SPTO in Open Quantum Systems

Caroline de Groot [1], Norbert Schuch [2], Alex Turzillo [1,3].

[1]Max Planck Institute of Quantum Optics
[2]University of Vienna
[3]Perimeter Institute for Theoretical Physics

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Topological phases in open systems

What are the topological phases of open systems? Of mixed states?

In particular, which phases of closed systems (pure states) are robust against coupling to an environment?

Defining phase equivalence

- In closed systems...

paths in the space of gapped Hamiltonians... or equivalently...

fast, local Hamiltonian evolution $\psi \xrightarrow{H} \psi'$

  - ... approximated by low depth circuits of local gates (e.g. polylog(L) depth of constant size gates)

- In open systems... Coser & Pérez-García [1810.05092] propose using

fast, local Lindbladian evolutions $\rho \xrightarrow{\mathcal{L}_1} \rho'$ and $\rho' \xrightarrow{\mathcal{L}_2} \rho$

  - motivation: local observables are analytic within phases.

Defining phase equivalence for SPT phases

- In closed systems...

  paths in the space of symmetric gapped Hamiltonians

  fast, local, symmetric Hamiltonian evolution

  - ... approximated by low depth circuits of local symmetric gates

- In open systems...?

  Not obvious what symmetry condition to impose on Lindbladian evolutions.

  Two natural guesses...

  - weak symmetry: $\mathcal{U}_g \circ \mathcal{L} \circ \mathcal{U}_g^\dagger = \mathcal{L}$.
  - strong symmetry: $\mathcal{L}^\dagger(U_g) = 0$.

  ...referred to as *average and exact symmetry* in Ma & Wang [2209.02723].

Dynamical and static definitions of phases

- Dynamical definition = which states can be evolved into which states
- Static definition = which states have the same values of order parameters

- For SPT orders in 1D... we consider string order parameters.

  ... more on these later ...

- We ask which symmetry condition on Lindbladian evolution makes it such that

  dynamical definition = static definition

Proposed SPT phase equivalence

Main claim

SPT order, as defined by string order parameters,
is preserved by a fast, local Lindbladian evolution $\mathcal{L}$

if and only if $\mathcal{L}$ satisfies the strong symmetry condition $\mathcal{L}^\dagger(U_g) = 0$.

Comment: "Average SPT order" is a slightly different notion. More on this later.

Interpretations

1. Simply a rule for determining whether or not string order parameters are robust against certain couplings to the environment.

2. A "good" definition of SPT phase equivalence in open systems.

   - Two mixed states belong to the same SPT phase if they are related by... fast, local, strongly symmetric Lindbladian evolutions $\rho \xrightarrow{\mathcal{L}_1} \rho'$ and $\rho' \xrightarrow{\mathcal{L}_2} \rho$.

Weak and Strong Symmetry Conditions

Weakly symmetric noise destroys 1D SPT order

- C&PG constructs a fast, local Lindbladian evolution that maps

$$|\psi\rangle \quad \longrightarrow \quad |\text{product}\rangle \ ,$$

for any 1D state $|\psi\rangle$.

This evolution destroys SPT order.

- Solution...? Maybe this Lindbladian is not symmetric.
- However it satisfies

$$\mathcal{U}_g \circ \mathcal{L} \circ \mathcal{U}_g^\dagger = \mathcal{L} \ . \qquad \text{(weak symmetry)}$$

- Is there a stronger symmetry condition on $\mathcal{L}$ that protects SPT order?

Channels

- Work with channels $\mathcal{E}$. Have in mind Lindbladian evolutions $\mathcal{E}_t = e^{t\mathcal{L}}$.
- $\mathcal{E}$ is a CPTP map.
- Kraus representation in terms of Kraus operators K_i:

$$\mathcal{E}(\rho) = \sum_i K_i \rho K_i^\dagger \ , \qquad \sum_i K_i^\dagger K_i = \mathbb{1} \ .$$

- Example: a reversible channel has a single unitary Kraus operator:

$$\rho \mapsto W \rho W^\dagger \ .$$

- Two Kraus representations of $\mathcal{E}$ are unitarily related.

Weak symmetry allows interference between trajectories

- $\mathcal{L}$ is weakly symmetric precisely when $\mathcal{E}_t = e^{t\mathcal{L}}$ satisfies, at all times t,

$$\mathcal{U}_g \circ \mathcal{E}_t \circ \mathcal{U}_g^\dagger = \mathcal{E}_t \ . \qquad \text{(weak symmetry)}$$

- Weak symmetry (WS) in terms of Kraus operators:

$$\sum (U_g K_i U_g^\dagger)\, \rho\, (U_g K_i U_g^\dagger)^\dagger = \sum K_i \rho K_i^\dagger \ , \quad \forall g \ .$$

$\Rightarrow$ For each g, there is a basis of Kraus operators K_i^g where

$$U_g K_i^g U_g^\dagger = e^{i\theta_i(g)} K_i^g \ , \quad \forall i \ .$$

- Different phases $\theta_i \neq \theta_j$ give rise to interference between trajectories.

Strong symmetry

- Strong symmetry (SS): trajectories transform with *the same* phase:

$$U_g K_i U_g^\dagger = e^{i\theta(g)} K_i \ , \quad \forall\, i, g \ .$$

- This condition is basis-independent, so the superscript g can be dropped.
- Neglect the weak invariant $\theta(g)$. It vanishes for $\mathcal{E} = e^{t\mathcal{L}}$.

- Comment: WS and SS are the same for reversible channels $\rho \mapsto W\rho W^\dagger$.

Strong symmetry of Lindbladians

- $\mathcal{L}$ may be expressed in terms of jump operators L_i:

$$\mathcal{L}(\rho) = -i[H, \rho] + \sum_i \left(L_i \rho L_i^\dagger - \frac{1}{2} L_i^\dagger L_i \rho - \frac{1}{2} \rho L_i^\dagger L_i \right) .$$

- Using the relation between the K_i and the jump operators L_i, we see that...

...families $\mathcal{E}_t = e^{t\mathcal{L}}$ of SS channels are generated by SS Lindbladians:

$$U_g H = H U_g \ , \qquad U_g L_i = L_i U_g \ , \qquad \forall\, i, g \ .$$

- The SS condition on $\mathcal{L}$ has appeared previously. [Buča , Prosen 12] [Albert , Jiang 13]

Strong symmetry in the Heisenberg picture

- The Heisenberg picture is convenient for studying the evolution of observables:

$$\mathrm{Tr}[\mathcal{E}(\rho)\,\mathcal{O}] = \mathrm{Tr}\left[\rho\,\mathcal{E}^\dagger(\mathcal{O})\right] \, .$$

- Strong symmetry of $\mathcal{E}$ means that $\mathcal{E}^\dagger$ fixes the symmetry operators:

$$U_g K_i U_g^\dagger = K_i \ , \quad \forall\, i, g \qquad \Longleftrightarrow \qquad \mathcal{E}^\dagger(U_g) = U_g \ , \quad \forall\, g \ .$$

- ... in other words, charge is conserved in the system alone.

- In terms of the Lindbladian, strong symmetry means...

$$\mathcal{L}^\dagger(U_g) = 0 \ , \quad \forall\, g \ .$$

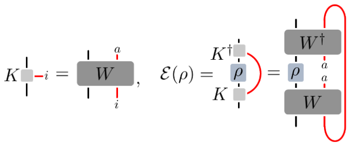
- The SPT-destroying Lindbladian of C&PG fails this condition.

Strong symmetry in terms of purifications

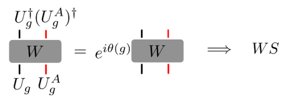
- Purification: unitary W on $\mathcal{H} \otimes A$ such that

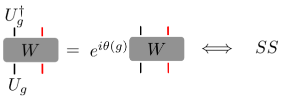
$$K_i = \langle e_i | W | a \rangle \ , \quad \text{i.e.}$$

$$\mathcal{E}(\rho) = \mathrm{Tr}_A \left(W(\rho \otimes |a\rangle\langle a|) W^\dagger \right)$$



- Claim: If $\exists$ a W symmetric with respect to some $U_g \otimes U_g^A$, the channel $\mathcal{E}$ is WS.





- Claim: The channel is SS if and only if there exists a W with $U_g^A = \mathbb{1}$.
- Like average and exact symmetries, but defined on channels rather than states.

- Strong symmetry means the system and bath couple by symmetric terms:

$$W = e^{-itH/\hbar} \ , \quad H = \sum_i H_i^S \otimes H_i^E \quad \stackrel{\text{SS}}{\Longrightarrow} \quad U_g H_i^S = H_i^S U_g \ , \forall \, i, g \ .$$

Review: String Order for SPT Phases

Symmetry fractionalization in one dimension

- SPT phases are characterized by patterns of symmetry fractionalization on the edge.

- Boundary action V_g may be projective

$$V_g V_h = \omega(g, h) V_{gh} \ , \qquad \omega : G \times G \to U(1)$$

[Chen, Gu, Wen 10] [Schuch, Perez-Garcia, Cirac 11] [Else, Nayak 14]

[Pollmann, Turner, Berg, Oshikawa 10]

- ω satisfies a cocycle condition and is defined up to coboundaries
$\Longrightarrow$ patterns of symmetry fractionalization are classified by group cohomology

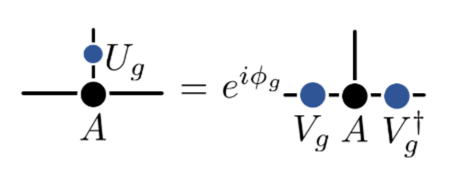
$$[\omega] \in H^2(G, U(1))$$

Fractionalization in MPS

- A symmetric state satisfies $U_g^N|\psi\rangle = |\psi\rangle$. Represent each side as an MPS.
- Since the tensors $(U_g)^{ij}A_j$ and A^i define the same state...

  the Fundamental Theorem of MPS say they related by a gauge trans.:

$$U_g^{ij}A_j = e^{i\phi_g}V_g A_i V_g^\dagger \ .$$

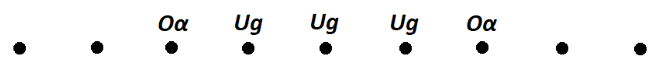


String operators

- For simplicity... assume the symmetry G is a finite abelian group.

- String operators: long strings of symmetry operators capped by end operators:

  [den Nijs, Rommelse 89] [Perez-Garcia, Wolf, Sans, Verstraete, Cirac 08] [Pollmann, Turner 12]



- End operators O^l_α and O^r_α are charged under G according to the irreps α and α^*

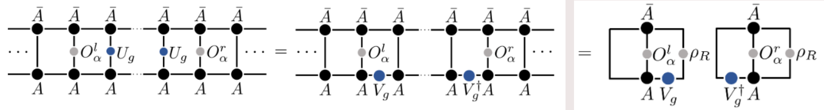
- Expectation values on states display a selection rule, aka "pattern of zeros"

$$\langle s(g, O_\alpha)\rangle = 0 \quad \text{unless} \quad \chi_\alpha(h) = \frac{\omega(h,g)}{\omega(g,h)} \text{ for all } h\ .$$

- The pattern determines the ratios ω/ω and therefore the class $[\omega]$.

Patterns of zeros of MPS states

- Evaluate $\langle\psi|s(g,\alpha)|\psi\rangle$ by applying relations (injectivity) of the MPS tensor:

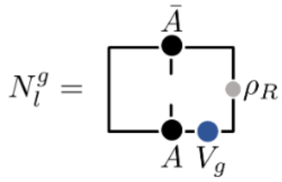


- Result is two end terms $\mathrm{Tr}\left[N_g^\dagger O_\alpha\right]$.

- O_α transforms as $\alpha(h)$.
- N_g transforms as $\omega(h,g)/\omega(g,h)$, i.e. the charge of V_g under the virtual symmetry V_h.
- String order vanishes unless these charges are equal:

$$\alpha = \omega/\omega$$

- If they are equal, it is *generically* nonvanishing.



Reconstructing the SPT invariant

- Represent the pattern as an array with columns g, rows α.
- For each g, there is a unique α with $\langle s(g, O_\alpha)\rangle \neq 0$.

$$\langle s(g, O_\alpha)\rangle_{\text{trivial}} = \begin{pmatrix} \star & \star & \star & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad , \quad \langle s(g, O_\alpha)\rangle_{\text{Haldane}} = \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix}$$

- Knowledge of all values $\langle s(g, O_\alpha)\rangle$ determines the ratios ω/ω.

- The ratios ω/ω completely determine the cohomology class $[\omega]$.
  - Follows from Schur's lemma.

- Comment: If G is non-abelian, other nonlocal order parameters may be needed to fully reconstruct $[\omega]$. [Pollmann, Turner 12]

Strong Symmetry and String Order

Strongly symmetric noise on string operators

- The following argument generalizes to causal channels.
- For simplicity, just consider uncorrelated noise

$$\mathcal{E} = \mathcal{E}_1 \otimes \cdots \otimes \mathcal{E}_L \ .$$

$\mathcal{E}$ is strongly symmetric if and only if the $\mathcal{E}_s$ are.

- The bulk and ends of the string transform as

$$\mathcal{E}_s^\dagger(U_g) = U_g \ , \qquad U_g^\dagger \mathcal{E}_s^\dagger(O_\alpha) U_g = \chi_\alpha(g) \mathcal{E}_s^\dagger(O_\alpha)$$

so the string operator evolves into one of the same label (g, α).

- Conversely, if a channel preserves all string operator labels, it must be SS.

- Next... what about the expectation values of the evolved strings?

Strong symmetry of $\mathcal{L}$ is necessary and sufficient

Fix any string order pattern of zeros.

A Lindbladian evolution preserves the pattern at all finite times if and only if the Lindbladian is strongly symmetric.

'If' direction

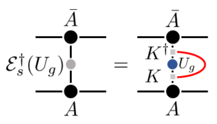
- Does SS protect the pattern of zeros in the expectation values $\langle s(g, O_\alpha) \rangle$?
- Yes. String operators $s(g, O_\alpha)$ and $\mathcal{E}^\dagger(s(g, O_\alpha))$ have the same labels, so they have the same pattern *for generic choices of end operators*...

  ...i.e. if O_α is not annihilated and is orthogonal to neither N_g nor $\mathcal{E}(N_g)$.
- Note: at $t \to \infty$, $\mathcal{E}_t$ may annihilate some end operators and spoil the pattern.

Transfer matrix argument

- Conversely... suppose string order is preserved and show strong symmetry.

- String order vanishes unless the following transfer matrix has $\lambda_{\text{max}} = 1$.



- $\lambda_{\text{max}} = 1$ implies the insertion is a symmetry. [Bridgeman, Chubb 17]

$$\implies \quad \mathcal{E}^\dagger(U_g) = U_{\sigma(g)} \ ,$$

- The family $\mathcal{E}_t = e^{t\mathcal{L}}$ defines a continuous path σ_t from 1 to σ.

- For finite G, this implies that $\sigma = 1$, which is strong symmetry.

- Beyond finite G... σ is an inner automorphism, and so $\mathcal{L}$ can involve rotation by a generator of a continuous symmetry under which ω is invariant.

Interpretation and Related Results

Recent related results

- Ma & Wang [2209.02723] consider "average SPT order" (ASPTO).
  - Systems evolve with subgroup H "exact symmetry" (strong symmetry) and G/H "average symmetry" (weak symmetry).
  - Nevertheless, some SPT order is robust.

- Lee, You, Xu [2210.16323] and Zhang, Qi, Bi [2210.17485] detect ASPTO with strange correlators.

- How can non-SS evolutions preserve SPTO?

- Resolution: these channels wash out some strings but are nevertheless preserve SPTO in some other sense.

New patterns for mixed states

Example: $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ under evolution that is

- Strongly/exactly symmetric under the first $\mathbb{Z}_2$
- only weakly/averagely under the second $\mathbb{Z}_2$.

Strings with average symmetries in the bulk are washed out:

$$\text{trivial phase:} \quad \begin{pmatrix} \star & \star & \star & \star \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{Haldane phase:} \quad \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star \end{pmatrix} \longrightarrow \begin{pmatrix} \star & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

These patterns never appear in pure states.

Here, the original SPTO can be reconstructed. Not the case in general.

Future directions and open questions

- Meaning of the symmetry conditions in realistic open quantum systems.
- Classification of mixed state SPT phases, including average SPT phases.
- Robustness of SPT and SET phases in higher dimensions. (May require the development of new non-local order parameters.)
- Mixed SPT states as steady states of Lindbladians.