Quantum mean estimation (QME)

Applications

Quantum advantage

Conclusion

Quantum mean estimation for lattice field theory arXiv:2303.00094

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2 Quantum mean estimation (QME)

3 Applications

- **4** Quantum advantage
- **5** Conclusion

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3 Applications

- 4 Quantum advantage
- **5** Conclusion

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QME for LFT	3 / 34

Application:

Quantum advantage

Conclusion

Traditional sampling



- Lattice field theory
 → Statistical mechanics
- Partition function

$$Z = \sum_{i} e^{-S(g_i)}$$

Probability

$$p_i = \frac{e^{-S(g_i)}}{Z}$$

• Probability \implies sampling

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Applications

Quantum advantage

Conclusion

Traditional sampling



Observables



Error

 $\sigma_{\langle \mathcal{O} \rangle} \sim \frac{1}{\sqrt{\textit{N}}}$

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Applications

Quantum advantage

Conclusion

Can we do better?

 Quantum Amplitude Amplification and Estimation: [Brassard et al., 2002]

$$\sigma_{\langle \mathcal{O} \rangle} \sim rac{1}{N}$$

• Fleshed out:

[Montanaro, 2015, Shyamsundar, 2021, Hamoudi, 2021, Kothari and O'Donnell, 2022]



2 Quantum mean estimation (QME)

3 Applications

4 Quantum advantage

5 Conclusion

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We use QME from [Shyamsundar, 2021], $\langle \mathcal{O}
angle \in [-1,1]$

1. Prepare a probability distribution state

$$\mathcal{A} \ket{0} = \ket{\psi_0} = \sum_i \sqrt{\frac{e^{-S(g_i)}}{Z}} \ket{g_i}$$

- 2. Encode $\sim \mathcal{O}_i$ into phases $\rightarrow \mathcal{Q}$
- 3. Quantum phase estimation (QPE)
- 4. Output is $\sim |\langle \mathcal{O} \rangle
 angle$

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Quantum mean estimation (QME)

Applications

Quantum advantage

Conclusion

What does QME calculate?

$$egin{aligned} U_arphi \equiv \sum_i e^{iarphi(g_i)} \ket{g_i}ig\langle g_i \ & & \ \Re[ig\langle\psi_0|\;U_arphi\,|\psi_0ig
angle] & \ ext{and} \quad \Im[ig\langle\psi_0|\;U_arphi\,|\psi_0ig
angle] \end{aligned}$$

$$egin{aligned} &\Re[\langle\psi_0| \ U_arphi \ |\psi_0
angle] = \Re[\langle\psi_0| \sum_i e^{iarphi(g_i)} \ |g_i
angle \ \langle g_i|\psi_0
angle] \ &= \sum_i | \ \langle\psi_0|g_i
angle \ |^2 \Re[e^{iarphi(g_i)}] \ &= \sum_i | \ \langle\psi_0|g_i
angle \ |^2 \cos(arphi(g_i)), \end{aligned}$$

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Applications

Quantum advantage

Conclusion

How does QME calculate $\langle \mathcal{O} \rangle$?

$$\Re [ig \langle \psi_0 | \; U_arphi \; | \psi_0
angle] \ = \sum_i |ig \langle \psi_0 | g_i
angle \; |^2 \cos(arphi(g_i))$$

$$\Re[\langle \psi_{0} | U_{\varphi} | \psi_{0} \rangle] \sim \langle \mathcal{O} \rangle$$

$$|\langle \psi_0 | g_i \rangle|^2 \sim p_i$$

$$\langle \mathcal{O} \rangle = \sum_{i} p_{i} \mathcal{O}_{i}$$

$$\cos(\varphi(g_i)) \sim \mathcal{O}_i$$

How does QPE enter?

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Application

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Conclusion

Quantum Fourier transform

$$|x
angle = \mathsf{QFT}^{-1}\left[\mathsf{QFT}\left[|x
angle
ight]
ight]$$

$$|x
angle = \mathsf{QFT}^{-1}\left[rac{1}{\sqrt{2}}\left(|0
angle + e^{i2\pi \mathbf{0}.x_1} \left|1
angle
ight)\cdotsrac{1}{\sqrt{2}}\left(|0
angle + e^{i2\pi \mathbf{0}.x_1x_2\cdots x_n} \left|1
angle
ight)
ight]$$

A way to put x into $|x\rangle$.

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Background Qua

Quantum mean estimation (QME)

Applications

Quantum advantage

Conclusion

Quantum phase estimation

Given $\mathcal Q$ and $|q\rangle$, QPE returns $\chi(q)$ in the form $|\chi(q)\rangle$ such that

$$\mathcal{Q}\ket{q}=e^{i\chi(q)}\ket{q}$$
 .

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Backgroui	nd

Quantum advantage

Conclusion

Quantum phase estimation



 $\left|\chi\right\rangle = \mathsf{QFT}^{-1}\left[\frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{i0.\chi_{1}}\left|1\right\rangle\right) \cdots \frac{1}{\sqrt{2}}\left(\left|0\right\rangle + e^{i0.\chi_{1}\chi_{2}\cdots\chi_{r}}\left|1\right\rangle\right)\right]\left|q\right\rangle$

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Using QPE

- We want $\chi(q) \sim \langle \mathcal{O}
 angle$
- Find a Q and $|q\rangle$ that do this 1. $|q\rangle \rightarrow |\psi_0\rangle = \mathcal{A} |0\rangle$ 2.

$$\mathcal{Q} \to \mathcal{S}_{\mathcal{A}} \mathbb{U}_{\varphi} = \mathcal{A} \underbrace{\mathcal{S}_{0}}_{0\text{-reflection}} \mathcal{A}^{\dagger} \underbrace{\mathbb{U}_{\varphi}}_{\text{phase oracle}}$$



- $|\psi_0
 angle$ is not an eigenvector of ${\cal Q}$
- a simple linear combination

$$\ket{\psi_0} = rac{1}{\sqrt{2}} \left(\ket{q^+} - \ket{q^-}
ight)$$

Nice eigenvalues

$$\mathcal{Q}\ket{q^{\pm}}=\mathrm{e}^{\pm i heta}\ket{q^{\pm}}$$

•
$$\theta \sim \langle \mathcal{O} \rangle$$



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Quantum advantage

Conclusion

Phase oracle

$$\mathcal{Q} = \mathcal{A}S_0\mathcal{A}^\dagger \underbrace{\mathbb{U}_{arphi}}_{\text{phase oracle}}$$

• \mathbb{U}_{φ} applies phases

•
$$\varphi(g_i) = \arccos \mathcal{O}_i$$

$$\Re[\langle \psi_0 | U_{arphi} | \psi_0
angle] = \sum_i |\langle \psi_0 | g_i
angle |^2 \cos(arphi(g_i))$$

•
$$\mathbb{U}_{arphi} \ket{g_i} \sim e^{i \arccos \mathcal{O}_i} \ket{g_i}$$



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Quantum advantage

Conclusion

QPE for QME

$$\Re[\langle \psi_0 | U_{arphi} | \psi_0
angle] = \sum_i | \langle \psi_0 | g_i
angle |^2 \cos(arphi(g_i))$$
 $\implies \cos(heta)$

• QPE with
$$|\psi_0\rangle$$
 and Q returns θ .
 $\rightarrow \cos(\theta) = \langle \mathcal{O} \rangle$

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1 Background

2 Quantum mean estimation (QME)

3 Applications

4 Quantum advantage

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Applications

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Conclusion

Computing π



 $|x\rangle$

$$\frac{\pi}{4} = \frac{1}{Z} \iint_{0}^{1} dx \, dy \, \Theta(1 - r^{2}(x, y)) \approx \frac{1}{Z} \sum_{x, y} \, \Theta(1 - r^{2})$$

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Conclusion

Computing π



Estimates of $\pi/4 \approx 0.785$

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Applications

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Computing π with a sign problem



$$\frac{\pi}{12} = \frac{1}{Z} \iint_0^1 dx \, dy \, \Theta(1-r^2) \left[2\Theta\left(\frac{x^2}{r^2} - \frac{1}{4}\right) - 1 \right].$$

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Computing π with a sign problem



Estimates of $\pi/12 \approx 0.262$

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Conclusion

Ising model

$$S = rac{eta}{2} \sum_{\langle ij
angle} (1 - s_i s_j) = eta \left(4 \sum_i n_i - 2 \sum_{\langle ij
angle} n_i n_j
ight)$$

 $s_i = \pm 1$, $s_i = 2n_i - 1$, $n_i = 0, 1$

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Ising model



 $\langle m^2
angle$ in 1D Ising model

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2 Quantum mean estimation (QME)

3 Applications

4 Quantum advantage

5 Conclusion

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Applications

Quantum advantage

Conclusion

Traditional sampling

$$\mathcal{A} \ket{0} = \ket{\psi_0} = \sum_i \sqrt{rac{e^{-S(g_i)}}{Z}} \ket{g_i}$$

Why not just sample?

- 1. Run \mathcal{A}
- 2. Measure $|g\rangle$ registers
- 3. Save the data
- 4. Repeat

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- Saved $|g\rangle$ s are uncorrelated
- Take $N = 2^k$ measurements
- When calculating $\langle \mathcal{O}
 angle$

$$\sigma_{\langle \mathcal{O} \rangle} \sim \frac{1}{\sqrt{N}} = \frac{1}{2^{k/2}}$$

For a fixed $\sigma_{\langle O \rangle}$, in QPE, we need k/2 qubits.



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Fixed precision



For a fixed $\sigma_{\langle \mathcal{O} \rangle}$ in QPE, we need k/2 qubits.

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Counting calls



- \mathcal{Q} gets called $\sum_{i=0}^{k/2-1} 2^i = 2^{k/2} 1$ times
- $\mathcal A$ the same

Backg	round

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Conclusion

Counting calls

Just sampling:

$$N = 2^k, \quad \sigma_{\langle \mathcal{O} \rangle} \sim 1/2^{k/2}$$

QME:

$$N=2^{k/2},~\sigma_{\langle {\cal O}
angle}\sim 1/2^{k/2}$$

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2 Quantum mean estimation (QME)

3 Applications

4 Quantum advantage



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QME for LFT	31 / 34

Conclusions

- Quantum circuit uses quadratically fewer resources asymptotically.
- Same precision.
- Don't have to use vanilla QPE.
- Can use Metropolis or heat bath for \mathcal{A} .
- Insensitive to sign problem.

Thank you!

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