Superfluid Fraction of the Inner Crust of Neutron Stars

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Outline

- Motivation
- Hartree-Fock-Bogoliubov and band theory
- Revised linear response formula
- Superfluid hydrodynamics and Leggett's upper bound
- Conclusions

Neutron stars

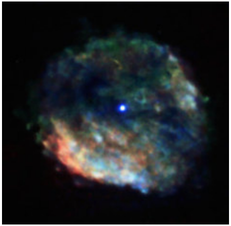
- Neutron star (NS) formed at the end of the "life" of an intermediate-mass star (supernova)
- $M \sim 1 - 2\ M_\odot$ in a radius of $R \sim 10 - 15$ km
  $\rightarrow$ average density $\sim 5 \times 10^{14}$ g/cm^3
  ($\sim 2\times$ nuclear matter saturation density)
- Cools down rapidly by neutrino emission within ~ 1 month: $T \lesssim 10^9$ K ~ 100 keV
- Internal structure of a neutron star:

  outer crust: Coulomb lattice of neutron rich nuclei in a degenerate electron gas

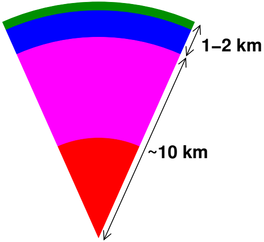
  inner crust: unbound neutrons form a neutron gas between the nuclei

  outer core: homogeneous matter $(n,\ p,\ e^-)$

  inner core: new degrees of freedom: hyperons? quark matter?

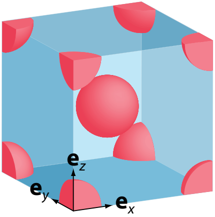


RCW103 [Chandra X-ray telescope]

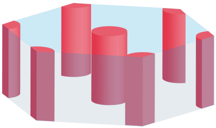


Structure of the inner crust

- For $n_B \sim 0.001 \ldots 0.08 \text{ fm}^{-3}$ $(1.7 \times 10^{12} \ldots 1.3 \times 10^{14} \text{ g/cm}^3)$, clusters made of neutrons and protons are surrounded by a dilute neutron gas



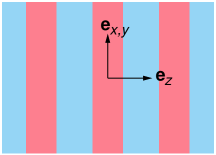
- For $n_B \lesssim 0.06 \text{ fm}^{-3}$ (10^{14} g/cm^3), clusters arrange in a BCC lattice (3D) to minimize the Coulomb energy



- At higher densities, one expects so-called "pasta phases":

  rods ("spaghetti", 2D hexagonal lattice)

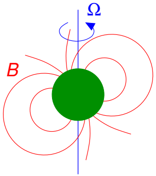
  or slabs ("lasagna", 1D)

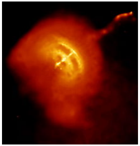


- Neutrons are supposed to be superfluid

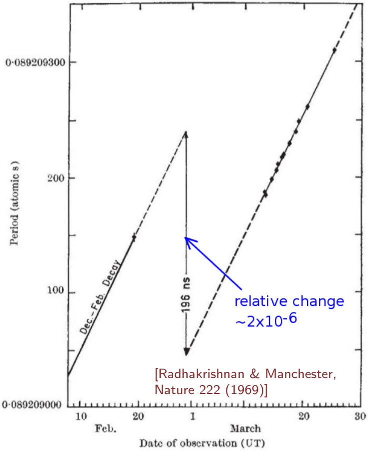
Pulsar glitches

- Neutron star: rotating magnetic dipole, period increases slowly with time
- Glitch = sudden speed-up of the rotation, followed by a slow relaxation
- First glitch observed 1969 in the Vela pulsar, since then 520 glitches in 180 different pulsars [Manchester (2017)]
- Possible explanation: pinning of quantized vortices to the clusters in the inner crust [Manchester & Itoh (1975)]
- While the normal part of the star is slowing down (Ω_n), the superfluid neutrons are spinning at constant frequency (Ω_s)
- When $\Omega_s - \Omega_n$ becomes too large, the vortices get unpinned and the superfluid transfers angular momentum to the normal part





[Chandra]



[Radhakrishnan & Manchester, Nature 222 (1969)]

Superfluid fraction ("entrainment")

- Question: how many neutrons in the inner crust are superfluid?
- Current in a uniform superfluid ($T = 0$):

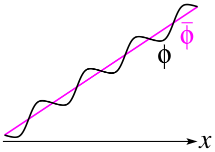
  $\mathbf{j} = n\frac{\hbar}{2m}\nabla\phi$ where $\Delta = |\Delta|e^{i\phi}$

  assuming that ϕ varies only on large enough length scales
- In an inhomogeneous system, define superfluid and normal densities n_S and n_N in terms of coarse grained quantities $\bar{\mathbf{j}}$, $\bar{\phi}$, $\bar{n}$ such that:

  $$\bar{\mathbf{j}} = n_S\frac{\hbar}{2m}\nabla\bar{\phi} + n_N\mathbf{v}_N \quad \text{with} \quad n_S + n_N = \bar{n}$$

  ($\mathbf{v}_N$ = velocity of the inhomogeneities)

  [see e.g. Pethick, Chamel & Reddy (2010)]



- If the system is non-uniform, then $n_S < \bar{n}$ even at $T = 0$

  [A. Leggett, J. Stat. Phys. 93, 927 (1998)]
- Some of the particles are "entrained" by the motion of the inhomogeneities
- In general (e.g., in pasta phases), n_S and n_N are matrices

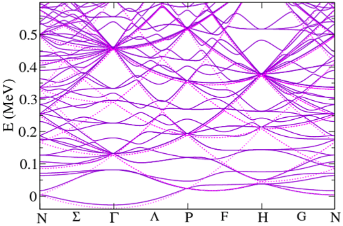
Band theory vs. hydrodynamics

- Normal band theory

  [Carter & Chamel (2004); Chamel (2005-. . .);
  Figure: Chamel & Haensel, Liv. Rev. Rel. 11 (2008)]

  analogous to band theory in solids

  valid for weak coupling ($\Delta \to 0$)



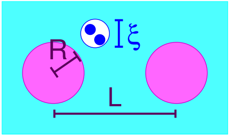
- Superfluid hydrodynamics

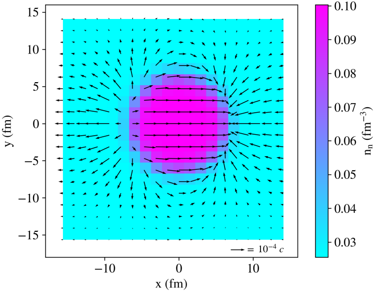
  [Sedrakian (1996); Magierski & Bulgac (2004);
  N. Martin & MU (2016); Th. Kaskitsi (Master student)]

  assume also microscopic current $\mathbf{j}$ and microscopic phase ϕ fulfil $\mathbf{j} = n\frac{\hbar}{2m}\boldsymbol{\nabla}\phi$

  valid for strong coupling:

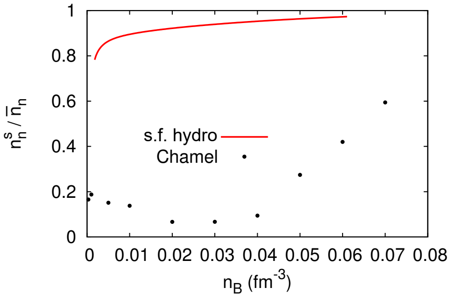
$$\xi \sim \frac{k_F}{\pi m \Delta} \ll R, L$$

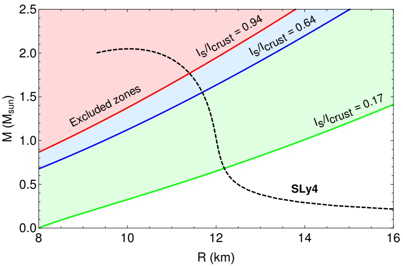




Vela glitches and superfluid fraction in the crust

- Contradictory predictions for the crust superfluid fraction from superfluid hydrodynamics [Martin & MU, PRC 94 (2016)] and normal band structure theory [Chamel, PRC 85 (2012)]
- Observed Vela glitches require substantial contribution I_s of superfluid neutrons to the moment of inertia I_{crust}
- Observed glitches incompatible with superfluid fraction $I_s/I_{\text{crust}} = 0.17$ from band theory [Chamel, PRL 110 (2013)]
- Do we need to include also the core? [Andersson et al. PRL 109 (2012)]
- Does normal band theory underestimate the superfluid fraction? [see also Watanabe & Pethick, PRL 119 (2017)]





Hartree-Fock-Bogoliubov (HFB) with periodicity

HFB can interpolate between normal band theory in weak coupling and superfluid hydrodynamics in strong coupling

$$\begin{pmatrix} h-\mu & -\Delta \\ -\Delta^\dagger & -\bar{h}+\mu \end{pmatrix} \begin{pmatrix} U_\alpha^* \\ -V_\alpha \end{pmatrix} = E_\alpha \begin{pmatrix} U_\alpha^* \\ -V_\alpha \end{pmatrix}$$

working in momentum space: $h_{\mathbf{pp}'} = \frac{\mathbf{p}^2}{2m}\delta_{\mathbf{pp}'} + U_{\mathbf{pp}'}$

mean field: $U_{\mathbf{pp}'} = -\sum_{\mathbf{qq}'} V_{\mathbf{pqp}'\mathbf{q}'}\,\rho_{\mathbf{q}'\mathbf{q}}$ (Skyrme functional)

gap: $\Delta_{\mathbf{pp}'} = -\sum_{\mathbf{qq}'} V_{\mathbf{pp}'\mathbf{q}'\mathbf{q}}\,\kappa_{\mathbf{q}'\mathbf{q}}$ (separable interaction $\sim V_{\text{low}-k}$)

Periodicity: example: 1D case (lasagna)

$$p_x = n_x \tfrac{2\pi}{L} + k_x\,, \quad \text{with} \quad n_x \in \mathbb{Z}, \quad k_x \in (-\tfrac{\pi}{L}, \tfrac{\pi}{L}]$$

$\rightarrow$ HFB matrix is diagonal in k_x (Bloch momentum), p_y, and p_z, non-diagonal only in discrete index n_x

Hartree-Fock-Bogoliubov (HFB) with periodicity

HFB can interpolate between normal band theory in weak coupling and superfluid hydrodynamics in strong coupling

$$\begin{pmatrix} h-\mu & -\Delta \\ -\Delta^\dagger & -\bar{h}+\mu \end{pmatrix} \begin{pmatrix} U^*_\alpha \\ -V_\alpha \end{pmatrix} = E_\alpha \begin{pmatrix} U^*_\alpha \\ -V_\alpha \end{pmatrix}$$

working in momentum space: $h_{\mathbf{pp'}} = \frac{\mathbf{p}^2}{2m}\delta_{\mathbf{pp'}} + U_{\mathbf{pp'}}$

mean field: $U_{\mathbf{pp'}} = -\sum_{\mathbf{qq'}} V_{\mathbf{pqp'q'}}\, \rho_{\mathbf{q'q}}$ (Skyrme functional)

gap: $\Delta_{\mathbf{pp'}} = -\sum_{\mathbf{qq'}} V_{\mathbf{pp'q'q}}\, \kappa_{\mathbf{q'q}}$ (separable interaction $\sim V_{\mathrm{low}-k}$)

Periodicity: 3D crystal with primitive reciprocal lattice vectors $\mathbf{b}_i$

$$\mathbf{p} = n_1\mathbf{b}_1 + n_2\mathbf{b}_2 + n_3\mathbf{b}_3 + \mathbf{k}\,, \quad \text{with} \quad n_i \in \mathbb{Z}, \quad \mathbf{k} \in \mathrm{BZ}$$

$\rightarrow$ HFB matrix is diagonal in $\mathbf{k}$ (Bloch momentum),
non-diagonal in discrete indices n_i

Band structure: example for a simple cubic cell

- In principle, diagonalization must be done for all $\mathbf{k} \in$ BZ

  (in practice only for a finite number of integration points)

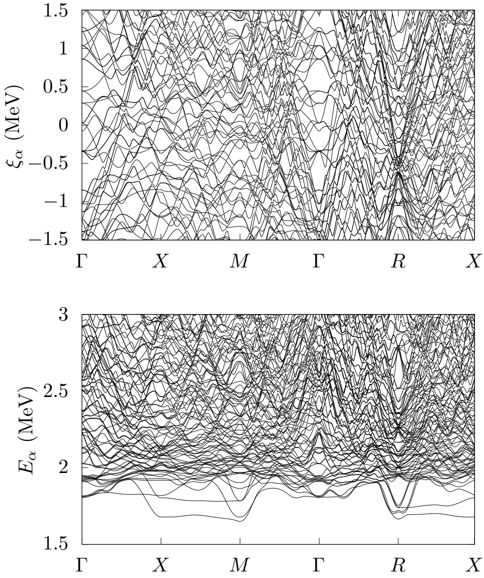
- Diagonalizing only h: single particle bands

  $\xi_{\alpha,\mathbf{k}} = \epsilon_{\alpha,\mathbf{k}} - \mu$

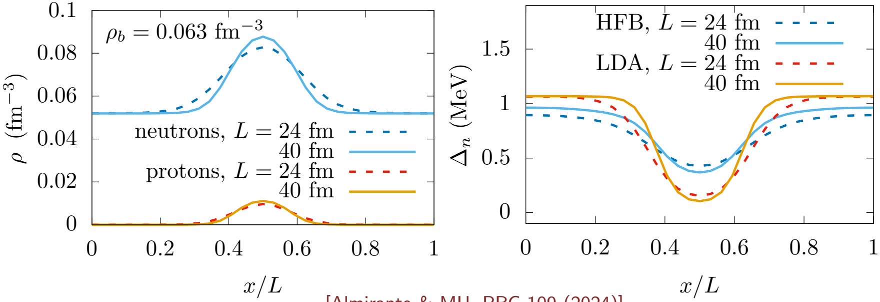
- Diagonalizing full HFB matrix: quasiparticle bands

  $E_{\alpha,\mathbf{k}} \gtrsim \Delta$

$\Gamma - X - M - \Gamma - R - X$ = path on symmetry lines between special points (with $|\mathbf{k}| = 0, \frac{\pi}{L}, \sqrt{2}\frac{\pi}{L}, 0, \sqrt{3}\frac{\pi}{L}, \frac{\pi}{L}$)



Density profile and gap: 1D example (lasagna)



[Almirante & MU, PRC 109 (2024)]

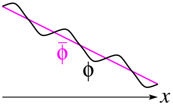
- gap inside the slab is smaller than in the neutron gas
- but this suppression is weaker than what one would get when using the local-density approximation (LDA)

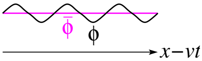
Introducing a stationary flow

- Consider relative velocity

  $\mathbf{v} = \mathbf{v}_N - \mathbf{v}_S \quad (\mathbf{v}_S = \frac{\hbar}{2m}\boldsymbol{\nabla}\bar{\phi})$

  between clusters and superfluid





- In the rest frame of the superfluid:
  - $\Delta = |\Delta| e^{i\phi}$ is periodic
  - Hamiltonian $h \to h - \mathbf{p}\cdot\mathbf{v}$ (additional term does not destroy periodicity)
  - $\mathbf{v}_S = 0, \; \mathbf{v}_N = \mathbf{v} \qquad \Rightarrow \qquad \bar{\mathbf{j}} = \rho_N \mathbf{v} = (\bar{\rho}_n - \rho_S)\mathbf{v}$

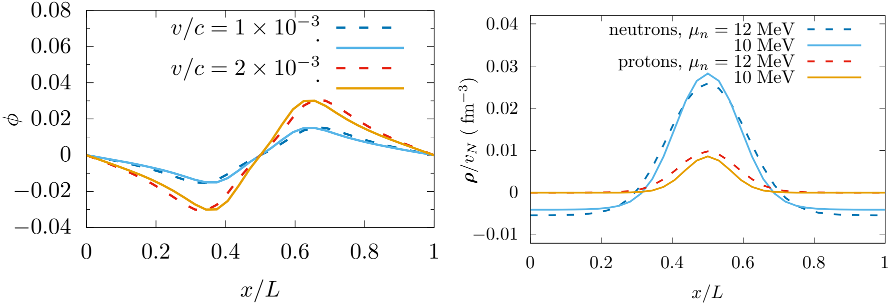
- Make sure that v is small enough to be in the linear regime (no pair breaking)

- Estimate v just before a Vela glitch ($\delta\Omega \simeq 10^{-2} - 10^{-1}\,\mathrm{s}^{-1}$ [Ruderman, ApJ 203 (1976)]):

$$v = R_{\mathrm{NS}}\delta\Omega \simeq \frac{R_{\mathrm{NS}}}{12\,\mathrm{km}} \times 4 \times (10^{-7} - 10^{-6})c$$

$$\ll v_{\mathrm{Landau}} \simeq \frac{\Delta}{\hbar k_F} \simeq \frac{\Delta}{1\,\mathrm{MeV}} \times \frac{1.3\,\mathrm{fm}^{-1}}{k_F} \times 4 \times 10^{-3} c$$

Phase of the gap and current: 1D example (lasagna)

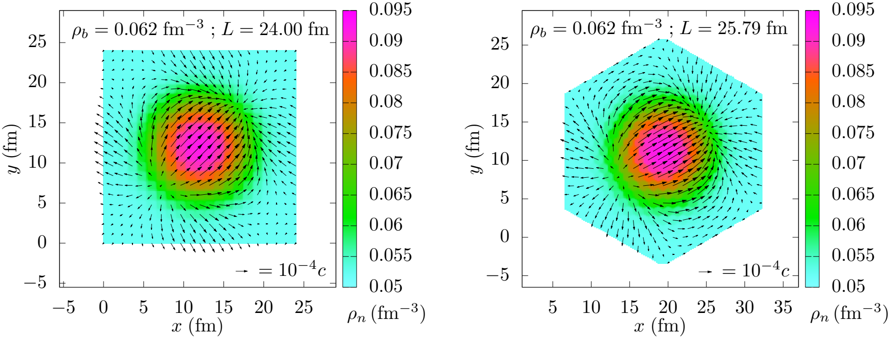


[Almirante & MU, Phys. Rev. C 109 (2024)]

- phase $\phi \propto v \;\longrightarrow\;$ linear regime
- proton current $= v\times$ proton density
- neutron current shifted down by a constant (superfluid part doesn't move)

Density and current in 2D (spaghetti)

Neutron density ρ_n and velocity $\mathbf{v}_n = \mathbf{j}_n/\rho_n$ in square and hexagonal lattices



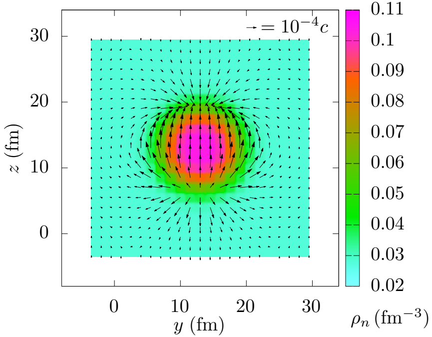
[Almirante & MU, Phys. Rev. C 110 (2024)]

$$\rho_b = 0.062\,\text{fm}^{-3}, \quad \rho_S/\bar{\rho}_n = 95\%$$

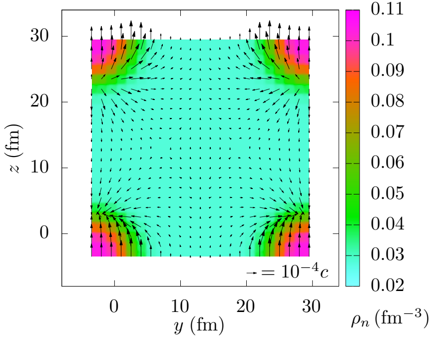
Density and current in 3D (BCC crystal)

Neutron density ρ_n and velocity $\mathbf{v}_n = \mathbf{j}_n/\rho_n$ in two cuts through the unit cell

$x = x_0$



$x = x_0 + L/2$



$$\rho_b = 0.033\,\mathrm{fm}^{-3}, \quad L = 33\,\mathrm{fm}, \quad \rho_S/\bar{\rho}_n = 92\%$$

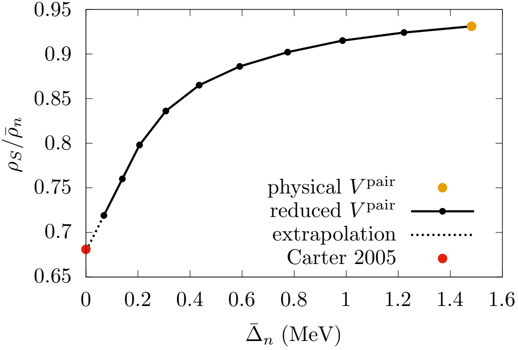
Results and comparison with normal band theory

	μ_n (MeV)	L (fm)	ρ_b (fm^{-3})	ρ_S/ρ_n (HFB %)	ρ_S/ρ_n (HF %)
crystal	9	33	0.0334	92.1	7
crystal	10	31	0.0425	92.8	9
crystal	11	29	0.0518	94.1	27
spaghetti	12	24	0.0619	94.5	75
spaghetti	12.5	24	0.0670	95.4	82
lasagna	13	20	0.0723	96.3	93
lasagna	13.5	20	0.0768	97.2	94

$\longrightarrow$ HFB superfluid fractions closer to the results of superfluid hydrodynamics than to the ones of normal band theory.

Band structure effect vs. pairing gap

- Normal band theory should be valid in the weak-coupling limit ($\Delta \to 0$)
- Superfluid hydrodynamics only valid for $\xi \ll L \;\to\; \Delta \gg \dfrac{k_F}{\pi m L}$
- HFB should be valid all the way between these two limits!
- Varying artificially the strength of the pairing interaction:



Rod phase,
$\bar{\rho}_n = 0.059$ fm^{-3}
$L = 27.17$ fm,
for same conditions as in Carter, Chamel & Haensel, NPA 748 (2005)

Linear response on top of BCS

Simpler approach than full HFB:

- linear response on top of BCS with constant gap
- need to diagonalize only h (additional approximation: $m^* = m$)
- treat $-\mathbf{p} \cdot \mathbf{v}$ term as perturbation

Notice the analogy between normal density $\rho_N = \bar{\rho}_n - \rho_S$ and moment of inertia

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SUPERFLUIDITY AND THE MOMENTS OF INERTIA OF NUCLEI

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Received 11 April 1959

$$\rho'_{\lambda\lambda'} = \int G'_{\lambda\lambda'} \frac{d\varepsilon}{2\pi i} = \frac{(\varepsilon_\lambda \varepsilon_{\lambda'} - E_\lambda E_{\lambda'}) V_{\lambda\lambda'} - \Delta^2 V^*_{\lambda\lambda'} + \Delta(\varepsilon_\lambda \Delta'_{\lambda\lambda'} + \varepsilon_{\lambda'} \Delta'^*_{\lambda\lambda'})}{2E_\lambda E_{\lambda'}(E_\lambda + E_{\lambda'})} . \quad (15)$$

Geometric contribution

- Final expression, neglecting the change of the gap (i.e., the phase ϕ):

$$\rho_S = \frac{1}{3m}\int_{BZ}\frac{d^3k}{(2\pi)^3}\sum_{\alpha\beta}\frac{2\Delta^2\,|\langle\alpha\mathbf{k}|\mathbf{p}|\beta\mathbf{k}\rangle|^2}{E_{\alpha\mathbf{k}}E_{\beta\mathbf{k}}(E_{\alpha\mathbf{k}}+E_{\beta\mathbf{k}})}$$

$\mathbf{k}$ = Bloch momentum,

α, β = band labels,

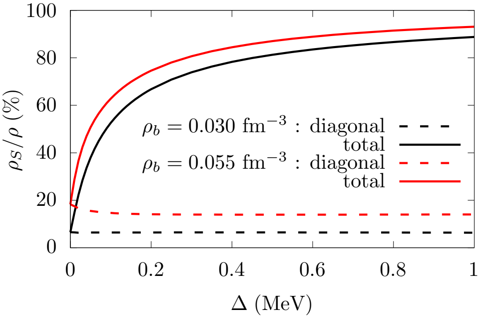
$E_{\alpha\mathbf{k}} = \sqrt{(\epsilon_{\alpha\mathbf{k}}-\mu)^2+\Delta^2}$

- Using $\langle\alpha\mathbf{k}|\mathbf{p}|\alpha\mathbf{k}\rangle = m\frac{\partial\epsilon_{\alpha\mathbf{k}}}{\partial\mathbf{k}}$, the contribution for $\alpha=\beta$ becomes:

$$\rho_S^{\text{diag.}} = \frac{m}{3}\int_{BZ}\frac{d^3k}{(2\pi)^3}\sum_{\alpha}\frac{\Delta^2}{E_{\alpha\mathbf{k}}^3}\left(\frac{\partial\epsilon_{\alpha\mathbf{k}}}{\partial\mathbf{k}}\right)^2$$

Notice: $\lim_{\Delta\to 0}\frac{\Delta^2}{E_{\alpha\mathbf{k}}^3} = 2\delta(\epsilon_{\alpha\mathbf{k}}-\mu)$

$\rightarrow$ Chamel's formula



[Almirante & MU, arXiv:2503.21635]

- The contribution $\alpha \neq \beta$ can be large and is called "geometric contribution" in condensed-matter physics
[e.g. Peotta & Törmä, Nature Comm. 6, 8944 (2015)]

- Potential problem at large Δ due to violation of continuity equation (neglected ϕ)

Superfluid hydrodynamics and Leggett's upper bound

- Assumption of superfluid hydrodynamics: $\mathbf{v}_n(\mathbf{r}) = \dfrac{\hbar}{2m}\boldsymbol{\nabla}\phi(\mathbf{r})$
- In the rest frame of the superfluid:
  - Continuity: $\boldsymbol{\nabla}\cdot\mathbf{j}_n(\mathbf{r}) = -\dot{\rho}_n(\mathbf{r}) \quad \Rightarrow \quad \boldsymbol{\nabla}\cdot\rho_n(\mathbf{r})\boldsymbol{\nabla}\phi(\mathbf{r}) = \mathbf{v}\cdot\boldsymbol{\nabla}\rho_n(\mathbf{r})$
  - Periodicity: $\phi(\mathbf{r}+\mathbf{a}_i) = \phi(\mathbf{r}) \qquad (\mathbf{a}_i = \text{primitive lattice vector})$

  $\phi(\mathbf{r})$ can be easily solved for any periodic density profile $\rho_n(\mathbf{r})$, e.g., by expanding ρ_n and ϕ in a Fourier series
- 1D case: solution can be written explicitly:
  - Continuity $(\rho_n\phi')' = v\rho_n' \quad \Rightarrow \quad \phi'(x) = v + \dfrac{C}{\rho_n(x)}$
  - Periodicity $\phi(L) = \phi(0) \quad \Rightarrow \quad C^{-1} = -\dfrac{1}{vL}\displaystyle\int_0^L \dfrac{dx}{\rho_n(x)} \equiv -v^{-1}\,\overline{\rho_n^{-1}}$
  - $\rho_S = -\dfrac{C}{v} = \left(\overline{\rho_n^{-1}}\right)^{-1}$: Leggett's upper bound [J. Stat. Phys. 93, 927 (1998)]
- Conjecture: also in 2D and 3D, superfluid hydrodynamics gives an upper limit for ρ_S

Conclusions

- Superfluid fraction important for glitches
  (also for cooling and star oscillations)

- HFB band theory interpolates between normal band theory (for $\Delta \to 0$) and superfluid hydrodynamics (for Δ large)

- $\rho_S/\bar{\rho}_n$ depends strongly on the gap if the gap is small, but reaches rapidly values of $\sim 90\%$ for realistic values of the gap

- Superfluid fraction of the crust is high enough to explain glitches without need for superfluidity in the core

- The strong gap dependence can be understood from the so-called "geometric contribution" in the linear response expression for ρ_S

- In inhomogeneous systems, the continuity equation imposes an upper limit on $\rho_S/\bar{\rho}_n$