Quantum mean estimation for lattice field theory

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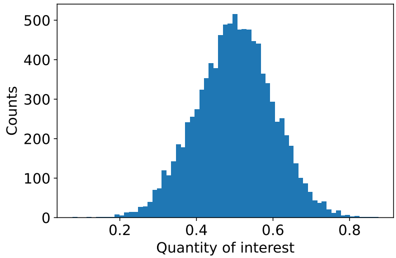
Traditional sampling



- Lattice field theory
  $\rightarrow$ Statistical mechanics
- Partition function

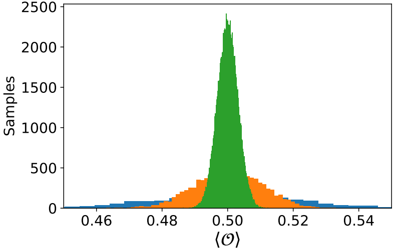
$$Z = \sum_i e^{-S(g_i)}$$

- Probability

$$p_i = \frac{e^{-S(g_i)}}{Z}$$

- Probability $\implies$ sampling

Traditional sampling



- Observables

$$\langle \mathcal{O} \rangle = \sum_i p_i \mathcal{O}_i$$
$$= \sum_i \frac{e^{-S(g_i)}}{Z} \mathcal{O}_i$$

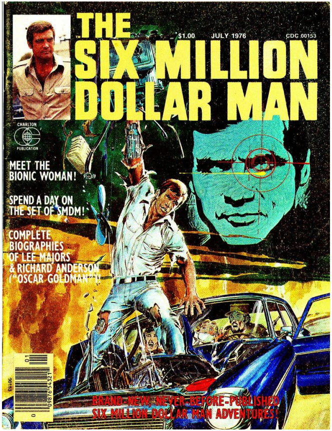
- Error

$$\sigma_{\langle \mathcal{O} \rangle} \sim \frac{1}{\sqrt{N}}$$

Can we do better?

- Quantum Amplitude Amplification and Estimation: [Brassard et al., 2002]
-
$$\sigma_{\langle \mathcal{O} \rangle} \sim \frac{1}{N}$$
- Fleshed out:

  [Montanaro, 2015,
  Shyamsundar, 2021,
  Hamoudi, 2021,
  Kothari and O'Donnell, 2022]



1. Background
2. Quantum mean estimation (QME)
3. Applications
4. Quantum advantage
5. Conclusion

Overview

We use QME from [Shyamsundar, 2021], $\langle \mathcal{O} \rangle \in [-1, 1]$

1. Prepare a probability distribution state

$$\mathcal{A} |0\rangle = |\psi_0\rangle = \sum_i \sqrt{\frac{e^{-S(g_i)}}{Z}} |g_i\rangle$$

2. Encode $\sim \mathcal{O}_i$ into phases $\to \mathcal{Q}$
3. Quantum phase estimation (QPE)
4. Output is $\sim |\langle \mathcal{O} \rangle|$

What does QME calculate?

$$U_\varphi \equiv \sum_i e^{i\varphi(g_i)} \left|g_i\right\rangle \left\langle g_i\right|$$

$$\Re[\langle\psi_0| U_\varphi |\psi_0\rangle] \quad \text{and} \quad \Im[\langle\psi_0| U_\varphi |\psi_0\rangle]$$

$$\begin{aligned}
\Re[\langle\psi_0| U_\varphi |\psi_0\rangle] &= \Re[\langle\psi_0| \sum_i e^{i\varphi(g_i)} \left|g_i\right\rangle \langle g_i|\psi_0\rangle] \\
&= \sum_i |\langle\psi_0|g_i\rangle|^2 \Re[e^{i\varphi(g_i)}] \\
&= \sum_i |\langle\psi_0|g_i\rangle|^2 \cos(\varphi(g_i)),
\end{aligned}$$

How does QME calculate $\langle \mathcal{O} \rangle$?

$$\Re[\langle \psi_0 | U_\varphi | \psi_0 \rangle] = \sum_i |\langle \psi_0 | g_i \rangle|^2 \cos(\varphi(g_i))$$

$$\Re[\langle \psi_0 | U_\varphi | \psi_0 \rangle] \sim \langle \mathcal{O} \rangle$$

$$|\langle \psi_0 | g_i \rangle|^2 \sim p_i$$

$$\langle \mathcal{O} \rangle = \sum_i p_i \mathcal{O}_i$$

$$\cos(\varphi(g_i)) \sim \mathcal{O}_i$$

How does QPE enter?

Quantum Fourier transform

$$|x\rangle = \mathrm{QFT}^{-1}\left[\mathrm{QFT}\left[|x\rangle\right]\right]$$

$$|x\rangle = \mathrm{QFT}^{-1}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi 0.x_1}|1\rangle\right)\cdots\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i2\pi 0.x_1 x_2 \cdots x_n}|1\rangle\right)\right]$$

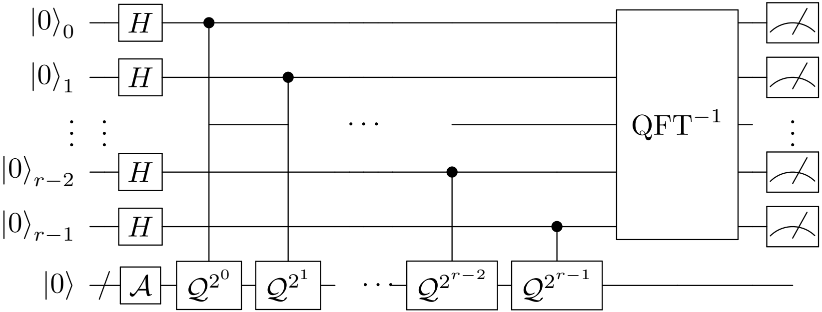
A way to put x into $|x\rangle$.

Quantum phase estimation

Given $\mathcal{Q}$ and $|q\rangle$, QPE returns $\chi(q)$ in the form $|\chi(q)\rangle$ such that

$$\mathcal{Q}\,|q\rangle = e^{i\chi(q)}\,|q\rangle\,.$$

Quantum phase estimation



$$|\chi\rangle = \mathrm{QFT}^{-1}\left[\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i0.\chi_1}|1\rangle\right)\cdots\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i0.\chi_1\chi_2\cdots\chi_r}|1\rangle\right)\right]|q\rangle$$

Using QPE

- We want $\chi(q) \sim \langle \mathcal{O} \rangle$
- Find a $\mathcal{Q}$ and $|q\rangle$ that do this
  1. $$|q\rangle \to |\psi_0\rangle = \mathcal{A}|0\rangle$$
  2. $$\mathcal{Q} \to S_{\mathcal{A}} \mathbb{U}_{\varphi} = \mathcal{A} \underbrace{S_0}_{\text{0-reflection}} \mathcal{A}^{\dagger} \underbrace{\mathbb{U}_{\varphi}}_{\text{phase oracle}}$$

Using QPE

- $|\psi_0\rangle$ is not an eigenvector of $\mathcal{Q}$
- a simple linear combination

$$|\psi_0\rangle = \frac{1}{\sqrt{2}}\left(|q^+\rangle - |q^-\rangle\right)$$

- Nice eigenvalues

$$\mathcal{Q}\,|q^\pm\rangle = e^{\pm i\theta}\,|q^\pm\rangle$$

- $\theta \sim \langle \mathcal{O} \rangle$

Phase oracle

$$\mathcal{Q} = \mathcal{A} S_0 \mathcal{A}^\dagger \underbrace{\mathbb{U}_\varphi}_{\text{phase oracle}}$$

- $\mathbb{U}_\varphi$ applies phases
- $\varphi(g_i) = \arccos \mathcal{O}_i$
- $\mathbb{U}_\varphi |g_i\rangle \sim e^{i \arccos \mathcal{O}_i} |g_i\rangle$

$$\Re[\langle\psi_0| U_\varphi |\psi_0\rangle] = \sum_i |\langle\psi_0|g_i\rangle|^2 \cos(\varphi(g_i))$$

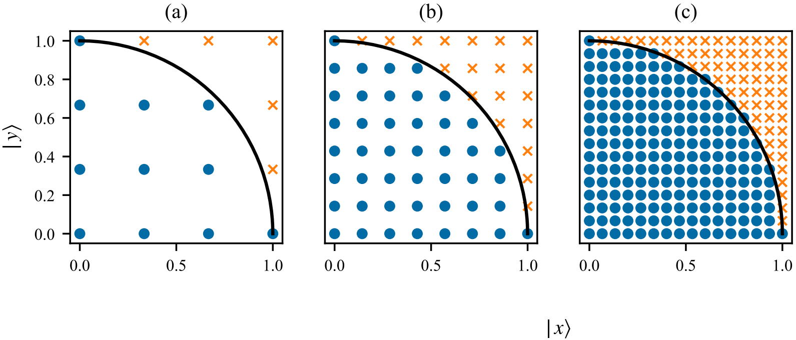
QPE for QME

$$\Re[\langle\psi_0| U_\varphi |\psi_0\rangle] = \sum_i |\langle\psi_0|g_i\rangle|^2 \cos(\varphi(g_i))$$
$$\implies \cos(\theta)$$

- QPE with $|\psi_0\rangle$ and $\mathcal{Q}$ returns θ.

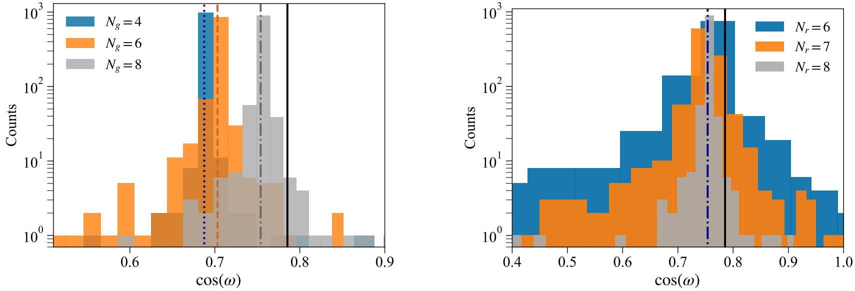
→ $\cos(\theta) = \langle\mathcal{O}\rangle$

Computing π



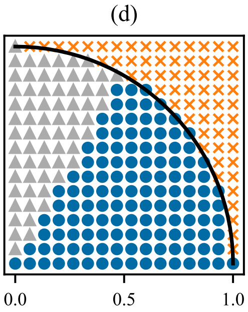
$$\frac{\pi}{4} = \frac{1}{Z}\iint_0^1 dx\, dy\, \Theta(1 - r^2(x,y)) \approx \frac{1}{Z}\sum_{x,y} \Theta(1 - r^2)$$

Computing π



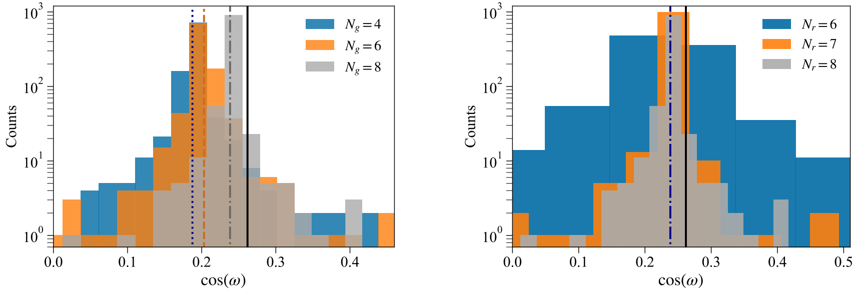
Estimates of $\pi/4 \approx 0.785$

Computing π with a sign problem



$$\frac{\pi}{12} = \frac{1}{Z} \iint_0^1 dx\, dy\, \Theta(1 - r^2) \left[2\Theta\left(\frac{x^2}{r^2} - \frac{1}{4} \right) - 1 \right].$$

Computing π with a sign problem



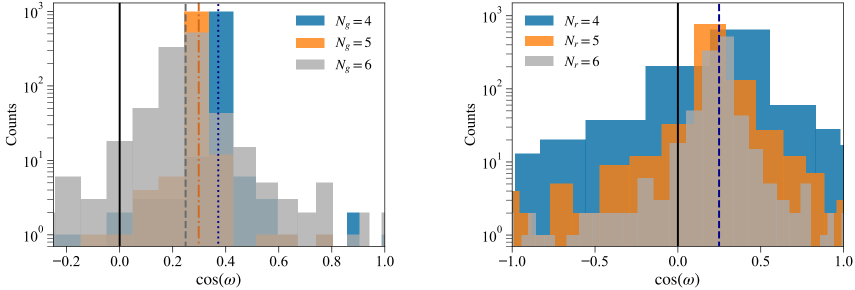
Estimates of $\pi/12 \approx 0.262$

Ising model

$$S = \frac{\beta}{2} \sum_{\langle ij \rangle} (1 - s_i s_j) = \beta \left(4 \sum_i n_i - 2 \sum_{\langle ij \rangle} n_i n_j \right)$$

$s_i = \pm 1,\ s_i = 2n_i - 1,\ n_i = 0, 1$

Ising model



$\langle m^2 \rangle$ in 1D Ising model

Traditional sampling

$$\mathcal{A}|0\rangle = |\psi_0\rangle = \sum_i \sqrt{\frac{e^{-S(g_i)}}{Z}}\,|g_i\rangle$$

Why not just sample?

1. Run $\mathcal{A}$
2. Measure $|g\rangle$ registers
3. Save the data
4. Repeat

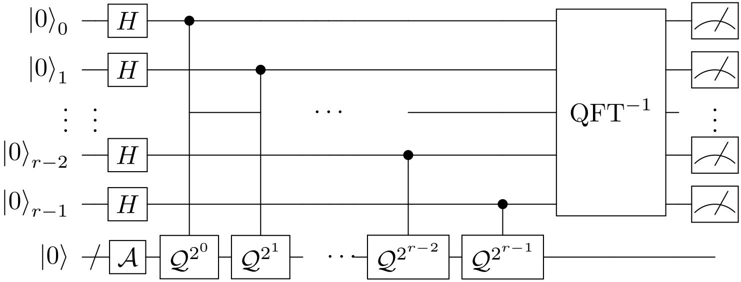
Fixed precision

- Saved $|g\rangle$s are uncorrelated
- Take $N = 2^k$ measurements
- When calculating $\langle \mathcal{O} \rangle$

$$\sigma_{\langle \mathcal{O} \rangle} \sim \frac{1}{\sqrt{N}} = \frac{1}{2^{k/2}}$$

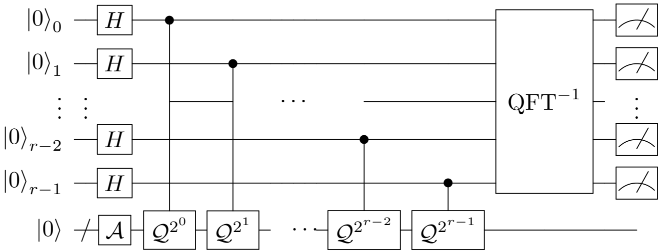
For a fixed $\sigma_{\langle \mathcal{O} \rangle}$, in QPE, we need $k/2$ qubits.

Fixed precision



For a fixed $\sigma_{\langle \mathcal{O} \rangle}$ in QPE, we need $k/2$ qubits.

Counting calls



- $\mathcal{Q}$ gets called $\sum_{i=0}^{k/2-1} 2^i = 2^{k/2} - 1$ times
- $\mathcal{A}$ the same

Counting calls

Just sampling:

$$N = 2^k, \quad \sigma_{\langle \mathcal{O} \rangle} \sim 1/2^{k/2}$$

QME:

$$N = 2^{k/2}, \quad \sigma_{\langle \mathcal{O} \rangle} \sim 1/2^{k/2}$$

Conclusions

- Quantum circuit uses quadratically fewer resources asymptotically.
- Same precision.
- Don't have to use vanilla QPE.
- Can use Metropolis or heat bath for $\mathcal{A}$.
- Insensitive to sign problem.

Thank you!

References I

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Mean estimation when you have the source code; or, quantum Monte Carlo methods.

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