Ground state degeneracy on torus in topologically ordered phases & in symmetry-broken phases

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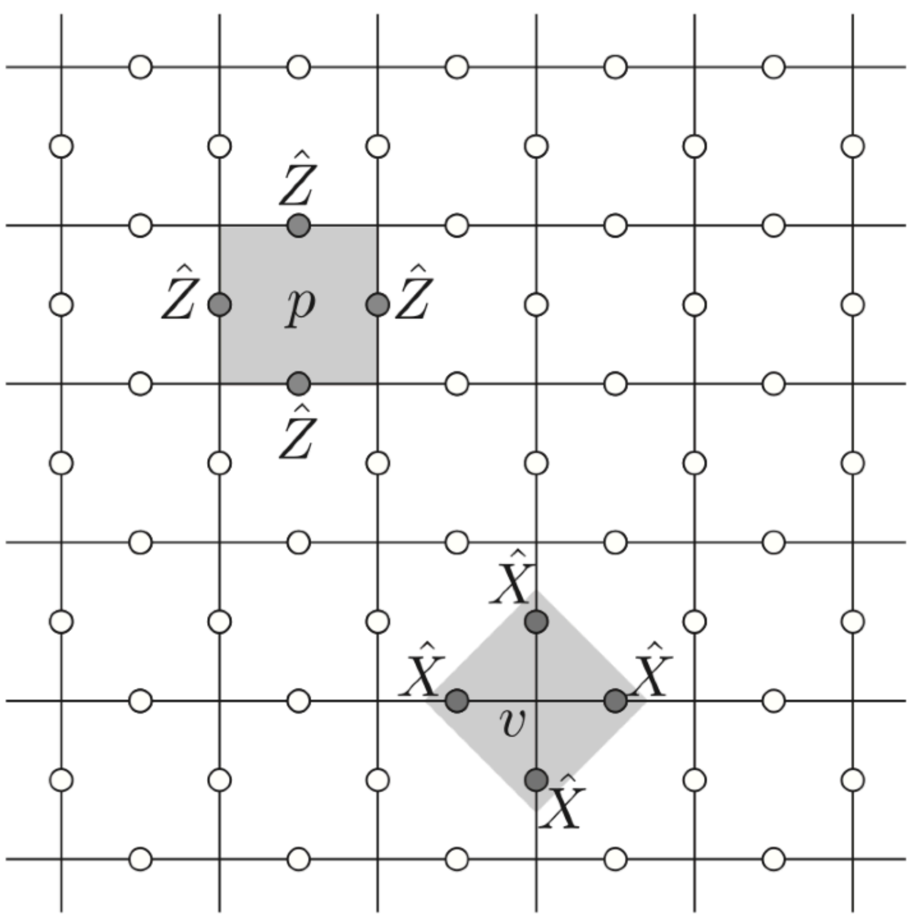
Collaborators: Yohei Fuji (U Tokyo), Meng Cheng (Yale)

Refs: HW, M. Cheng, Y. Fuji, arXiv:2211.00299

Y. Hu and HW, arXiv:2302.01207



the standard $\mathbb{Z}_2$ toric code

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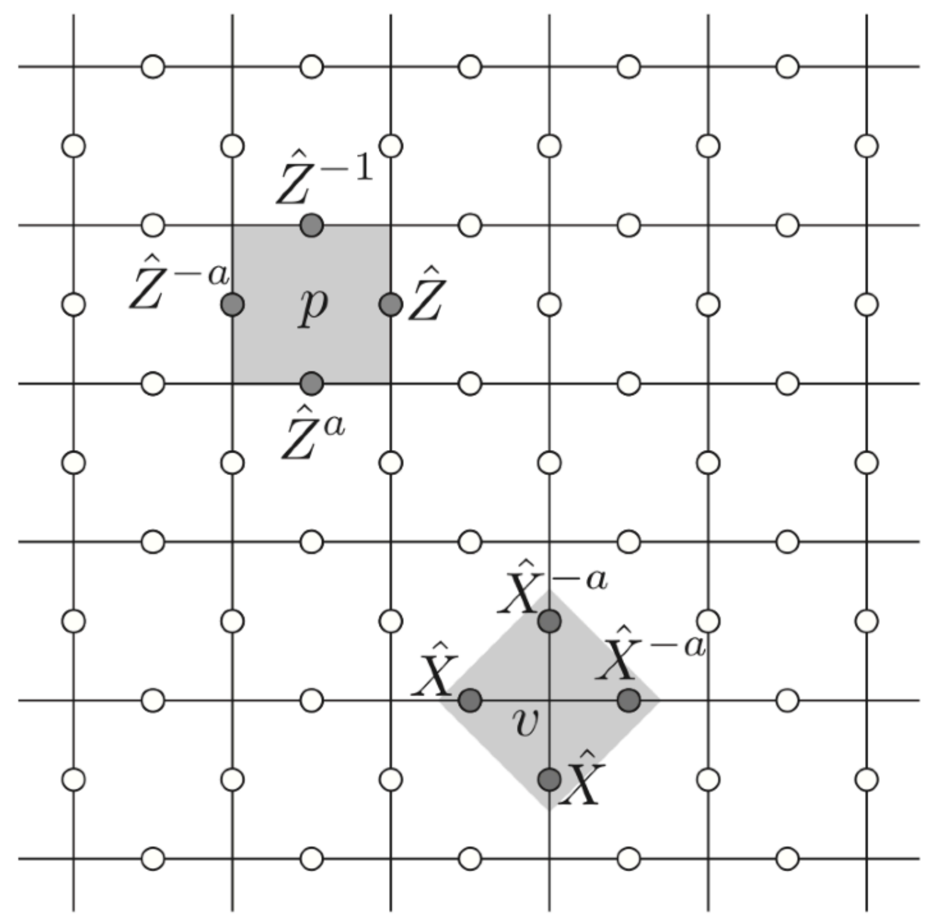

$\mathbb{Z}_N$ toric code with parameter a

Introduction

Defining properties of phases (textbook)

e.g. B. Zeng, X. Chen, D.-L. Zhou, X.-G.Wen, Springer (2019)

Symmetry-protected topological/trivial phases

Topologically-ordered phases

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- Unique GS with excitation gap.
- Stable against symmetry-preserving perturbations.
- Can be connected to product states by local unitaries if symmetries are broken.
  (Exceptions: 1D Kitaev chain, integer quantum Hall systems, …)
- Trivial excitations in the bulk.
  Anomalous states on symmetry-preserving boundaries.

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This standard understanding of phases will be questioned by our new example.

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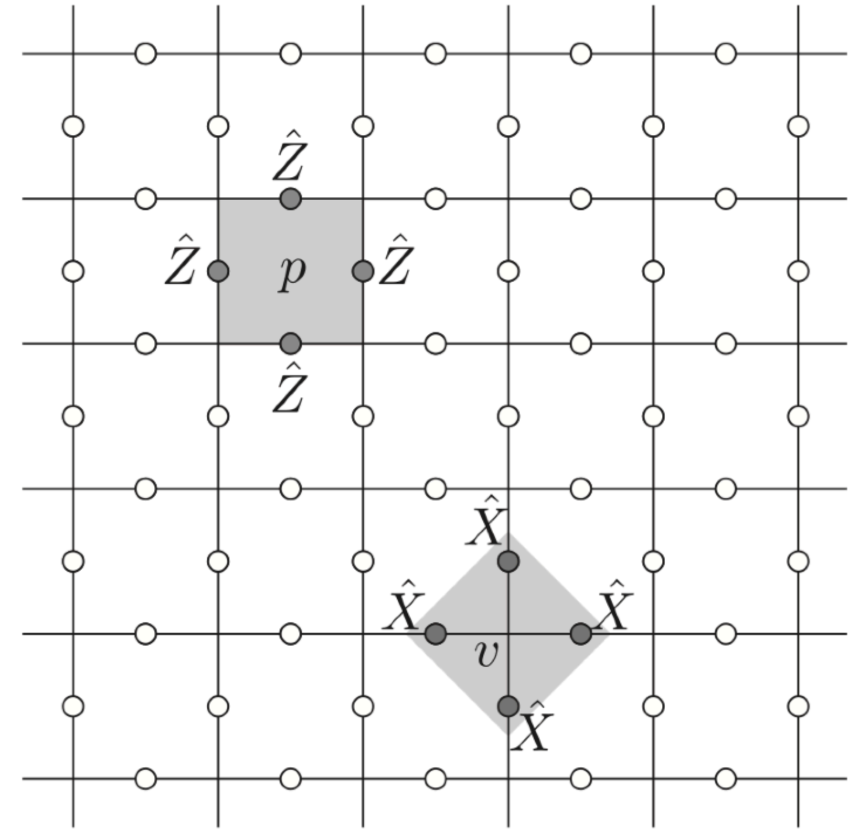

- Hamiltonian is the sum of commuting operators.

$$\hat{H} = -\sum_{v \in \mathscr{V}} \hat{A}_v - \sum_{p \in \mathscr{P}} \hat{B}_p$$

  - $\mathscr{V} = \{(m_1, m_2) \,|\, m_i = 0,1,\cdots, L_i - 1\}$
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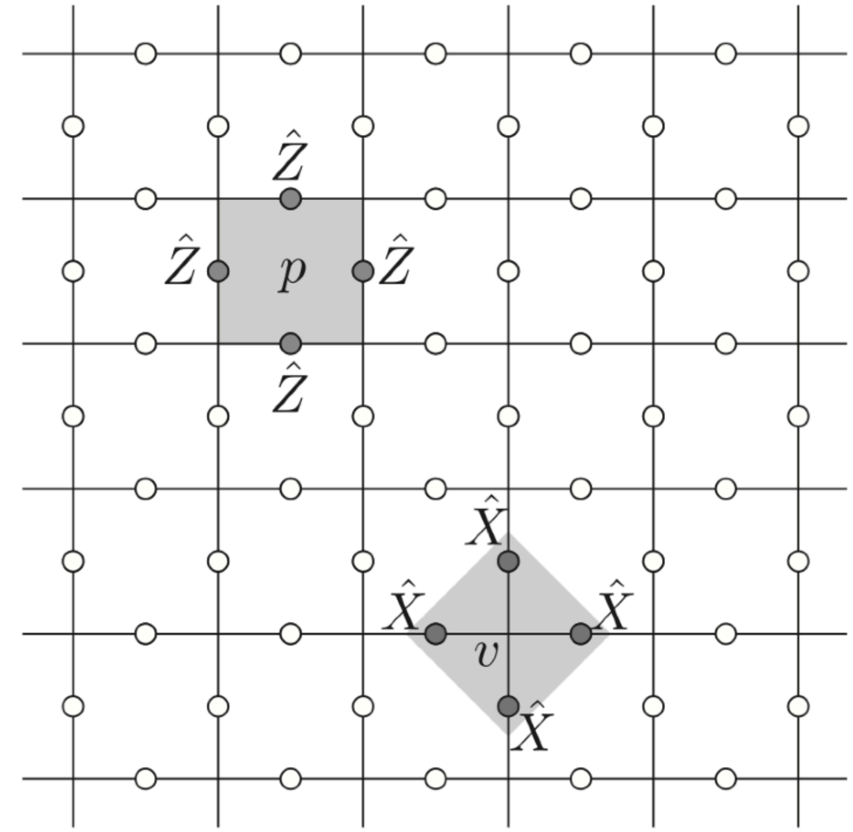

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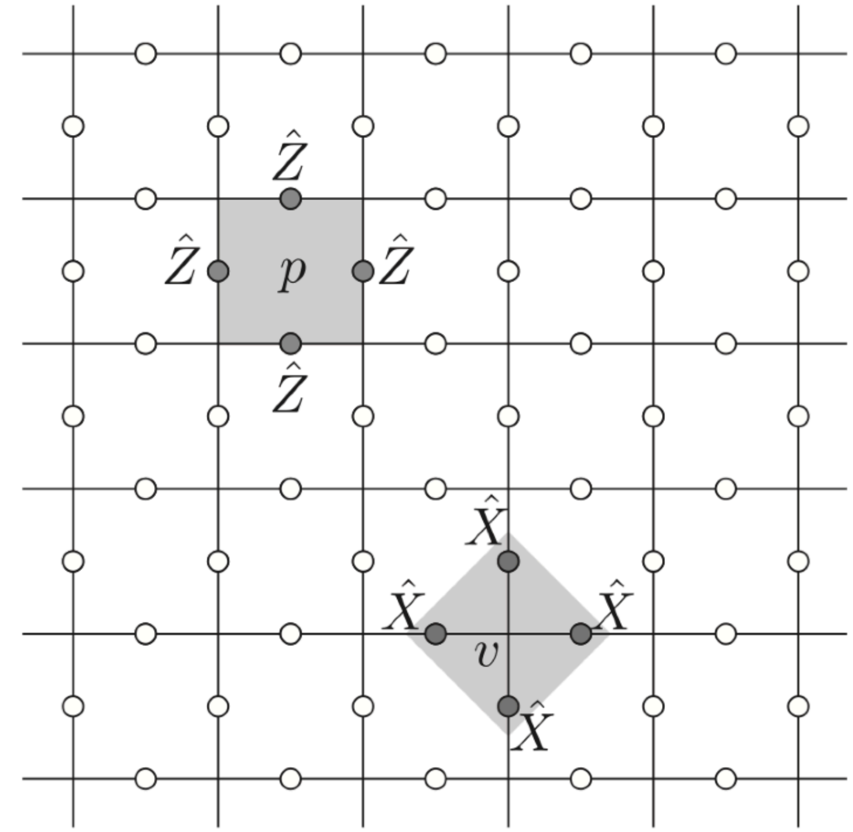

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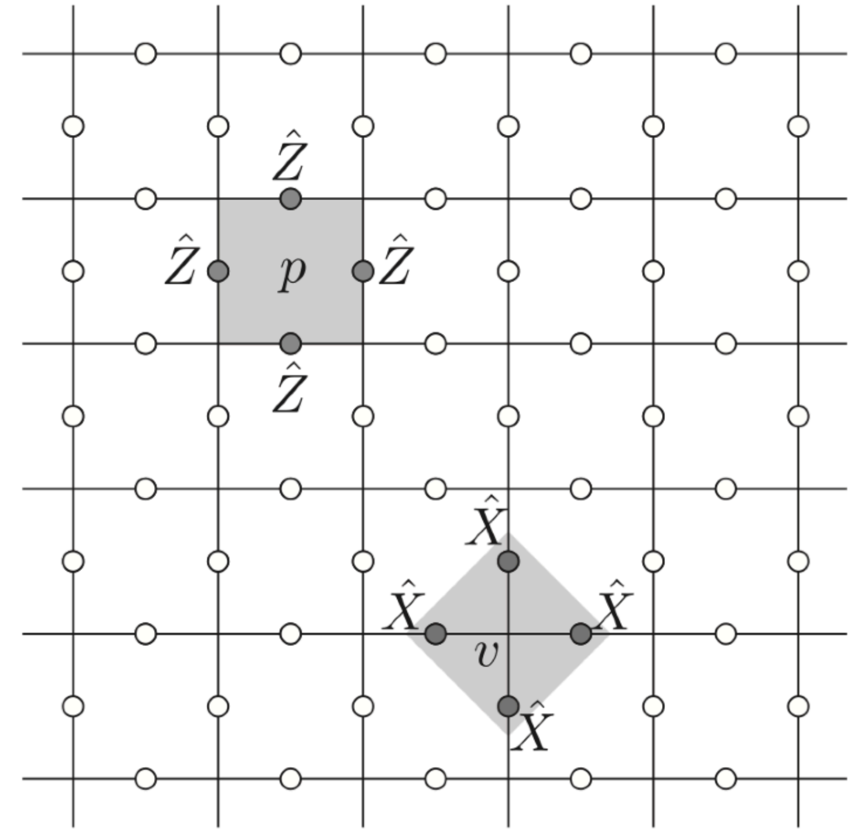

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- Since $\hat{A}_v^2 = \hat{B}_p^2 = 1$, the eigenvalues of $\hat{A}_v$ and $\hat{B}_p$ are ± 1. Any state with eigenvalues +1 for all vertices and plaquettes is a ground state.

Topological ground state degeneracy

- Global constraints $\prod_{v \in \mathscr{V}} \hat{A}_v = \prod_{p \in \mathscr{P}} \hat{B}_p = 1$ → Not all stabilizers are independent.

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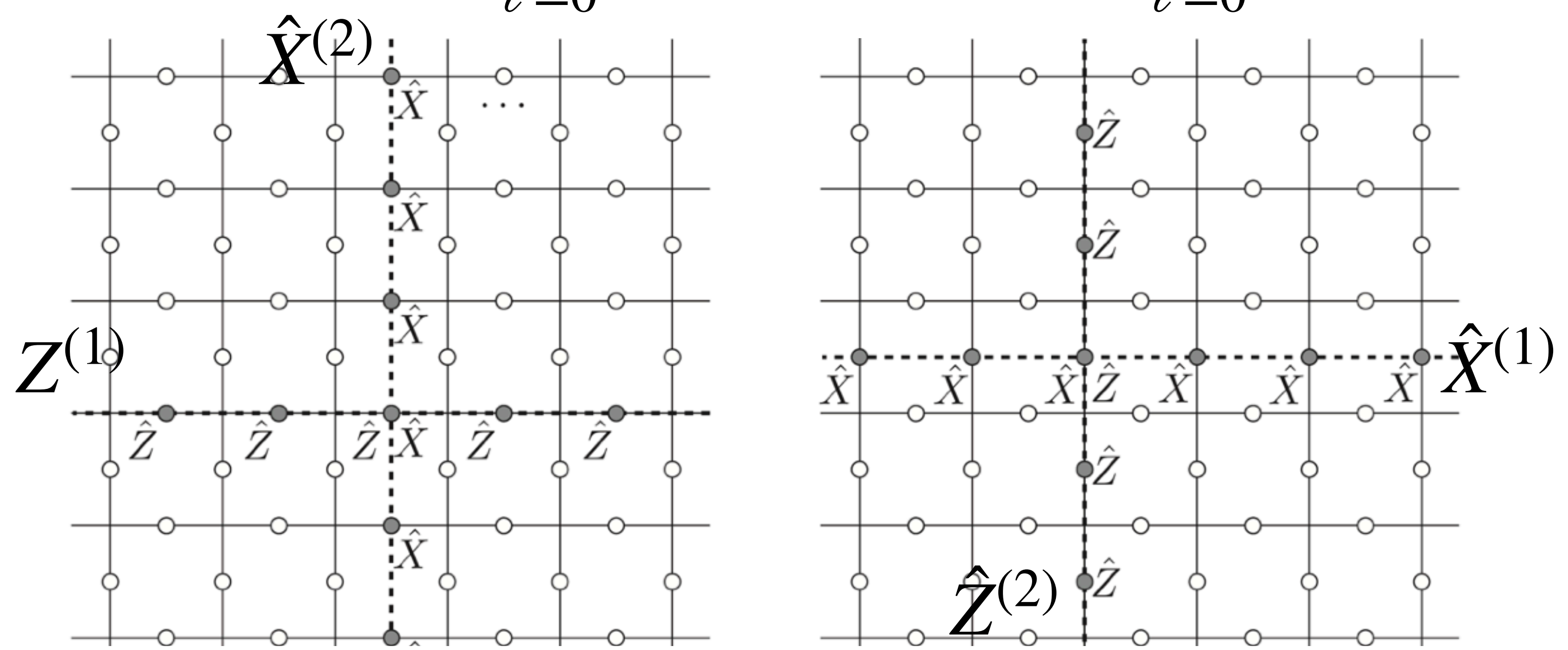

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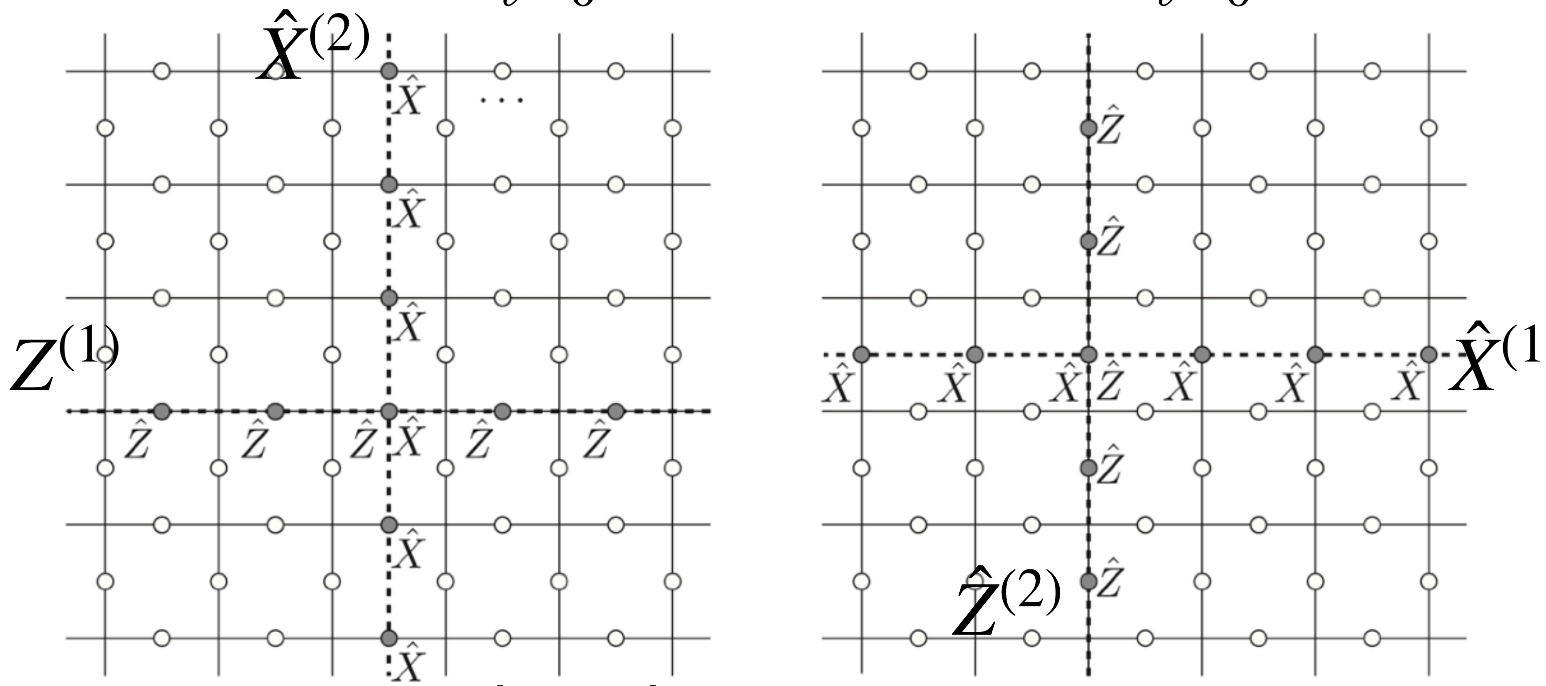

- Starting from a GS $|\Phi_0\rangle$ with $+1$ eigenvalues of $\hat{Z}^{(1)},\hat{Z}^{(2)}$, we can generate four degenerate GSs:

$$|\Phi_0\rangle,\quad \hat{X}^{(1)}|\Phi_0\rangle,\quad \hat{X}^{(2)}|\Phi_0\rangle,\quad \hat{X}^{(1)}\hat{X}^{(2)}|\Phi_0\rangle$$

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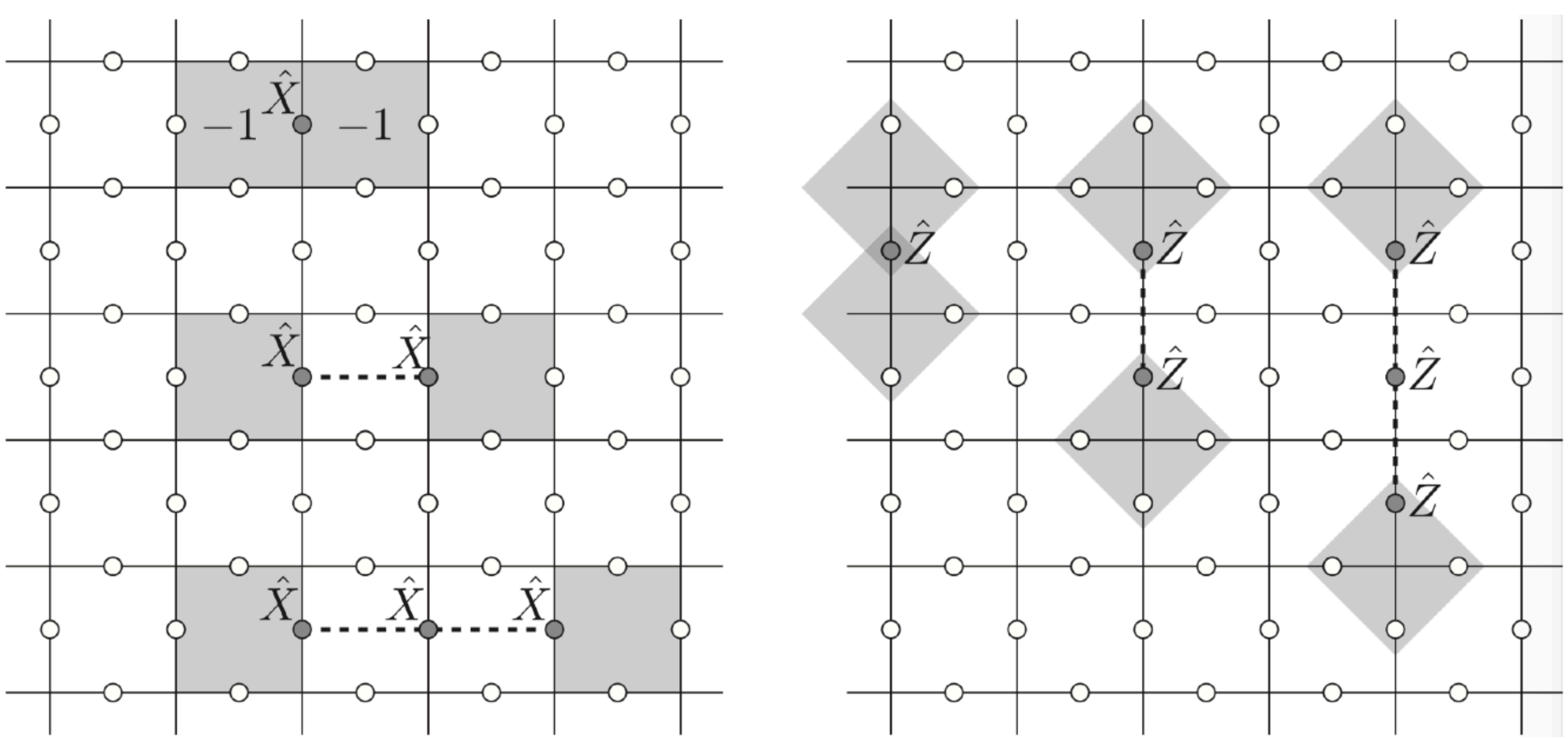
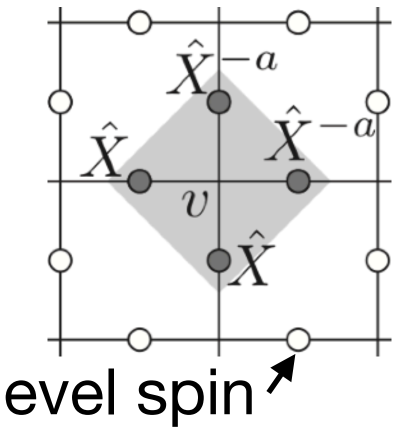
3. Anyons: 2^2 species. $\{1, e\} \times \{1, m\} = \{1, e, m, f\}$



Features of new model



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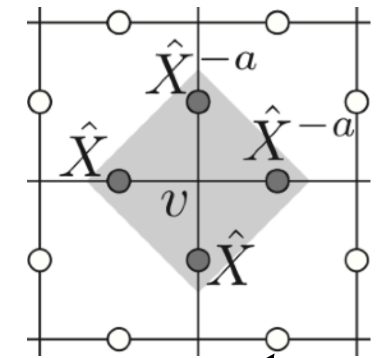

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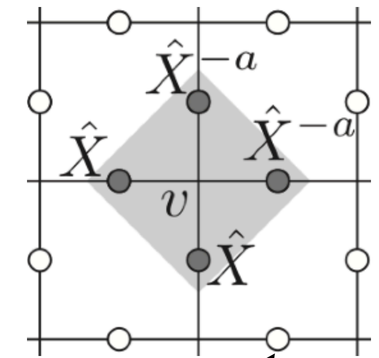

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e.g. $N = 11$ and $a = 2$.
To observe $N_{\mathrm{deg}} > 1$, both L_1 and L_2 need to be multiples of 10.
The minimum system size is 10×10; the total Hilbert space dimension = 11^{200}.
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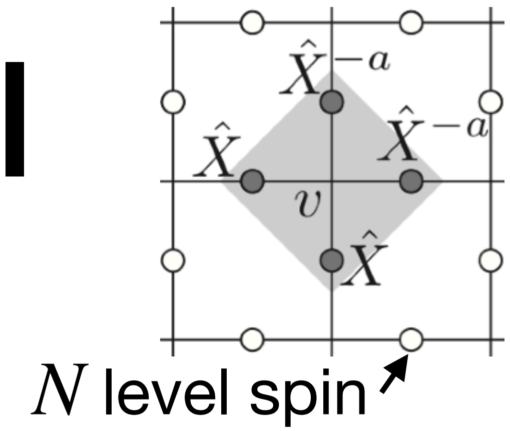

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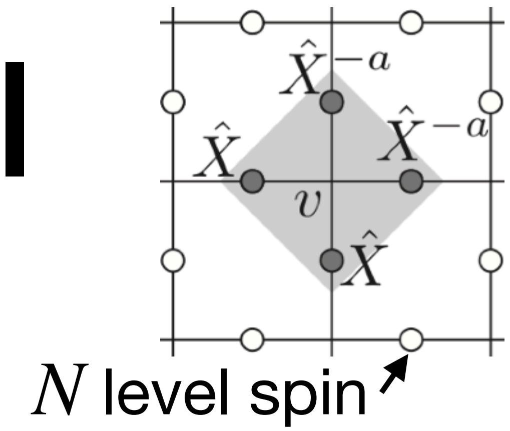

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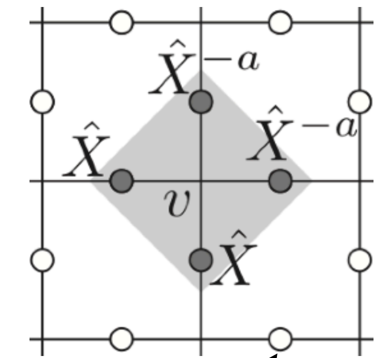

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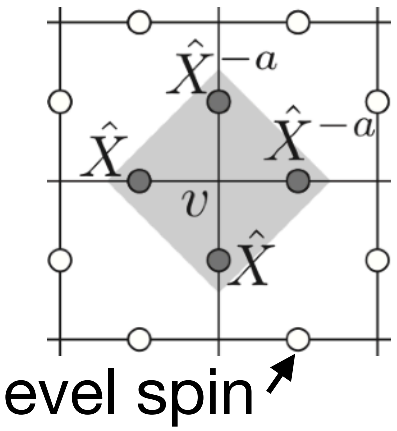

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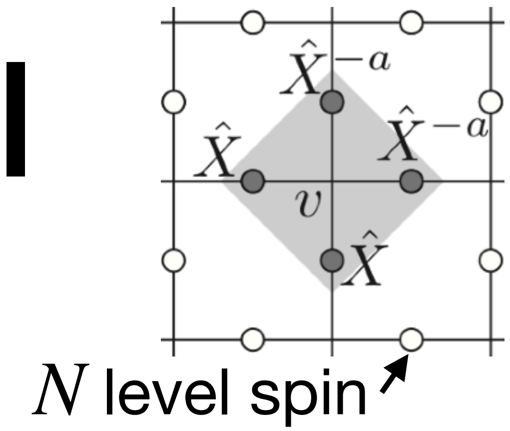

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   Translation permutes anyons as $T_i : (q_e, q_m) \to (aq_e, a^{-1}q_m)$.

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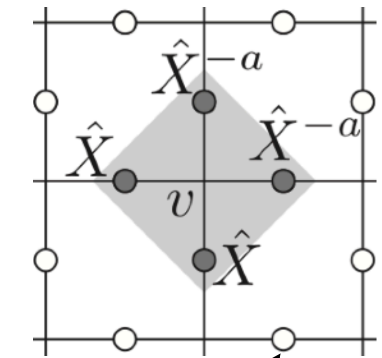

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- $N_a =$ the largest divisor of N that is coprime to a.
  - $N_a \neq 1$ if a is not a multiple of $\text{rad}(N)$ → Topologically-ordered phases
  - $N_a = 1$ if a is a multiple of $\text{rad}(N)$ → SPT phases
- Prime factorization $N = \prod_{j=1}^{n} p_j^{r_j} = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$.
- Radical of N: $\text{rad}(N) = \prod_{j=1}^{n} p_j = p_1 p_2 \cdots p_n$.

Defining properties of phases (updated)

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Defining properties of phases (updated)

Symmetry-protected topological/trivial phases

- Unique GS with excitation gap **for any sequence of L_1 and L_2.**
- Stable against symmetry-preserving perturbations.
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Talk plan

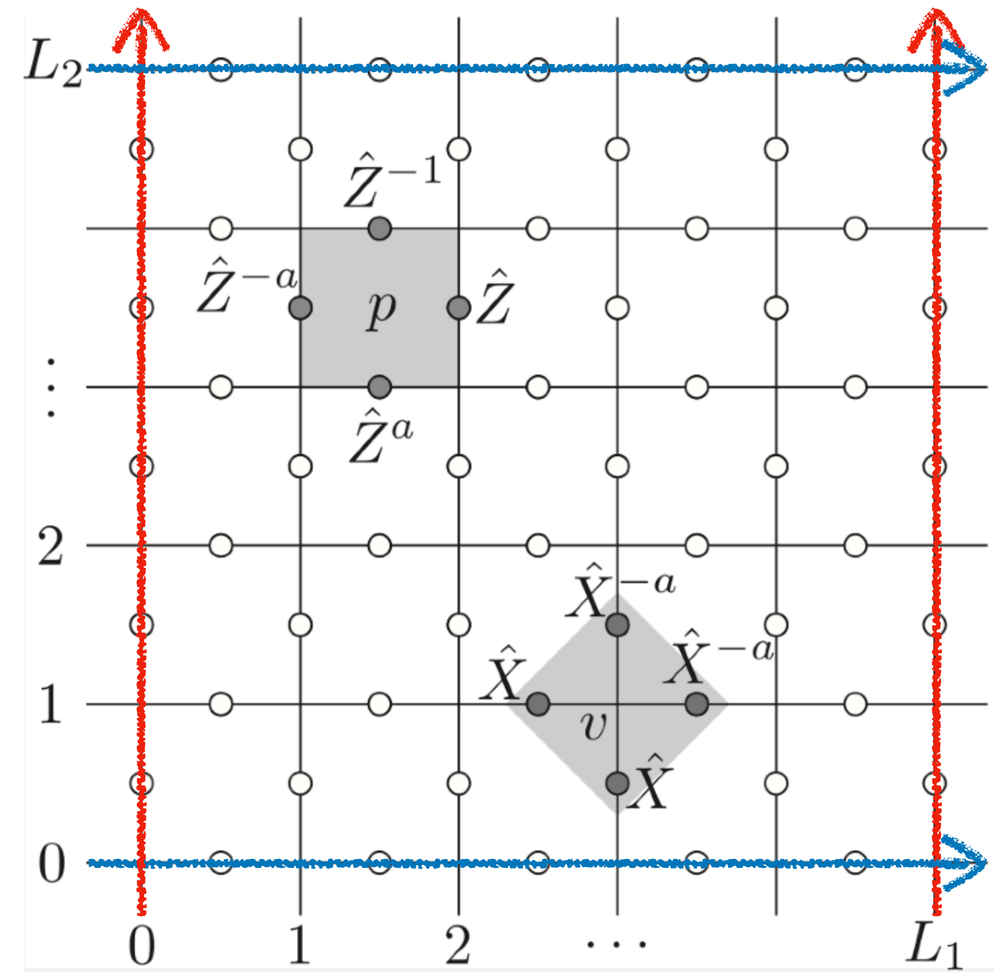
- Introduction ✔
- Definition & basic properties of $\mathbb{Z}_N$ toric code model with parameter a
  - Case 1: $a = 1$ (the standard choice)
  - Case 2: $a = N$ (a product state)
- More general cases
  - Case 3: N is a prime (e.g. $N = 11$ and $a = 2$)
  - Formulas for the general case
  - Case 4: $a^2 = N$ (a subsystem-symmetry protected topological phase)
- Another example: quantum Ising model (on-going)
- Summary

Definition & basic properties of our model

N level spins

- Generalization of Pauli matrices to $N \geq 2$ level spins

$$X = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & \end{pmatrix} \quad Z = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}$$



- $\mathbb{Z}_N \times \mathbb{Z}_N$ group with phase $\omega = e^{\frac{2\pi i}{N}}$

$$ZX = \omega XZ = \begin{pmatrix} & & & 1 \\ \omega & & & \\ & \omega^2 & & \\ & & \ddots & \\ & & & \omega^{N-1} \end{pmatrix}$$

$$Z^N = X^N = 1$$

- A N level spin is placed on every bond. They satisfy $\hat{Z}_{\boldsymbol{r}}\hat{X}_{\boldsymbol{r}'} = \omega^{\delta_{\boldsymbol{r},\boldsymbol{r}'}}\hat{X}_{\boldsymbol{r}'}\hat{Z}_{\boldsymbol{r}}$ and $\hat{Z}_{\boldsymbol{r}}^N = \hat{X}_{\boldsymbol{r}}^N = 1$.
- Periodic boundary condition (PBC) with system size L_1 and L_2 → System is put on torus.
- Total Hilbert space dimension is $N^{2L_1L_2}$.

$\mathbb{Z}_N$ toric code with parameter a

- Vertex and plaquette operators. All commute regardless of the choice of a.
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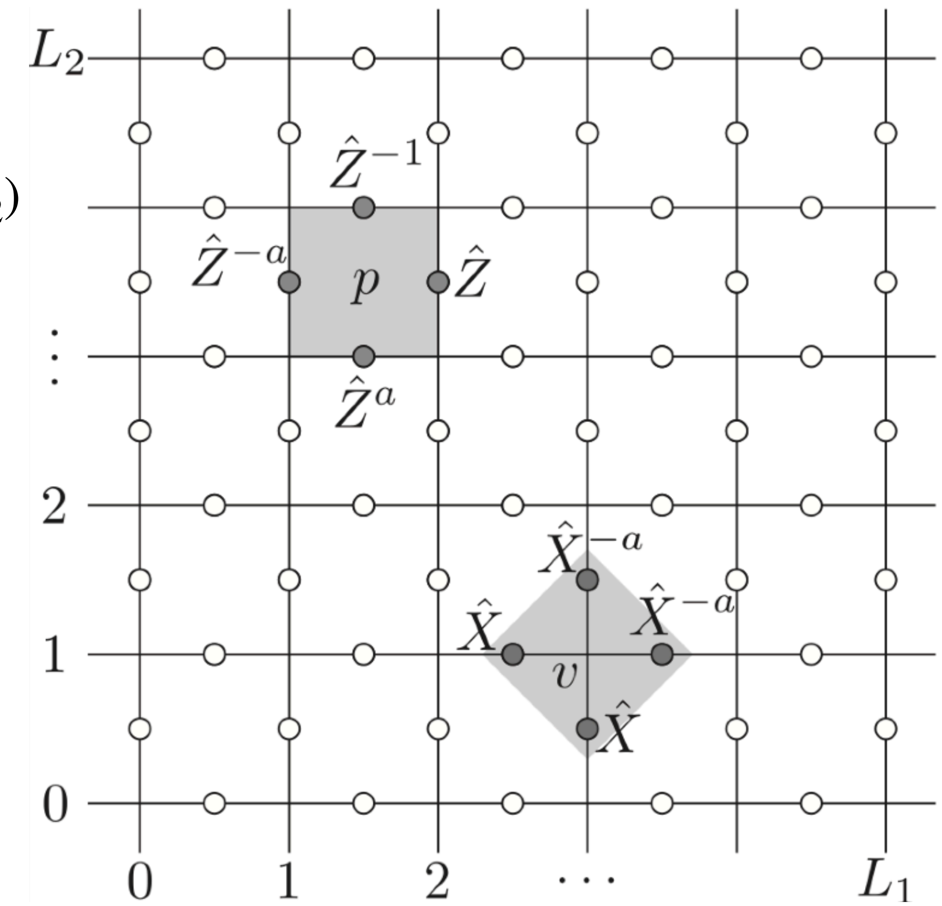


- Hamiltonian is the sum of stabilizers:

$$\hat{H} = -\sum_{v\in\mathscr{V}}\frac{1}{2}(\hat{A}_v + \text{h.c.}) - \sum_{p\in\mathscr{P}}\frac{1}{2}(\hat{B}_p + \text{h.c.})$$

- Translation symmetry: $\hat{T}_i\hat{X}_{\boldsymbol{r}}\hat{T}_i^\dagger = \hat{X}_{\boldsymbol{r}+\boldsymbol{e}_i}$ and $\hat{T}_i\hat{Z}_{\boldsymbol{r}}\hat{T}_i^\dagger = \hat{Z}_{\boldsymbol{r}+\boldsymbol{e}_i}$.

A. Kitaev, Ann. Phys. (2003).

- $a = 1,2,\cdots,N$ is a very important parameter.
  - $a = 1$: the standard choice.
  - $a = N-1$: discussed previously but GSD on torus was not investigated.

M. D. Schulz, et al, New J. Phys (2012).
M. Barkeshli et al Math. Phys. (2020).
J. C. Bridgeman et al, PRB (2017).
Y. Fuji, PRB (2019)

A ground state

- Since $\hat{A}_v^N = \hat{B}_p^N = 1$, the eigenvalues of $\hat{A}_v$ and $\hat{B}_p$ are N-fold: $1, \omega, \cdots, \omega^{N-1}$. Any state with eigenvalues +1 for all vertices and plaquettes is a ground state.
- A ground state can be constructed by
(i) starting from the ferromagnetic state $\hat{Z}_{\boldsymbol{r}}|\phi_0\rangle = |\phi_0\rangle \quad (\forall \boldsymbol{r} \in \Lambda)$
(ii) applying the projector $\hat{P} = \frac{1}{N^{L_1 L_2}} \prod_{v \in \mathscr{V}} \sum_{\ell=0}^{N-1} \hat{A}_v^{\ell}$ with the following properties.
    - $\hat{P}^2 = \hat{P}$
    - $\hat{A}_v \hat{P} = \hat{P} \hat{A}_v = \hat{P}$
    - $\hat{B}_p \hat{P} = \hat{P} \hat{B}_p$
    - $\hat{T}_i \hat{P} = \hat{P} \hat{T}_i$

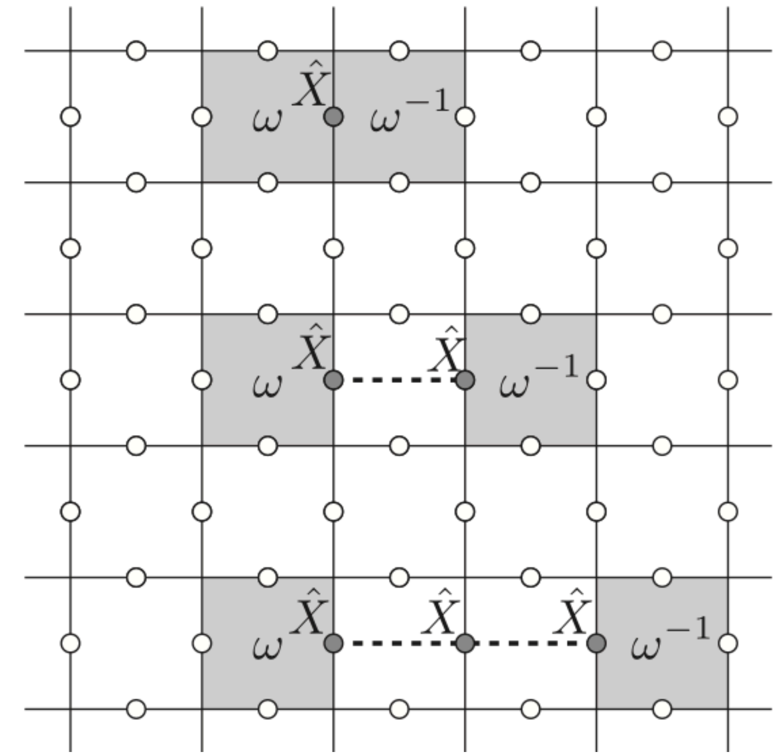
Then $|\Phi_0\rangle \propto \hat{P}|\phi_0\rangle$ has eigenvalues +1 for all vertices and plaquettes.
This extends the standard discussion in literature.

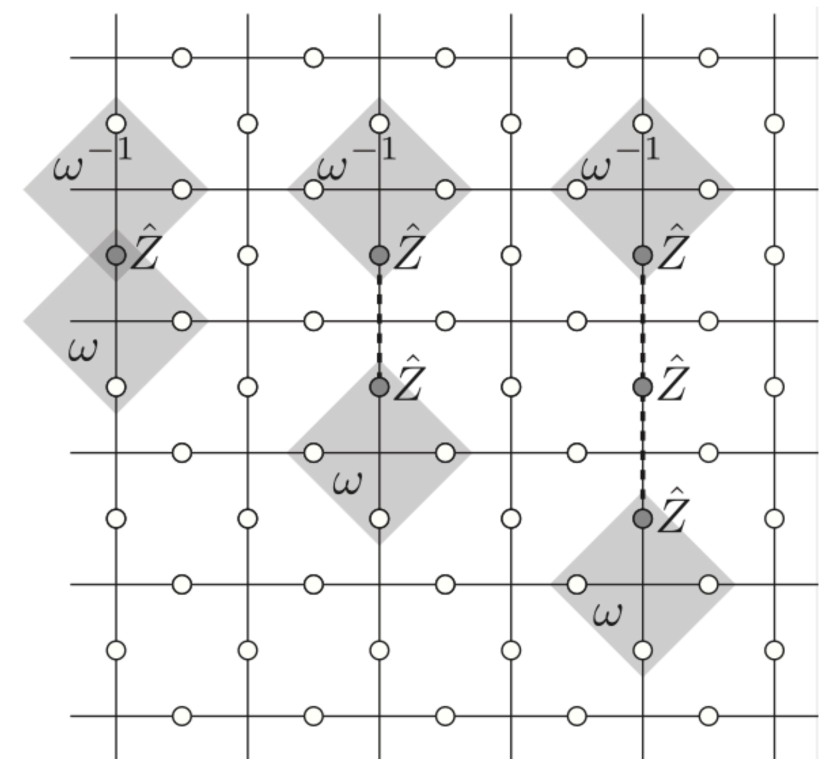
e.g. H. Tasaki's textbook, Springer (2020)

Case 1: $a = 1$

This case is the standard choice. Basically the same as the original toric code.

- Global constraints $\prod_{v\in\mathscr{V}} \hat{A}_v = \prod_{p\in\mathscr{P}} \hat{B}_p = 1.$
- X loops: $\hat{X}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{X}_{(\ell, L_2-\frac{1}{2})}, \hat{X}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{X}_{(L_1-\frac{1}{2}, \ell)}.$
- Z loops: $\hat{Z}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{Z}_{(\ell+\frac{1}{2}, 0)}, \hat{Z}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{Z}_{(0, \ell+\frac{1}{2})}.$
  - $\hat{Z}^{(1)}\hat{X}^{(2)} = \omega\,\hat{X}^{(2)}\hat{Z}^{(1)}.$
  - $\hat{Z}^{(2)}\hat{X}^{(1)} = \omega\,\hat{X}^{(1)}\hat{Z}^{(2)}.$
  - $[\hat{Z}^{(1)}, \hat{Z}^{(2)}] = [\hat{X}^{(1)}, \hat{X}^{(2)}] = 0.$
  - $[\hat{Z}^{(1)}, \hat{X}^{(1)}] = [\hat{X}^{(2)}, \hat{Z}^{(2)}] = 0.$
- GSD: $N_{\mathrm{deg}} = N^2$. TEE: $S_{\mathrm{top}} = -\log N$. Anyons: N^2 species





Proof of GSD via explicit construction of all states

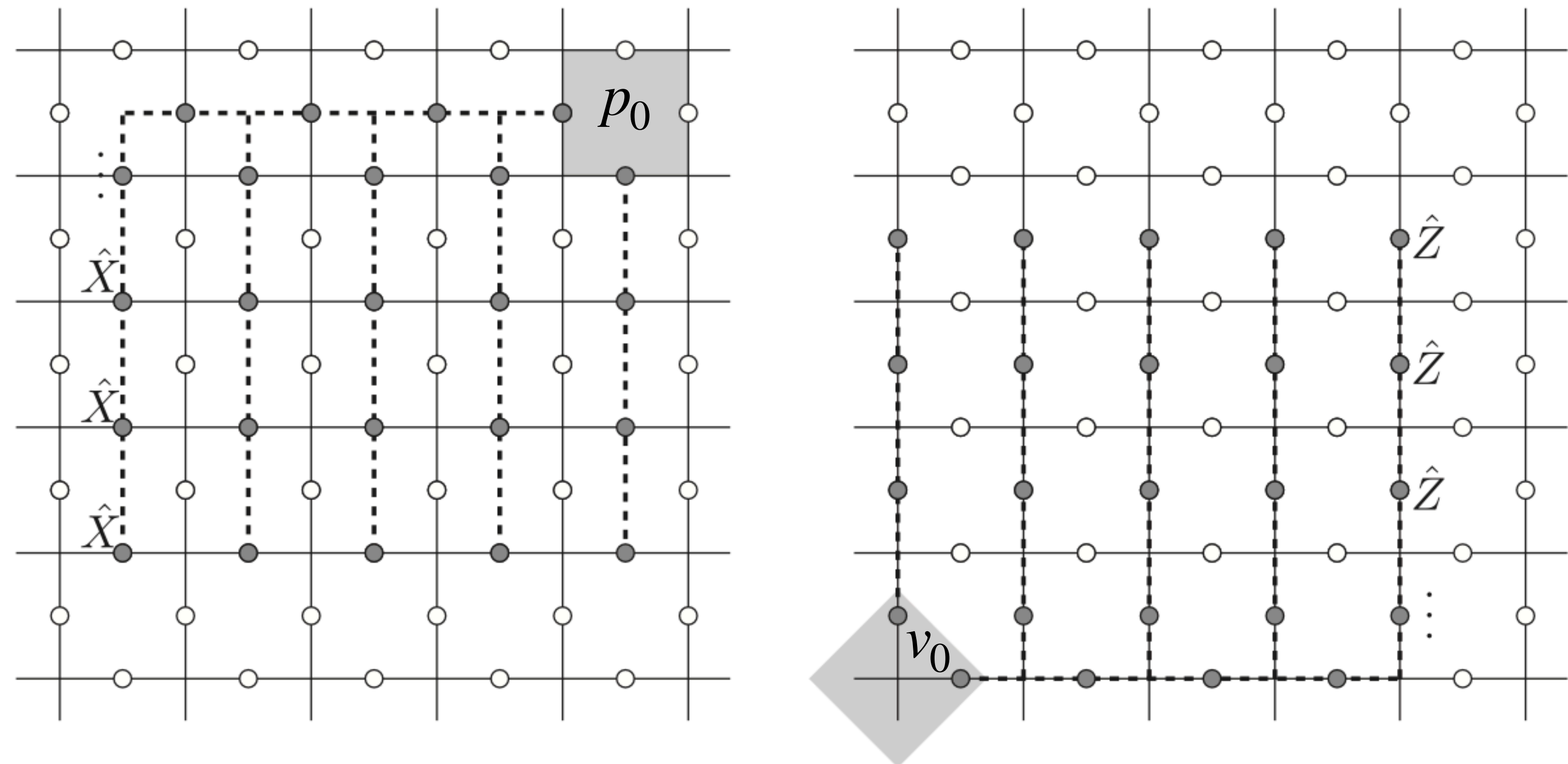
$\hat{A}_{v_0}$ and $\hat{B}_{p_0}$ are fixed by global constraints.

<u>Open string operators</u>

Control the eigenvalues of

- $\hat{A}_v \ (v \in \mathscr{V}, v \neq v_0)$
- $\hat{B}_p \ (p \in \mathscr{P}, p \neq p_0)$

$$\hat{Z}_{\boldsymbol{r}} \hat{X}_{\boldsymbol{r}'} = \omega^{\delta_{r,r'}} \hat{X}_{\boldsymbol{r}'} \hat{Z}_{\boldsymbol{r}}$$

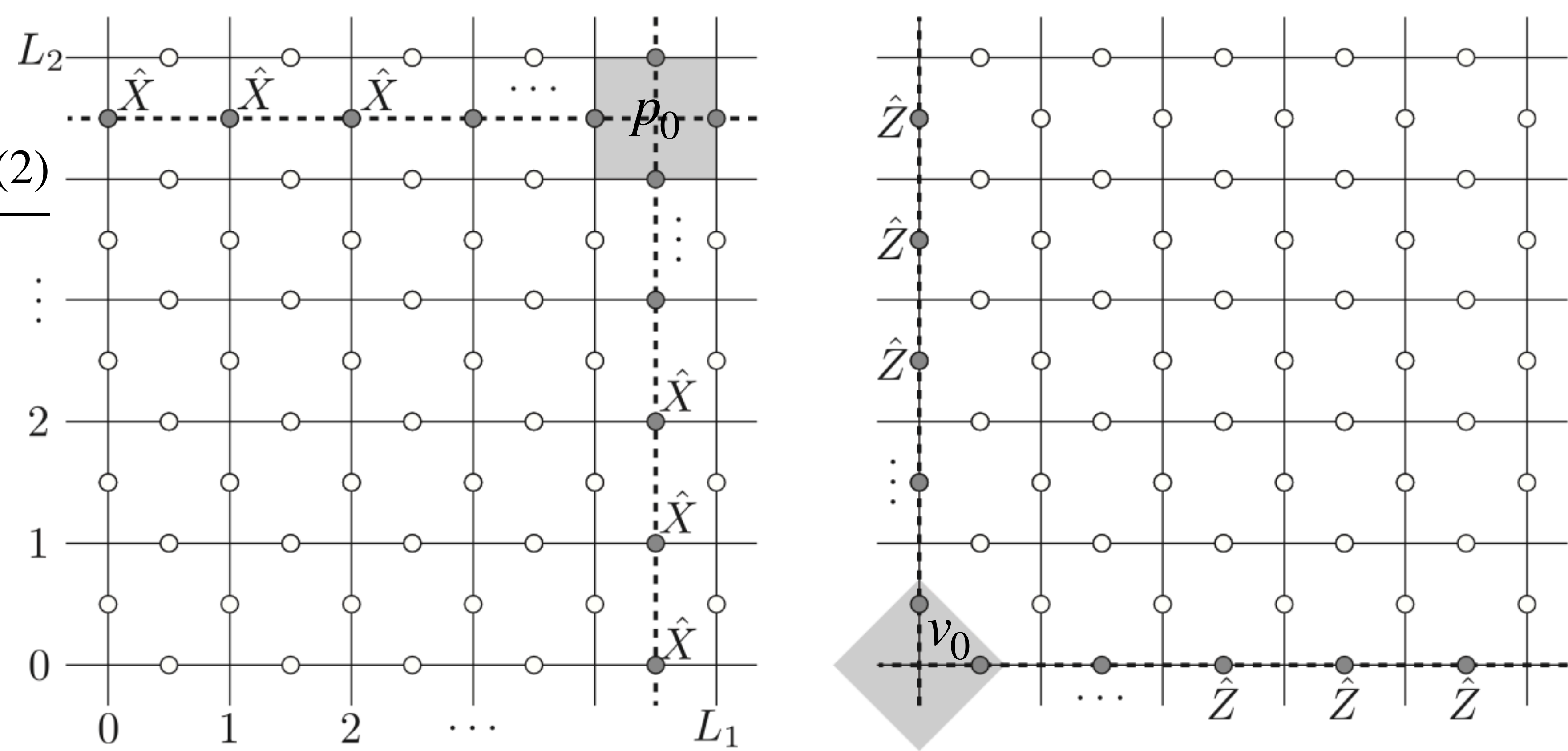


<u>Closed string operators $\hat{X}^{(2)}, \hat{Z}^{(2)}$</u>

Control the eigenvalues of

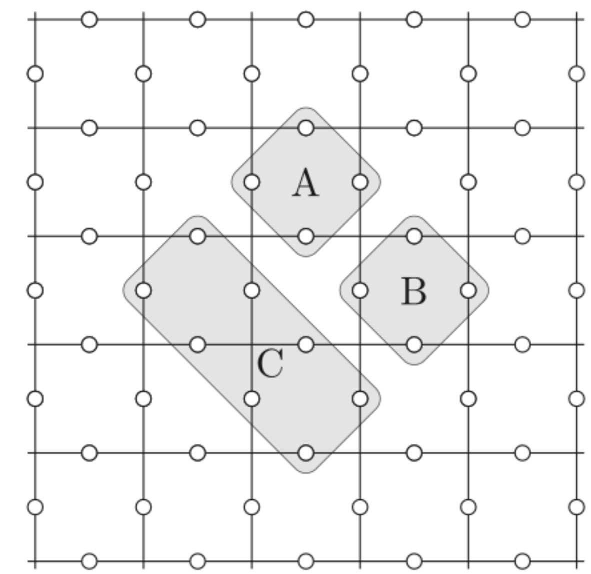
- $\hat{X}^{(1)}$
- $\hat{Z}^{(1)}$

$$\hat{Z}^{(2)} \hat{X}^{(1)} = \omega \, \hat{X}^{(1)} \hat{Z}^{(2)}$$
$$\hat{Z}^{(1)} \hat{X}^{(2)} = \omega \, \hat{X}^{(2)} \hat{Z}^{(1)}$$



Calculation of TEE

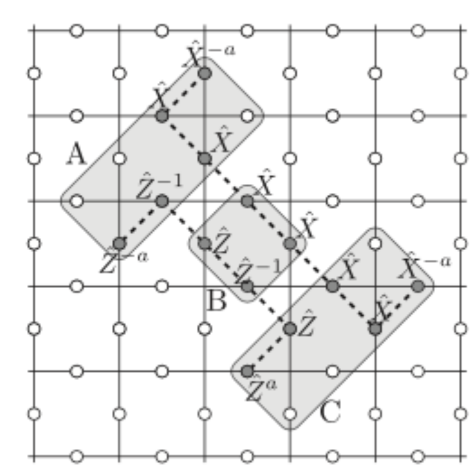
- Kitaev-Preskill prescription — A. Kitaev and J. Preskill, PRL (2006)
  - $S_{\text{topo}} = (S_{\text{A}} + S_{\text{B}} + S_{\text{C}}) - (S_{\text{AB}} + S_{\text{BC}} + S_{\text{CA}}) + S_{\text{ABC}}$
  - $S_{\text{R}} = -\operatorname{tr}[\hat{\rho}_{\text{R}} \log \hat{\rho}_{\text{R}}]$



- Useful formula: $S_{\text{R}} = n_{\text{R}} \log N - \log |G_{\text{R}}|$
  - n_{R} is the number of N-level spins in R.
  - G_{R} is the subgroup of G supported in R.
  - G is the multiplicative group generated by all $\hat{A}_v$'s ($v \in \mathscr{V}$), $\hat{B}_p$'s ($p \in \mathscr{P}$), and possible closed string operators for which $|\Phi_0\rangle$ has the eigenvalue +1.

N. Linden et al TQC (2013).
L. Zou and J. Haah, PRB (2016).

- When N and a are coprime: further simplified to $S_{\text{R}} = (n_{\text{R}} - m_{\text{R}}) \log N$.
  - m_{R} is the number of generators of G supported in R.



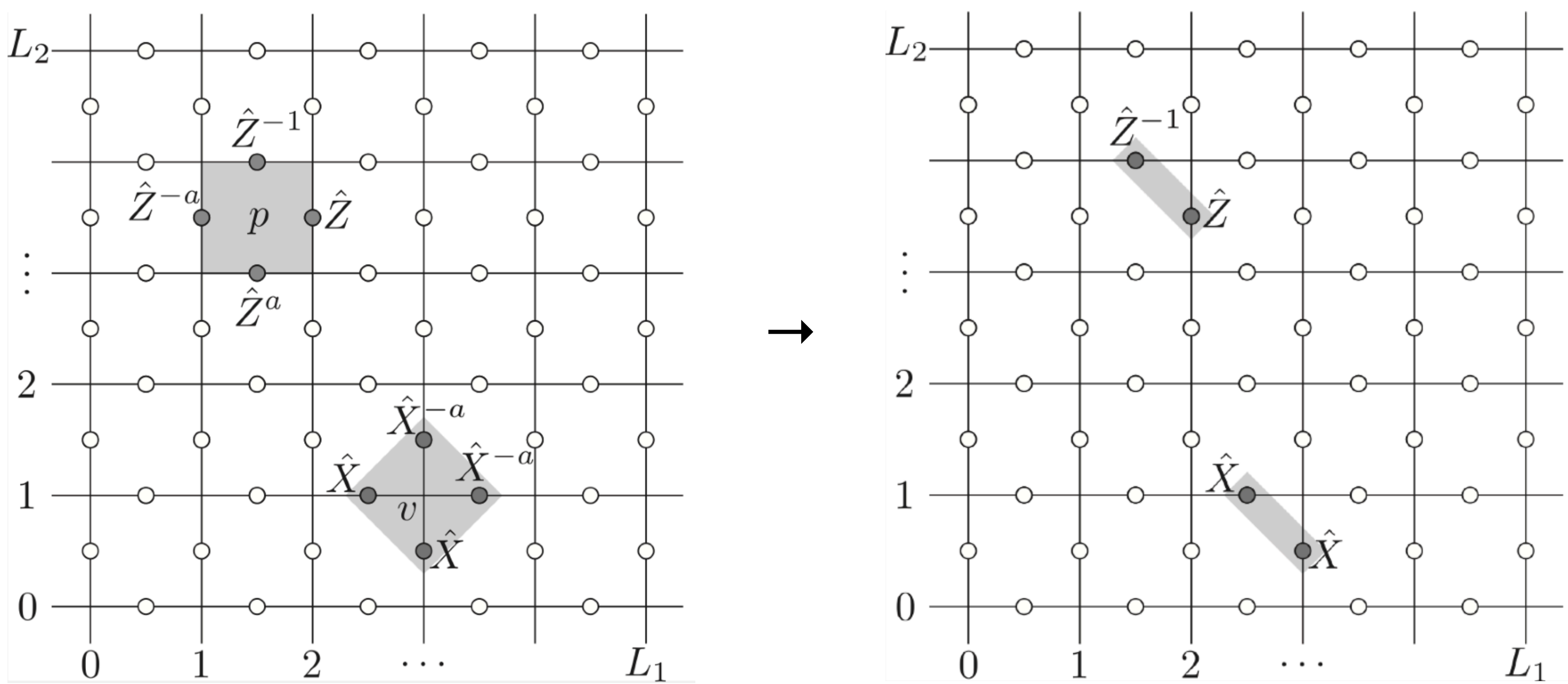
- There can be spurious contribution, which characterize SSPT phase.

D. J. Williamson et al, PRL (2019)
D. T. Stephen et al, PRB (2019)

Case 2: $a = N$

This is the other extreme case.

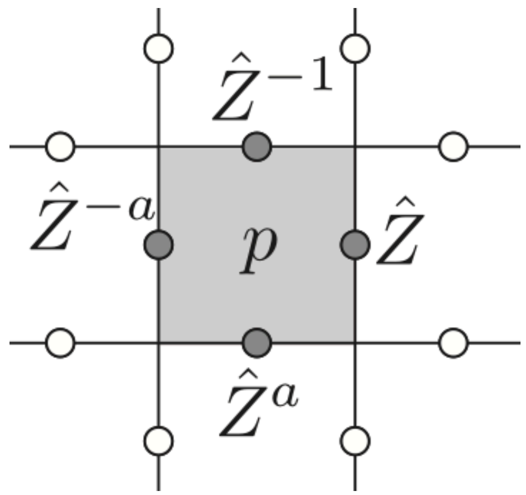
- $\hat{H} = \sum_{\boldsymbol{r}\in\Lambda} \hat{h}_{\boldsymbol{r}}$ with
$$\hat{h}_{(m_1,m_2)} = -\frac{1}{2}\left(\hat{X}_{(m_1-\frac{1}{2},m_2)}\hat{X}_{(m_1,m_2-\frac{1}{2})} + \text{h.c.}\right) - \frac{1}{2}\left(\hat{Z}_{(m_1,m_2-\frac{1}{2})}\hat{Z}^{-1}_{(m_1-\frac{1}{2},m_2)} + \text{h.c.}\right).$$
- Completely decoupled → product state.
- GSD: $N_{\text{deg}} = 1$. TEE: $S_{\text{top}} = 0$. No anyons.



More general cases

Loop operators for general a

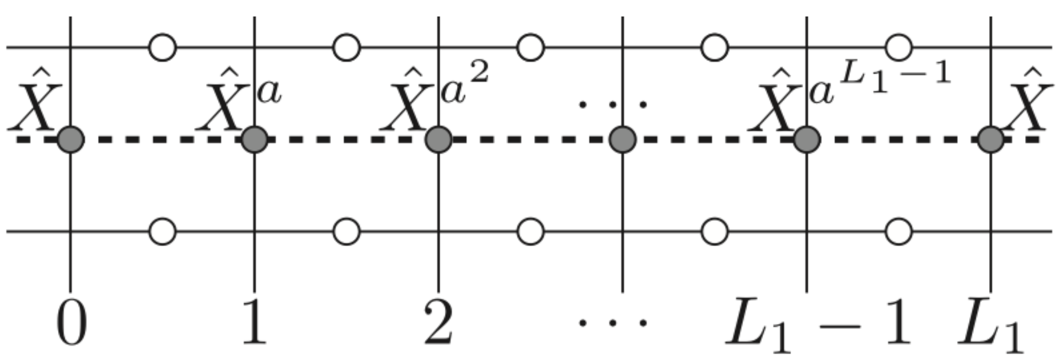
- Unless $a = 1$, simple loop operator do not commute with stabilizers
$\hat{X}^{(1)} = \hat{X}_{(0,m_2+\frac{1}{2})}\hat{X}_{(1,m_2+\frac{1}{2})}\hat{X}_{(2,m_2+\frac{1}{2})}\cdots\hat{X}_{(L_1-1,m_2+\frac{1}{2})}$.
The same is true for Z loops.



- Modified, trial loop operator
$\hat{X}^{(1)} = \hat{X}_{(0,m_2+\frac{1}{2})}\hat{X}^{a}_{(1,m_2+\frac{1}{2})}\hat{X}^{a^2}_{(2,m_2+\frac{1}{2})}\cdots\hat{X}^{a^{L_1-1}}_{(L_1-1,m_2+\frac{1}{2})}$

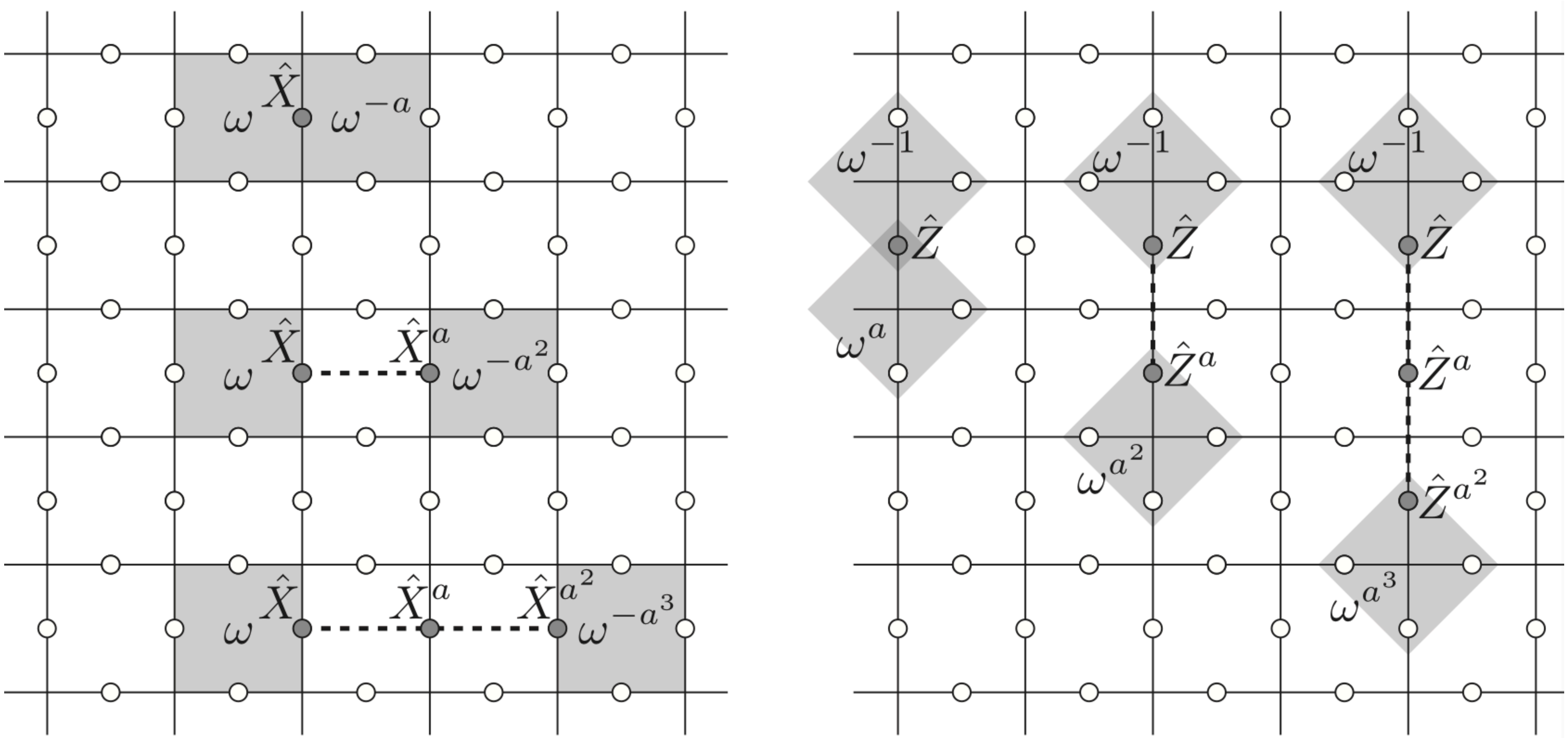
Needs to be connected to $\rightarrow \hat{X}^{a^{L_1}}_{(0,m_2+\frac{1}{2})}$ under PBC.

Commutes with the boundary plaquette only when $a^{L_1} - 1 = 0 \mod N$.



Excited states for general a

- Applying open strings of $\hat{X}$ and $\hat{Z}$ creates a pair of electric and magnetic particles.
- Translation permutes anyons (i.e. changes the electric and magnetic charges).



$$\hat{X}^{(1)}_{(m_1-\frac{1}{2},m_2+\frac{1}{2}),(m_1'+\frac{1}{2},m_2+\frac{1}{2})} = \prod_{\ell=0}^{m_1'-m_1} \hat{X}^{a^\ell}_{(m_1+\ell,m_2+\frac{1}{2})}$$

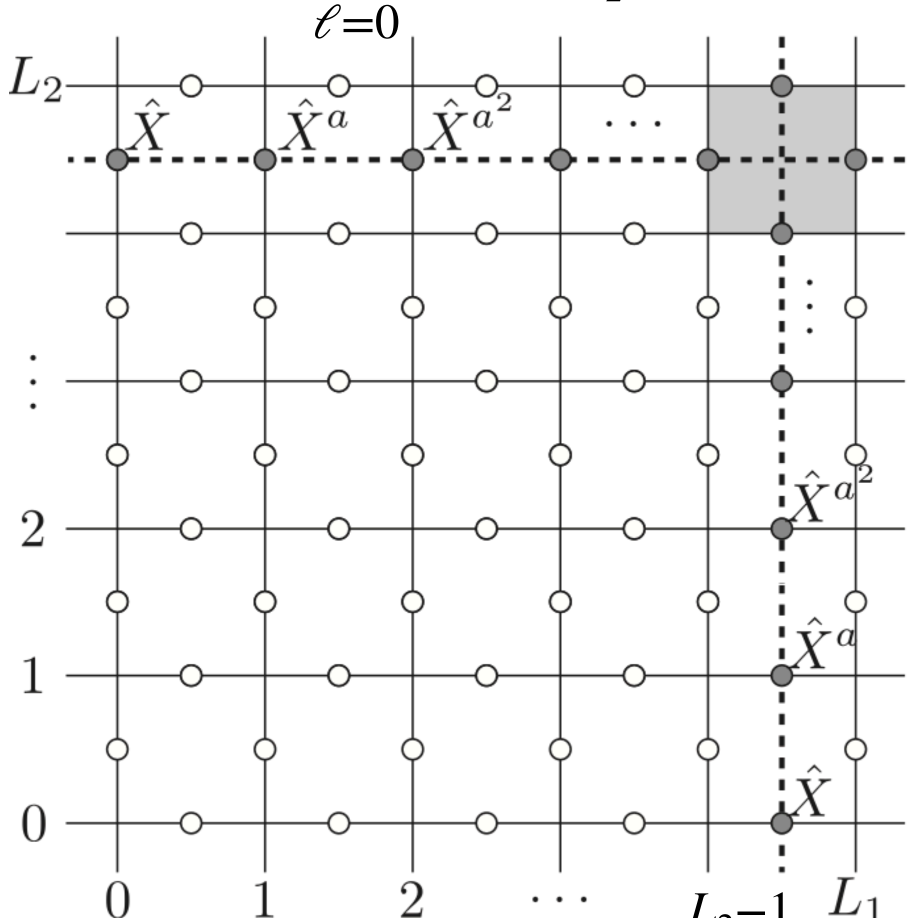
$$\hat{Z}^{(2)}_{(m_1,m_2),(m_1,m_2'+1)} = \prod_{\ell=0}^{m_2'-m_2} \hat{Z}^{a^{m_2'-m_2-\ell}}_{(m_1,m_2+\ell+\frac{1}{2})}$$

Single anyon

- Mismatch at the boundary implies a single anyon excitation without a pair.

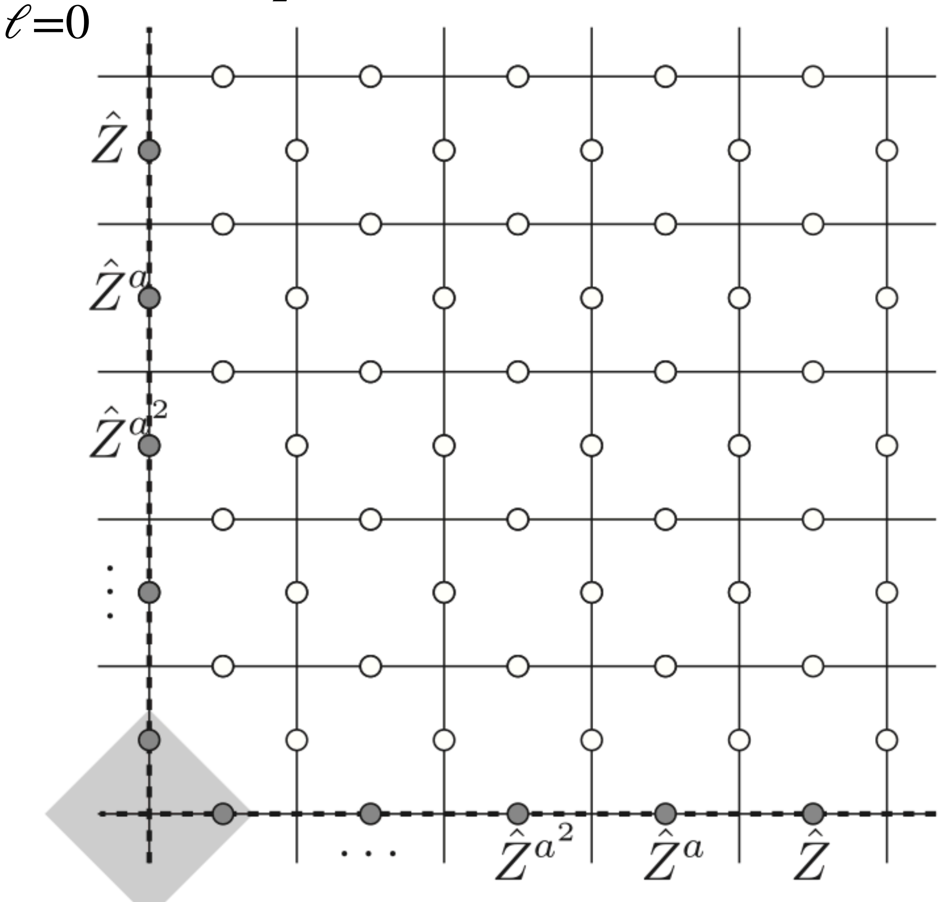
$$\hat{X}^{(i)}\hat{B}_{p_0} = \omega^{a^{L_i}-1}\,\hat{B}_{p_0}\hat{X}^{(i)} \text{ and } \hat{Z}^{(i)}\hat{A}_{v_0} = \omega^{1-a^{L_i}}\,\hat{A}_{v_0}\hat{Z}^{(i)}.$$

$$\hat{X}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{X}^{a^{\ell}}_{(\ell, L_2-\frac{1}{2})}$$



$$\hat{X}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{X}^{a^{\ell}}_{(L_1-\frac{1}{2}, \ell)}$$

$$\hat{Z}^{(2)} = \prod_{\ell=0}^{L_2-1} \hat{Z}^{a^{L_2-1-\ell}}_{(0, \ell+\frac{1}{2})}$$



$$\hat{Z}^{(1)} = \prod_{\ell=0}^{L_1-1} \hat{Z}^{a^{L_1-1-\ell}}_{(\ell+\frac{1}{2}, 0)}$$

Global constraints

- GSD is given by $N_{\rm deg} = N_C^2$ (Recall the original toric code. Can be proven.)
    - N_C is the number of global constraints:

$$\prod_{v\in\mathscr{V}} \hat{A}_v^{\ell_v} = 1 \; (0 \le \ell_v \le N-1).$$

    - There will be an equal number of constraints:

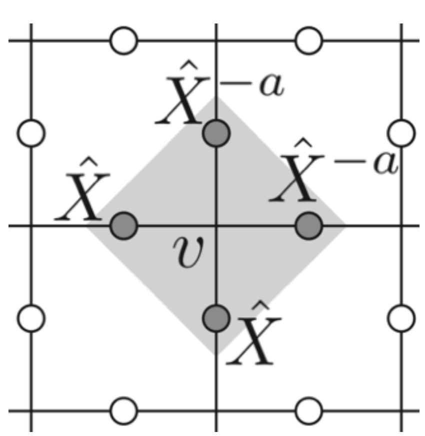
$$\prod_{p\in\mathscr{P}} \hat{B}_p^{\ell_p} = 1 \; (0 \le \ell_p \le N-1).$$

Global constraints

- Trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{A}_{(m_1,m_2)}^{a^{m_1+m_2}} = \prod_{m_1=0}^{L_1-1}\hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}(a^{L_1}-1)}.$$
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{a^{(L_1-1-m_1)+(L_2-1-m_2)}} = \prod_{m_1=0}^{L_1-1}\hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}(a^{L_1}-1)}.$$

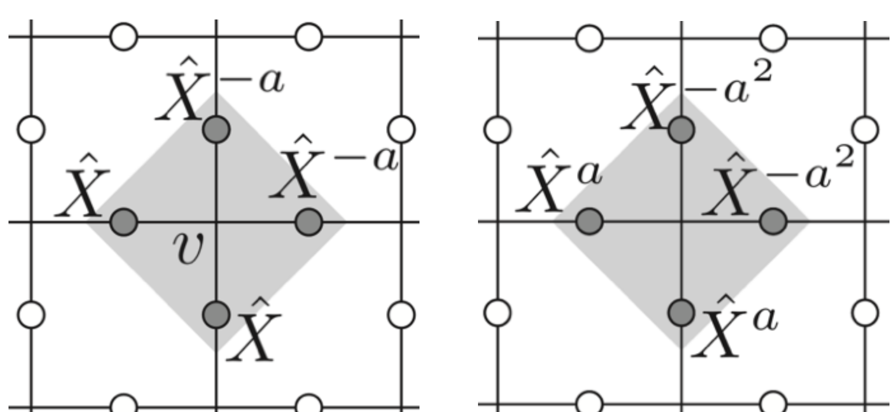
Global constraints

- Trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{A}_{(m_1,m_2)}^{a^{m_1+m_2}} = \prod_{m_1=0}^{L_1-1}\hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}(a^{L_1}-1)}.$$
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{a^{(L_1-1-m_1)+(L_2-1-m_2)}} = \prod_{m_1=0}^{L_1-1}\hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}(a^{L_1}-1)}.$$



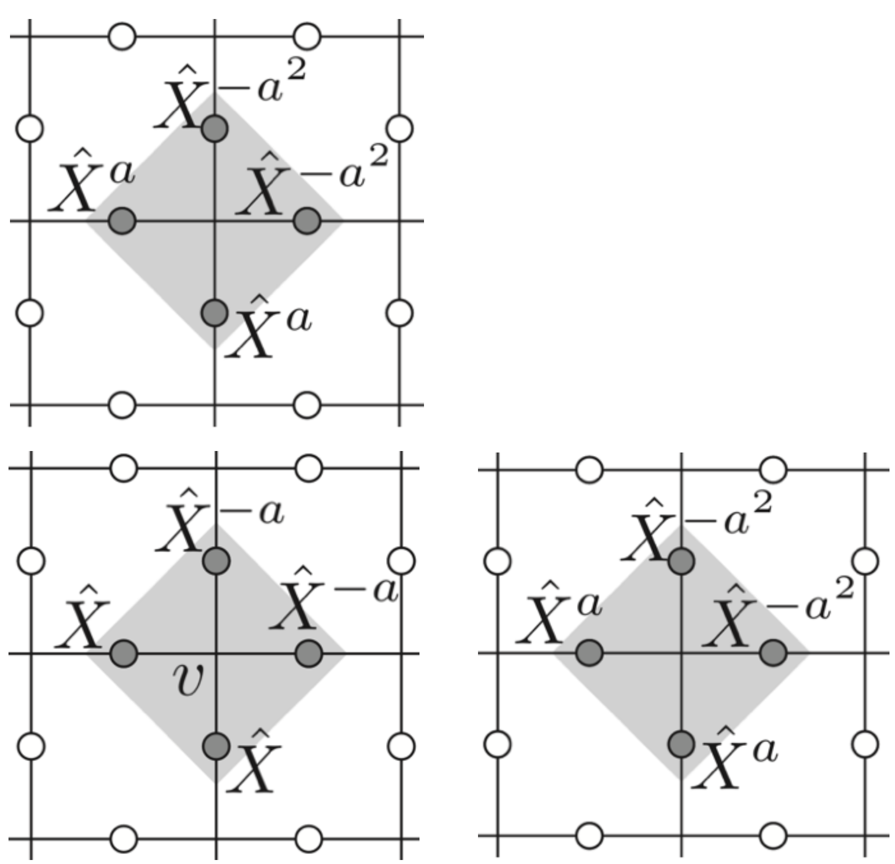
Global constraints

- Trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{A}_{(m_1,m_2)}^{a^{m_1+m_2}} = \prod_{m_1=0}^{L_1-1}\hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}\underline{(a^{L_2}-1)}}\prod_{m_2=0}^{L_2-1}\hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}\underline{(a^{L_1}-1)}}.$$
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{a^{(L_1-1-m_1)+(L_2-1-m_2)}} = \prod_{m_1=0}^{L_1-1}\hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}\underline{(a^{L_2}-1)}}\prod_{m_2=0}^{L_2-1}\hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}\underline{(a^{L_1}-1)}}.$$



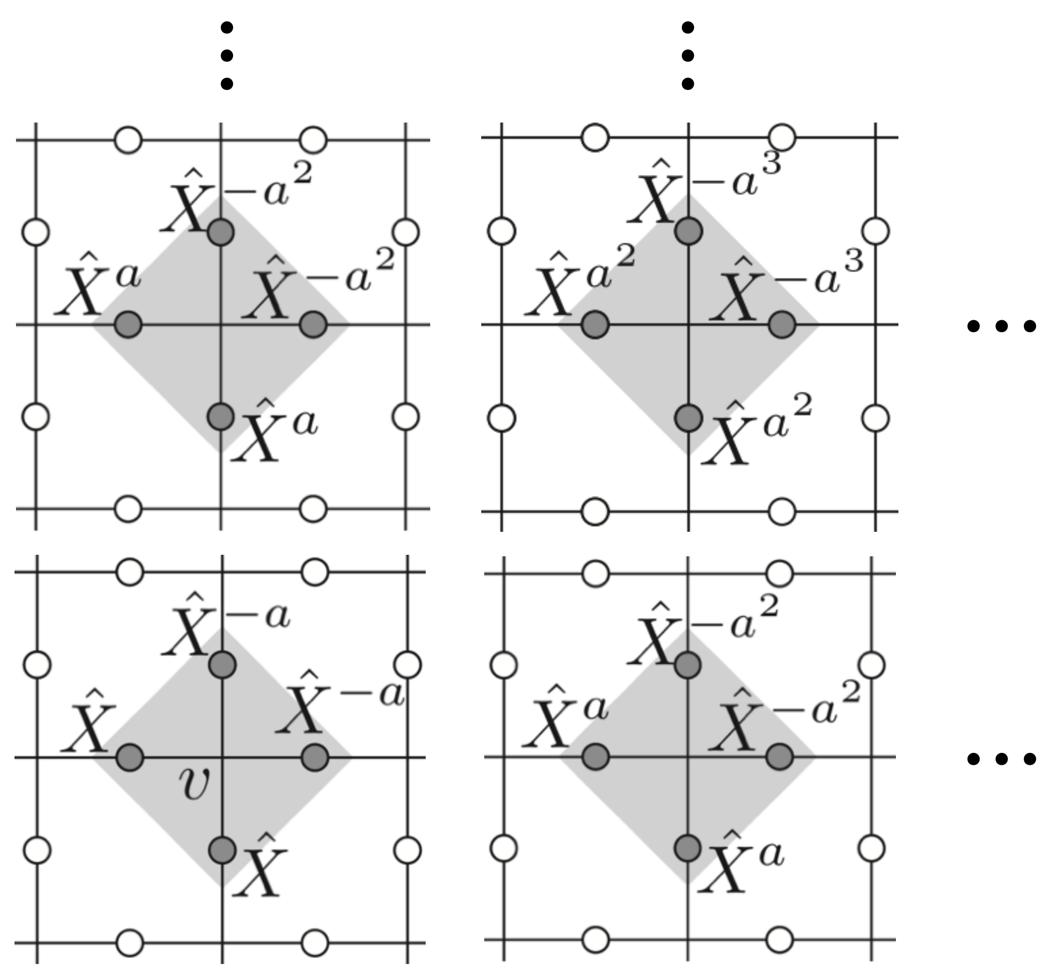
Global constraints

- Trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{A}_{(m_1,m_2)}^{a^{m_1+m_2}}=\prod_{m_1=0}^{L_1-1}\hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}(a^{L_1}-1)}.$$
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{a^{(L_1-1-m_1)+(L_2-1-m_2)}}=\prod_{m_1=0}^{L_1-1}\hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}(a^{L_1}-1)}.$$



Global constraints

- Trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{A}_{(m_1,m_2)}^{a^{m_1+m_2}}=\prod_{m_1=0}^{L_1-1}\hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}(a^{L_1}-1)}.$$
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{a^{(L_1-1-m_1)+(L_2-1-m_2)}}=\prod_{m_1=0}^{L_1-1}\hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}(a^{L_2}-1)}\prod_{m_2=0}^{L_2-1}\hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}(a^{L_1}-1)}.$$



General case

- Prime factorization $N = \prod_{j=1}^{n} p_j^{r_j} = p_1^{r_1} p_2^{r_2} \cdots p_n^{r_n}$.
- Radical of N: $\mathrm{rad}(N) = \prod_{j=1}^{n} p_j = p_1 p_2 \cdots p_n$.
- $N_a =$ the largest divisor of N that is coprime to a.
  - $N_a \neq 1$ if a is not a multiple of $\mathrm{rad}(N)$ → Topologically-ordered phases
  - $N_a = 1$ if a is a multiple of $\mathrm{rad}(N)$ → SPT phases
- GSD $N_{\mathrm{deg}} = d_a^2$ with $d_a = \gcd(a^{L_1} - 1, a^{L_2} - 1, N)$ in general.
  → $N_{\mathrm{deg}} = 1$ regardless of L_1, L_2 when a is a multiple of $\mathrm{rad}(N)$.
- Anyons: N_a^2 species characterized by the electric charge $q_e = 1,2,\cdots,N_a$ and the magnetic charge $q_m = 1,2,\cdots,N_a$.
  Translation $T_i : (q_e, q_m) \to (aq_e, a^{-1}q_m)$.

Justification of $N_{\text{deg}} = d_a^2$

- Consider their n-th power of the trial global constraints:
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{A}_{(m_1,m_2)}^{na^{m_1+m_2}} = \prod_{m_1=0}^{L_1-1}\hat{X}_{(m_1,-\frac{1}{2})}^{-a^{m_1}(a^{L_2}-1)n}\prod_{m_2=0}^{L_2-1}\hat{X}_{(-\frac{1}{2},m_2)}^{-a^{m_2}(a^{L_1}-1)n}.$$
  - $$\prod_{m_1=0}^{L_1-1}\prod_{m_2=0}^{L_2-1}\hat{B}_{(m_1+\frac{1}{2},m_2+\frac{1}{2})}^{na^{(L_1-1-m_1)+(L_2-1-m_2)}} = \prod_{m_1=0}^{L_1-1}\hat{Z}_{(m_1+\frac{1}{2},0)}^{a^{(L_1-1-m_1)}(a^{L_2}-1)n}\prod_{m_2=0}^{L_2-1}\hat{Z}_{(0,m_2+\frac{1}{2})}^{-a^{(L_2-1-m_2)}(a^{L_1}-1)n}.$$
- We need $(a^{L_1}-1)n = (a^{L_2}-1)n = 0 \mod N$.

  Solution: $n = \dfrac{N}{d_a}$ with $d_a = \gcd(a^{L_1}-1, a^{L_2}-1, N_a)$.

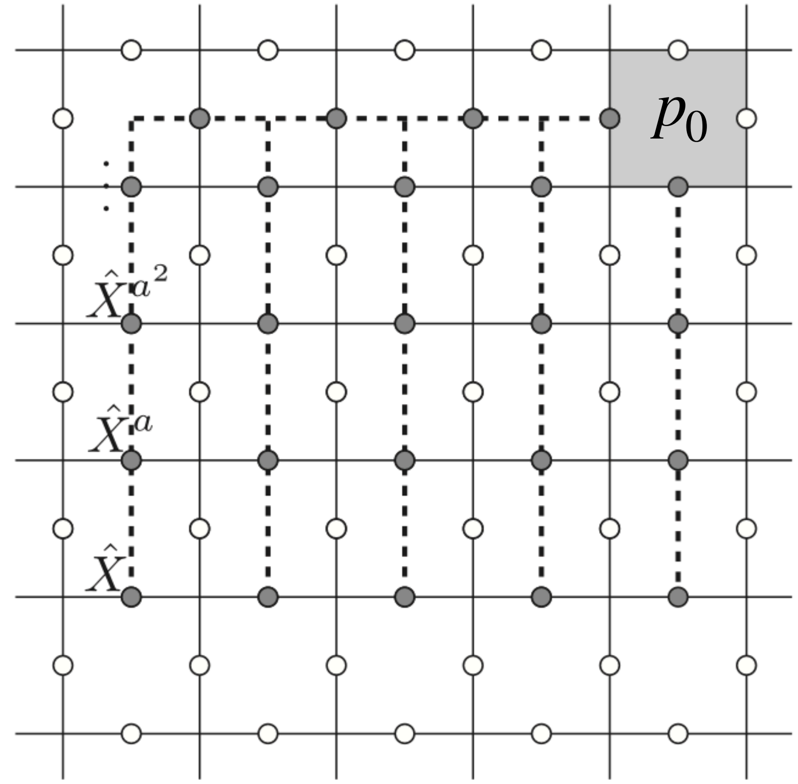
  The eigenvalues of $\hat{A}_{v_0}$ and $\hat{B}_{p_0}$ can be written as $\omega^{x+d_a\ell}$ $(\ell = 0,1,\cdots,n_a-1)$.

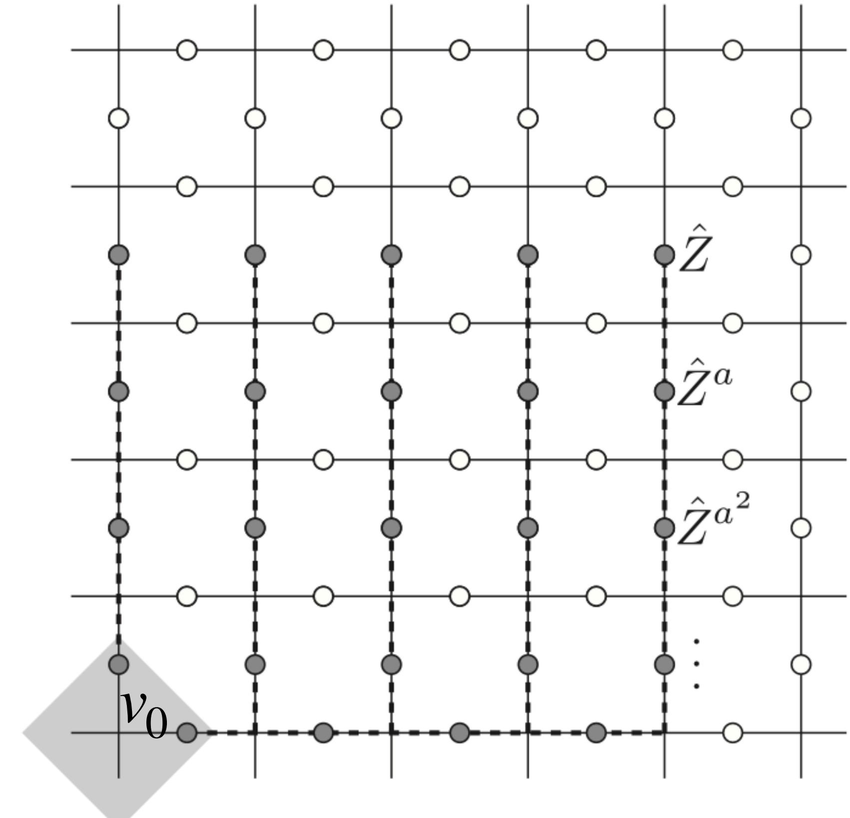
  Only the value of x $(x = 0,1,\cdots,d_a)$ is automatically fixed by global constraints.

  → The part associated with "genuine" closed strings is $N_{\text{deg}} = d_a^2$.

Proof of GSD via explicit construction of all states

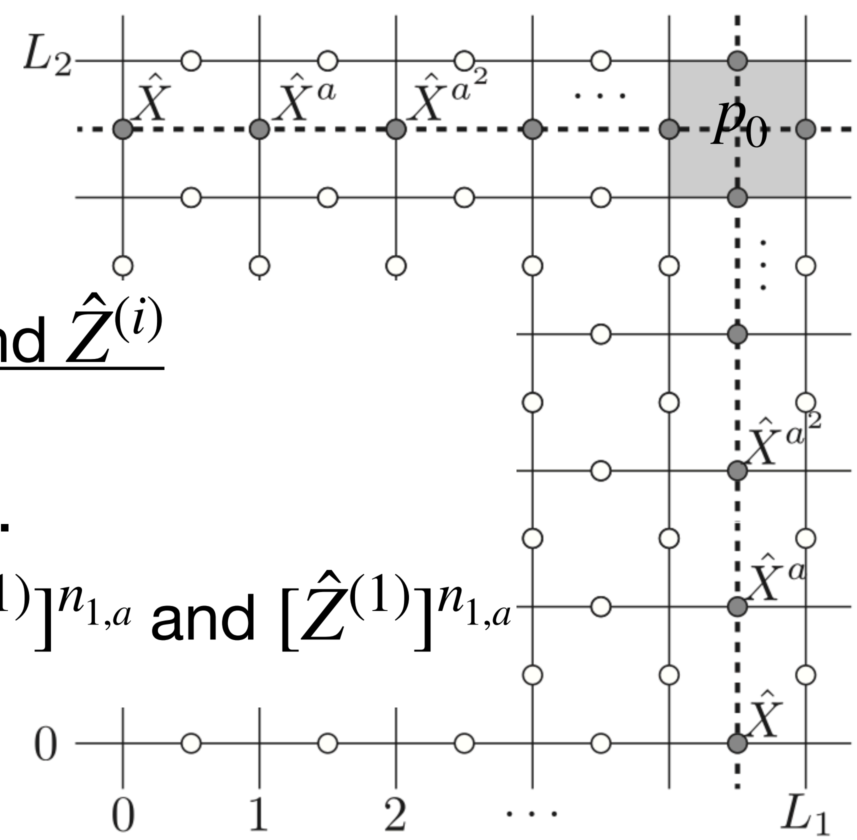
$[\hat{A}_{v_0}]^{\frac{N}{d_a}}$ and $[\hat{B}_{p_0}]^{\frac{N}{d_a}}$ are fixed by global constraints.

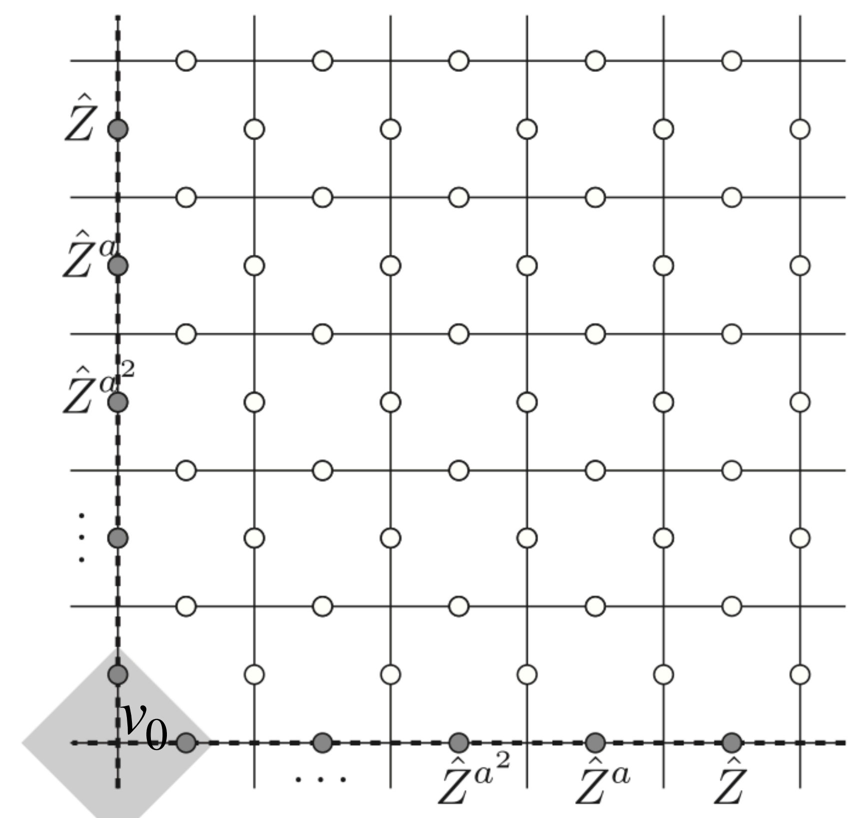




<u>Open string operators</u>
Control the eigenvalues of

- $\hat{A}_v$ $(v \in \mathscr{V}, v \neq v_0)$
- $\hat{B}_p$ $(p \in \mathscr{P}, p \neq p_0)$

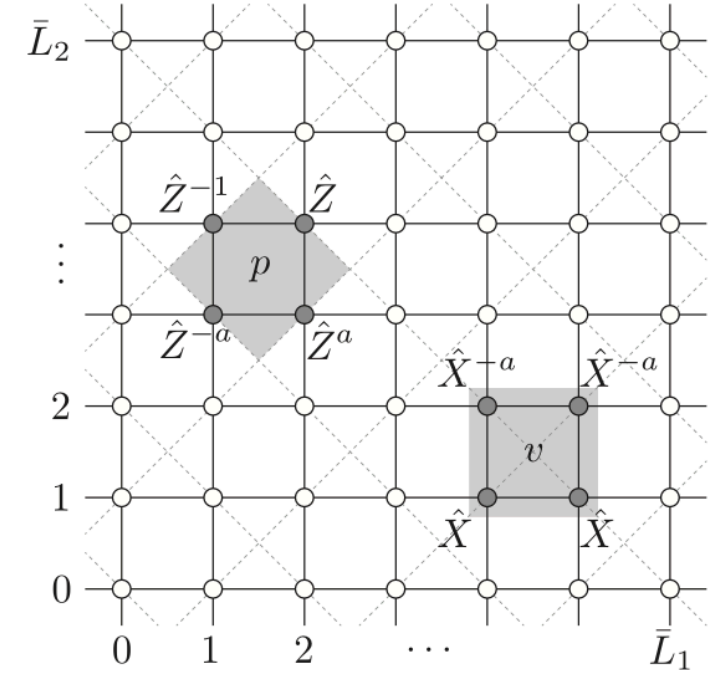




<u>Closed string operators $\hat{X}^{(i)}$ and $\hat{Z}^{(i)}$</u>
Control the eigenvalues of

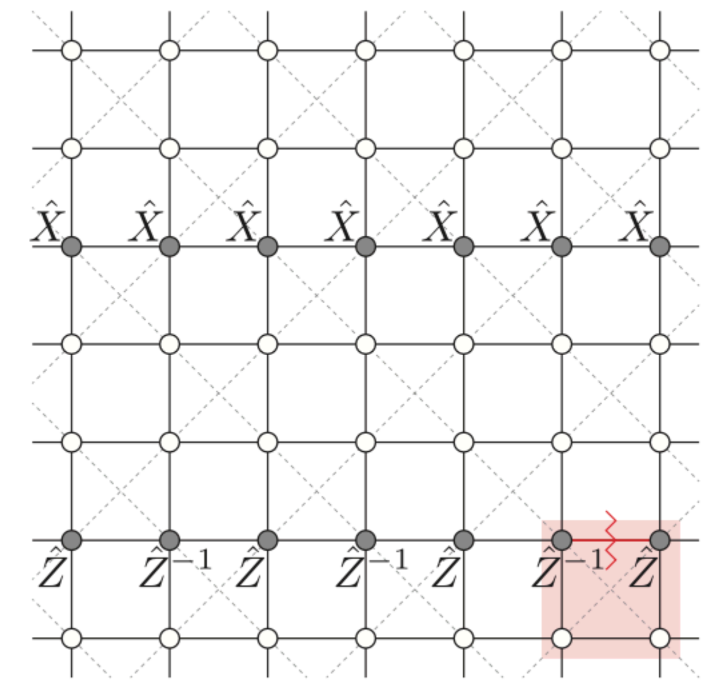
- Residual part of $\hat{A}_{v_0}$ and $\hat{B}_{p_0}$.
- Genuine closed strings: $[\hat{X}^{(1)}]^{n_{1,a}}$ and $[\hat{Z}^{(1)}]^{n_{1,a}}$

Case 3: $a^2 = N$

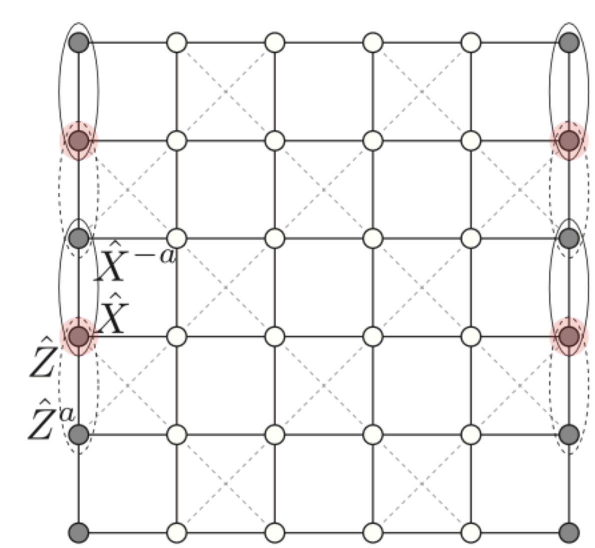


GSD: $N_{\text{deg}} = 1$. No anyons. But $S_{\text{top}} \neq 0$ & $S_{\text{spurious}} \neq 0$.
Here we consider model rotated by 45 degree.

- Subsystem symmetries for each row
  - $$\hat{X}_{\bar{m}_2} = \prod_{\bar{m}_1=0}^{\bar{L}_1-1} \hat{X}_{(\bar{m}_1,\bar{m}_2)} = \prod_{j_1=0}^{\bar{L}_1/2} \hat{A}_{(2j_1+1,\bar{m}_2)} \hat{A}^{a}_{(2j_1,\bar{m}_2+1)}$$
  - $$\hat{Z}_{\bar{m}_2} = \prod_{\bar{m}_1=0}^{\bar{L}_1-1} \hat{Z}^{(-1)^{\bar{m}_1}}_{(\bar{m}_1,\bar{m}_2)} = \prod_{j_1=0}^{\bar{L}_1/2} \hat{B}_{(2j_1-1,\bar{m}_2-1)} \hat{B}^{a}_{(2j_1,\bar{m}_2-2)}$$



- This phase turns out to be
  a subsystem-symmetry protected topological (SSPT) phase
  We confirmed
  - Charge pumping under subsystem-symmetry flux insertion.
  - Zero energy edge states under open boundary condition.



Generalized Ising model

Yaozong Hu and HW
arXiv:2302.01207

Defining properties of phases (textbook)

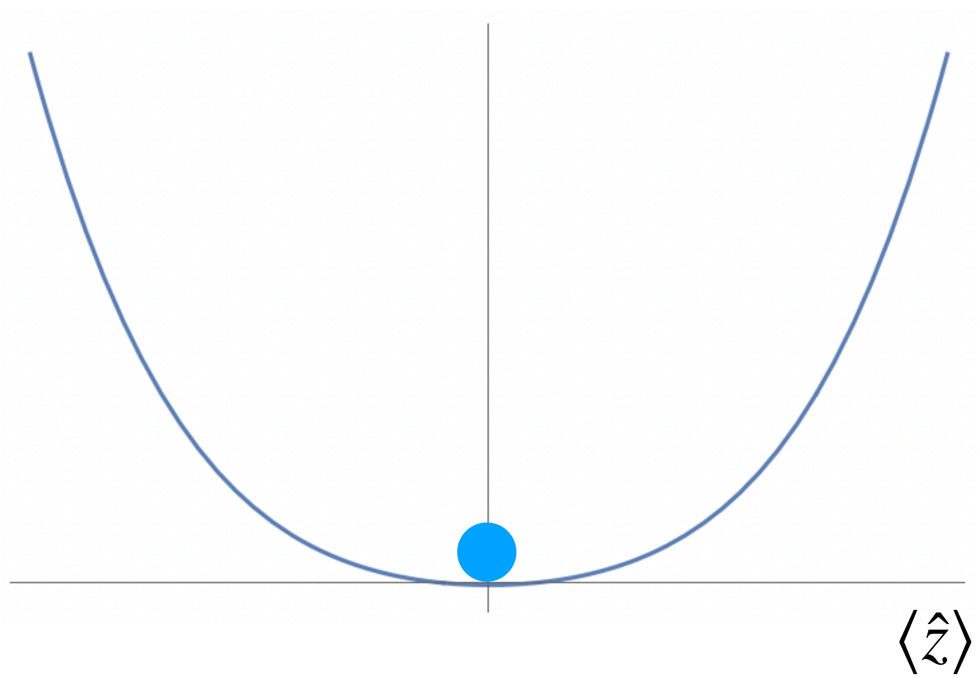
Symmetry breaking phase of discrete symmetries

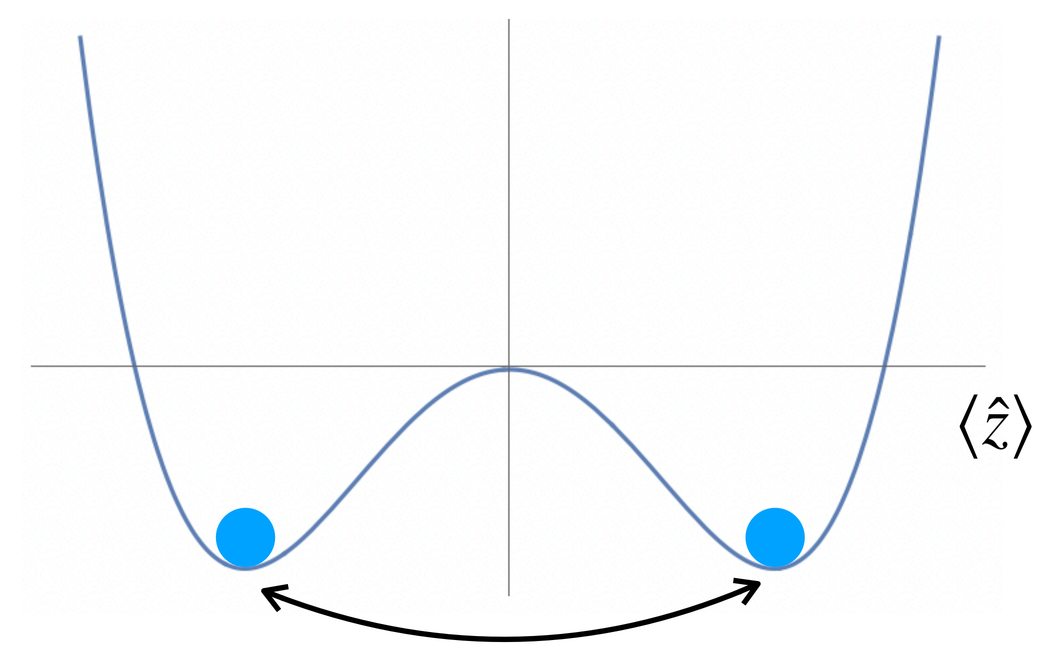
- Degenerate ground states characterized by order parameter.
- Stable against symmetry-preserving perturbations.
- The large volume limit and the vanishing field limit do not commute!

$$\hat{H}(\epsilon) = \hat{H} - \epsilon V \hat{z}$$

$$\lim_{V\to\infty} \lim_{\epsilon\to+0} \langle \hat{z} \rangle = 0$$

$$\lim_{\epsilon\to+0} \lim_{V\to\infty} \langle \hat{z} \rangle \neq 0$$





N-state clock model (generalized transverse-field Ising model)

- $\hat{H}(g) = -\dfrac{1}{2}\sum_{i=0}^{L-1}\left[(\hat{Z}_i^\dagger \hat{Z}_{i+1} + \text{h.c.}) + g(\hat{X}_i + \text{h.c.})\right]$
  - N level spin: $X = \begin{pmatrix} & & & 1 \\ 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & 1 & \end{pmatrix}, \; Z = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}.$
  - Symmetry: $\hat{X} = \prod_{i=0}^{L-1} \hat{X}_i = \hat{X}_0 \hat{X}_1 \cdots \hat{X}_{L-1}.$ → $\hat{X}\hat{z}\hat{X}^\dagger = \omega \hat{z}$
  - Order parameter: $\hat{z} = \dfrac{1}{L}\sum_{i=0}^{L-1} \hat{Z}_i = \dfrac{1}{L}(\hat{Z}_0 + \hat{Z}_1 + \cdots + \hat{Z}_{L-1}).$
- Phases
  - $1 \gg g \geq 0$: Ordered phase. Spontaneous breaking of $\mathbb{Z}_N$ symmetry.
  - $1 \ll g$: Disordered phase. No symmetries are broken.

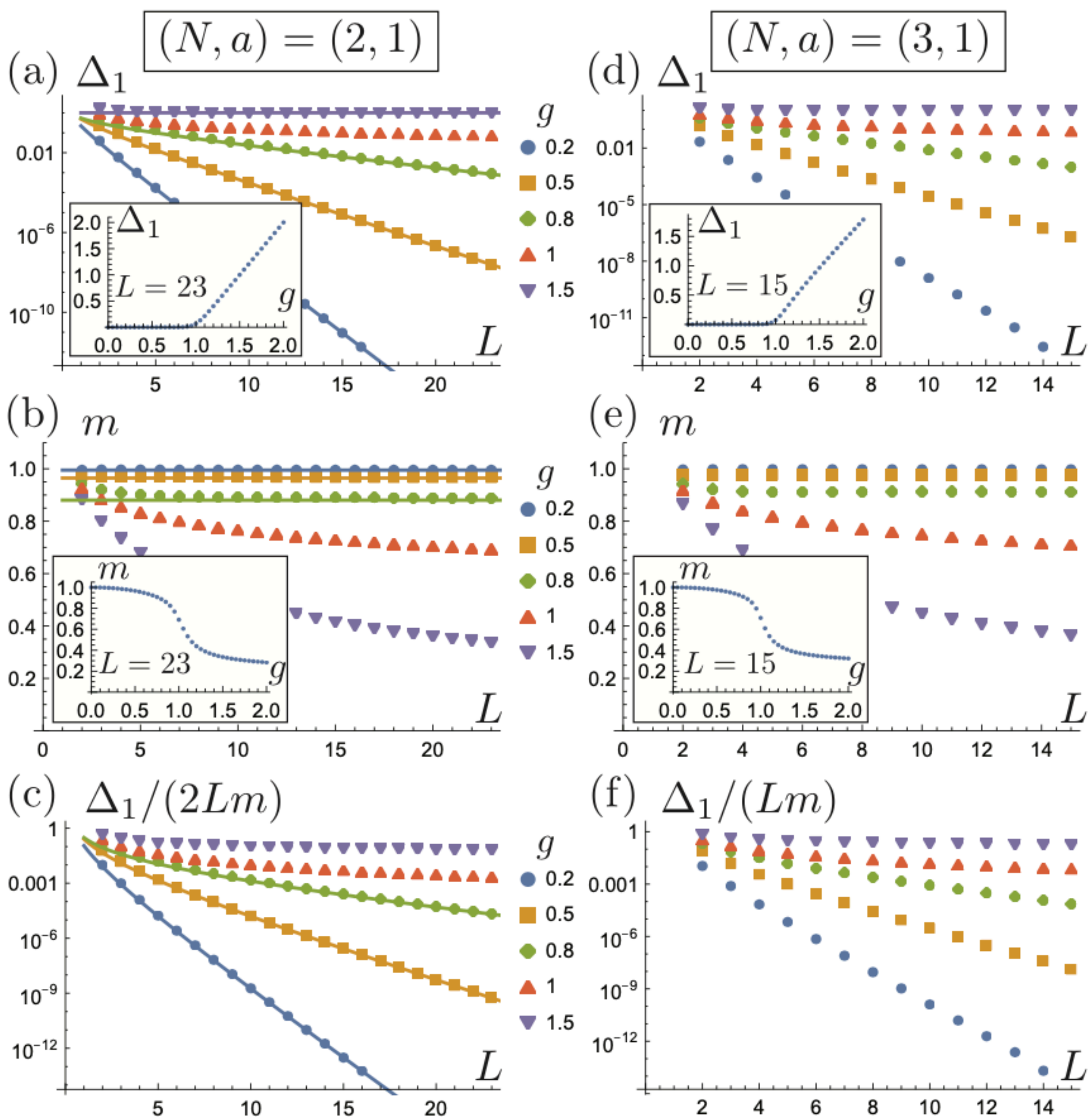


FIG. 1. Exact-diagonalization results for the standard N-state clock model with $N = 2$ [(a)–(c)] and $N = 3$ [(d)–(f)]. (a),(d): The energy difference $\Delta_1 := E_1 - E_0$ between the ground state and the first excited state in a finite system. (b),(e): The long-range order $m := \sqrt{\langle \Phi_0 | \hat{z}^\dagger \hat{z} | \Phi_0 \rangle}$. (c),(f): $\Delta_1/(2Lm)$ [(c)] and $\Delta_1/(Lm)$ [(f)] that approximate $\epsilon_*(L)$. The insets in (a),(b),(d),(e) show the g dependence. The curves in (a)-(c) are the analytic expressions in Eqs. (20),(21), and (27).

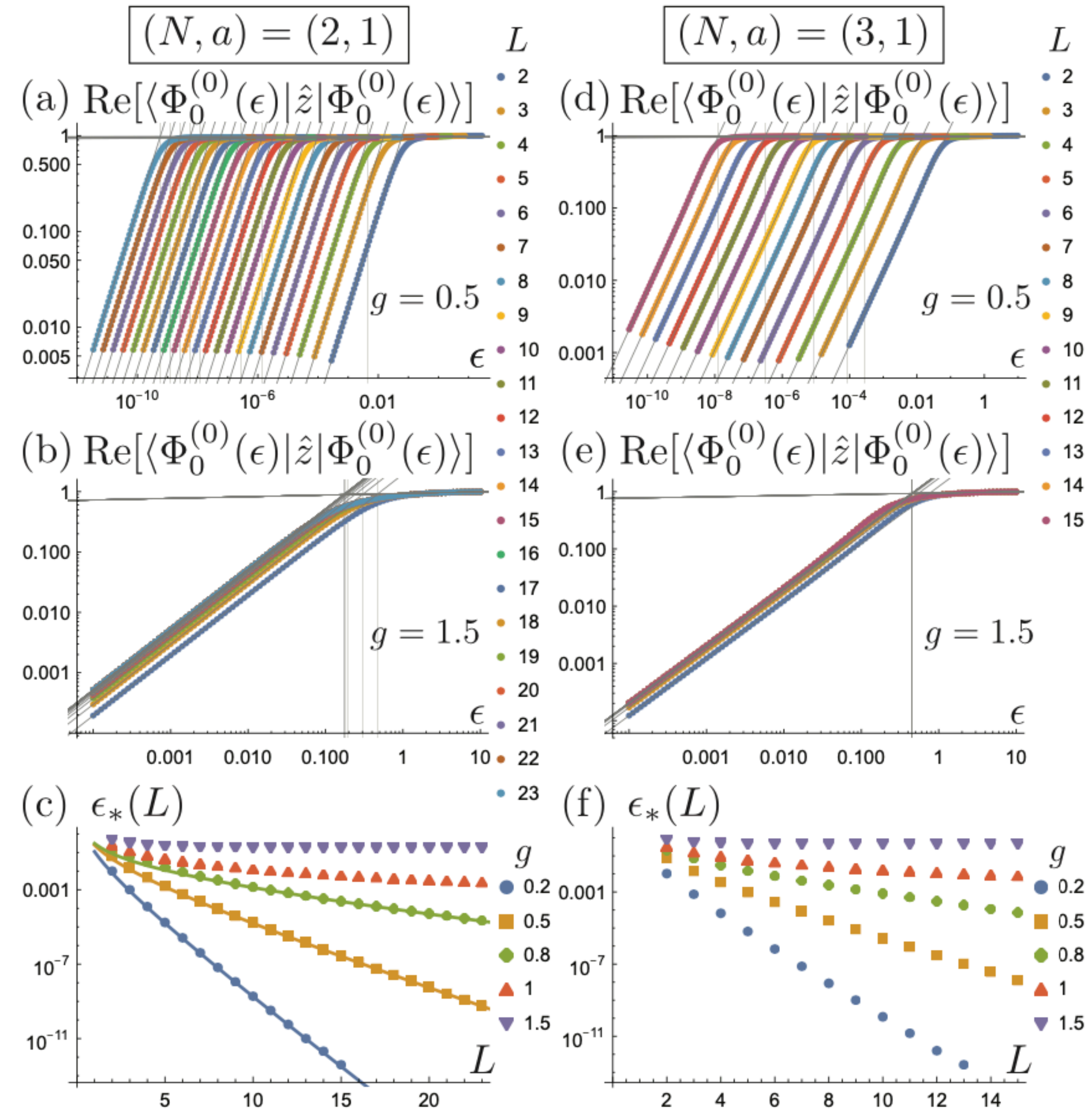


FIG. 2. Exact-diagonalization results for the standard N-state clock model with symmetry-breaking field ϵ ($\ell_0 = 0$) for $N = 2$ [(a)–(c)] and for $N = 3$ [(d)–(f)]. (a),(b),(d),(e): The order parameter $\mathrm{Re}[\langle \Phi_0^{(0)}(\epsilon)|\hat{z}|\Phi_0^{(0)}(\epsilon)\rangle]$ for $g = 0.5$ [(a),(c)] and $g = 1.5$ [(b),(e)]. (c),(f): The magnetic field $\epsilon_*(L)$ at the transition point, which is determined by the crossing point of two fitting lines (gray lines) in the log-log plot of $\mathrm{Re}[\langle \Phi_0^{(0)}(\epsilon)|\hat{z}|\Phi_0^{(0)}(\epsilon)\rangle]$. The curves in (c) are the analytic expression in Eq. (27).

$$\hat{H}^{(0)}(\epsilon) = \hat{H} - \frac{1}{2}\epsilon L\,(\hat{z} + \text{h.c.})$$

Generalized N-state clock model

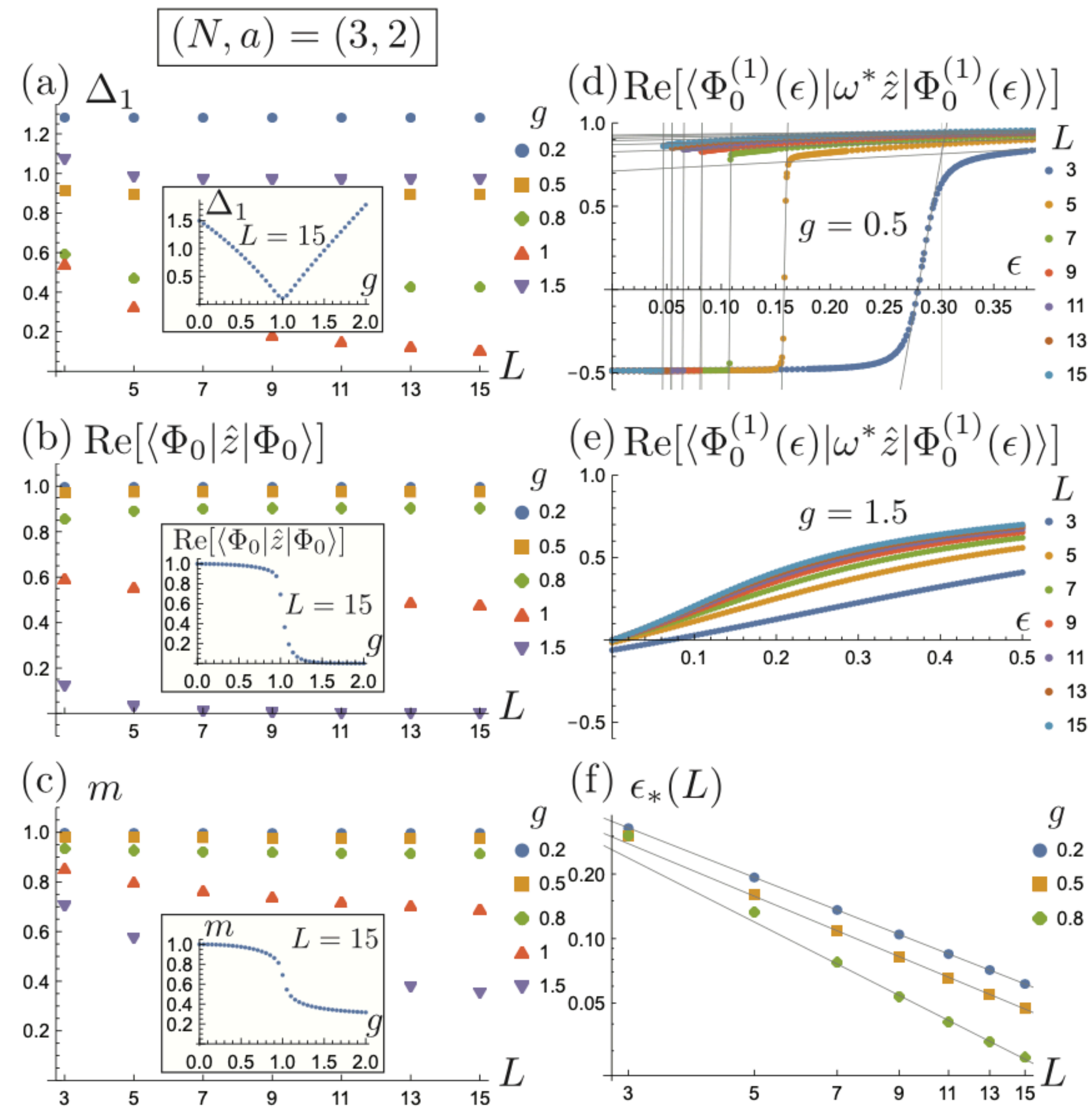
- $$\hat{H}(g) = -\frac{1}{2}\sum_{i=0}^{L-1}\left[(\hat{Z}_i^{-a}\hat{Z}_{i+1} + \text{h.c.}) + g(\hat{X}_i + \text{h.c.})\right]$$
  - N level spin: $$X = \begin{pmatrix} & & & & 1 \\ 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \end{pmatrix},\ Z = \begin{pmatrix} 1 & & & & \\ & \omega & & & \\ & & \omega^2 & & \\ & & & \ddots & \\ & & & & \omega^{N-1} \end{pmatrix}.$$
  - Symmetry: $\hat{X} = \prod_{i=0}^{L-1}\hat{X}_i^{a^i} = \hat{X}_0\hat{X}_1^a\cdots\hat{X}_{L-1}^{a^{L-1}}$ if $a^L = 1 \mod N$.
  - Order parameter: $\hat{z} = \frac{1}{L}\sum_{i=0}^{L-1}\hat{Z}_i^{a^{L-i}} = \frac{1}{L}(\hat{Z}_0^{a^L} + \hat{Z}_1^{a^{L-1}} + \cdots + \hat{Z}_0^{a})$.
  - $\hat{X}\hat{z}\hat{X}^\dagger = \omega\hat{Z}$ if $a^L = 1 \mod N$

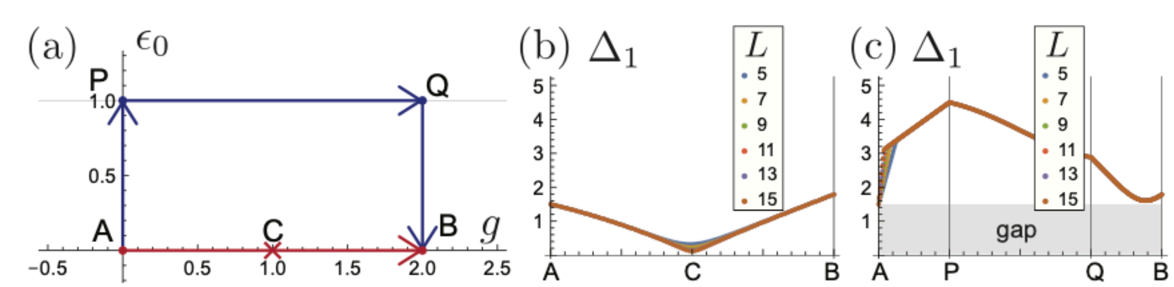
Example: $(N, a) = (3,2)$

- $\mathbb{Z}_N$ symmetry is absent.
- No ground state degeneracy.
- Still a gap closing at $g = 1$!
- Still the large L limit and vanishing ϵ limit do not commute!!

$$\hat{H}^{(\ell_0)}(\epsilon) = \hat{H} - \frac{1}{2}\epsilon L \left(\omega^{-\ell_0}\hat{z} + \text{h.c.}\right)$$

- Spontaneous symmetry breaking without exact symmetry or ground state degeneracy…





Summary 1

Ref: HW, M. Cheng, Y. Fuji, arXiv:2211.00299

- We introduced a family of $\mathbb{Z}_N$ toric code with an integer parameter a.
  - $a = 1$: the standard choice.
  - $a = N - 1$: discussed previously but GSD on torus was not investigated.
  - $a = 2,3,\cdots, N-1$ showed interesting behavior.
- $N_a =$ the largest divisor of N that is coprime to a.
  - $N_a \neq 1$ if a is not a multiple of $\mathrm{rad}(N)$ → Topologically-ordered phases
  - $N_a = 1$ if a is a multiple of $\mathrm{rad}(N)$ → (S)SPT phases
- GSD $N_{\mathrm{deg}} = d_a^2$ with $d_a = \mathrm{gcd}(a^{L_1} - 1, a^{L_2} - 1, N)$.
- Anyons: N_a^2 species characterized by the electric charge $q_e = 1,2,\cdots, N_a$ and the magnetic charge $q_m = 1,2,\cdots, N_a$.
  Translation $T_i : (q_e, q_m) \to (a q_e, a^{-1} q_m)$.

Summary 2

Ref: HW, M. Cheng, Y. Fuji, arXiv:2211.00299

Symmetry-protected topological/trivial phases

- Unique GS with excitation gap **for any sequence of L_1 and L_2.**
- Stable against symmetry-preserving perturbations.
- Can be connected to product states by local unitaries if symmetries are broken.
  (Exceptions: 1D Kitaev chain, integer Quantum Hall systems, …)
- Trivial excitations in the bulk.
  Anomalous states on symmetry-preserving boundaries.

Topologically-ordered phases

- Degenerate GS with excitation gap **for a sequence of L_1 and L_2.**
  Degeneracy does not originate from spontaneous breaking of symmetries.
- Stable against any local perturbations.
- Cannot be connected to product state by any local unitaries.
  Topological entanglement entropy.
- Fractionalized (anyonic) excitations in the bulk. ~~Anyons must appear in pairs.~~

Summary 3

Symmetry breaking phase of discrete symmetries

- Degenerate ground states **for a sequence of L_1 and L_2** characterized by order parameter.
- Stable against symmetry-preserving perturbations.
- The large volume limit and the vanishing field limit do not commute!

$$\hat{H}(\epsilon) = \hat{H} - \epsilon V \hat{z}$$

$$\lim_{V\to\infty} \lim_{\epsilon\to+0} \langle \hat{z} \rangle = 0$$

$$\lim_{\epsilon\to+0} \lim_{V\to\infty} \langle \hat{z} \rangle \neq 0$$