# Fast Scrambling in the Hyperbolic Ising Model

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Based on: Work done in colloboration with M. Assaduzaman, S. Catterall, Y. Meurice and R. Sakai.

April 4, 2023

# Motivations

- Our main goal in this talk is to develop classical and quantum simulations to probe AdS/CFT correspondence.
- Hyperbolic Ising chain is a great candidate for this since it can be easily simulated using DMRG and TEBD algorihtms
- Ising Hamiltonian matches very closely to Rydberg Hamiltonian which opens posibilities to use Rydberg Arrays for quantum simulating this model.
- Information spread in hyperbolic spaces has many applications both in physics and general information sciences.

# Hyperbolic Space & AdS/CFT

- The AdS/CFT correspondence is a very powerful tool that provides a duality between strongly coupled $d$-dimensional critical systems and weakly coupled $d+1$ dimensional gravitational theories on a negatively curved background
- The the $d$-dimensional non-gravitational conformal theory resides on the boundary of $AdS_{d+1}$ which makes this duality holographic in it's nature.

# *AdS* Space

- Euclidean $AdS_{d+1}$ with curvature radius $\ell$ is a space of constant negative curvature defined as follows

$$-X_0^2 + \vec{X} \cdot \vec{X} = -X_0 X_0 + \sum_{i=1}^{d+1} X_i X_i = -\ell^2 \tag{1}$$

This is embedded in $\mathbb{R}^{1,d+1}$ and has the same isometries as the Euclidean conformal group of $SO(1, d+1)$.

- The geodesic between any two points is given by

$$\ell^2 \cosh(\sigma(X, X`)) = X_0 X_0` - \vec{X} \cdot \vec{X} \geq 0 \tag{2}$$

Which gives a positive spacelike distance when projected on to a hyperbolic surface

# $AdS$ Space

- There are 3 conventional choice of coordinates for hyperbolic surfaces,

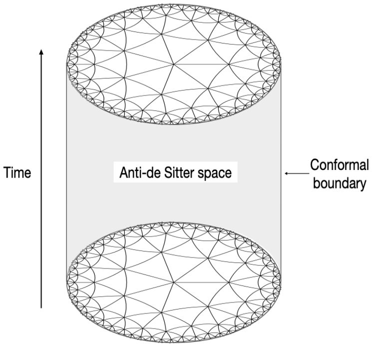
1. Upper-Half Plane $\rightarrow$ Euclidean $\mathbb{R}^d$
2. The Poincare ball $\rightarrow \mathbb{S}^d$
3. $AdS$ cylinder $\rightarrow \mathbb{R} \times \mathbb{S}^{d-1}$

- For Each choice of coordinates the hyperbolic manifold remains the same but the corresponding boundary $CFT$ maps to different manifolds.

# $AdS$ Space

Euclidean $AdS_{d+1}$ has the following metric which has the topology of a cylinder $\mathbb{R} \times \mathbb{H}^d$. [1]

$$ds^2 = g_{00}dt^2 + ds^2_{\mathbb{H}^d} \tag{3}$$




[1] Image taken from arxiv:2202.03464

# *AdS* Space

Under a particular choice of coordinates this metric can be expressed as

$$ds^2 = \pm\ell^2 \cosh^2 \rho dt^2 + \ell(d\rho^2 + \sinh^2 \rho d\Omega_{d-1}^2) \tag{4}$$

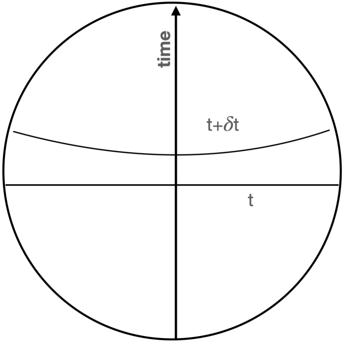
- Where $\rho$ is the geodesic from the origin of $\mathbb{H}^d$ and $g_{00}(\rho) = \ell^2 \cosh(\rho)$ at fixed time
- $d\Omega_{d-1}^2$ is the line element of the sphere $\mathbb{S}^{d-1}$

# $AdS_2$ Ising Hamiltonian

- For $AdS_2$ where $d = 1$ this cylindirical form reduces to a strip with $1D$ conformal quantum mechanics at the each end and leads to the following Ising Hamiltonian

$$H_{AdS} = -J \sum_{<ij>} \cosh(\rho_i)\sigma_i^z \sigma_j^z - h \sum_i \cosh(\rho_i)\sigma_i^x - m \sum_i \cosh(\rho_i)\sigma_i^z$$

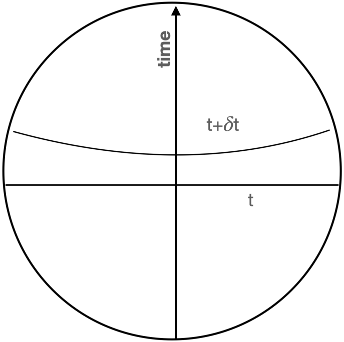
# $AdS_2$ Ising Hamiltonian




- All the geodesics that are perpendicular to the time axis can be regarded as equal time curves.
- Consider a small time evolution for a quantum state $|\Psi(t)\rangle$

$$|\Psi(t+\delta t)\rangle = \hat{U}(\delta t)\,|\Psi(t)\rangle \qquad (5)$$

# $AdS_2$ Ising Hamiltonian



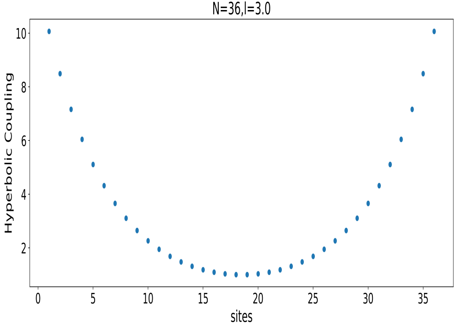

- Even though $|\Psi(t)\rangle$ and $|\Psi(t+\delta t)\rangle$ are translationally invariant. $\hat{U}(t)$ is not.
- The distance between two points $(x,t)$ and $(x,t+\delta t)$ is an increasing function of $x$ and can be written as $\cosh(\rho x)\delta t$
- Which means that the Hamiltonian responsible for generating this time evolution can be written as

$$\hat{H} = \int \cosh(\rho x)\hat{h}(x)dx \qquad (6)$$

# DMRG simulation of the $AdS_2$ Hamiltonian

- For simulations of this model using DMRG, we need to find a way to control the hyperbolic deformation $\cosh(\rho_i)$ for any given chain size $N$

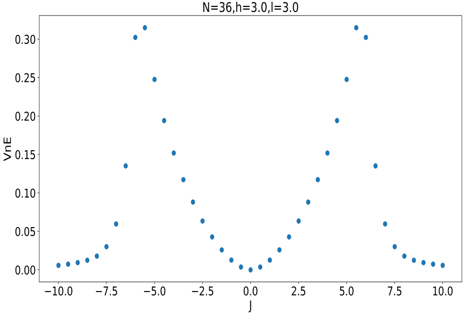
1. Replace $\cosh(\rho_i)$ with $\cosh(l_i)$ where $l_i$ ranges from $-l$ to $l$ from the first site to the last one.
2. We start at the first site with $\cosh(-l)$ Then we increase $l_i$ in increments of $\delta_l = 2l/(N-1)$ until we reach $\cosh(l_{max})$ at the last site.



# Ground State Properties

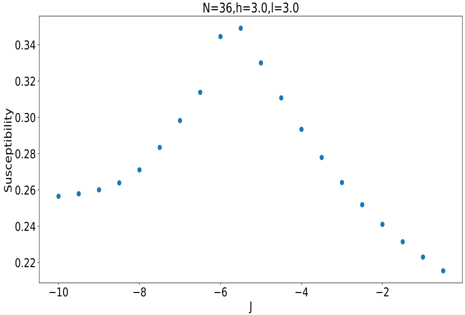
Using the DMRG algorithm we can investigate the ground state properties of the Hyperbolic Ising Model,

- First we calculate the half-chain Von-Neumann entropy for $N = 36, l = 3.0, h = 3.0, m = 0.25$



# Magnetic Susceptibility

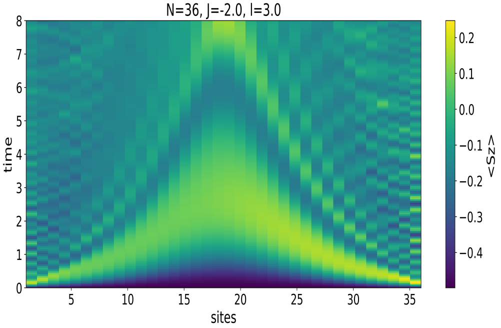
Next, we calculate the Magnetic Susceptibility for the same parameters,




We see that both the entropy and susceptibility peaks around $J = -6$ signaling a phase transition in the model

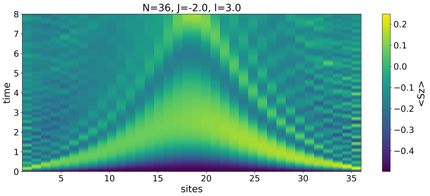
# Time-Evolution of the Model

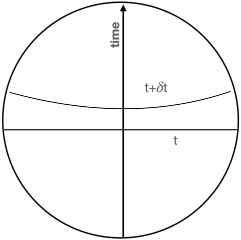
- To obtain the time evolution we use the TEBD algorithm.
- Below we plot the time evolution of $\langle S_z \rangle$ for $N = 36, I = 3.0, h = 2.0,$ and $J = -2.0$



# Time-Evolution of the Model

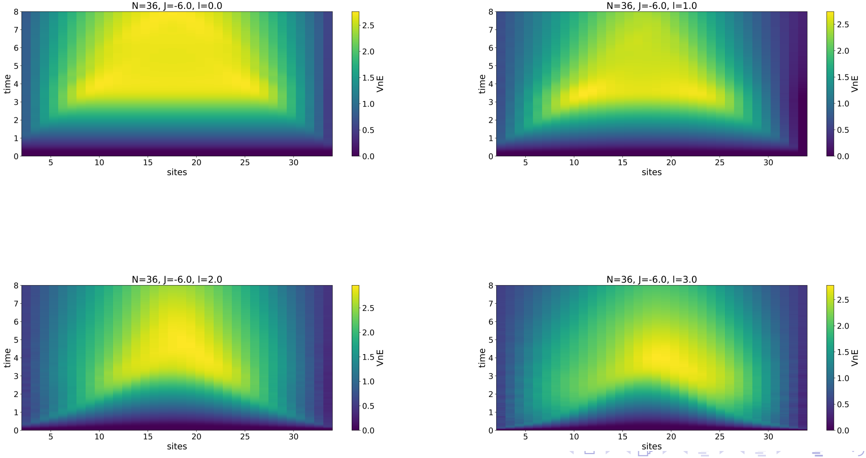
- This interesting warping effect in the bulk can be related to the time-dilation of the coefficient $g_{00}(\rho)$



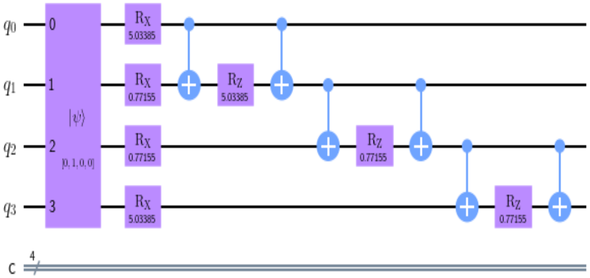
# Time Evolution of the Model

We can also look at time dependence of the Von Neumann entropy of the system, which has the same warping effects.



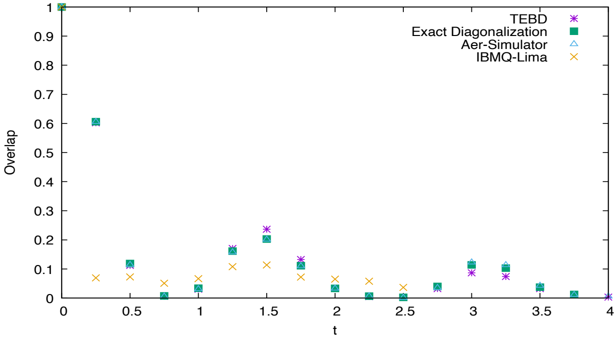
# Quantum Simulation of Hyperbolic Ising Model

- This formulation can be easily generalized to obtain the Suzuki-Trotter evolution on a Universal Quantum Computer
- For 4 qubits we get the following circuit for the time-evolution

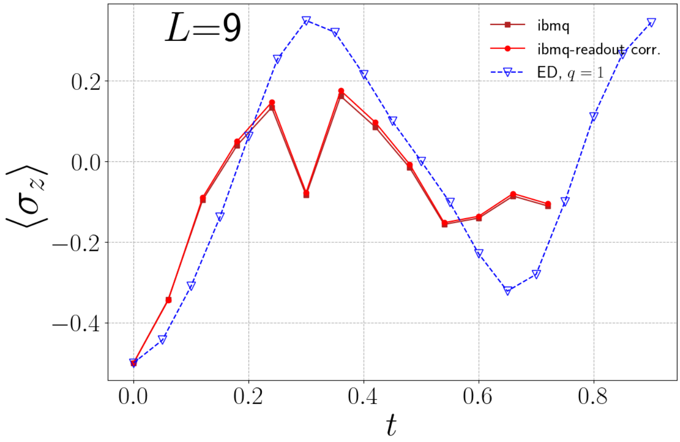



Notice that gates at different sites have different phases this results in the hyperbolic deformation we want.

Now we can compare the results of this circuit with results we obtained from the TEBD algorithm. We choose our initial state to be $|0010\rangle$ and fix $l = 2$.



We can look at the $\langle S_z \rangle$ again,



# Rydberg Simulation of the Model

- As can be seen from the previous plot quantum computers based on superconducting qubits are highly restricted in the number of avaliable qubits for simulations.
- Quantum computers that use trapped ions or Rydberg arrays offer some help with quantum simulations that require a large number of qubits.
- Especially quantum computers based on Rydberg arrays are very usfeul for us since the Rydberg Hamiltonian can be easily mapped to the Ising Hamiltonian

# Rydberg Hamiltonian

We have the following Hamiltonian for the Rydberg atoms,

$$\hat{H}_R(t) = \sum_j \frac{\Omega_j(t)}{2}\left(e^{i\phi_j(t)} |g_j\rangle \langle r_j| + e^{-i\phi_j(t)} |r_j\rangle \langle g_j|\right) - \sum_j \Delta_j(t)\hat{n}_j + \sum_{j<k} V_{jk}\hat{n}_j\hat{n}_k$$

Where $V_{jk} = C_6/|r_j - r_k|^6$ and $C_6 = 2\pi \times 862690 \mathrm{MHz \mu m}^6$

# Rydberg Hamiltonian

The Rydberg Hamiltonian can be matched to the Ising Hamiltonian in the following way.

$$\hat{H}_R(t) = \sum_j \frac{\Omega_j(t)}{2} \underbrace{\left(e^{i\phi_j(t)} |g_j\rangle \langle r_j| + e^{-i\phi_j(t)} |r_j\rangle \langle g_j|\right)}_{\sigma_x} - \sum_j \Delta_j(t) \underbrace{\hat{n}_j}_{\sigma_z} + \sum_{j<k} V_{jk} \underbrace{\hat{n}_j \hat{n}_k}_{\sigma_z \sigma_z}$$

# Rydberg Hamiltonian

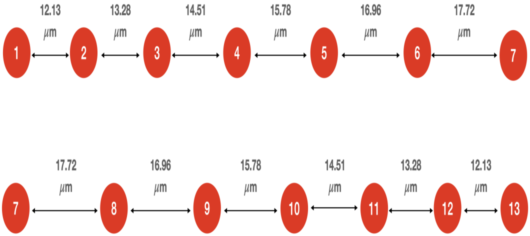
- To get a hyperbolic Ising like model with this Hamiltonian we need to adjust the separation between the atoms such that $V_{jk}$ matches the form of the hyperbolic deformation we had.
- This can be done iteratively once we calculate the hyperbolic coefficients for a given number of sites $N$ and hyperbolic coupling $l$.
- With the hyperbolic couplings at hand we can start building our chain with placing the first atom at $(0, 0)$ and solve the following equation to get the distance between consecutive atoms

$$\delta_{i+1} = (A/\eta_i)^{1/6} + r_i \tag{7}$$

Where $\eta$ denotes the hyperbolic couplings and $A$ is used as constant to determine the separation between atoms and is set to $A = 2\pi \times 512$.

# Rydberg Hamiltonian

Doing this procedure for $l = 3$ results with $N = 13$ results in distances that range in between 12.13µm to 17.72µm, and the furthest atom being located at 180.77µm.

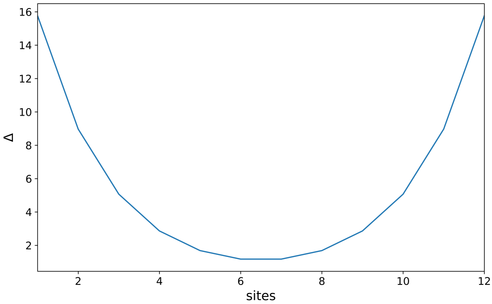



The placement of the Rydberg atoms determine the form of the potential $V_{jk}$

# Rydberg Hamiltonian

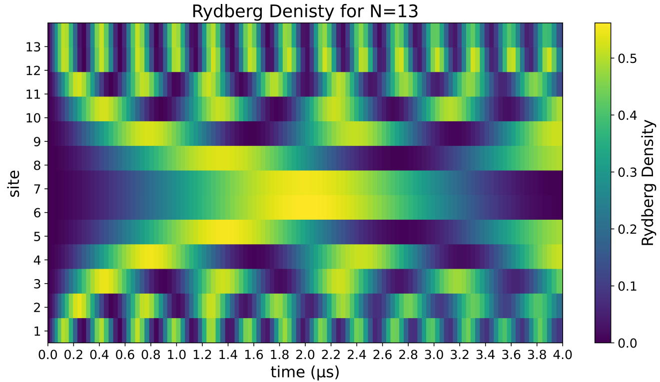
Now that we have a hyperbolic form for the $V_{jk}$. Next step is to find the necessary $\Delta_j(t)$ and $\Omega_j(t)$ at every site. To do this we calculate the following at every site

$$\Delta_j, \Omega_j = \frac{C_6 * 10}{(r_{j+1} - r_j)^6} \tag{8}$$



# Simulation of the Rydberg Hamiltonian

- With all the ingredients of the Hamiltonian set we can calculate the corresponding Rydberg density $\langle n_i \rangle$ for the system.
- For Rydberg Simulations we used the Bloqade Software package developed by QuEra.



# Out of Time Ordered Correlators & Information Spread

- Now we focus on the question of how information spreads in the Hyperbolic Ising chain, for that we calculate Out of Time Ordered Correlators (OTOCs)
- Which are one of the main observables used to measure scrambling,information spread and quantum chaos.

In general OTOCs have the following form

$$F_r(t) = Tr(W(t)^\dagger V_r^\dagger W(t) V_r) \tag{9}$$

For the Ising case $W$ and $V$ can be taken as local Pauli operators.

# OTOCs & Scrambling

The connection between OTOC and operator growth can be made explicit by introducing the squared commutator.

$$C(r,t) = \frac{1}{Tr\mathbb{I}} Tr([W(t), V_r]^\dagger [W(t), V_r]) = \frac{1}{Tr\mathbb{I}} ||W(t), V_r||^2 \quad (10)$$

$$C(r,t) = -F(r,t) - F^*(r,t) \quad (11)$$

$$+ \frac{1}{Tr\mathbb{I}} (Tr(W^\dagger V_r^\dagger V_r W(t)) + Tr(V_r^\dagger W^\dagger(t) W(t) V_r)) \quad (12)$$

- The last 2 terms are local observables that thermalize to a constant after a short time
- This relation further simplifies when $W, V$ are Unitary and/or Hermitian.
- For example for Pauli operator this can be simplified to $C(r,t) = 2 - 2F(r,t)$

# OTOCs & Scrambling

- The squared commutator depends on the number of the degrees of freedom $W(t)$ acts on.
- At $t = 0$ $W(t)$ acts only on one site and commutes with $V_r$ that is located away from $W$ so $C(r, t) = 0$
- As the system evolves under time, $W(t)$ becomes more and more non-local and starts to overlap with $V_r$ which results in an increase in $C(r, t)$

# OTOCs & Scrambling

- So by changing the location of $V_r$ we can probe how $C(r,t)$ changes
- If $V_r$ doesn't overlap with $W(t)$ $C(r,t)$ remains small.
- If $V_r$ overlaps with $W(t)$ $C(r,t)$ grows large.

This means that we can use $C(r,t)$ to determine the number of degrees of freedom $W(t)$ acts on at time $t$ which also means that $C(r,t)$ can be used to track the lightcone of information dynamics

# OTOCs & Scrambling

- In our calculations for the OTOC using TEBD we used this specific form for the OTOC operator.

$$O(t) = Tr(\rho W(t) V^{\dagger} W(t) V) / Tr(\rho W(t)^2 V^{\dagger} V) \tag{13}$$

This definition ensures that $O(t) = 1$ when $W(t)$ and $V$ commute

- Taking $\rho \sim \mathbb{I}/D$ one can easliy take the infinite temperature limit or calculate the expectation value of this operator on any state to obtain finite temperature results.
- We obtain $W(t)$ by expressing $W$ as an MPO state and applying Heisenberg time evolution using TEBD.

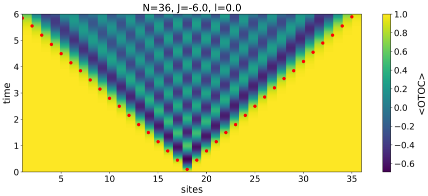
# OTOCs & Scrambling

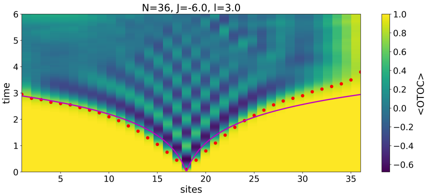
- How $O(t)$ spreads through the chain distinguishes between different kinds of scrambling

1. $O(t) \sim log(\lambda d) \rightarrow$ fast scramblers like the SYK model and Black Holes.
2. $O(t) \sim \lambda d^n \rightarrow$ systems with infinite/long range interactions
3. $O(t) \sim \lambda d \rightarrow$ systems that saturate the Lieb-Robinson bound

# OTOCs in Hyperbolic Ising Model

Let's start our discussion with OTOC calculations at the infinite temperature limit.



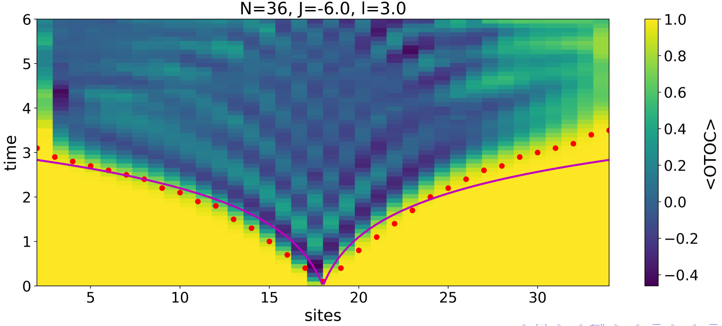



# OTOCs in Hyperbolic Ising Model

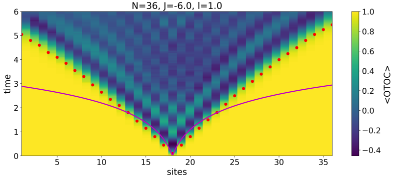
For finite temperature results we measure the following,

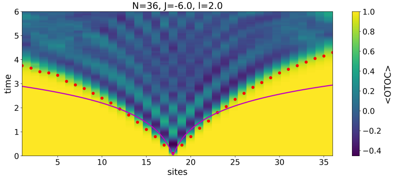
$$O(t) = \frac{\langle\psi| \left(W(t) V^\dagger W(t) V\right) |\psi\rangle}{\langle\psi| \left(W(t)^2 V^\dagger V\right) |\psi\rangle} \tag{14}$$

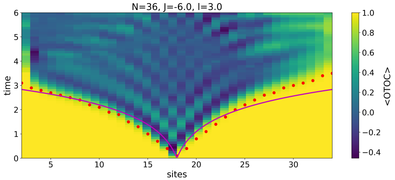
Where $|\psi\rangle = \rho^{1/2} |0\rangle$ we will take $|\psi\rangle$ to consist of all down state for the rest of this talk.

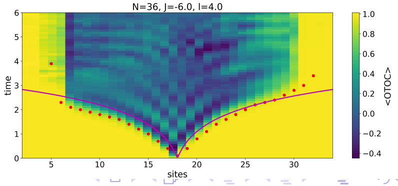


Now turning on the Hyperbolic deformation $I$ for fixed $J = -6.0$ and other parameter are tuned to their corresponding critical values.



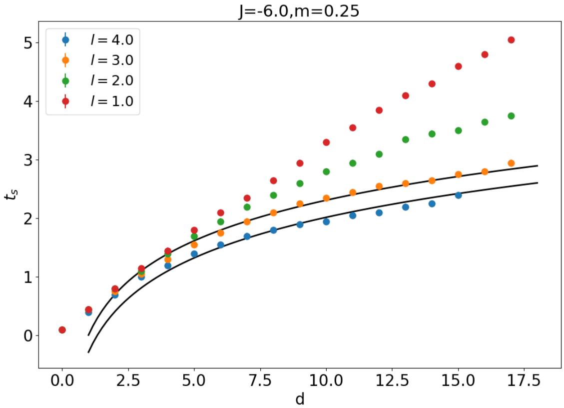



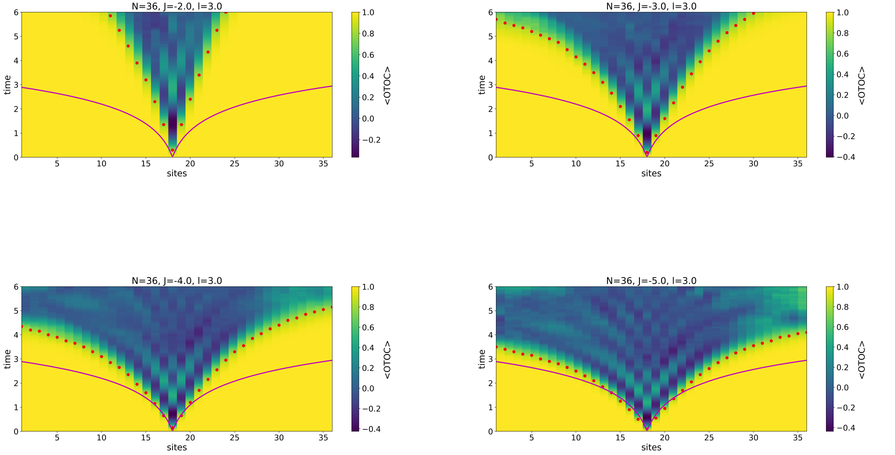
# OTOCs in Hyperbolic Ising Model

Plotting the $d$ dependence of the light-cone



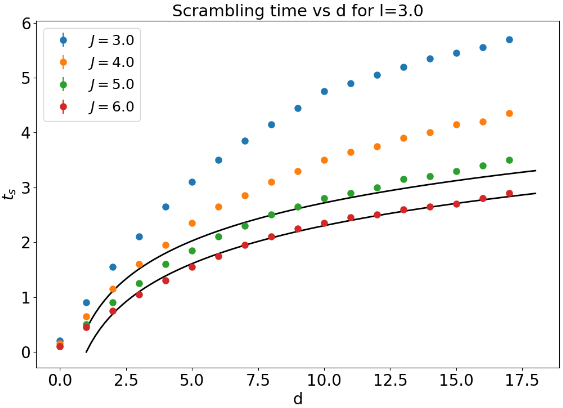
# OTOCs in Hyperbolic Ising Model

Alternatively, we can fix $I = 3.0$ and see the dependence on $J$ of OTOCs.



# OTOCs in Hyperbolic Ising Model

Again, plotting the $d$ dependence of the light-cone



# OTOCs in Hyperbolic Ising Mode

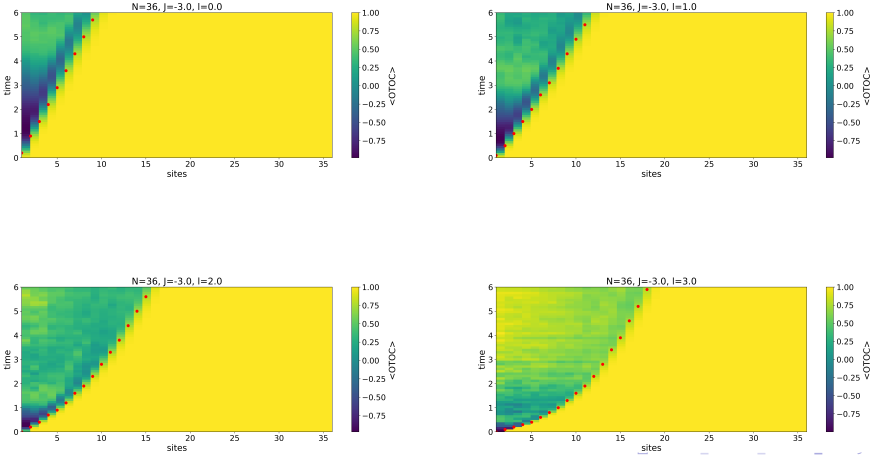
We see that for suitable parameters Hyperbolic Ising Model acts like fast scrambler and propagates information in logarithmic time.

This is important for a few reasons

1. Models like SYK that exhibit fast scrambling are very hard to simulate.
2. Models with infinite range interactions requires long range entanglement which is problematic for quantum simulations
3. Our model is easy to simulate both on classical and quantum computers which makes it a very rare model in the class of fast scramblers.

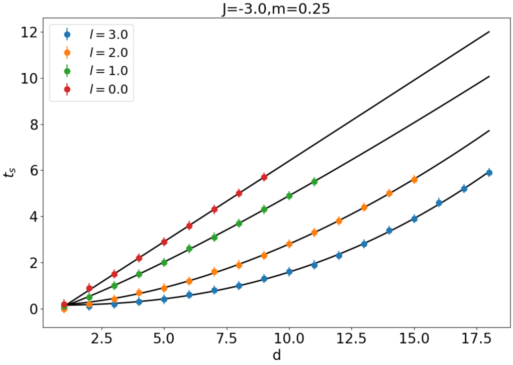
# Boundary to Bulk Correlators

- Up to now we focused on bulk correlators but what happens if we start from the boundary.



# Boundary to Bulk Correlators

We see that a correlator that starts out in the boundary spreads through the chain in a power-law like manner.




- This is interesting because it shows us that depending on the location of deformation $W(t)$ we can have different behaviours for the scrambling.

# Conclusions

- We developed classical and quantum simulations for the Hyperbolic Ising Model.
- Calculated the OTOCs and showed that Hyperbolic Ising model is a fast scrambler
- Which makes the Hyperbolic Ising Model one of the few models that is a fast scrambler and can be simulated easily.

Thanks for listening.

Fitting results for the logarithmic fits for $N = 36, l = 3.0$

| $T$ | $a + \log(bd)$ |
|---|---|
| 4 | $0.94 + \log(1.10d)$ |
| 5 | $0.38 + \log(1.03d)$ |
| 6 | $0.67 + \log(0.50d)$ |

Fitting results for the power law fits for $N = 36, J = -3.0$

| $l$ | $a + bd^c$ |
|---|---|
| 0 | $-0.6 + 0.7 * d^1$ |
| 1 | $-0.23 + 0.33 * d^{1.18}$ |
| 2 | $0.12 + 0.04 * d^{1.77}$ |
| 3 | $0.14 + 0.006 * d^{2.36}$ |