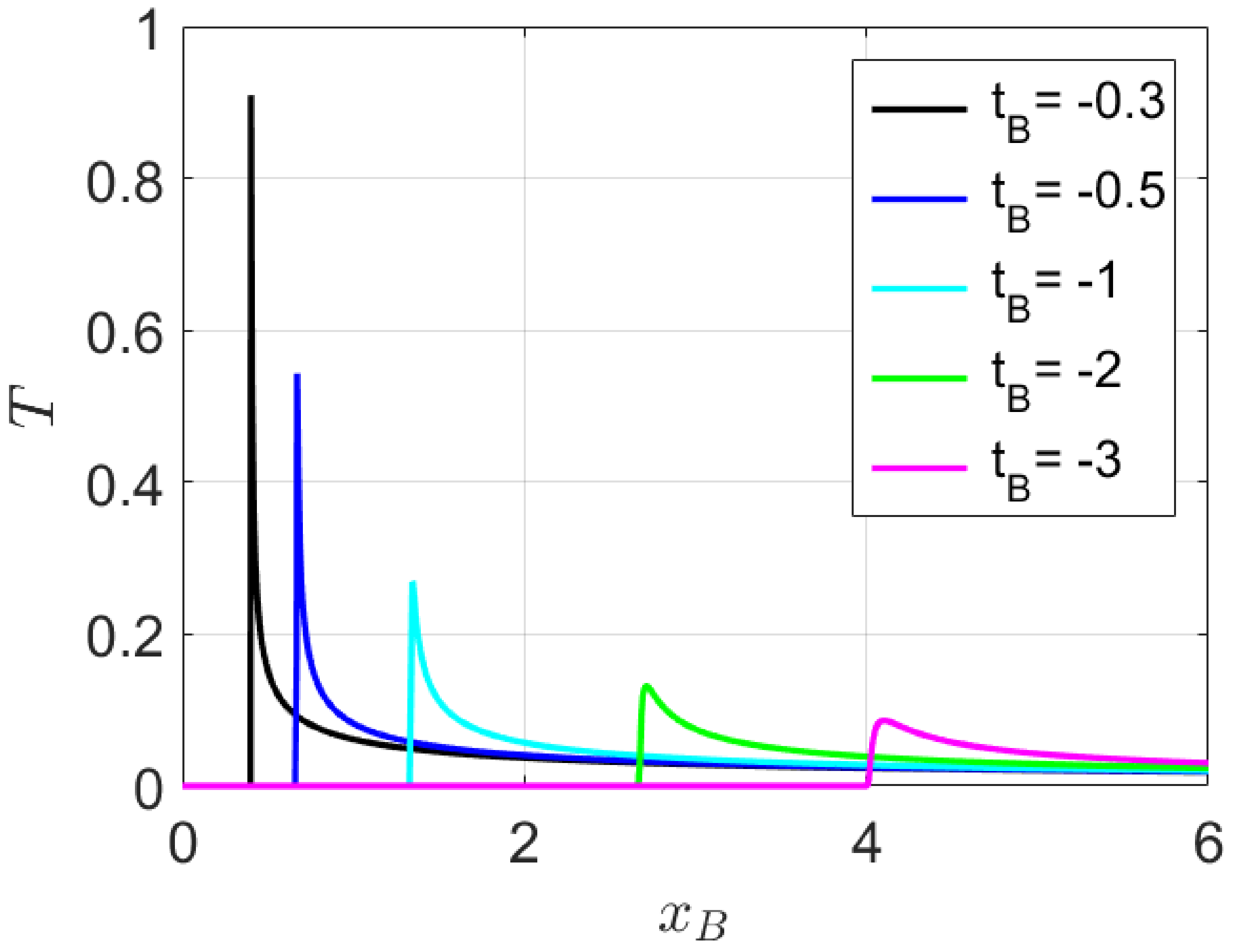


Violation of causality: the heat is there before Alice injects it!

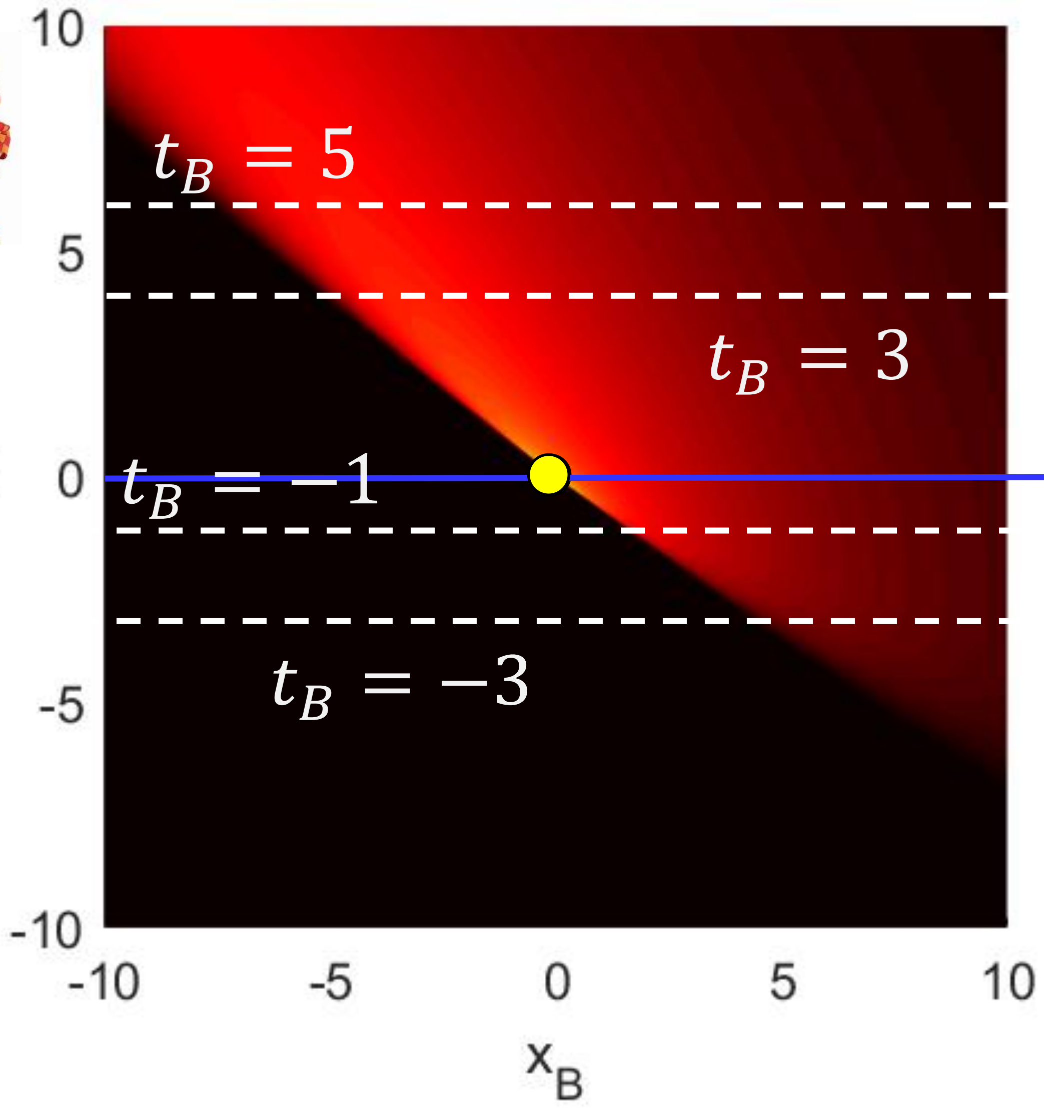
Some snapshots



Looks like " $-\partial_t T = \partial_x^2 T$ " (unstable)



$Recall: \quad T(t',x') = T_D(t',x') + T_U(t',x')$



Pure T_D

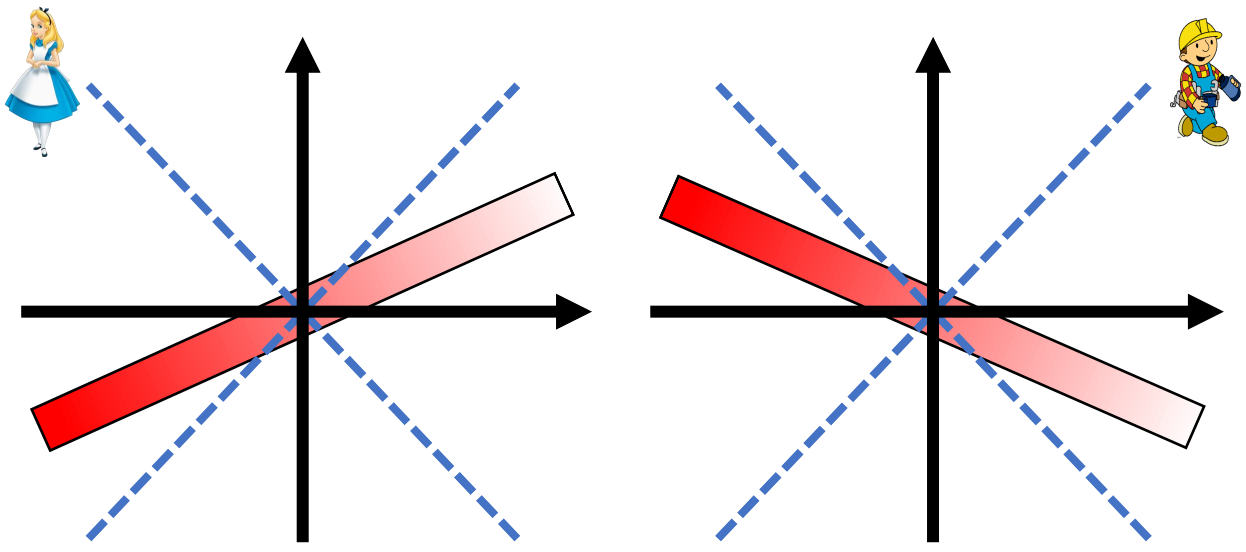
Looks like "$\partial_t T = \partial_x^2 T$" (dissipative)

Pure T_U

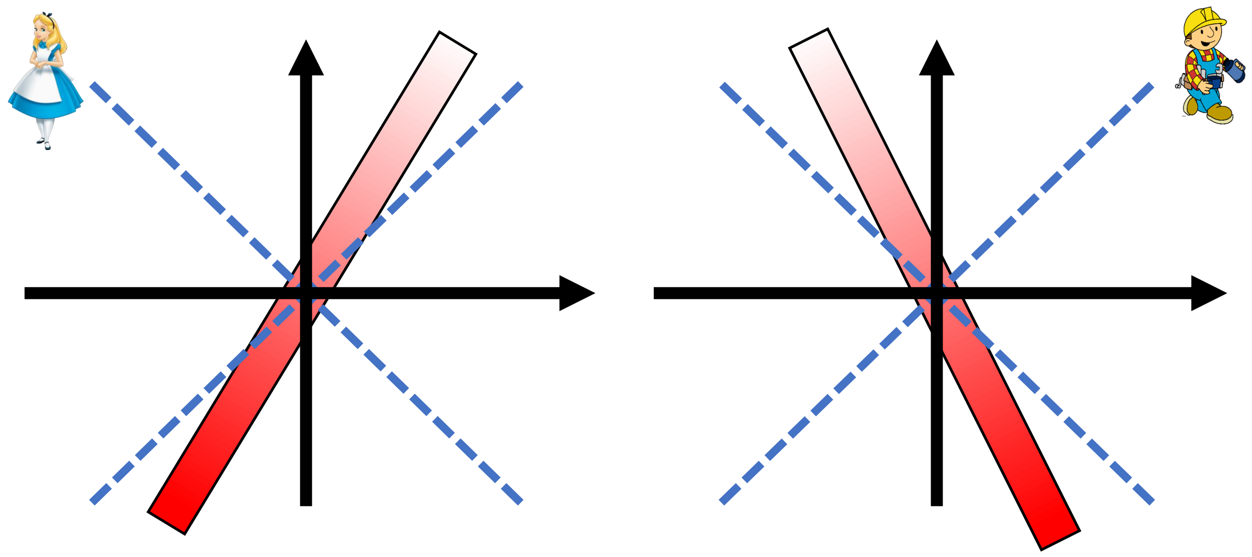
Looks like "$-\partial_t T = \partial_x^2 T$" (unstable)

Main message

1) If you break causality, then dissipation cannot be Lorentz-invariant.



2) If causality holds, then all observers agree on whether the fluid is dissipative or unstable.



A useful theorem

Theorem: If a causal linear theory presents a growing Fourier mode in some reference frame, then it is unstable in all reference frames.

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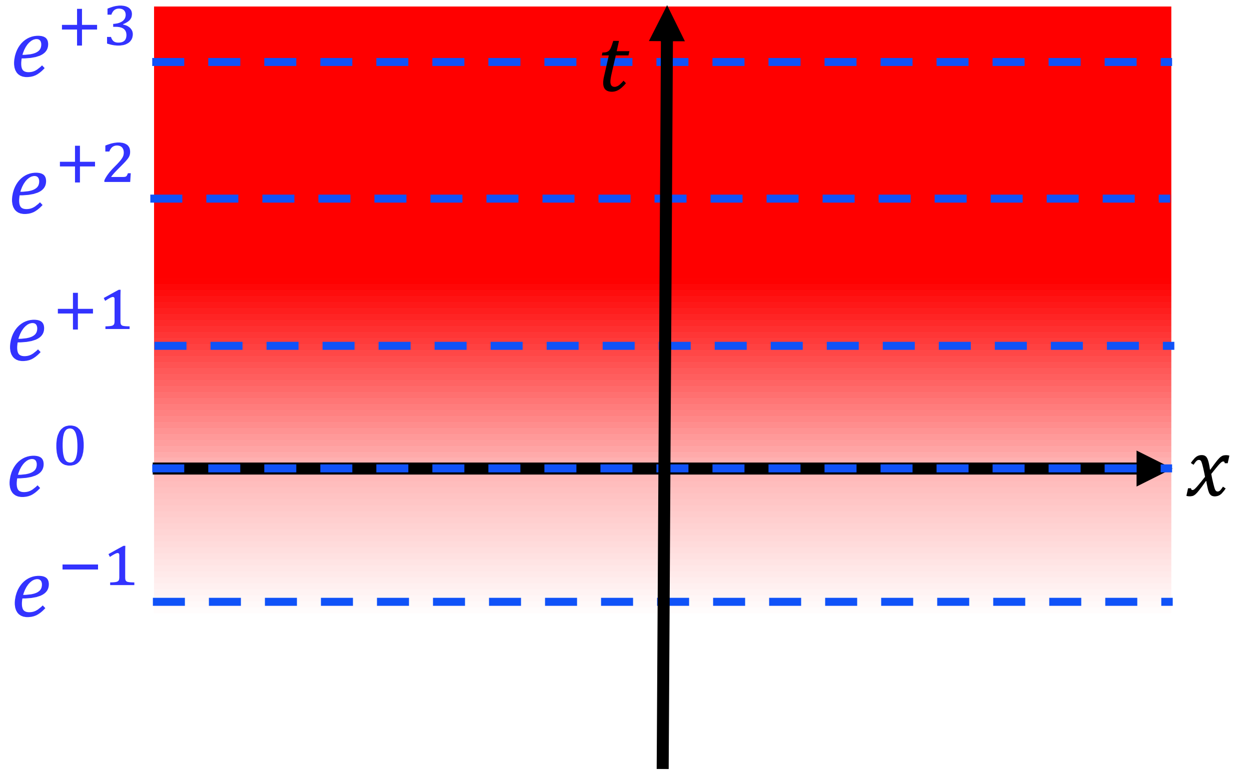
Proof: Suppose that $\varphi(t,x) = e^t \sin(kx - \omega t)$



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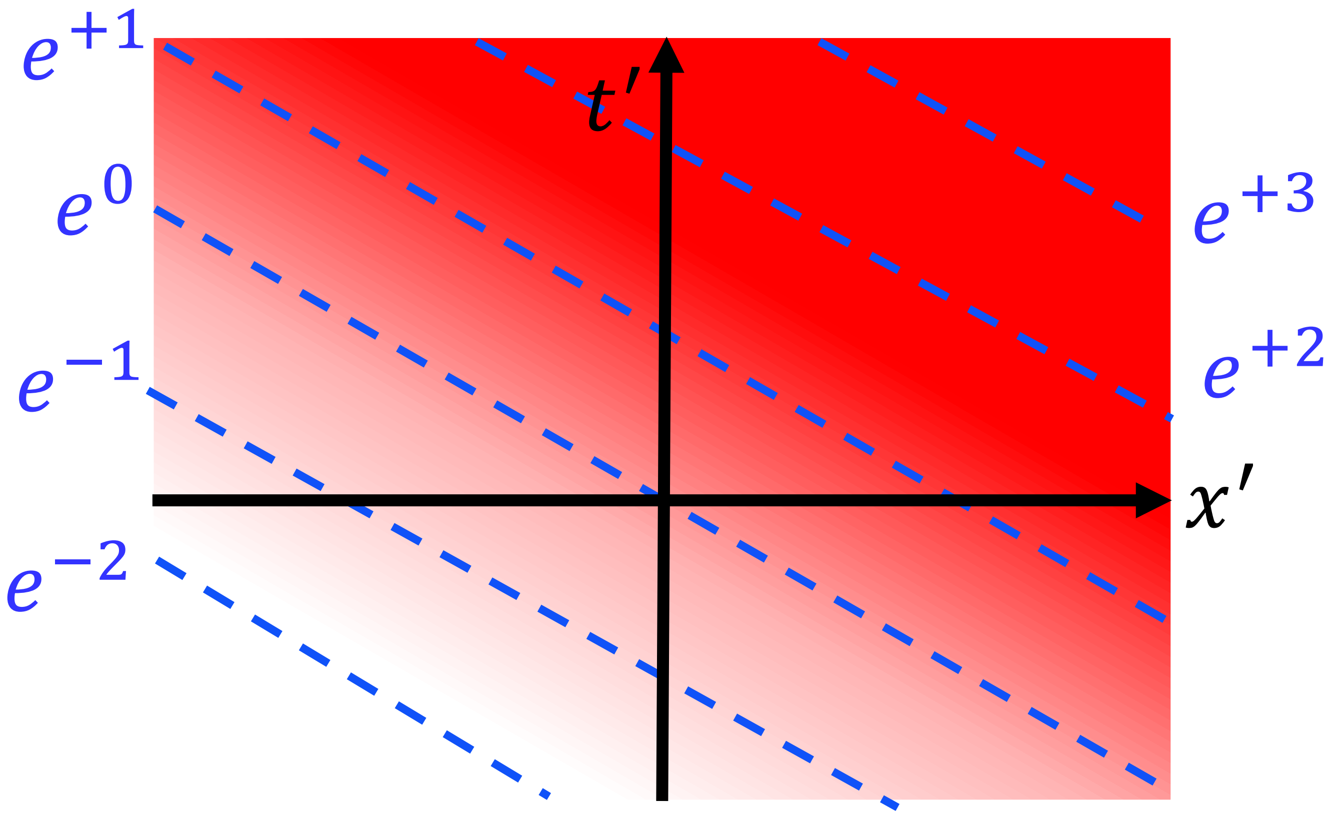
Proof: Boost it $\varphi(t',x') = e^{\gamma(t'+vx')} \sin(k'x' - \omega' t')$



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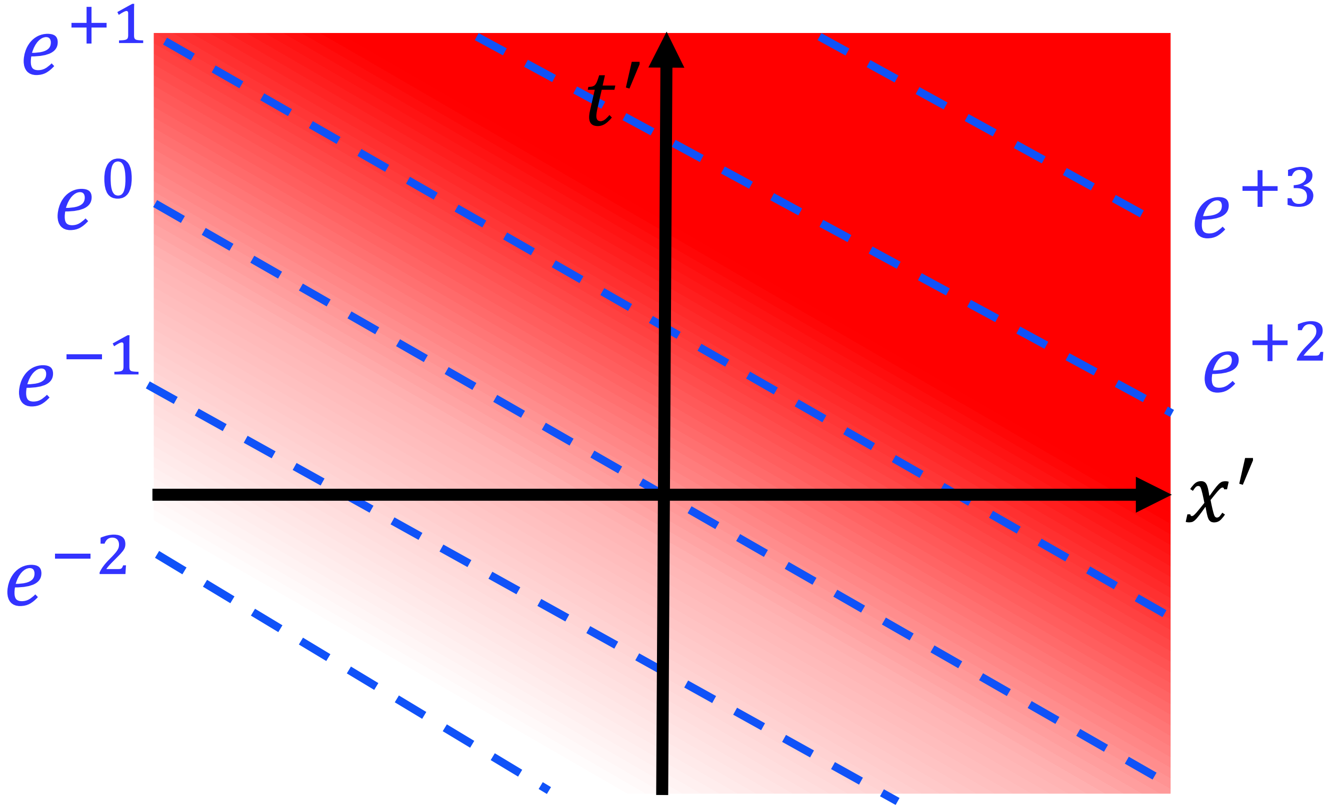
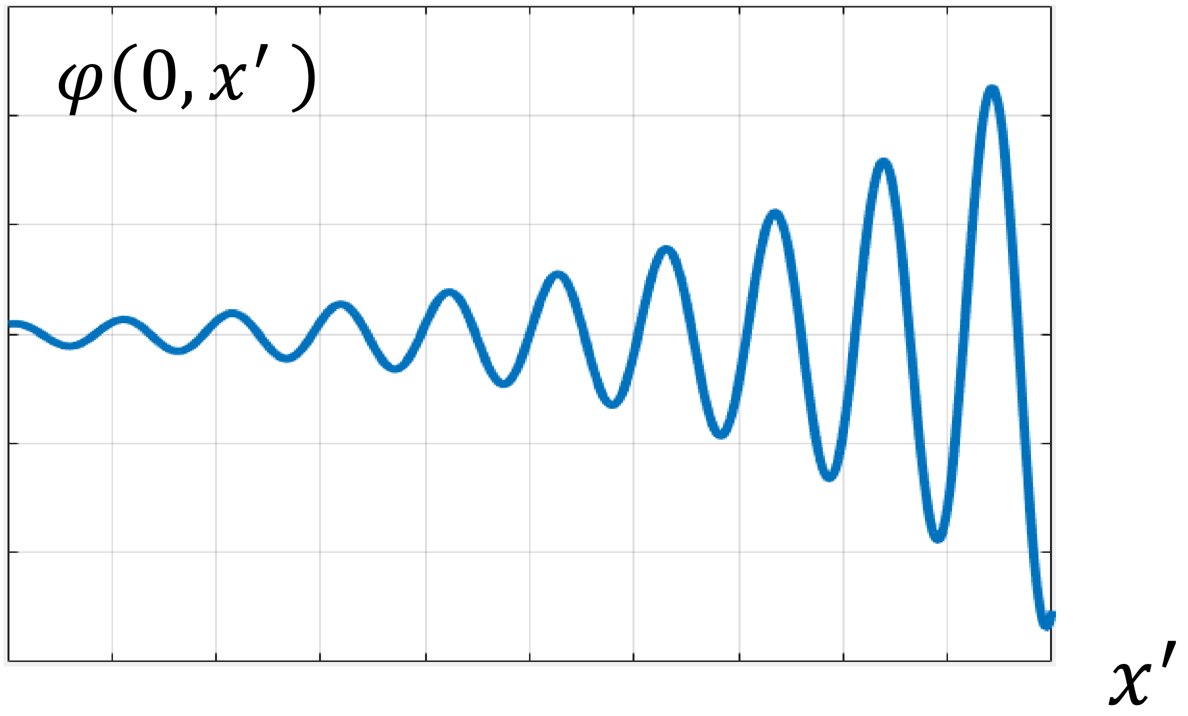


Note: it diverges in space as $x' \to +\infty$



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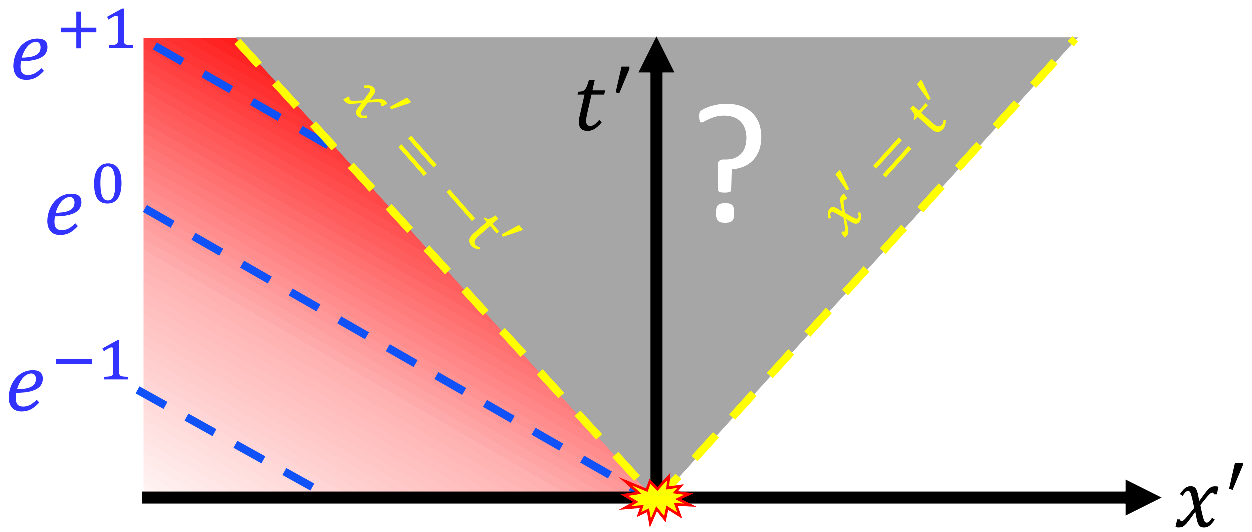
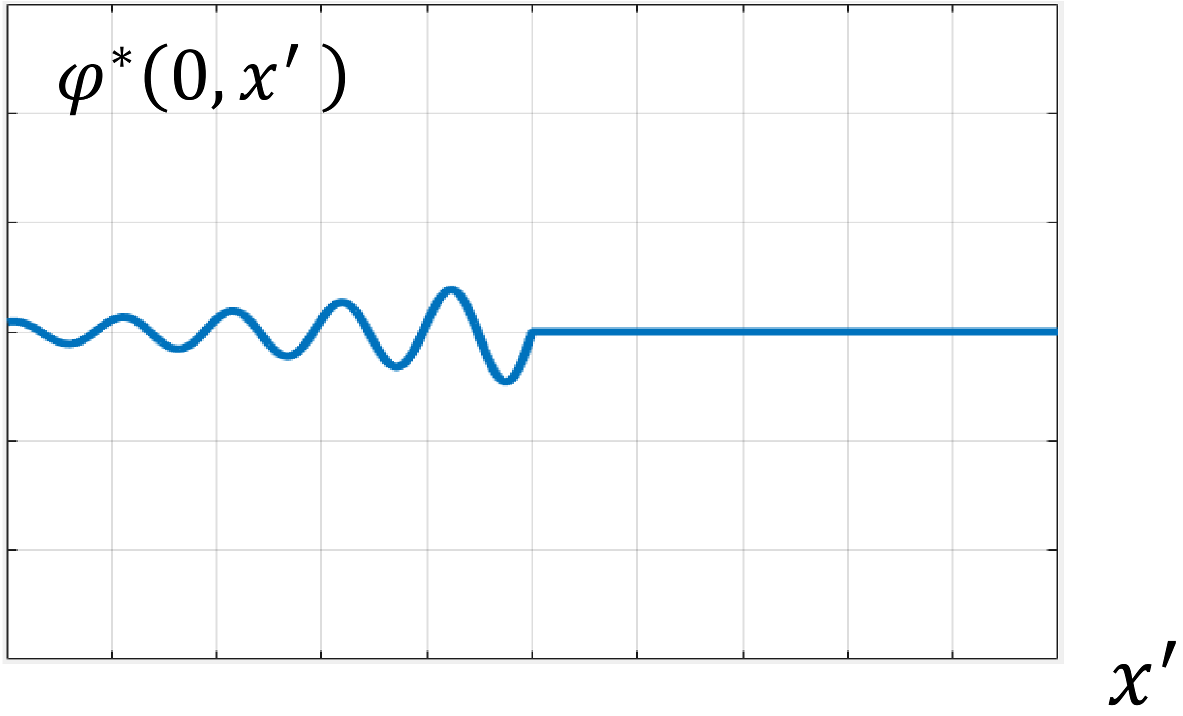
Proof: Construct a new solution by "gluing" initial data.



Causality implies that we can "glue" the solutions in the respective Cauchy developments.

Now the initial state is "innocent".



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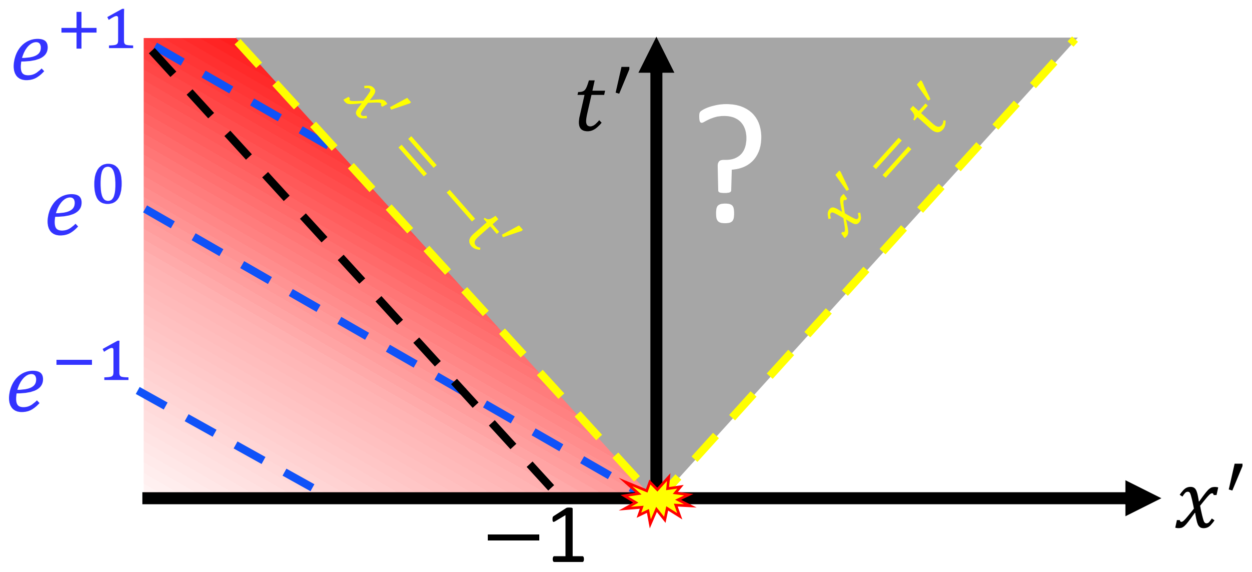
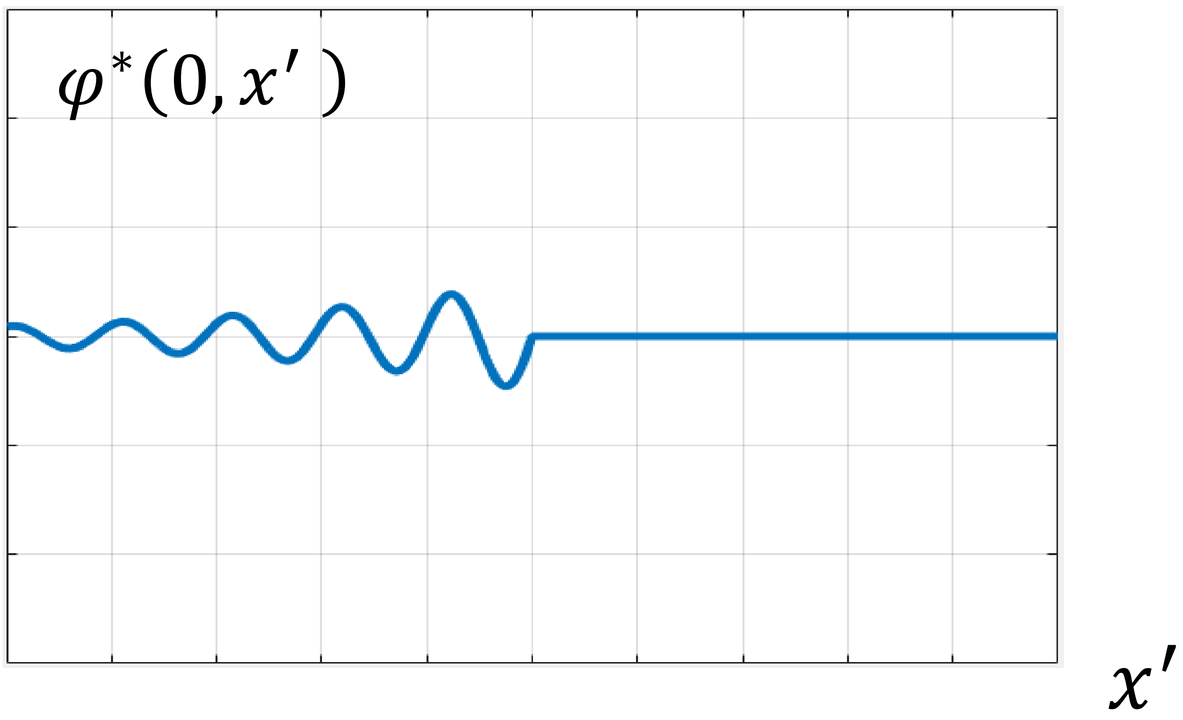


$\varphi^*(t', -t' - 1) = \varphi(t', -t' - 1) \propto e^{\gamma(1-v)t'} \to +\infty$. **Unstable!** Q.E.D.

Now the initial state is "innocent".



In conclusion

Dissipation is Lorentz-invariant if and only if information cannot travel faster than light.

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Can We Make Sense of Dissipation without Causality?

L. Gavassino

Phys. Rev. X **12**, 041001 – Published 3 October 2022

Physics See Viewpoint: Seeking Stability in a Relativistic Fluid

VIEWPOINT

Seeking Stability in a Relativistic Fluid

Gabriel Denicol
Department of Physics, Fluminense Federal University, Niterói, Brazil

October 3, 2022 • *Physics* 15, 149

A fluid dynamics theory that violates causality would always generate paradoxical instabilities—a result that could guide the search for a theory for relativistic fluids.

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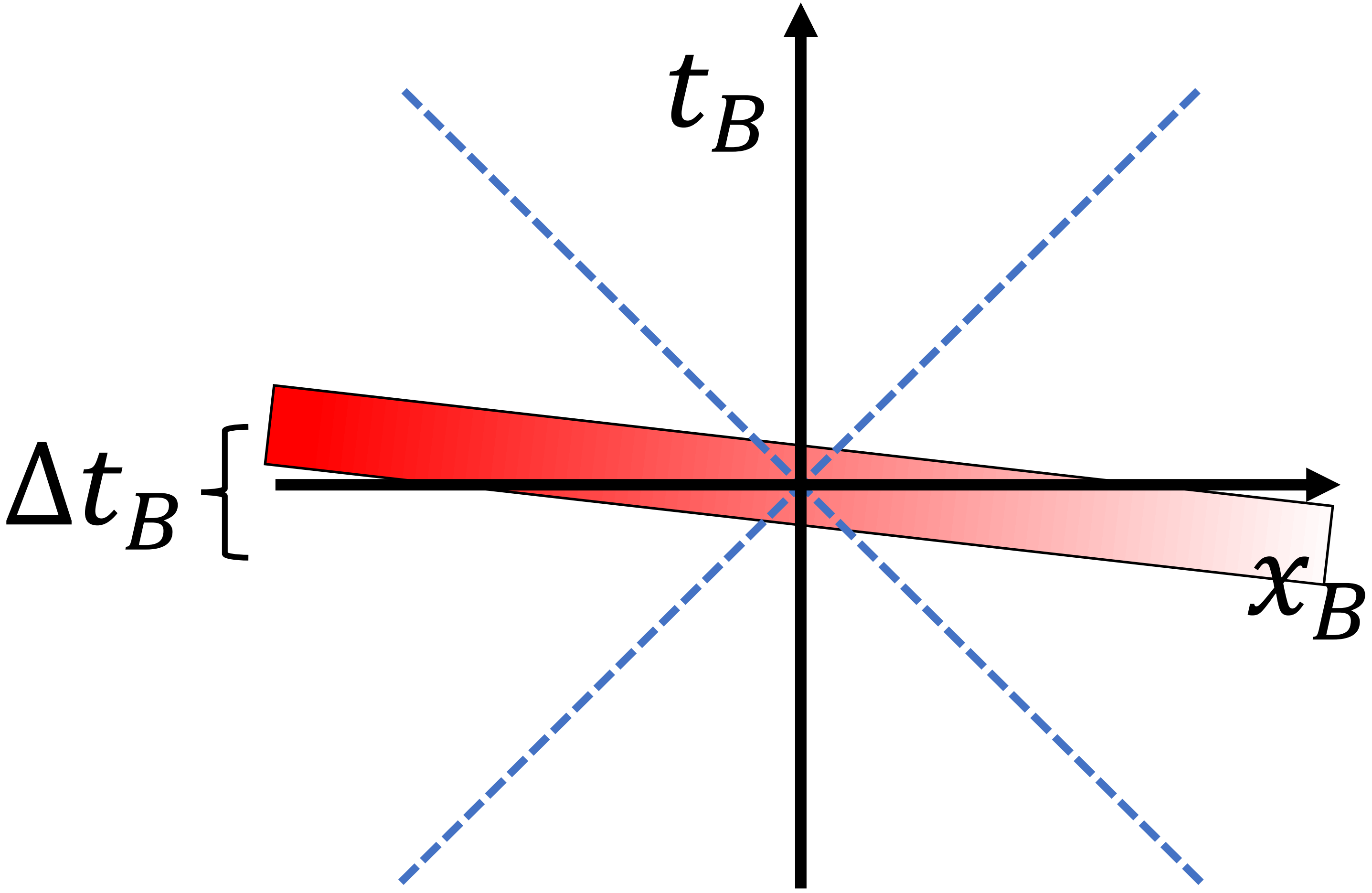
Appendices

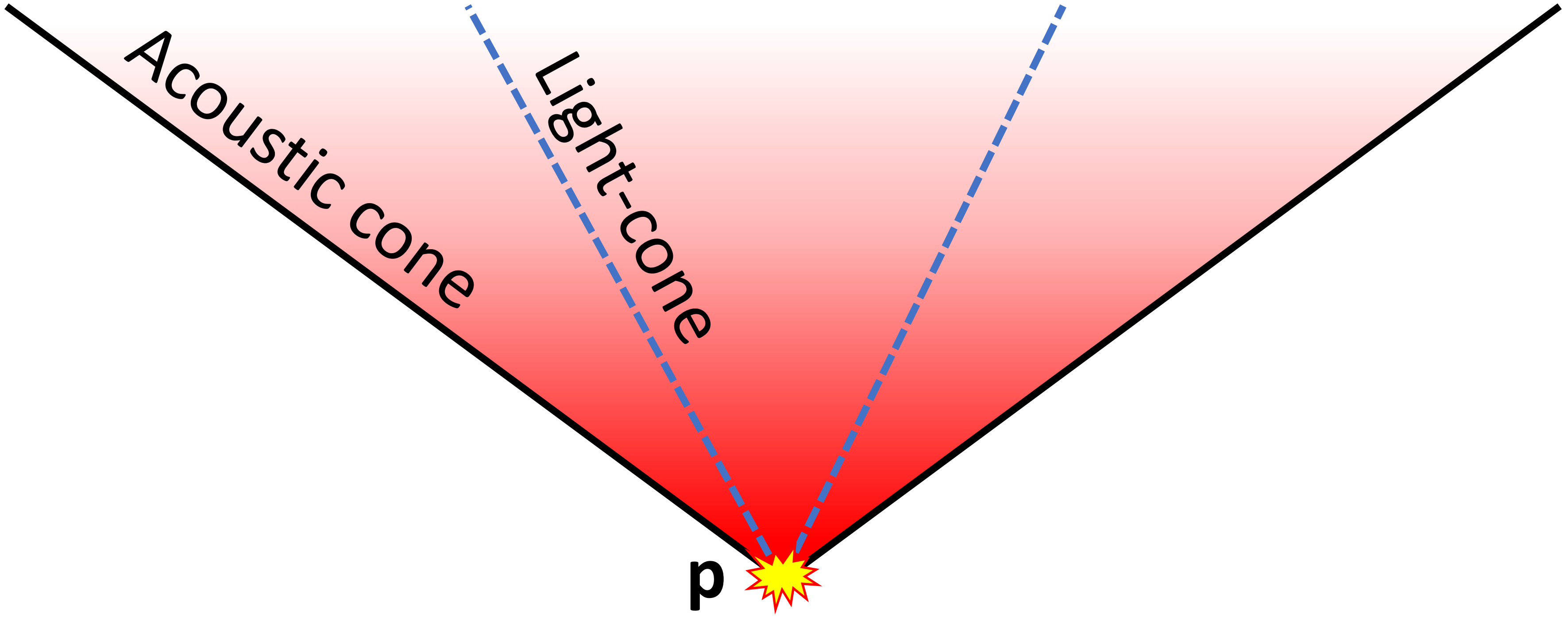
The instability is violent!



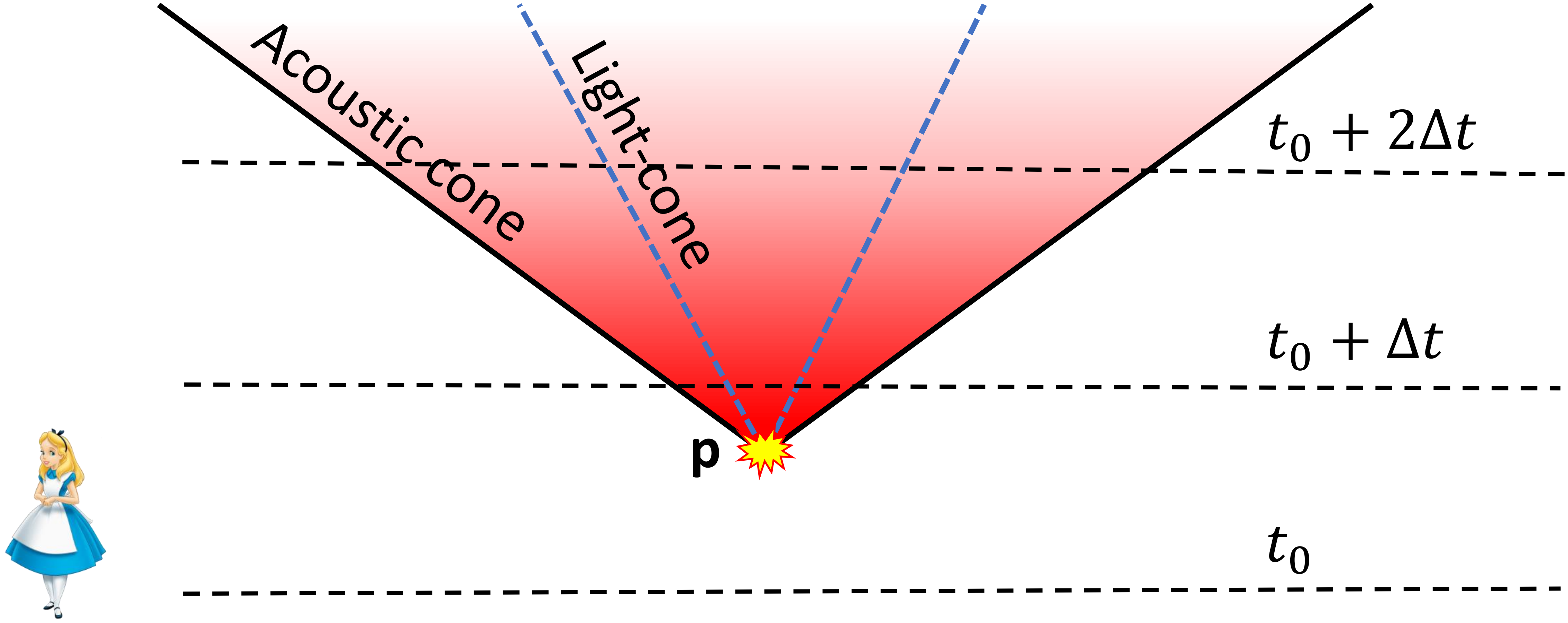
It is the growth **time** (not the growth rate!) that changes sign smoothly when we go from the stable to the unstable reference frames.

In some frame, the instability is infinitely fast!

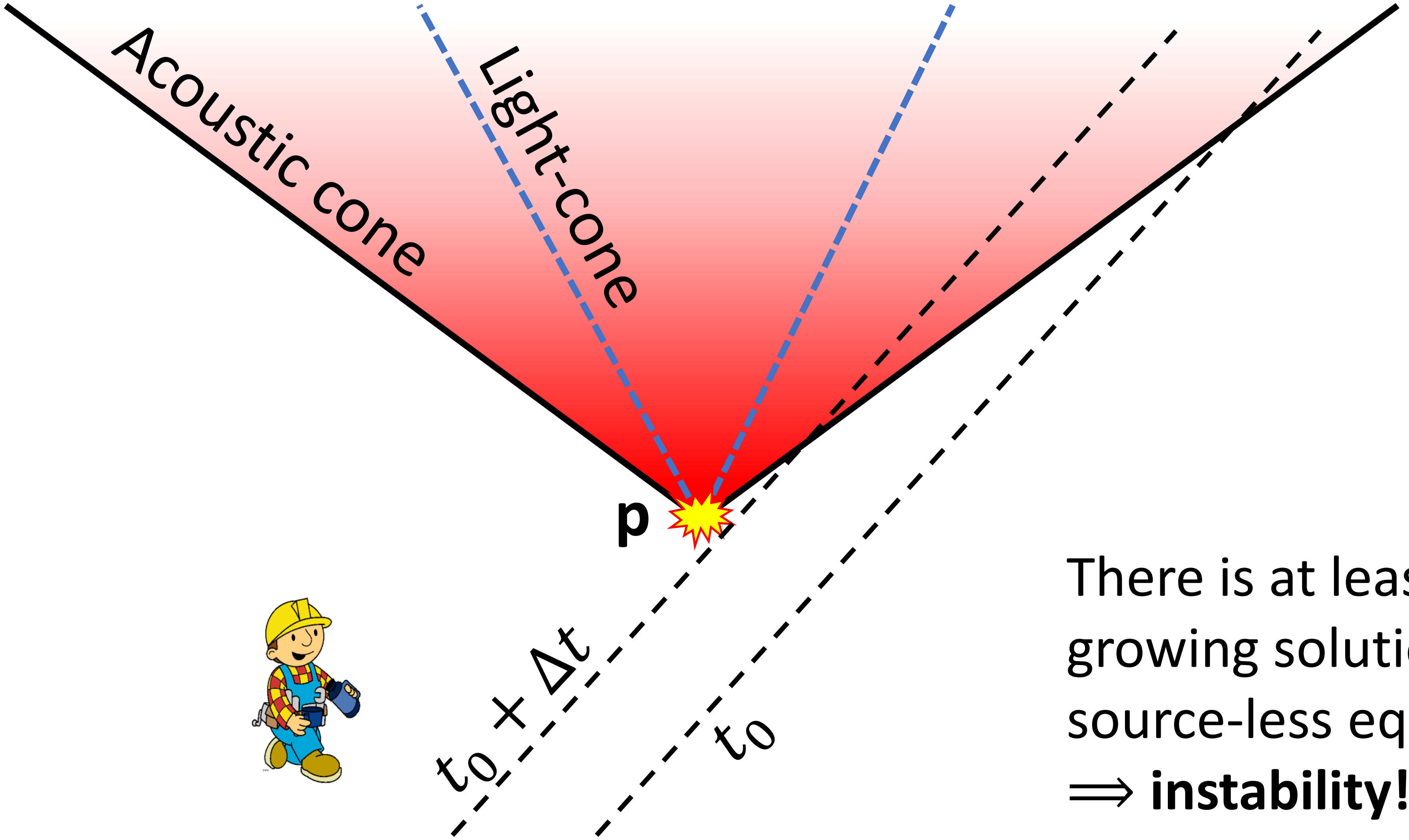
The result is general



Alice's point of view



Bob's point of view

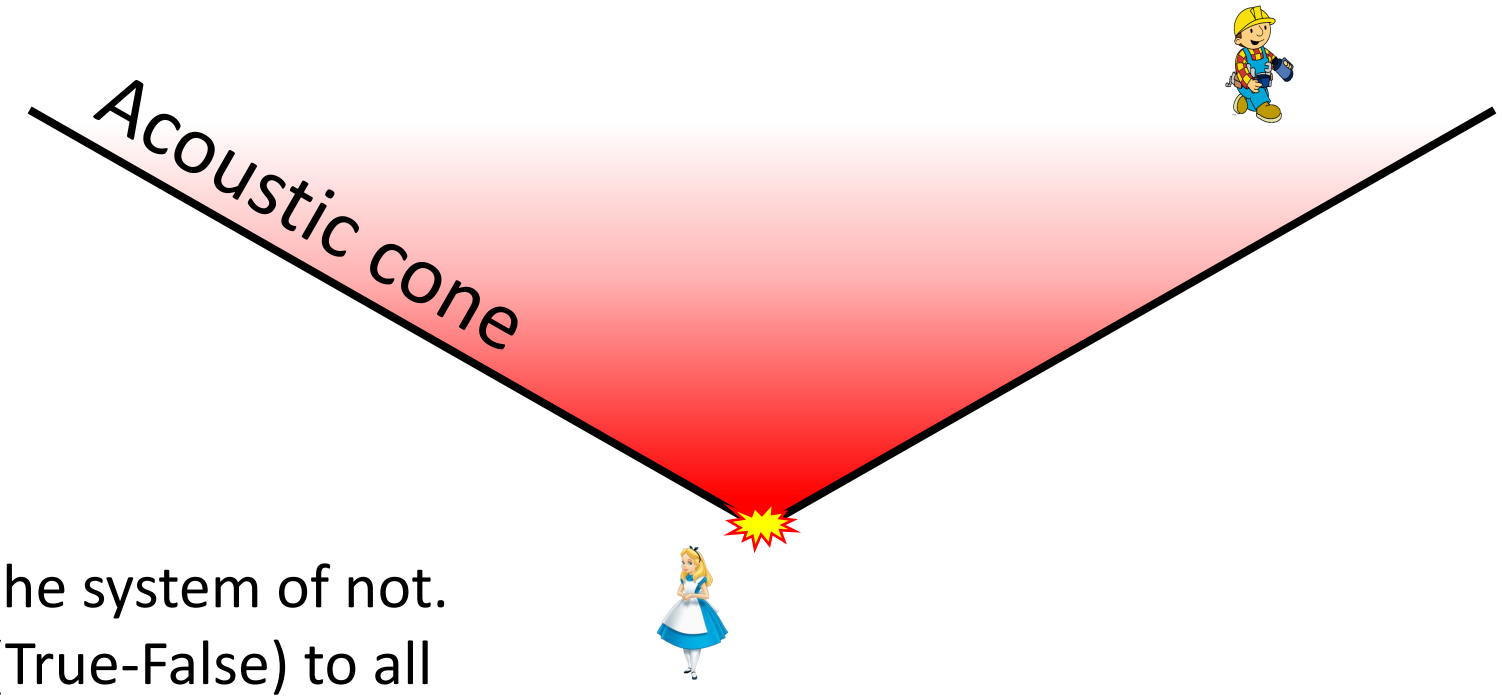


There is at least one growing solution of the source-less equations $\Rightarrow$ **instability!**

In summary

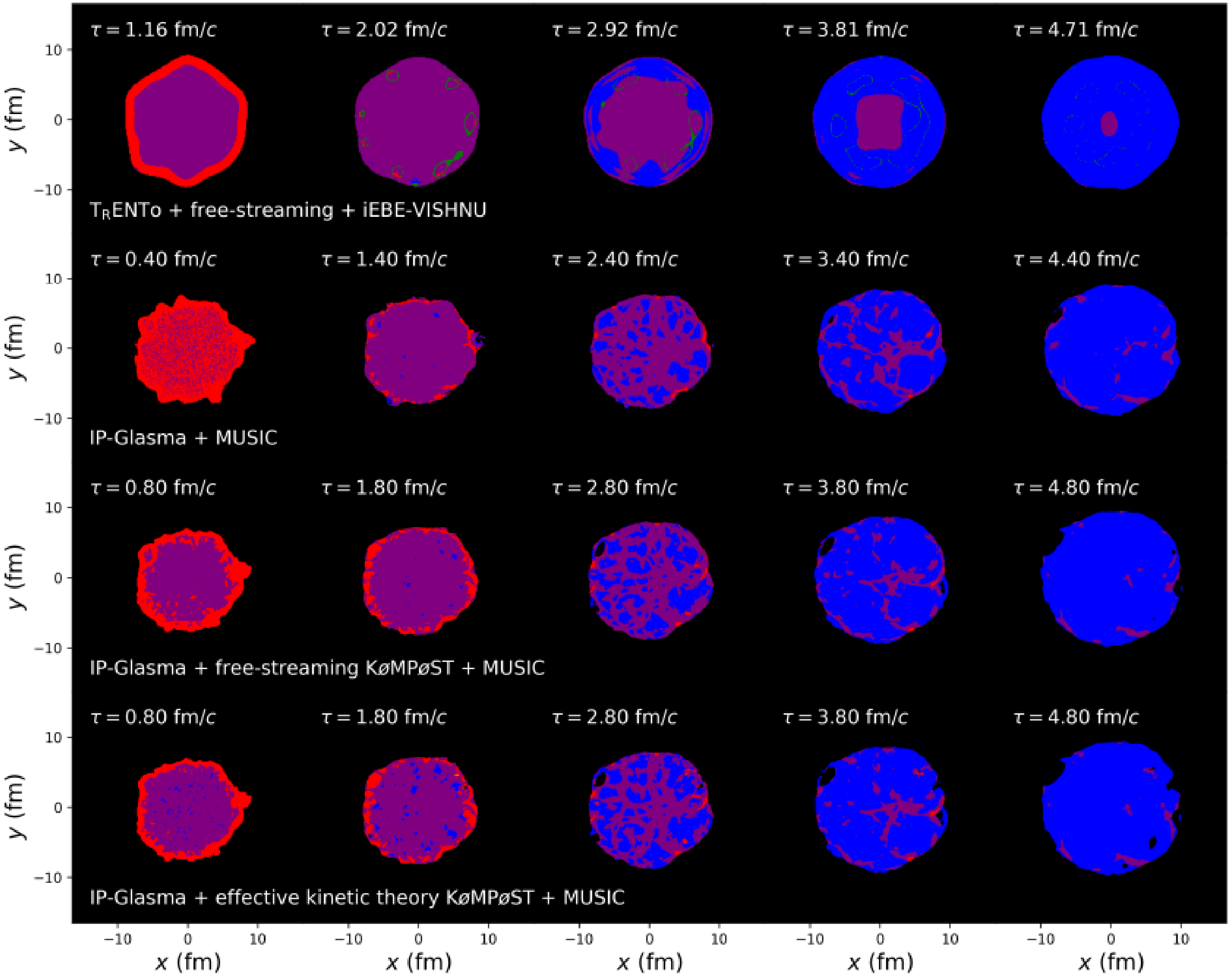
Dissipation is compatible with Special Relativity
only if information cannot travel than light (i.e. causality).

Why "information"?



Alice can choose whether to perturb the system of not.
Thus, she can send a binary message (True-False) to all observes who sit inside the acoustic cone!

Causality is a tricky beast!



Simulations of the QGP in a heavy-ion collision:

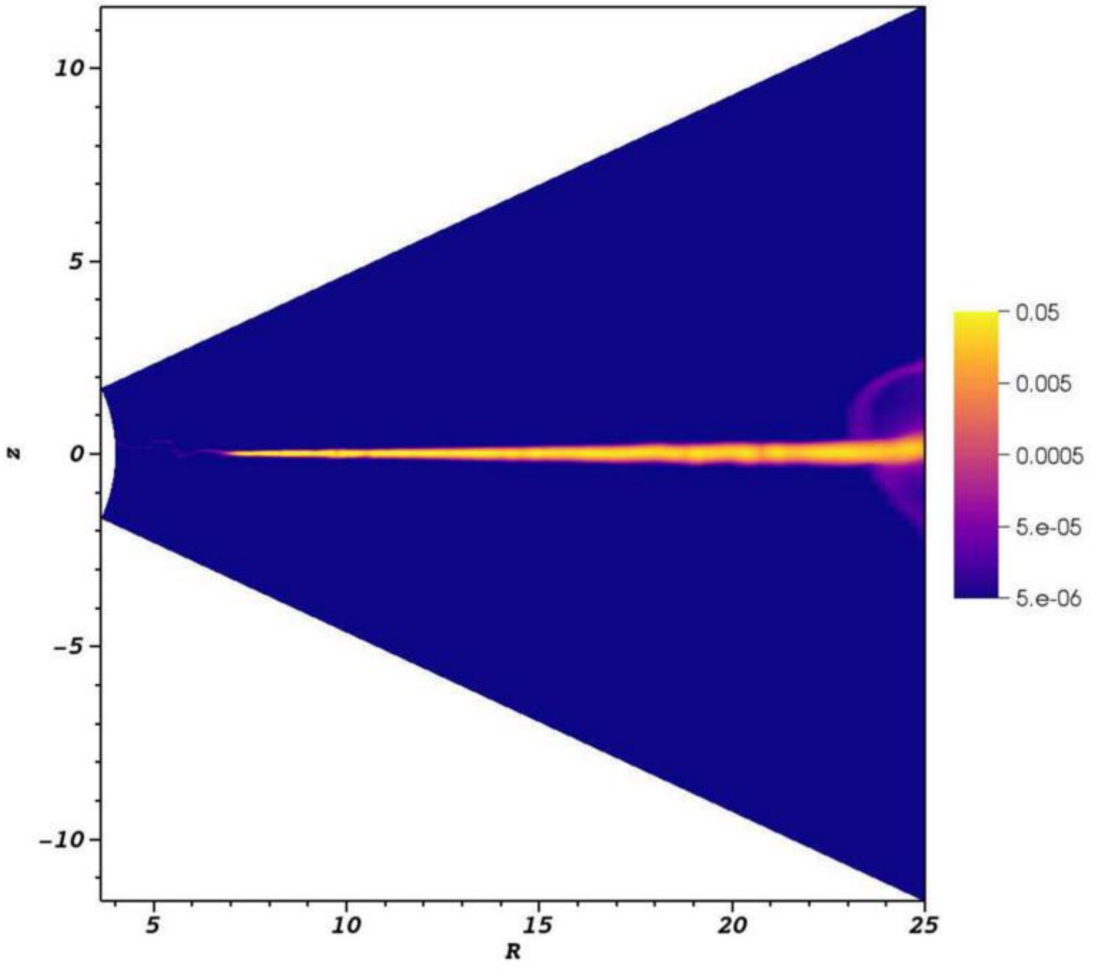
- Red = acausal;
- Purple = unknown;
- Blue = causal.

The best simulations currently available propagate information faster than light!

If you do not believe me, ask the authors ☺

(Phys. Rev. C **105**, L061901, 2022)

Acausal simulations are everywhere

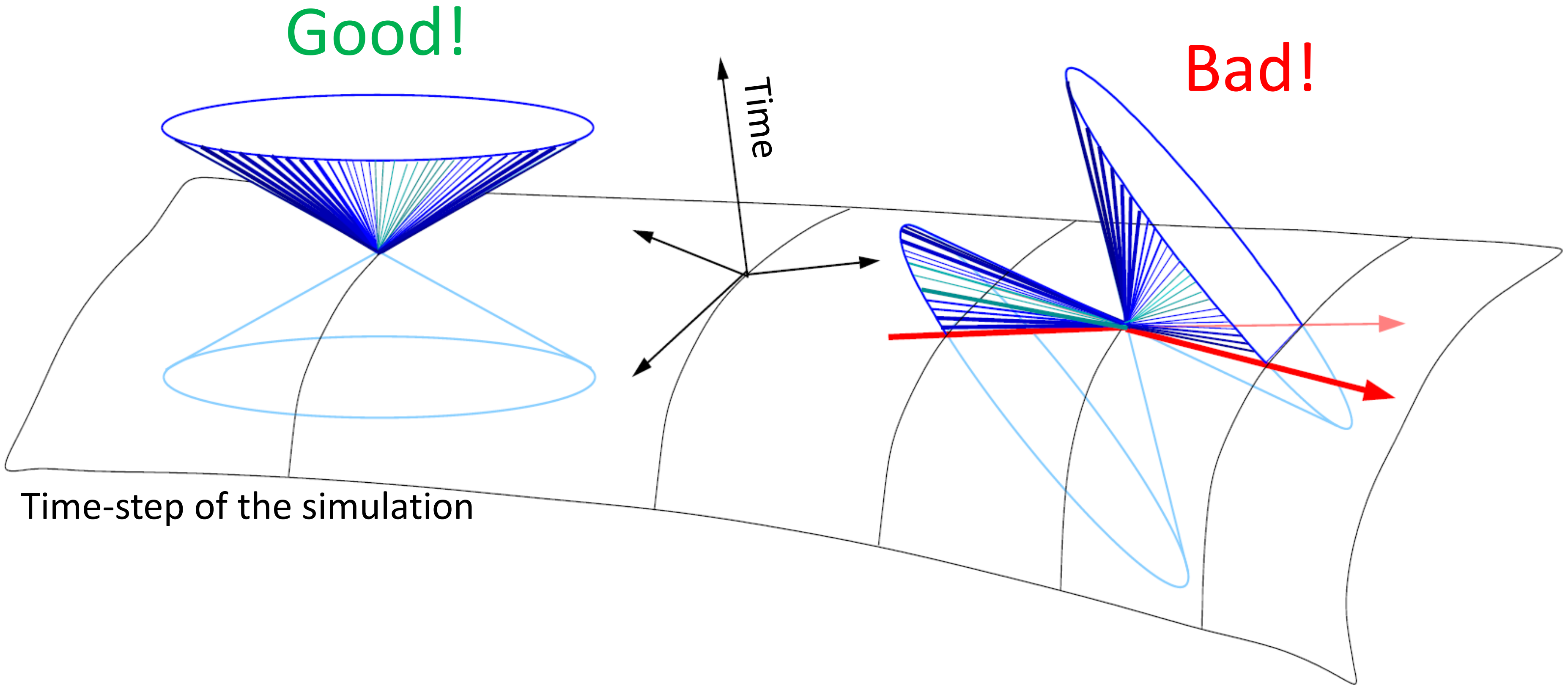


This accretion disk, if simulated correctly, should explode within one numerical timestep.

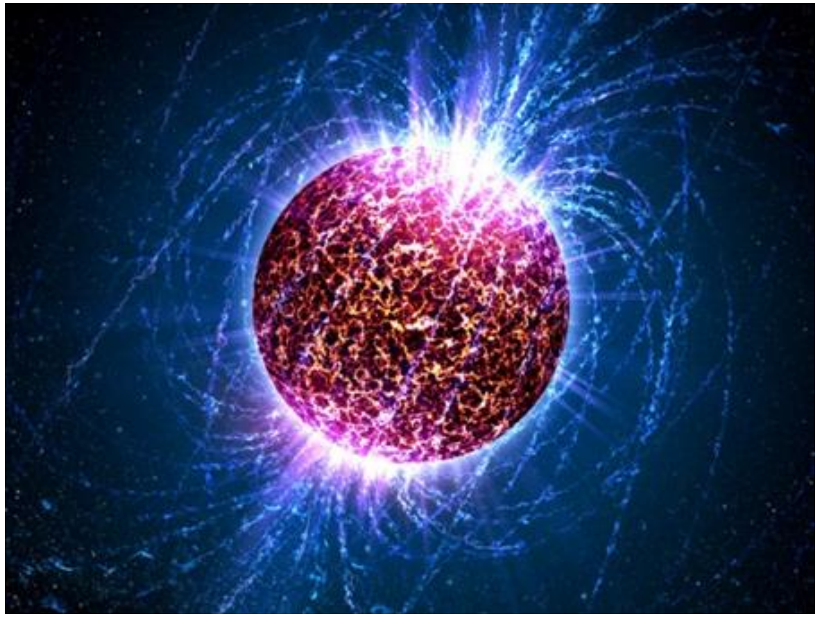
(P. Chris Fragile *et al* 2018 *ApJ* **857** 1)

What if you have simulated an acausal theory?

Plot the acoustic cone…



Cold Neutron Stars



Cold neutron stars are superfluid.
We need a relativistic generalization of Landau's two-fluid model.

Stability and causality of Carter's multifluid theory

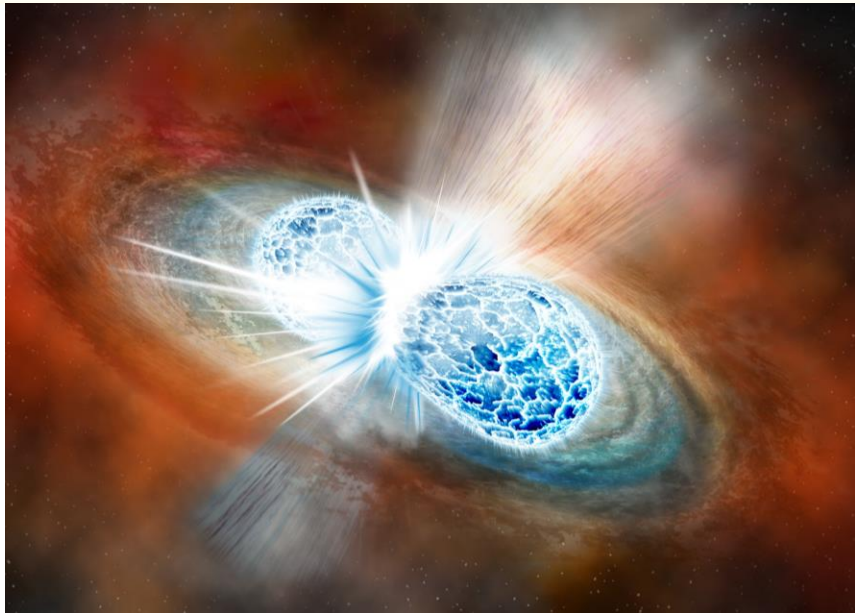
L Gavassino[2,1]

Published 23 August 2022 • © 2022 IOP Publishing Ltd
Classical and Quantum Gravity, Volume 39, Number 18
Citation L Gavassino 2022 *Class. Quantum Grav.* **39** 185008

Extending Israel and Stewart hydrodynamics to relativistic superfluids via Carter's multifluid approach

L. Gavassino, M. Antonelli, and B. Haskell
Phys. Rev. D **105**, 045011 – Published 16 February 2022

Neutron Mergers



Bulk-viscous effects due to nuclear reactions may be important.

arXiv > astro-ph > arXiv:2207.00442

Astrophysics > High Energy Astrophysical Phenomena

[Submitted on 1 Jul 2022]

Emergence of microphysical viscosity in binary neutron star post-merger dynamics

Elias R. Most, Alexander Haber, Steven P. Harris, Ziyuan Zhang, Mark G. Alford, Jorge Noronha

arXiv > gr-qc > arXiv:2204.11809

General Relativity and Quantum Cosmology

[Submitted on 25 Apr 2022]

Simulating bulk viscosity in neutron stars I: formalism

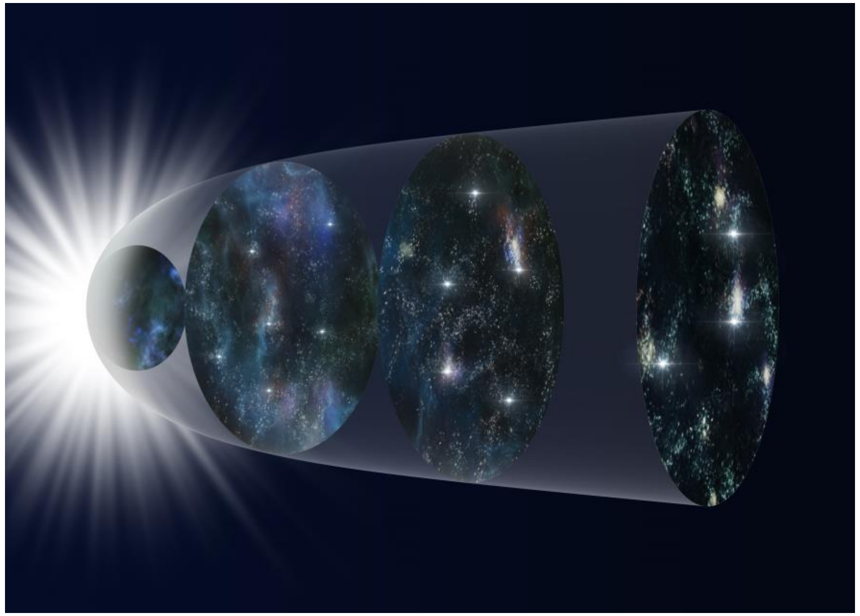
Giovanni Camelio, Lorenzo Gavassino, Marco Antonelli, Sebastiano Bernuzzi, Brynmor Haskell

Open Access

First-Order General-Relativistic Viscous Fluid Dynamics

Fábio S. Bemfica, Marcelo M. Disconzi, and Jorge Noronha
Phys. Rev. X **12**, 021044 – Published 24 May 2022

Cosmology



Also the Universe is bulk-viscous!

arXiv > gr-qc > arXiv:2210.13372

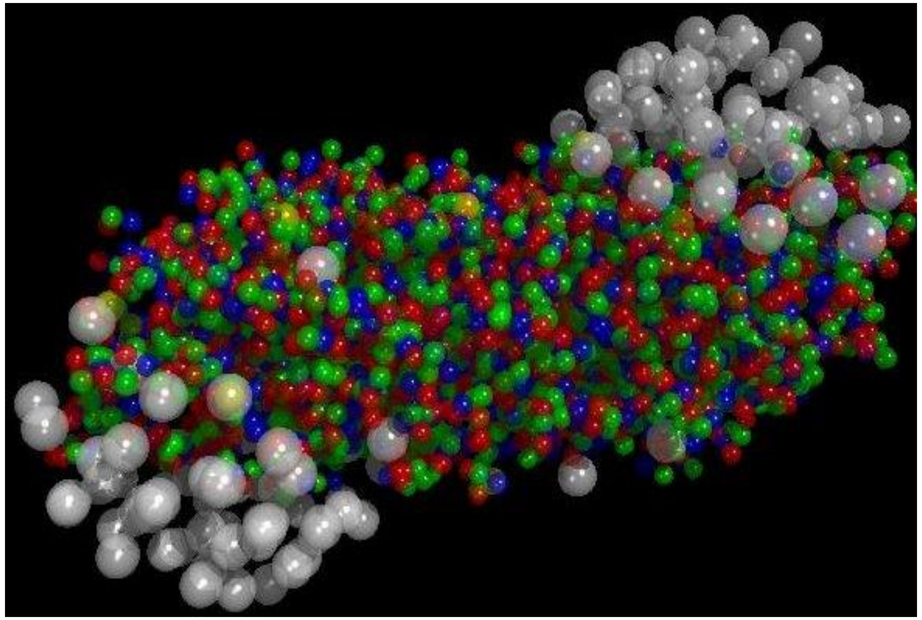
General Relativity and Quantum Cosmology

[Submitted on 24 Oct 2022 (v1), last revised 2 Nov 2022 (this version, v2)]

Cosmological consequences of first-order general-relativistic viscous fluid dynamics

Fábio S. Bemfica, Marcelo M. Disconzi, Jorge Noronha, Robert J. Scherrer

And, of course, the quark-gluon plasma!



But I leave this topic to better people…

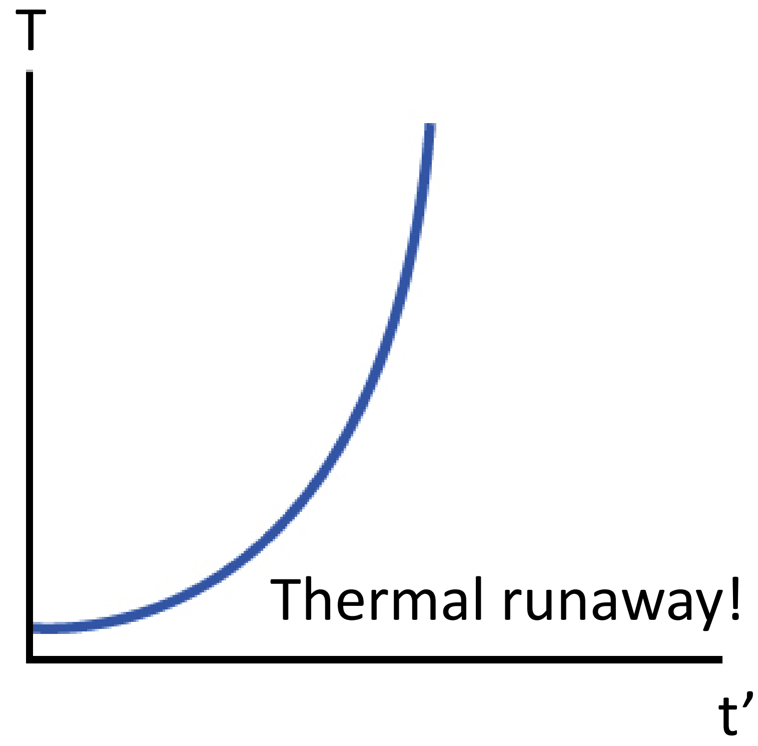
Example: boosted heat equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

$$\frac{\partial T}{\partial t'} - v\frac{\partial T}{\partial x'} = D\gamma\left(\frac{\partial^2 T}{\partial x'^2} - 2v\frac{\partial^2 T}{\partial x'\partial t'} + v^2\frac{\partial^2 T}{\partial t'^2}\right)$$

Homogeneous limit: $$\frac{\partial T}{\partial t'} = D\gamma v^2\frac{\partial^2 T}{\partial t'^2}$$

$$T = T_0 + \frac{\dot{T}_0}{\Gamma_+}\left(e^{\Gamma_+ t'} - 1\right) \qquad \Gamma_+ = \frac{1}{D\gamma v^2} > 0$$



The rate diverges when we approach $v = 0$... as expected.

The curious case of bulk viscosity

Navier-Stokes: $\Pi = -\zeta \nabla_\mu u^\mu$

Bad! Acausal, unstable, breaks the arrow of time.

Experiment: mixture of two species $\{a, b\}$ with a reaction

$$a + a \leftrightarrow b + b$$

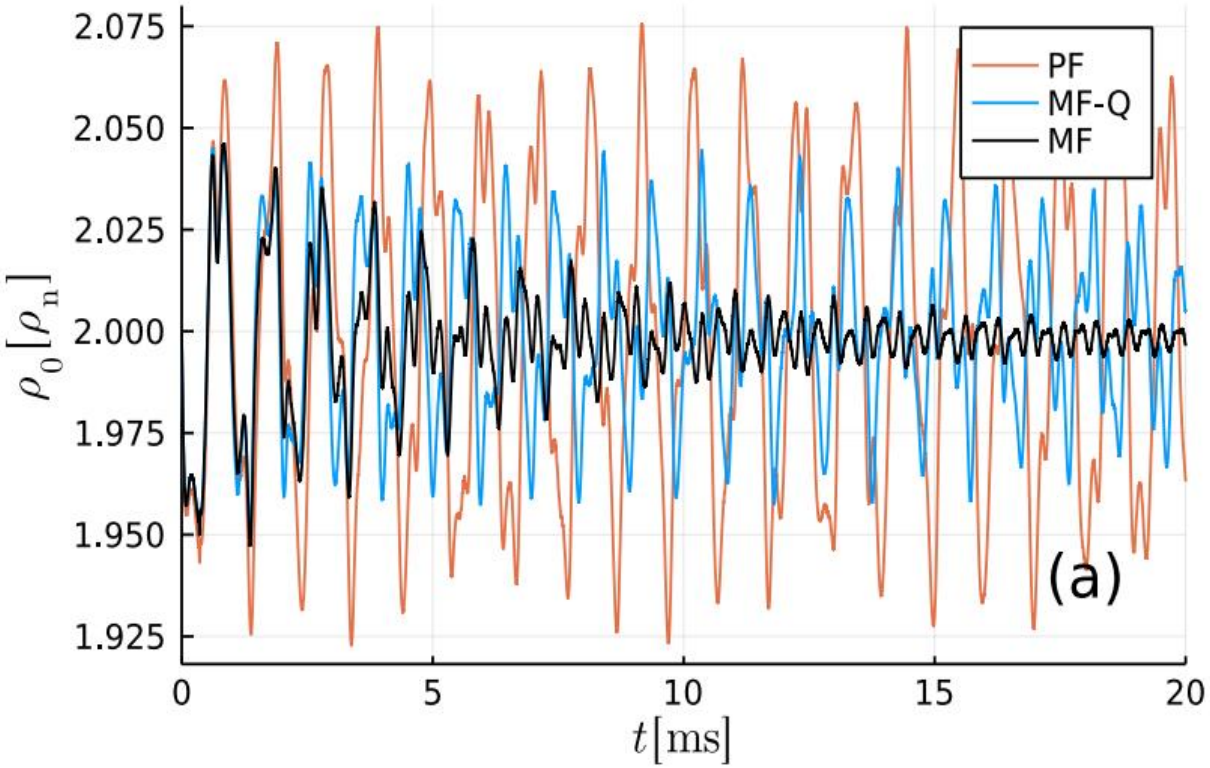
This is a thermodynamically consistent theory!

Introduce the affinity $A = \mu_b - \mu_a$ ($= 0$ at equilibrium)

Expand in the affinity. Eventually you get

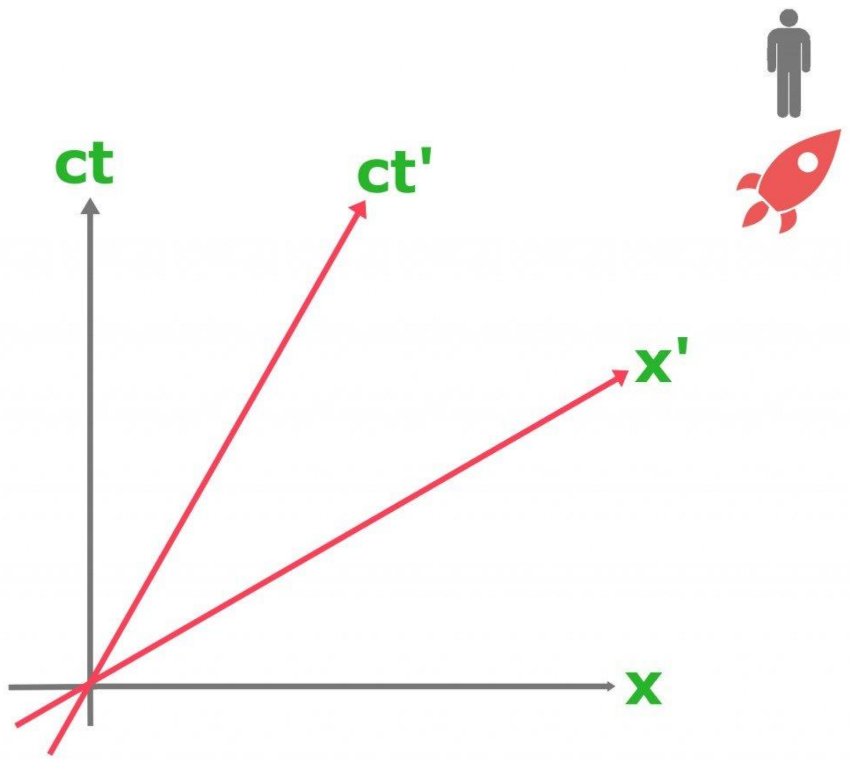
$$\tau\dot{\Pi} + \Pi = -\zeta \nabla_\mu u^\mu$$

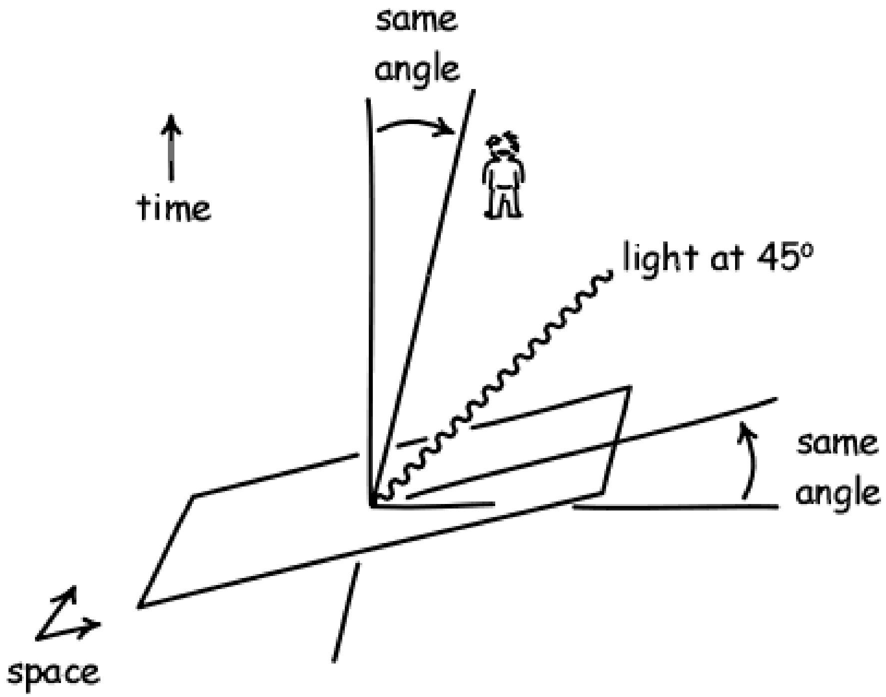
Now yes! Causal, stable and consistent with the arrow of time.



First fully General-Relativistic bulk-viscous simulation of neutron star oscillations (Camelio et al. arXiv:2204.11810)

Brief recap: Lorentz boost





“True” equation does not imply “true” solution

Newton’s second law

$$m\dot{v} = -\gamma v$$

A “true” fact about the system

A reliable tool to predict the future (i.e. an equation of motion)

Solutions:

$$v(t) = v(0)e^{-\gamma t/m}$$

Its derivative

$$m\ddot{v} = -\gamma\dot{v}$$

A “true” fact about the system

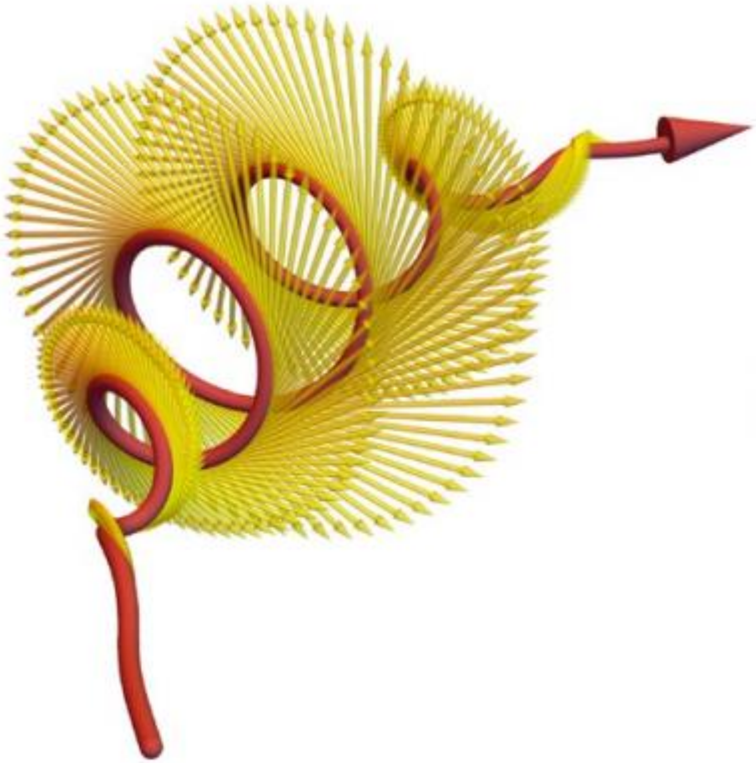
A non-reliable tool to predict the future (it is not a proper equation of motion)

You have an unphysical solution: $v(t) = v(0)$.

A well-known example: radiation reaction

Abraham-Lorentz-Dirac radiation reaction force:

$$m\ddot{x} = F_e(x, \dot{x}) + b\dddot{x}$$



Space of possible states: $\{x, \dot{x}, \ddot{x}\}$

The phase space is larger than the usual space $\{x, \dot{x}\}$ of dynamical systems. You have many more solutions.
Most of them blow up!

Dirac's proposal: the only physical solutions are those that don't blow up. Use future knowledge to set

$$\ddot{x} = \ddot{x}(x, \dot{x}).$$

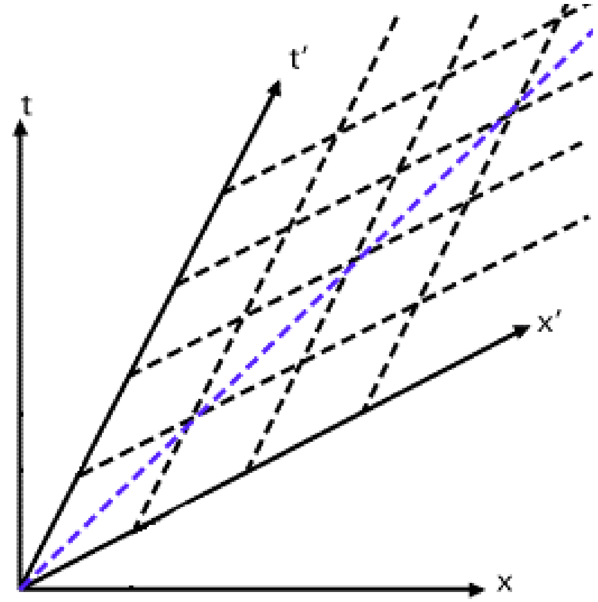
Another example: the boosted heat equation

$$\frac{\partial T}{\partial t} = D\frac{\partial^2 T}{\partial x^2}$$

Boost it!

$$\begin{cases} t' = \gamma(t - vx) \\ x' = \gamma(x - vt) \end{cases}$$

$$\frac{\partial T}{\partial t'} - v\frac{\partial T}{\partial x'} = D\gamma\left(\frac{\partial^2 T}{\partial x'^2} - 2v\frac{\partial^2 T}{\partial x'\partial t'} + v^2\frac{\partial^2 T}{\partial t'^2}\right)$$



Let's take a look at the state space:
In the rest frame $\{T(x)\}$;
In the boosted frame $\{T(x'), \partial_{t'}T(x')\}$;
There are many more solutions. **Most of them blow up!** Example: $T(t', x') = \exp(t'/D\gamma v^2)$

I **define** $T^{-1} := \sqrt{-\beta_\nu \beta^\nu}$, so I do not need to "transform" T

Dirac-like solution: use future knowledge to set $(\partial_{t'}T)_{t'=0} = \partial_{t'}T[T_{t'=0}]$. Not very practical…