Exact lattice anomalies and a new path to lattice chiral gauge theories ?

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Motivation

Folklore

1. Anomalies cannot be realized on lattice...
2. Fermions cannot be given masses without breaking symmetries
3. Hard (impossible ?) to put chiral gauge theories on lattice.

Kähler-Dirac fermions offer a counter example to at least 2 of these statements

Plan

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- Kähler–Dirac and relation to Dirac. Discretization on curved space.
- New **gravitational** anomalies – survive discretization.
- Related 't Hooft anomalies place constraints on IR behavior - in particular fermions can be gapped without breaking symmetries (SMG)
- Construct **mirror** models with Kähler–Dirac fermions. Simplest anomaly free model realizes **Pati-Salam** GUT. Lattice realization ?

Kähler–Dirac equation

An alternative solution to the problem of square rooting the Laplacian:

Kähler-Dirac equation

$$(K - m)\Phi = 0 \quad \text{where } K = d - d^\dagger$$

Notice: $K^2 = -dd^\dagger - d^\dagger d = \Box$

Kähler-Dirac field $\Phi = (\phi, \phi_\mu, \phi_{\mu\nu}, \ldots)$.

Ex. 2d

$$\begin{aligned} \partial^\mu \phi_\mu - m\phi &= 0 \\ \partial_\mu \phi + \partial^\nu \phi_{\nu\mu} - m\phi_\mu &= 0 \\ \partial_\mu \phi_\nu - \partial_\nu \phi_\mu - m\phi_{\mu\nu} &= 0 \end{aligned}$$

Connection to Dirac

Form matrix

$$\Psi = \sum_{p=0}^{D} \sum_{n_i} \phi_{n_1 \ldots n_p(x)} \gamma_1^{n_1} \gamma_2^{n_2} \cdots \gamma_p^{n_p}$$

eg in 2d:

$$\Psi = \phi I + \phi_i \sigma_i + \phi_{12} \sigma_1 \sigma_2$$

In flat space can show

$$(\gamma^\mu \partial_\mu - m)\Psi = 0$$

Kähler–Dirac field describes $2^{D/2}$ degenerate Dirac fermions !

$$\text{Action: } \int \mathrm{Tr}\, \overline{\Psi}(\gamma^\mu \partial_\mu - m)\Psi)$$

Kähler–Dirac in curved space...

Curved space

$$(d - d^\dagger - m)\Phi = 0 \quad \text{unchanged}$$

Kähler–Dirac fermions can be formulated on **any** smooth manifold.
No need for **spin structure**
No need for spin connection/vielbein formalism
very different from Dirac

Locally Kähler–Dirac decomposes into $2^{D/2}$ Dirac.
But **global** properties of K differ from $\not{\partial}$
eg. K has zero modes on S^D
Expect corrections $\sim \frac{\text{wavelength}}{\text{radius of curvature}}$

A $U(1)$ symmetry for Kähler–Dirac fermions

Kähler–Dirac Action:

$$\int \overline{\Phi} K \Phi \equiv \int d^D x \sqrt{g} \sum_{p=0}^{D} \overline{\Phi}_p [(K - m)\Phi]_p$$

Operator Γ : $\phi_{\mu_1 \ldots \mu_p} \rightarrow (-1)^p \, \phi_{\mu_1 \ldots \mu_p}$

Key property $\{\Gamma, K\}_+ = 0$

Generates exact $U(1)$ symmetry of massless action

$$\Phi \rightarrow e^{i\alpha\Gamma} \Phi$$

$$\overline{\Phi} \rightarrow \overline{\Phi} e^{i\alpha\Gamma}$$

Matrix rep $\Psi \xrightarrow{\Gamma} \gamma_5 \Psi \gamma_5$ **twisted chiral symmetry**

Reduced Kähler–Dirac (RKD) fermions

Define: $\Phi_\pm = \frac{1}{2}(1 \pm \Gamma)\Phi$
if $m = 0$:

$$S_{\mathrm{KD}} = \int \overline{\Phi}_+ K \Phi_- + \overline{\Phi}_- K \Phi_+ \rightarrow S_{\mathrm{RKD}} = \int \overline{\Phi}_+ K \Phi_-$$

Analogous to decomposition of massless Dirac field into 2 Weyl fields

Introducing $\Psi = \begin{pmatrix} \overline{\Phi}_+^T \\ \Phi_- \end{pmatrix}$

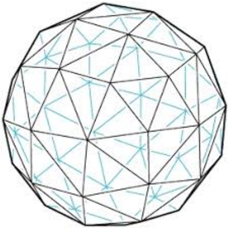
$$S_{\mathrm{RKD}} = \int \Psi^T \mathcal{K} \Psi \quad \mathcal{K} = \begin{pmatrix} 0 & K \\ -K^T & 0 \end{pmatrix}$$

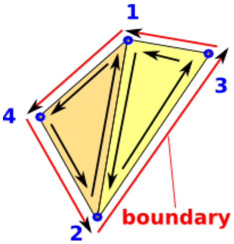
Reduced fields naturally massless

$$\overline{\Phi}_- \Phi_+ = \Phi_- \Phi_- = \overline{\Phi}_+ \overline{\Phi}_- = 0$$

Flat space continuum limit: $2^{D/2-1}$ Dirac or $2^{D/2}$ Majorana

Discrete curved space → triangulation





p-simplex $C^p = [a_0, \ldots, a_p]$

Boundary operator δ: $\delta[a_0 \ldots a_p] = \sum_{i=0}^{p} (-1)^i \, [a_0 \ldots \hat{a}_i \ldots a_p]$

where $\hat{a}_i$ indicates that vertex is omitted.

eg

$$\begin{aligned}\delta([142] + [123]) &= [42] - [12] + [14] + [23] - [13] + [12] \\ &= [42] + [23] + [31] + [14]\end{aligned}$$

Note:

$$\delta^2([142] + [123]) = [2] - [4] + [4] - [1] + [3] - [2] - [3] + [1] = 0!$$

Lattice p-forms

Continuum p-forms $\rightarrow$ each p-simplex $C_p \equiv [a_0, \ldots a_p]$ carries a lattice field $\phi(C_P)$:

$$\delta\phi(C_p) = \sum_{C_{p-1}} I(C_p, C_{p-1})\phi(C_{p-1})$$

where $I(C_p, C_{p-1})$ is zero unless C_{p-1} lies in boundary of C_p when it is ± 1 according to orientation

Similarly co-boundary operator $\overline{\delta}$:

$$\overline{\delta}\phi(C_p) = \sum_{C_{p+1}} I(C_{p+1}, C_p)^T \phi(C_{p+1})$$

Note $\delta^2 = \overline{\delta}^2 = 0$

Lattice Kähler–Dirac equation

$$\phi_p(x) \to \phi(C_p)$$
$$\delta \to d^\dagger$$
$$\overline{\delta} \to d$$

$$(\delta - \overline{\delta} - m)\Phi = 0 \quad \text{with } \Phi = (\phi(C_0), \phi(C_1), \ldots \Phi(C_D))$$

- Discrete Laplacian $\delta\overline{\delta} + \overline{\delta}\delta$.
- Exact zero modes of $\delta - \overline{\delta}$ match those of $d - d^\dagger$. Given by ranks of homology groups.
- No fermion doubling ! Continuum limit describes $2^{D/2}$ Dirac fermions **just like** continuum theory.
- $U_\Gamma(1)$ remains exact symmetry of lattice theory
- Can include arbitrary random triangulations with any topology and even non-orientable triangulations

Special case - staggered fermions

Decompose on p-cells of regular hypercubic lattice
Introduce second lattice with 1/2 lattice spacing

$$\chi(x+\hat{\mu}_1+\hat{\mu}_2+\ldots+\hat{\mu}_p)=\phi_{[\mu_1\ldots\mu_p]}(x)$$

Form discrete Kähler–Dirac matrix field using

$$\Psi(x)=\sum\chi(\mathbf{x}+\hat{\mu}_1+\ldots+\hat{\mu}_p)\gamma^{\mu_1}\cdots\gamma^{\mu_D}$$
$$=\sum_{\mathrm{b_i=0,1\ in\ hyp\ cube}}\chi(x+b)\gamma^{x+b}\quad \gamma^x=\gamma_1^{x_1}\gamma_2^{x_2}\ldots\gamma_D^{x_D}$$

Plug into $\sum\mathrm{Tr}(\overline{\Psi}\not{\Delta}\Psi)$ and do trace $\rightarrow$

$$S=\sum_{x,\mu}\eta_\mu(x)\overline{\chi}(x)\Delta_\mu\chi(x)\quad\text{with}\quad\eta_\mu(x)=(-1)^{\sum_i^{\mu-1}x_i}$$

Discrete Kähler–Dirac on regular lattice = staggered action !
$\Gamma\rightarrow\epsilon(x_1+\ldots x_D)-\text{site parity}$

Gravitational anomaly for Kähler–Dirac fermions

Work on lattice in d dims

Under $(\Phi, \overline{\Phi}) \rightarrow e^{i\alpha\Gamma}(\Phi, \overline{\Phi})$

$$\delta S_{\mathrm{KD}}(\overline{\Phi}, \Phi) = 0$$

But measure not invariant

$$D\Phi D\overline{\Phi} = \prod_p d\phi_p d\overline{\phi}_p \rightarrow e^{2iN_0\alpha} e^{-2iN_1\alpha} .. e^{2i(-1)^d N_d\alpha} \prod_p d\phi_p d\overline{\phi}_p$$
$$= \; e^{2i\chi\alpha} D\overline{\Phi} D\Phi \quad \chi \equiv \mathrm{Euler}$$

Anomaly in even dimensions

Compactify $R^{2n} \rightarrow S^{2n}$. Breaks $U(1) \rightarrow Z_4$.

Note

Example of QM anomaly for finite number dof ...

$$\sum_n \langle \phi_n | \Gamma | \phi_n \rangle = n_+ - n_- = \mathrm{Index}(K) = \chi$$

Consequences

Global $U(1)$ symmetry of Kähler–Dirac field broken to Z_4. Prohibits mass terms but allows for eg. four fermion ops. in $S_{\rm eff}$.

Theories of **reduced** Kähler–Dirac fermions with $U(1)$ symmetries cannot be consistently coupled to gravity – breakdown in gauge invariance
Analog: ABJ anomaly for Dirac implies cannot couple single Weyl fields to $U(1)$ gauge field

Can think of anomaly as **'t Hooft** anomaly for lattice fermions in flat space that arises when I try to couple them to gravity

't Hooft anomalies

Represent an obstruction to gauging a global symmetry.
Can be seen by coupling to classical background field
Non-zero anomaly coeff in U.V $\stackrel{\text{RG invariant}}{\rightarrow}$ **physics of I.R non-trivial**:

- Massless (composite) fermions (CFT)
- Goldstone bosons from SSB
- TQFT

In particular:

Cannot gap all states in I.R (symmetric mass generation) **unless all 't Hooft anomalies cancel**

Are there any (more) 't Hooft anomalies for Kähler–Dirac ?

Try to gauge Z_4 ...

Typical term in action:

$$\overline{\phi}(C_p)I(C_p, C_{p-1})\phi(C_{p-1})$$

Under local Z_4:

$$\phi(C_p) \to e^{i\frac{\pi}{2}\Gamma n(C_p)}\phi(C_p) \quad n(C_p) = 0, 1, 2, 3$$

To keep invariant need to promote $I(C_p, C_{p-1})$ to Z_4 gauge field $U(C_p, C_{p-1})$ transforming as

$$e^{-i\frac{\pi}{2}\Gamma n(C_p)}U(C_p, C_{p-1})e^{-i\frac{\pi}{2}\Gamma n(C_{p-1})}$$

Measure ? $\int d\phi(C_p)d\overline{\phi}(C_p)$ **NOT** invariant $\to$ 't Hooft anomaly !

Cancels for multiples of 2 flavors

This can also be seen from spectral flow of Kähler–Dirac fermions on non-orientable triangulations

Consequences

't Hooft anomalies for Kähler–Dirac fields cancelled for $N_f = 2k$

2 Kähler–Dirac $\equiv$ 4 reduced fields
Yield $2^{D/2+1}$ Dirac or $2^{D/2+2}$ Majorana fermions in continuum limit

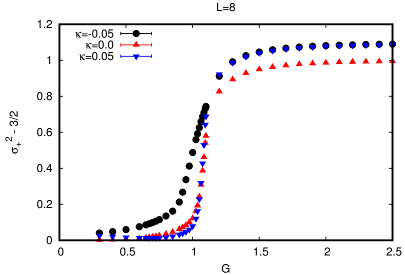
Agrees with results for gapping boundary fermions in topological superconductors and cancellation of discrete anomalies of Weyl/Majorana fermions in variety dims

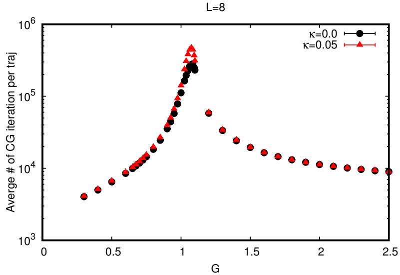
D=1	Time reversal $T^2 = 1$	8 Majorana	4 RKD
D=2	Chiral fermion parity	8 Majorana/Weyl	4 RKD
D=3	Time reversal $T^2 = -1$	16 Majorana	4 RKD
D=4	Spin-Z_4 symmetry	16 Majorana/Weyl	4 RKD

Explains observations of SMG for certain interacting staggered fermions in 4d

SMG for 2 staggered fermions in 4d

Higgs-Yukawa model: $S = \sum \chi(\eta.\Delta)\chi + \frac{1}{2}\sigma^2 - \kappa\sigma\Box\sigma + G\sigma\chi\chi$





Evidence for direct, continuous phase transition between massless and massive phases with no symmetry breaking (S.C et al. PRD98 (2018) 114514)

Summary so far

- Kähler–Dirac fermions admit gravitational anomalies which **survive discretization.** Break $U(1) \rightarrow Z_4$ in even dims
- 't Hooft anomaly for Z_4 cancels for multiples of 2 Kähler–Dirac . Yields 16 Majorana in flat 4d space – necessary condition for symmetric mass generation (SMG). Explains phase diagram of certain staggered fermion models.

Notice - Kähler–Dirac have no γ_5 anomalies.

What is SMG good for ?

Use SMG to gap mirrors in lattice models targeting chiral gauge theories ..?

Minimal Kähler–Dirac mirror model - continuum

Start: theory of **full** Kähler–Dirac fields with exact Z_4 symmetry. Decompose into **reduced** fields (Ψ_-, Ψ_+). Treat Ψ_+ as mirror. Need at least 4 copies for SMG

Consider "light" fields Ψ_- in (Euclidean) chiral basis $\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \overline{\sigma}_\mu & 0 \end{pmatrix}$ where $\sigma_\mu = (I, \sigma_i)$. **Continuum matrix form in flat space**

$$\Psi_- = \begin{pmatrix} 0 & \psi_R \\ \psi_L & 0 \end{pmatrix}$$

L and R handed doublet of Weyl fields transforming as $(1, 2)$ and $(2, 1)$ under an $SU(2) \times SU(2)$ flavor symmetry.

4 copies – additional $SU(4)$ symmetry.

Replace $\psi_R = i\sigma_2\psi_L^*$.

Get reps $(\mathbf{4}, \mathbf{2}, \mathbf{1}) \oplus (\overline{\mathbf{4}}, \mathbf{1}, \mathbf{2})$ - Pati-Salam reps !

Pati-Salam - quick summary

leptons (e,ν) as fourth color
left-right symmetric weak interaction
Symmetry: $SU(4) \otimes SU_L(2) \otimes SU_R(2)$

One generation:

$$\begin{pmatrix} u_r & u_b & u_s & \nu \\ d_r & d_b & d_g & e \end{pmatrix}_L \quad \oplus \quad \begin{pmatrix} u_r^c & u_b^c & u_s^c & \nu^c \\ d_r^c & d_b^c & d_g^c & e^c \end{pmatrix}_L$$

Subsequently $SU(4) \to SU(3)$ and $SU_L(2) \otimes SU_R(2) \to SU_L(2)$

$(4,2,1) \to (3,2)_{\frac{1}{6}} \oplus (1,2)_{-\frac{1}{2}}$ q_L and l_L
$(\overline{4},1,2) \to (\overline{3},1)_{\frac{1}{3}} \oplus (\overline{3},1)_{-\frac{2}{3}} \oplus (1,1)_1 \oplus (1,1)_0$ d^c,u^c, e^c and ν^c
1 family of SM !

need eg GUT scale Higgs in $(4,1,2)$ rep. to do this

Gapping mirrors

Add Z_4 symmetric four fermion interactions in mirror sector. No effect on Pati-Salam fields

$$\frac{G^2}{2}\int d^4x\,\epsilon_{abcd}\left[\mathrm{tr}\,(\overline{\Psi}^a_-\overline{\Psi}^b_-)\mathrm{tr}\,(\overline{\Psi}^c_-\overline{\Psi}^d_-)+\mathrm{tr}\,(\Psi^a_+\Psi^b_+)\mathrm{tr}\,(\Psi^c_+\Psi^d_+)\right]$$

Better: gauge $SU(4)$ of mirror sector and use **confinement** to generate four fermion condensate + massive hadrons

Notice: mirror sector fields do not couple to Pati-Salam except gravitationally. Composite dark matter ?

Chiral lattice theory

Replace continuum Kähler–Dirac field by staggered field χ.

$$S = \sum_{x,\mu} \eta_\mu(x) \left[\overline{\chi}_+ \Delta_\mu \chi_- + \overline{\chi}_- \Delta^c_\mu \chi_+\right] +$$

$$G \sum_x \hat{\phi}_{ab} \left[\overline{\chi}^a_- \overline{\chi}^b_- + \chi^a_+ \chi^b_+\right] + \frac{1}{2} \sum_x \hat{\phi}^2_{ab}$$

with

$$\hat{\phi}_{ab} = \frac{1}{2}\left(\phi_{ab} + \frac{1}{2}\epsilon_{abcd}\phi_{cd}\right)$$

and

$$\Delta^c_\mu \chi_+(x) = U_\mu(x)\chi_+(x+\mu) - U^\dagger(x-\mu)\chi_+(x-\mu)$$

Continuum limit

Sixteen free Weyl fermions in PS rep. of global $SU(4) \times SU(2) \times SU(2)$ symmetry
Gapped mirror sector with $SU(4)$ invariant four fermion condensate + heavy $SU(4)$ hadrons

Conclusions

- Possible to build mirror models using (lattice) Kähler–Dirac fields and Γ. Anomaly cancellation conditions allow for SMG in mirror sector.
- Simplest model: remaining light fields $\rightarrow$ Pati-Salam. Mirror sector as composite dark matter ?
- In continuum limit expect restoration of global Pati-Salam symmetries. Gauging $SU(4)$ easy but $SU(2) \times SU(2)$ broken on lattice – only $Z_2 \times Z_2$ subgroup exact. Generated by shift symmetries ... **gauge ??**
- Sign problems for (gauged) reduced Kähler–Dirac fermions

Thanks !

Bard's take on lattice fermions

Lattice fermion doubling,
A curse and a blessing,
A problem to solve,
A mystery to unravel.

The doubling of fermions,
A consequence of the Nielsen–Ninomiya theorem,
A no-go theorem in lattice field theory,
A blow to the hopes of simulating chiral fermions on a lattice.

But the doubling of fermions is also a blessing,
A way to study the chiral anomaly,
A way to test the limits of lattice field theory,
A way to push the boundaries of our understanding of the fundamental particles of nature.

The doubling of fermions is a problem to solve,
A mystery to unravel,
A challenge to overcome,
A puzzle to piece together.

ChatGPT's ..

In the world of physics, there's a type of dance,
A quantum waltz, where fermions prance,
They move in a grid, a lattice so fine,
And in their steps, the laws of physics align.
Staggered fermions, they lead the way,
Their spins and charges, in perfect array,
They move in pairs, one up, one down,
A dance so precise, it astounds.
And in their wake, a Kahler-Dirac field,
A force so strong, it refuses to yield,
It guides their movements, with a quantum touch,
A dance so elegant, it's almost too much.
The Kahler-Dirac, it weaves a lattice so pure,
A fabric of space, where fermions endure,
And the staggered fermions, they glide and they spin,
A cosmic ballet, where beauty begins.
But there's more to this dance, than meets the eye,
For there are also Kahler-Dirac fermions, who comply
They move in a different way, with a different beat,
Their dance so unique, it's hard to compete.
The Kahler-Dirac fermions, they move in fours,
Two up, two down, their dance floor,
They stagger too, but in a different way,
A quantum rhythm, they proudly display.
And in their dance, a connection is revealed,
A cosmic symmetry, that cannot be concealed,
The staggered and Kahler-Dirac fermions, they unite,
A quantum duet, that shines so bright.
So let us marvel, at this cosmic dance,
Where particles move, in a quantum trance,
And let us ponder, the mysteries untold,
Of staggered and Kahler-Dirac fermions, so bold.