

# Truncated partial-wave analysis for photoproduction using Bayesian inference

work done in collaboration with:

**Philipp Kroenert, Farah Afzal and Annika Thiel**

[arXiv publication will follow in the (near) future ...]

Yannick Wunderlich

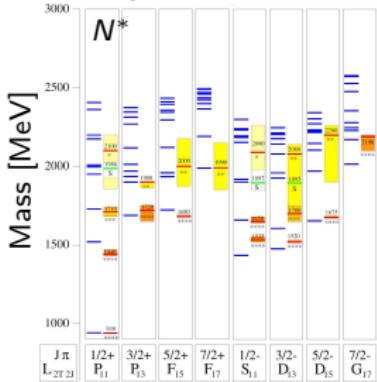
HISKP, University of Bonn

March 23, 2023



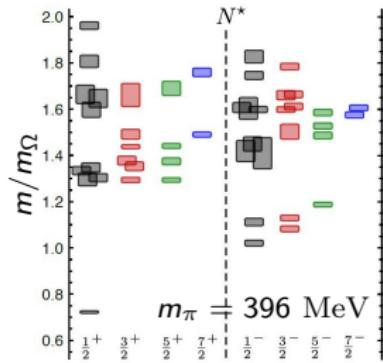
# Introduction: Why baryon spectroscopy?

## Quark models



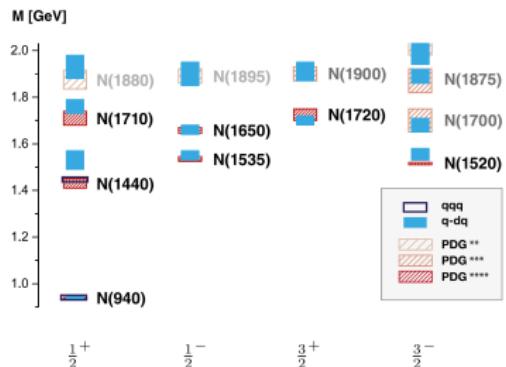
[U. Löring et al.,  
EPJ A **10**, 395 (2001)]

## Lattice QCD



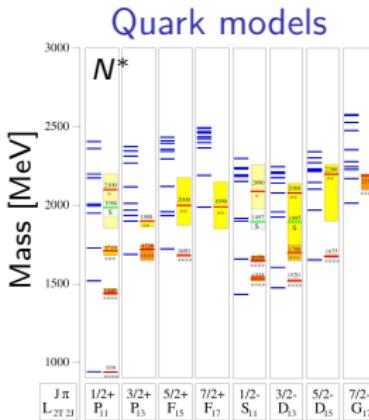
[R. Edwards et al.,  
Phys. Rev. D **84**, 07450 (2011)]

## DSE-/BSE framework

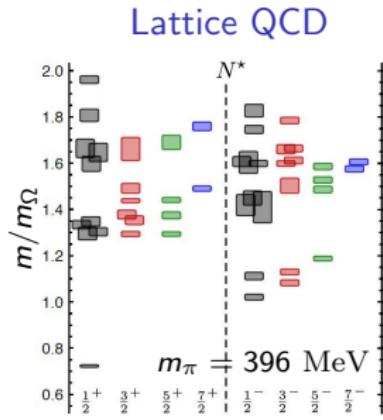


[G. Eichmann and Ch. S. Fischer,  
Few Body Syst. **60** (2019) 1, 2]

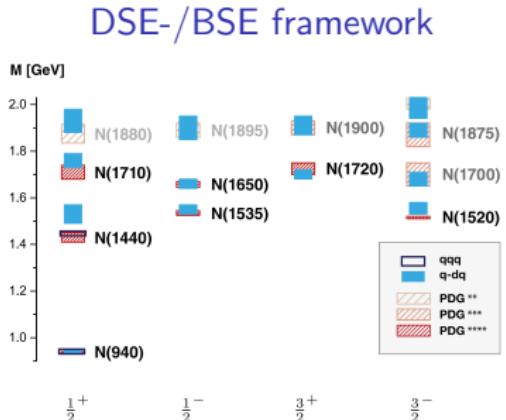
# Introduction: Why baryon spectroscopy?



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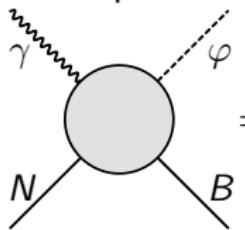
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Extract resonances from scattering-data via partial-wave analysis (PWA)

- \* ) Energy-dependent PWA: Reaction theoretic models; extract resonance poles in complex energy-plane; [BnGa], [JüBo], ...
- \* ) Single-energy PWA: Extract partial waves themselves at discrete energies on the real energy axis → This work (!):  $\eta$  photoproduction  $\gamma N \rightarrow \eta N$

# Photoproduction amplitudes

Photoproduction amplitude in the CMS:

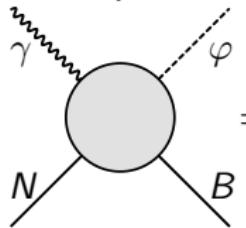

$$= \mathcal{T}_{fi} = \mathcal{C} \chi_{m_{sf}}^\dagger \left[ i \vec{\sigma} \cdot \hat{\epsilon} F_1 + \vec{\sigma} \cdot \hat{q} \vec{\sigma} \cdot (\hat{k} \times \hat{\epsilon}) F_2 + i \vec{\sigma} \cdot \hat{k} \hat{q} \cdot \hat{\epsilon} F_3 + i \vec{\sigma} \cdot \hat{q} \hat{q} \cdot \hat{\epsilon} F_4 \right] \chi_{m_{si}}$$

[Chew, Goldberger, Low & Nambu (1957)]

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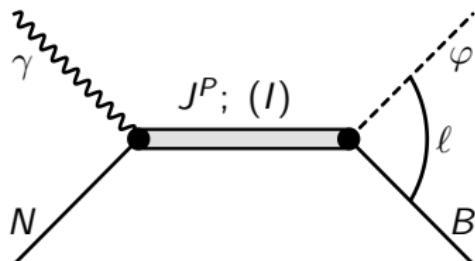
[Chew, Goldberger, Low & Nambu (1957)]

→ Process fully described by 4 complex amplitudes  $F_i(W, \theta)$ .

Important concept: expansion of full amplitudes into partial waves:

$$F_1(W, \theta) = \sum_{\ell=0}^{\infty} \left\{ [\ell M_{\ell+} + E_{\ell+}] P'_{\ell+1}(\cos(\theta)) + [(\ell+1) M_{\ell-} + E_{\ell-}] P'_{\ell-1}(\cos(\theta)) \right\}$$

$$F_2(W, \theta) = \dots$$



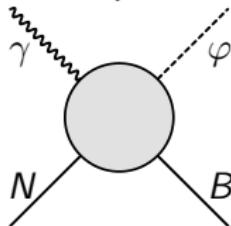
$$*) J = |\ell \pm 1/2|, P = (-)^{\ell+1}.$$

$$*) s\text{-chn. resonance } J^P; (I)$$

$$\updownarrow \text{multipole } E_{\ell\pm}^{(I)}, M_{\ell\pm}^{(I)}$$

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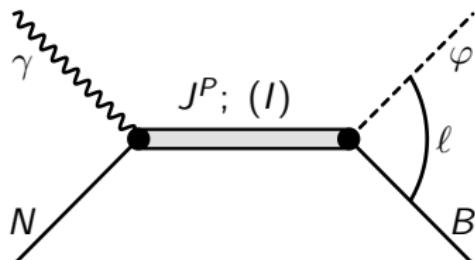
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In practice:

Truncate at some finite  $\ell_{\max}$

→ Try to extract the  $4\ell_{\max}$  complex multipoles in a fit to the data.

# Photoproduction amplitudes - II

The multipole-series fully written out ( $x = \cos \theta$ ):

$$F_1(W, \theta) = \sum_{\ell=0}^{\cancel{\ell_{\max}}} [\ell M_{\ell+}(W) + E_{\ell+}(W)] P'_{\ell+1}(x) + [(\ell+1) M_{\ell-}(W) + E_{\ell-}(W)] P'_{\ell-1}(x),$$

$$F_2(W, \theta) = \sum_{\ell=1}^{\cancel{\ell_{\max}}} [(\ell+1) M_{\ell+}(W) + \ell M_{\ell-}(W)] P'_{\ell}(x),$$

$$F_3(W, \theta) = \sum_{\ell=1}^{\cancel{\ell_{\max}}} \{ [E_{\ell+}(W) - M_{\ell+}(W)] P''_{\ell+1}(x) + [E_{\ell-}(W) + M_{\ell-}(W)] P''_{\ell-1}(x) \},$$

$$F_4(W, \theta) = \sum_{\ell=2}^{\cancel{\ell_{\max}}} [M_{\ell+}(W) - E_{\ell+}(W) - M_{\ell-}(W) - E_{\ell-}(W)] P''_{\ell}(x),$$

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\*)  $4\ell_{\max}$  complex multipoles present in every truncation-order  $\ell_{\max} \geq 1$ :

$$\{E_{0+}, E_{1+}, M_{1+}, M_{1-}, E_{2+}, E_{2-}, \dots, M_{\ell_{\max}-}\}.$$

\*) Multipoles determined up to 1 overall phase (!):

$$\Rightarrow 2 \times 4\ell_{\max} - 1 = \underline{8\ell_{\max} - 1} \text{ parameters in the TPWA.}$$

# Polarization observables

Generic definition of an observable

$$\check{\Omega} = \left[ \left( \frac{d\sigma}{d\Omega} \right)^{(B_1, T_1, R_1)} - \left( \frac{d\sigma}{d\Omega} \right)^{(B_2, T_2, R_2)} \right]$$

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\*) In total, 16 non-redundant observables

$$\check{\Omega}^\alpha(W, \theta) = \frac{1}{2} \frac{q}{k} \sum_{i,j} F_i^* \hat{A}_{ij}^\alpha F_j, \quad \alpha = 1, \dots, 16$$

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→ Example: Beam-asymmetry  $\check{\Sigma}$  is bilinear in the  $F_i$ :

$$\begin{aligned} \check{\Sigma} &= \left( \frac{d\sigma}{d\Omega} \right)^{(\perp, 0, 0)} - \left( \frac{d\sigma}{d\Omega} \right)^{(\parallel, 0, 0)} \\ &= -\frac{q}{k} \sin^2 \theta \operatorname{Re} \left[ |F_3|^2 + |F_4|^2 + 2 \{ F_1^* F_4 + F_2^* F_3 + \cos \theta F_3^* F_4 \} \right]. \end{aligned}$$

# Polarization observables - II

| Observable       | Transversity representation                            | Type                     |
|------------------|--|--------------------------|
| $\sigma_0$       | $\frac{1}{2} ( b_1 ^2 +  b_2 ^2 +  b_3 ^2 +  b_4 ^2)$  |                          |
| $\check{\Sigma}$ | $\frac{1}{2} (- b_1 ^2 -  b_2 ^2 +  b_3 ^2 +  b_4 ^2)$ | $S$                      |
| $\check{T}$      | $\frac{1}{2} ( b_1 ^2 -  b_2 ^2 -  b_3 ^2 +  b_4 ^2)$  |                          |
| $\check{P}$      | $\frac{1}{2} (- b_1 ^2 +  b_2 ^2 -  b_3 ^2 +  b_4 ^2)$ |                          |
| $\check{G}$      | $\text{Im} [-b_1 b_3^* - b_2 b_4^*]$                   |                          |
| $\check{H}$      | $-\text{Re} [b_1 b_3^* - b_2 b_4^*]$                   | $B\mathcal{T}$           |
| $\check{E}$      | $-\text{Re} [b_1 b_3^* + b_2 b_4^*]$                   |                          |
| $\check{F}$      | $\text{Im} [b_1 b_3^* - b_2 b_4^*]$                    |                          |
| $\check{O}_{x'}$ | $-\text{Re} [-b_1 b_4^* + b_2 b_3^*]$                  |                          |
| $\check{O}_{z'}$ | $\text{Im} [-b_1 b_4^* - b_2 b_3^*]$                   | $B\mathcal{R}$           |
| $\check{C}_{x'}$ | $\text{Im} [b_1 b_4^* - b_2 b_3^*]$                    |                          |
| $\check{C}_{z'}$ | $\text{Re} [b_1 b_4^* + b_2 b_3^*]$                    |                          |
| $\check{T}_{x'}$ | $-\text{Re} [-b_1 b_2^* + b_3 b_4^*]$                  |                          |
| $\check{T}_{z'}$ | $-\text{Im} [b_1 b_2^* - b_3 b_4^*]$                   | $\mathcal{T}\mathcal{R}$ |
| $\check{L}_{x'}$ | $-\text{Im} [-b_1 b_2^* - b_3 b_4^*]$                  |                          |
| $\check{L}_{z'}$ | $\text{Re} [-b_1 b_2^* - b_3 b_4^*]$                   |                          |

\*) Transversity amplitudes:

$$b_i = \sum_j M_{ij} F_j.$$

(Different scheme of spin-quantization)

\*) Observables simplify:

$$\check{\Omega}^\alpha = \frac{1}{2} \sum_{i,j} b_i^* \tilde{\Gamma}_{ij}^\alpha b_j.$$

# Ambiguities of the TPWA problem

\*) Consider  $\{\sigma_0, \check{\Sigma}, \check{T}, \check{P}\}$ , i.e.  $\check{\Omega}^{\alpha s} \propto \pm |b_1|^2 \pm |b_2|^2 \pm |b_3|^2 + |b_4|^2$ .

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- \*) Instead: Use  $t := \tan\left(\frac{\theta}{2}\right)$  and write linear factorizations (for finite  $\ell_{\max}$ ):

$$b_1(\theta) \propto \frac{\exp\left(-i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \prod_{j=1}^{2\ell_{\max}} (t + \beta_j), \quad b_2(\theta) \propto \frac{\exp\left(i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \prod_{j=1}^{2\ell_{\max}} (t - \beta_j),$$
$$b_3(\theta) \propto \frac{\exp\left(-i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \prod_{k=1}^{2\ell_{\max}} (t + \alpha_k), \quad b_4(\theta) \propto \frac{\exp\left(i\frac{\theta}{2}\right)}{(1+t^2)^{\ell_{\max}}} \prod_{k=1}^{2\ell_{\max}} (t - \alpha_k),$$

with  $4\ell_{\max}$  complex roots  $\{\alpha_k, \beta_j\}$  equivalent to multipoles, i.e.

$$\{E_{\ell\pm}, M_{\ell\pm}\} \Leftrightarrow \{\alpha_k, \beta_j\},$$

if and only if the constraint:  $\prod_{k=1}^{2\ell_{\max}} \alpha_k = \prod_{j=1}^{2\ell_{\max}} \beta_j$  is satisfied.

[A. S. Omelaenko, Sov. J. Nucl. Phys. 34, 406 (1981)]

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↪ The complex conjugation of roots leaves the group  $\mathcal{S}$  invariant:

$$(t - \alpha^*) (t - \alpha) \xrightarrow{\alpha \rightarrow \alpha^*} (t - [\alpha^*]^*) (t - \alpha^*) = (t - \alpha^*) (t - \alpha).$$

⇒ Mechanism can result in *multiple 'solutions' (or 'modes')* of single-energy fit!

# Bayes' Theorem and Bayesian inference

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$$\Rightarrow \underbrace{p(\theta|y)}_{\text{'posterior'}} = \frac{\overbrace{p(y|\theta)}^{\text{'likelihood'}} \overbrace{p(\theta)}^{\text{'prior'}}}{\underbrace{\int d^D\theta p(y|\theta)p(\theta)}_{\text{'Bayesian evidence'}}}.$$

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\*) In our case, i.e. for the TPWA:

- $\theta = \{E_{\ell\pm}, M_{\ell\pm}\}$  (option: *nuisance parameters* to model syst. errors),
- $\mathbf{y} = [\mathbf{y}^{\sigma_0}, \mathbf{y}^{\check{G}}, \dots, \mathbf{y}^{\check{F}}]^T$ : values of measured observables,
- Likelihood has ' $\exp[-\frac{1}{2}\chi^2]$ -form', with a *correlated chisquare*:  
$$p(\mathbf{y}|\theta) = \frac{1}{\sqrt{(2\pi)^N |\Lambda|}} \exp \left[ -\frac{1}{2} (\mathbf{y} - \boldsymbol{\mu})^T \Lambda^{-1} (\mathbf{y} - \boldsymbol{\mu}) \right], \{\Lambda\}: \text{cov.-matrix}$$
- Prior  $p(\theta)$  is chosen *flat*, within the 'physically allowed' region for multipoles [e.g.: YW, arXiv:2008.00514]  $\rightarrow$  other choices possible (!)

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⇒ We want to obtain individual distribution for each parameter  $\theta_i$  from the *unnormalized posterior*

- Marginalize:  $p(\theta_i|y) = \int \dots \int d\theta_1 \dots d\theta_{i-1} d\theta_{i+1} \dots d\theta_D p(\theta_1 \dots \theta_D | y)$ ,
- Complicated integrals! Use (Markov chain) Monte Carlo methods ...

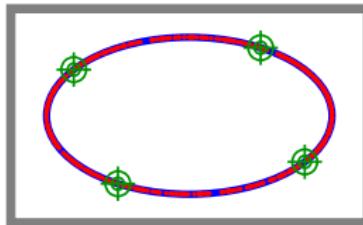
# Solution strategy for multimodal posteriors

## I. Monte Carlo maximum a posteriori estimation:

minimize the function  $\chi^2(\theta) + P(\theta)$  ('P': contains possible penalty-terms for nuisance parameters).

Employ Monte Carlo sampling, with  $N_{MC}$  initial conditions and equally many fits ( $N_{MC} \simeq 10000$ ).

⇒ Filter non-redundant sol.'s via clustering-algorithm



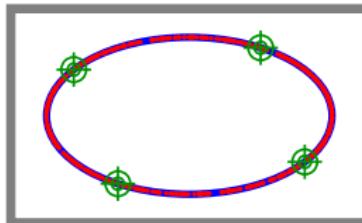
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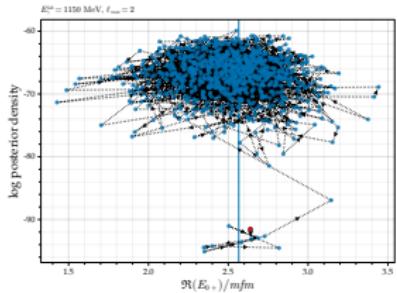


[Y.W. arXiv:2008.00514]

## II. Sampling the posterior:

Each solution found in step 'I.' is used as initial condition for the *Hamilton Monte Carlo (HMC)* algorithm as implemented in Stan.

For each solution, use  $N_c \simeq 10$  Markov chains and  $S \simeq 100000$  sampling points (50000 'warmup'-pts).



[Fig. courtesy of P. Kroenert]

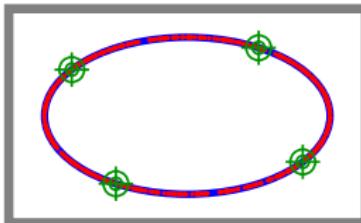
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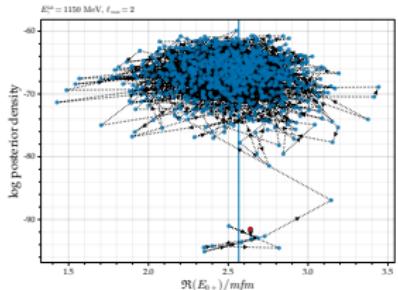


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[Y.W. courtesy of P. Kroenert]

## III. Monte Carlo convergence diagnostics:

- Cluster the chains into groups according to their sampled parameter space (calculate vector of quantiles for each chain → evaluate distance-matrix among quantile-vectors → input to DBSCAN-algorithm),  
→ Apply MC convergence diagnostics ( $\hat{R}$ -statistic, MC standard error) to each group individually.

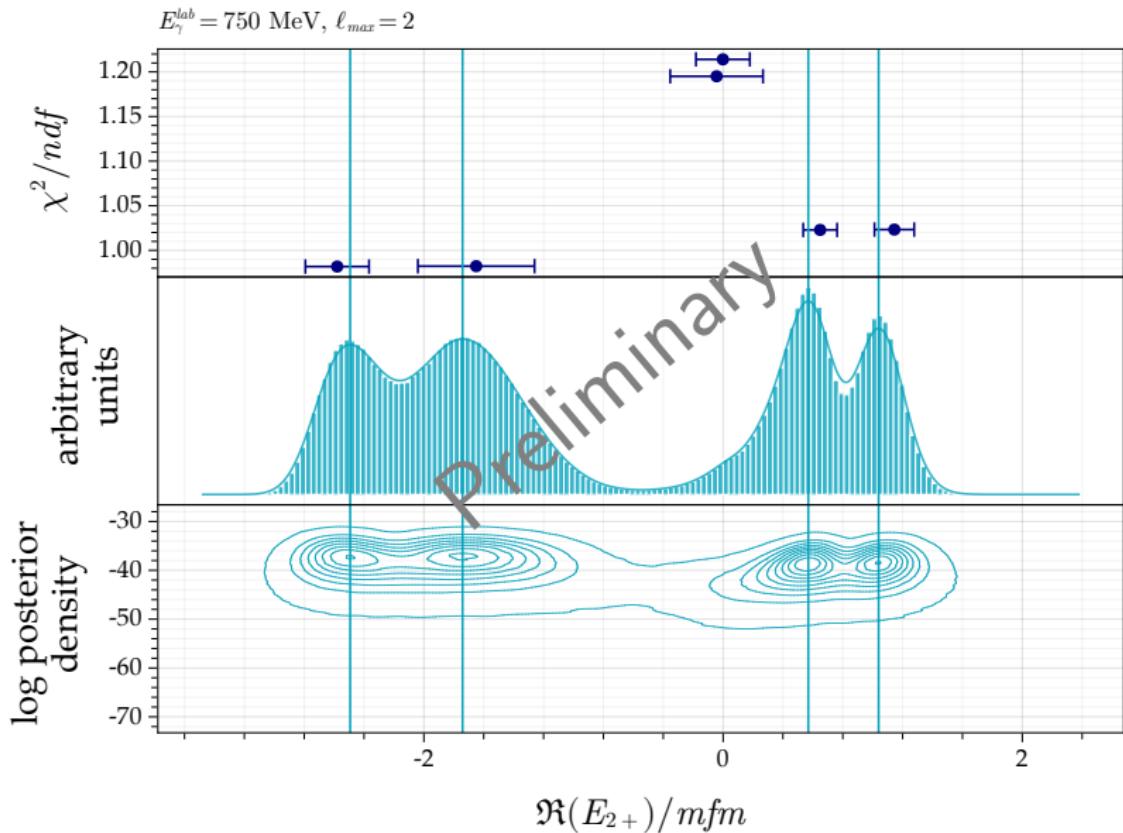
# Solution strategy for multimodal posteriors

- I. Monte Carlo maximum a posteriori estimation,
  - II. Sampling the posterior,
  - III. Monte Carlo convergence diagnostics,
- Apply the above-described procedure to a selected set of 6 polarization observables for the process  $\gamma p \rightarrow \eta p$ :

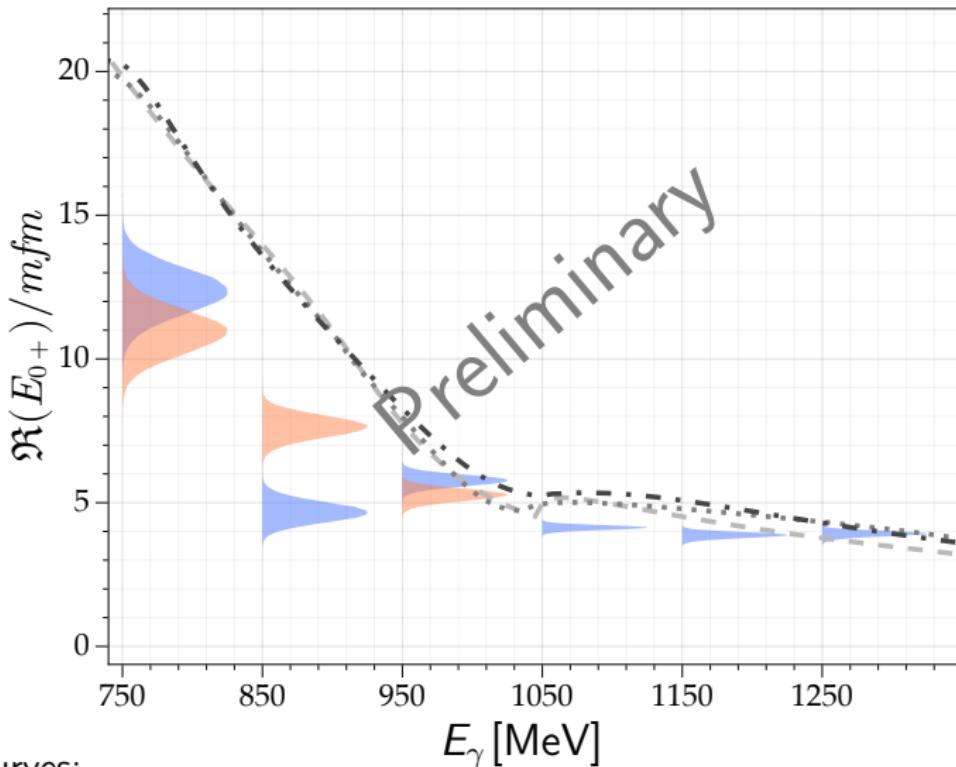
| Observable | No. of datapoints | Exp. facility | References  |
|------------|-------------------|---------------|---|
| $\sigma_0$ | 5736              | MAMI          | [Kashevarov et al., PRL <b>118</b> , 212001 (2017)]       |
| $T, F$     | 144               | MAMI          | [Akondi et al., PRL <b>113</b> , 102001 (2014)]           |
| $\Sigma$   | 140               | GRAAL         | [Bartalini et al., EPJ A <b>33</b> , 169 (2007)]          |
| $E$        | 84                | MAMI          | [F. Afzal, PhD thesis, ULB Bonn (2019)]                   |
| $G$        | 47                | CBELSA/TAPS   | [Müller et al., Phys. Lett. B <b>803</b> , 135323 (2020)] |

[cf.: P. Kroenert, YW, F. Afzal and A. Thiel, paper coming soon]

# Example: full results for one 'fit-parameter'



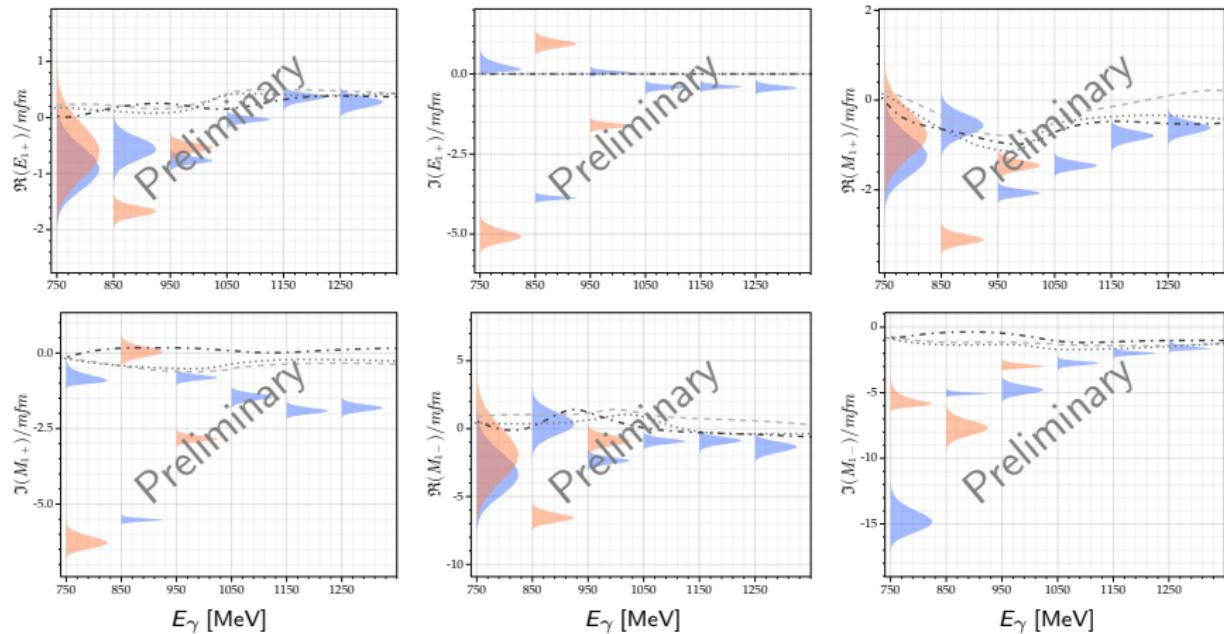
# $\ell_{\max} = 1$ : results for multipoles



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

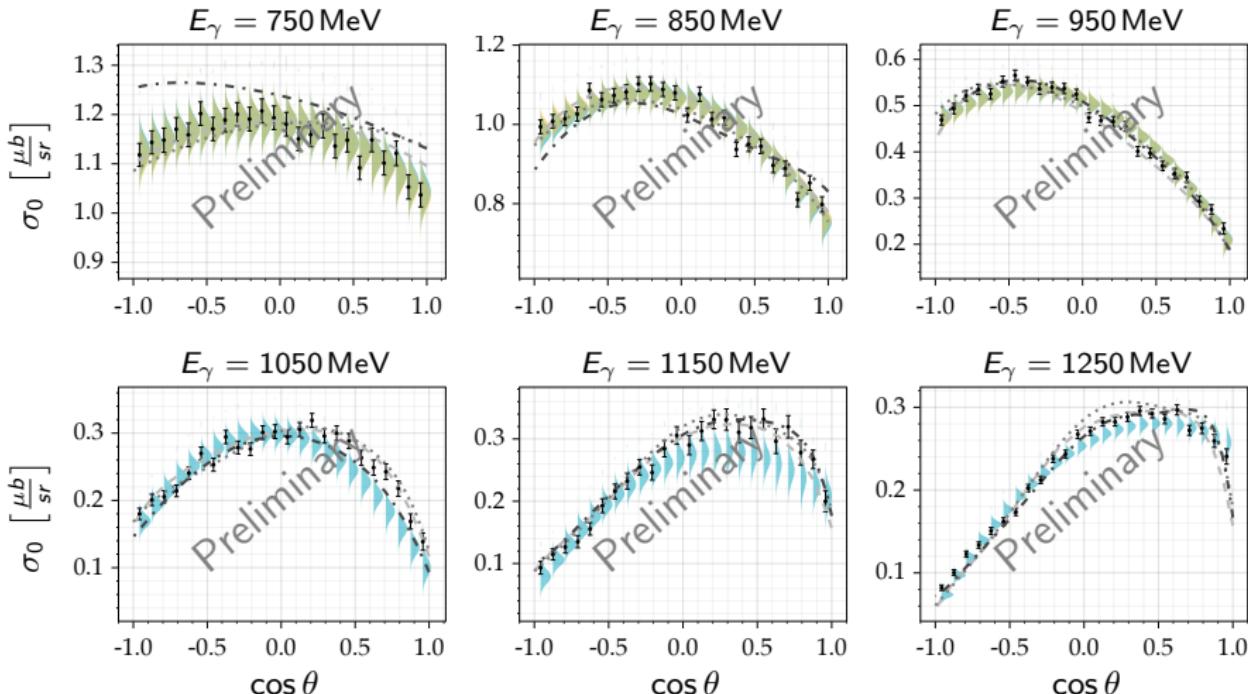
# $\ell_{\max} = 1$ : results for multipoles



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EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

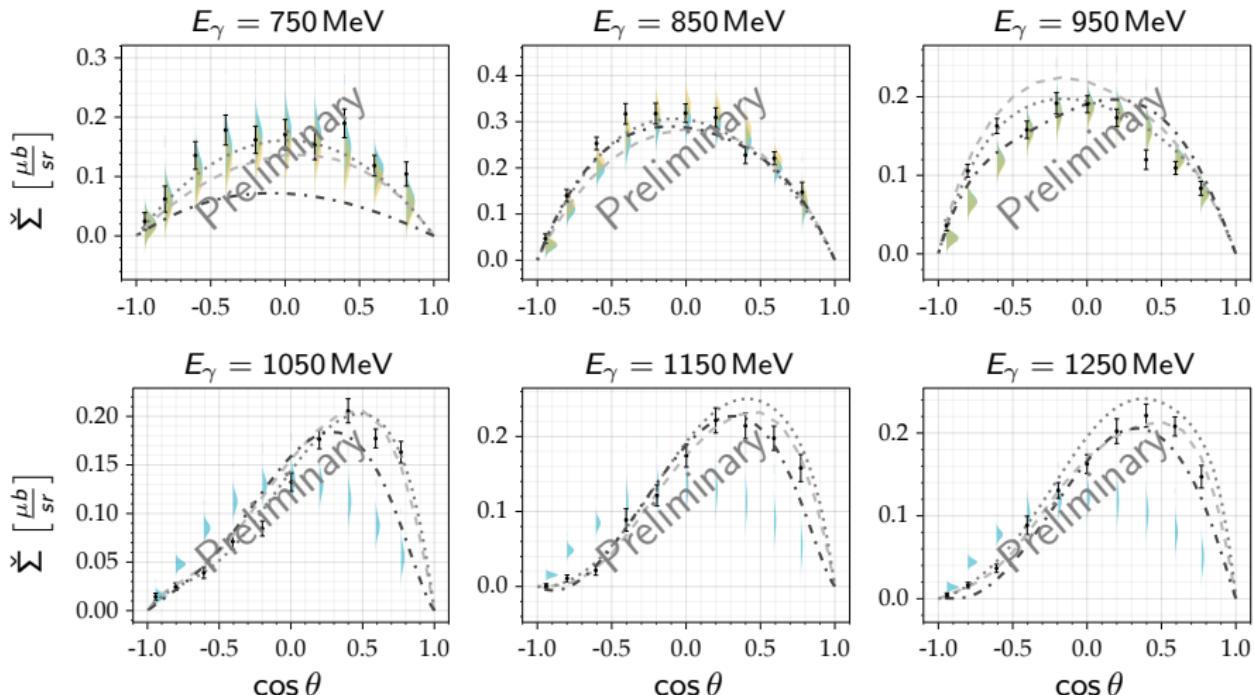
# $\ell_{\max} = 1$ : reproduced data



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

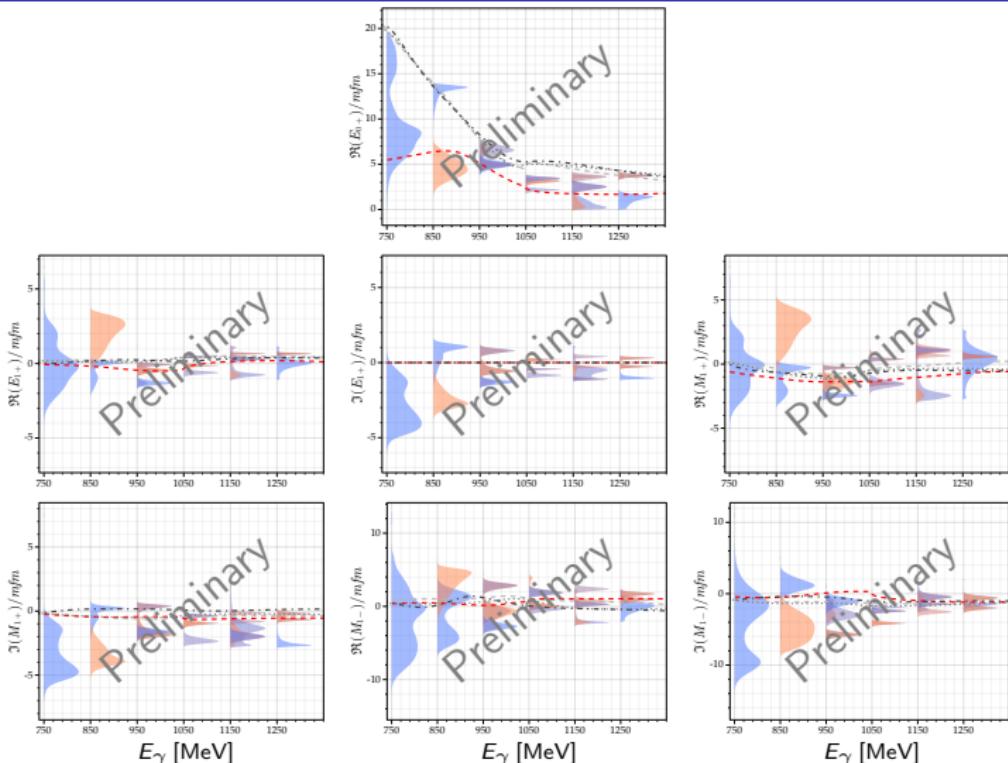
# $\ell_{\max} = 1$ : reproduced data



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

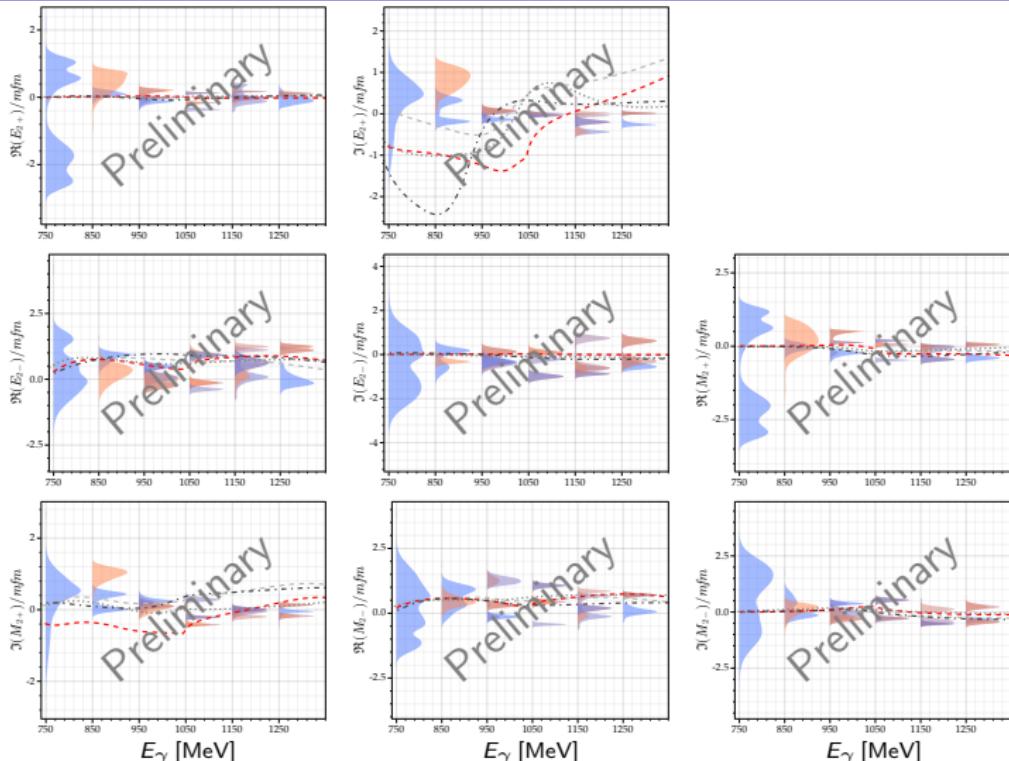
# $\ell_{\max} = 2$ : results for multipoles



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted);  
EtaMAID2018 {without  $S_{11}(1535)$ } (dashed).

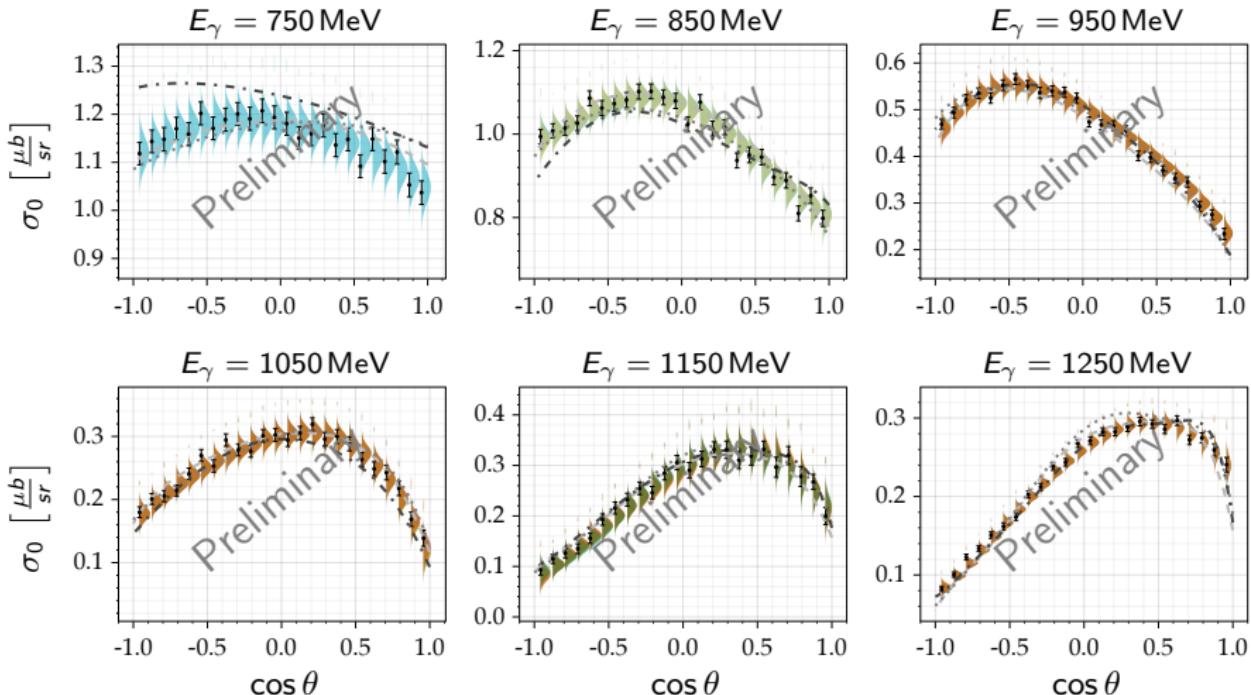
# $\ell_{\max} = 2$ : results for multipoles



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted);  
EtaMAID2018 {without  $S_{11}(1535)$ } (dashed).

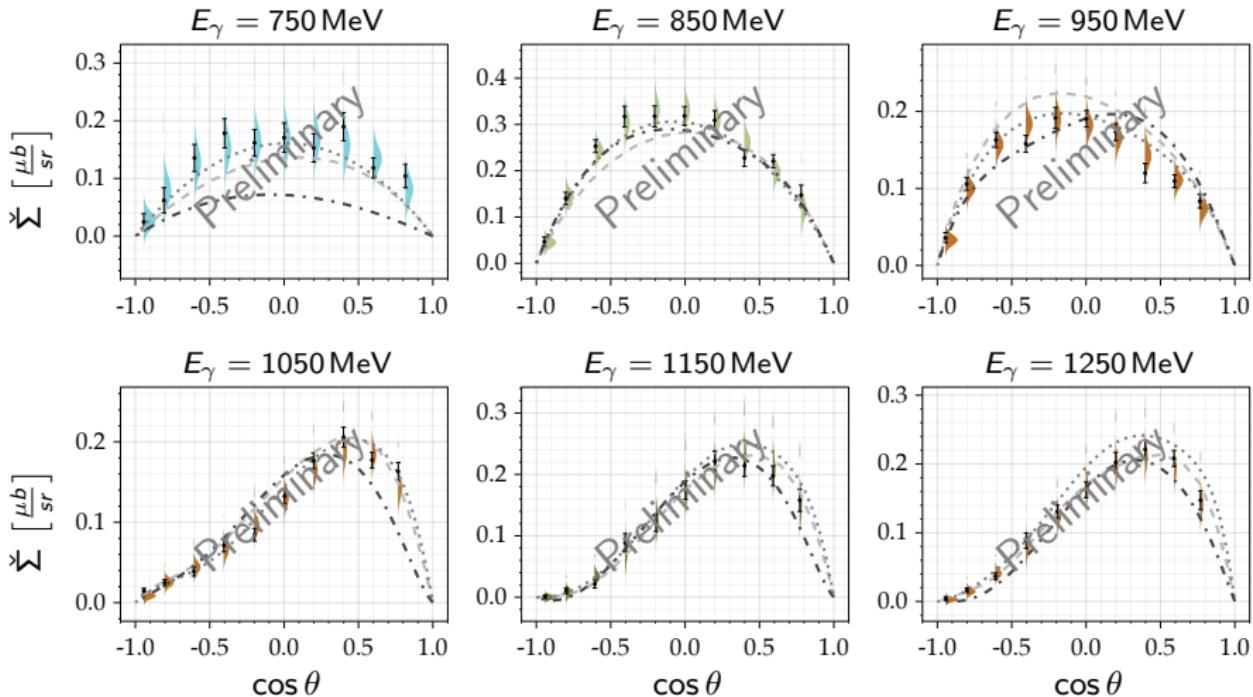
# $\ell_{\max} = 2$ : reproduced data



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

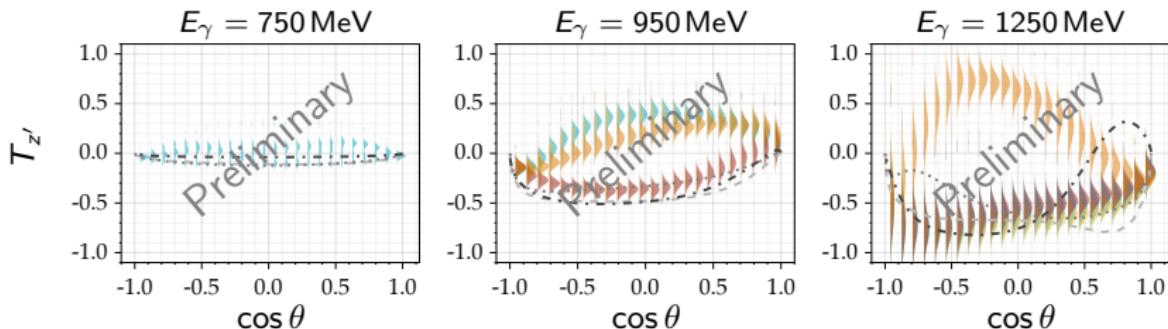
# $\ell_{\max} = 2$ : reproduced data



PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

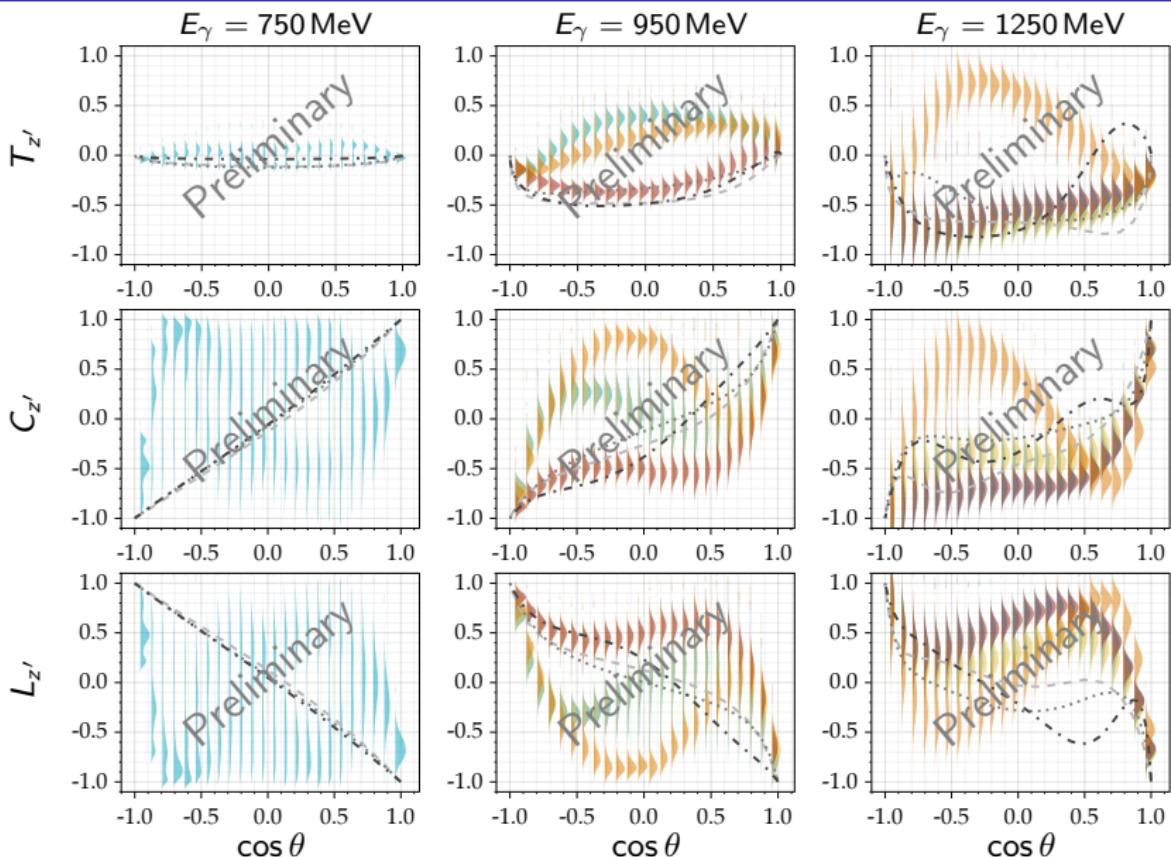
# $\ell_{\max} = 2$ : predicted data-distributions



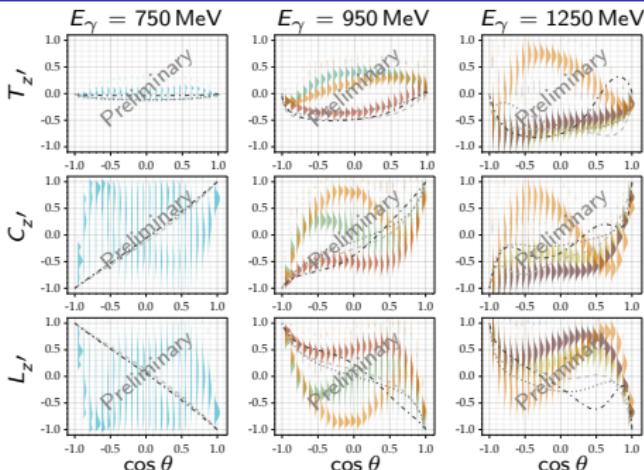
PWA-curves:

EtaMAID2018 (dashed); BnGa-2019 (dotted); Jülich-Bonn-2022 (dash-dotted).

# $\ell_{\max} = 2$ : predicted data-distributions



# $\ell_{\max} = 2$ : predicted data-distributions



→ Predict promising *candidate observables* for resolving discrete ambiguities:

| $E_{\gamma}^{\text{lab}} / \text{MeV}$ | Observables                                      |
|--|--|
| 750                                    | $C_{x'}, C_{z'}, L_{x'}, L_{z'}$                 |
| 850                                    | $C_{x'}, C_{z'}, L_{x'}, L_{z'}, T_{x'}, T_{z'}$ |
| 950                                    | $C_{x'}, C_{z'}, L_{x'}, L_{z'}, T_{z'}$         |
| 1050                                   | $C_{x'}, C_{z'}, L_{x'}, O_{z'}, T_{z'}$         |
| 1150                                   | $C_{z'}, O_{x'}, T_{x'}, T_{z'}$                 |
| 1250                                   | $C_{z'}$   |

cf.: [P. Kroenert, YW, F. Afzal and  
A. Thiel, paper coming soon (...)]

Thank You!