Energy Transfer and Scale-Locality in Compressible Shocks: Revisiting the Cascade Picture

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Compressible Turbulence: From Cold Atoms to Neutron Star Mergers

Summer 2025

Outline:

Introduction and/or Motivation

Conceptual Framework

Kinetic Energy Budget At Different Scales

Burgers Shock and Energy Cascade

Normal Shock: A Real Compressible System

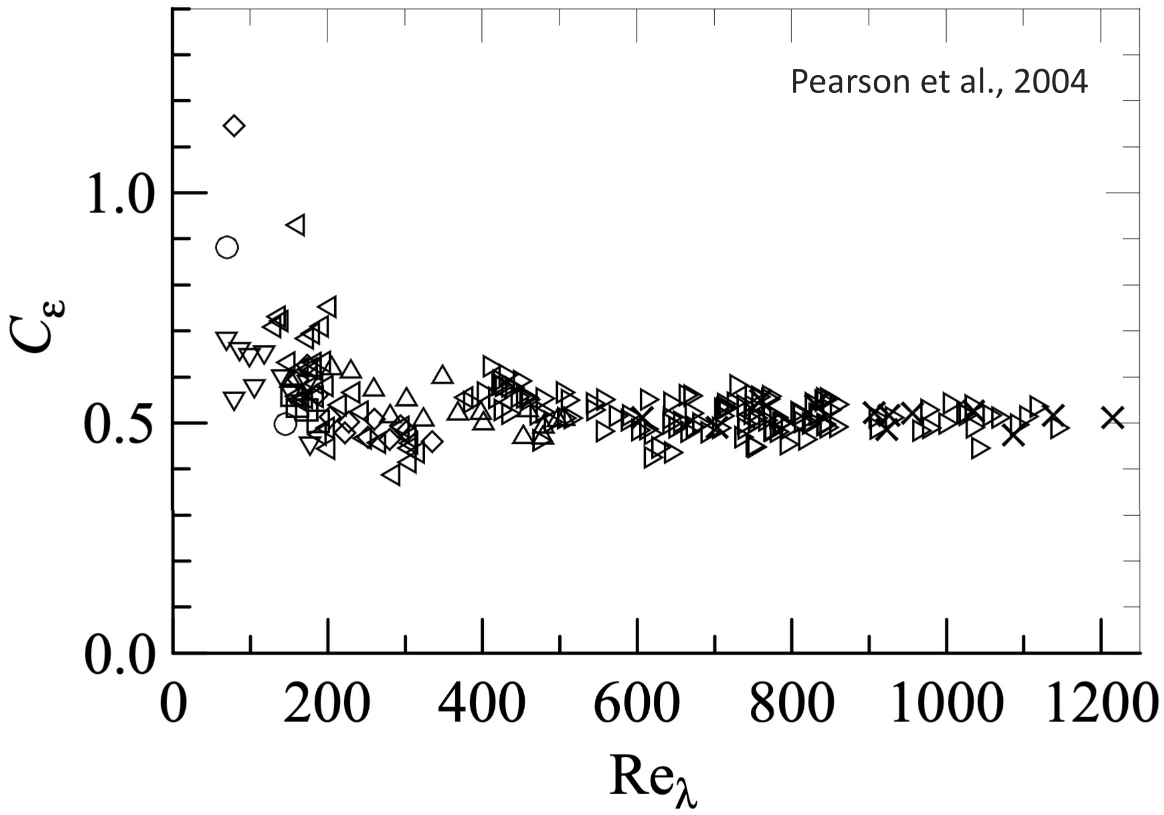
Incompressible Turbulence Theory

1. Relied in a central way on an empirical observation:
   The “Zeroth Law” of Turbulence [G. I. Taylor (1935)]

2. The 4/5-th Law

3. Inertial Range:
   Assuming self-similarity (scale-invariance), and scale-locality [Kolmogorov (1941, 1946), Onsager (1945)]

Zeroth Law of Turbulence



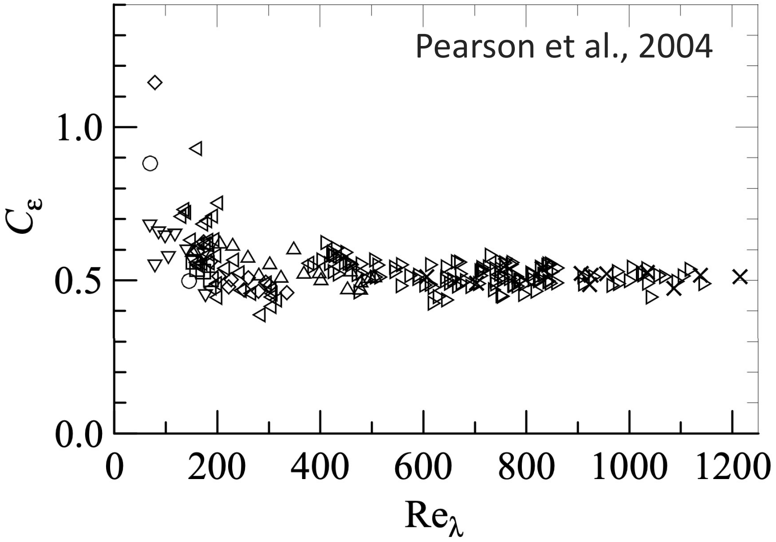
Empirical Observation[1]:

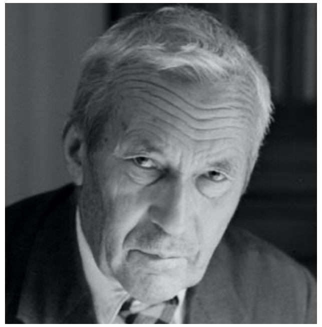
$$C_\varepsilon = \frac{\varepsilon_{dissip}}{U_{rms}^3/L}$$

G.I. Taylor,
Proc. Roy. Soc. Lond., (1935)

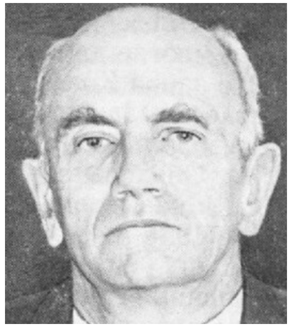
Dissipation is independent of microphysical viscosity.

Zeroth Law of Turbulence



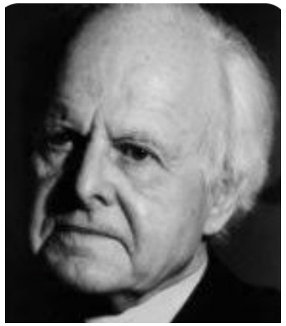


A. Kolmogorov



AM. Obukhov

W. Heisenberg



CF Weizsacker

L. Onsager

"turbulent fluids could be described by singular (weak) solutions of incompressible Euler equations whose kinetic-energy balance would be afflicted with an anomaly due to the nonlinear cascade mechanism[1]."

1. Eyink & Drivas, 2018, Phys. Rev. X.
2. A voyage through Turbulence, 2011

Incompressible Turbulence Theory

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Kolmogorov's 4/5-th Law

Traditional derivation (e.g. Frisch's 1995 book)

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

Point-splitting KE budget (a form of scale decomposition)

$$\partial_t \langle \boldsymbol{u}(x) \cdot \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}) \rangle = \frac{1}{4} \nabla_{\boldsymbol{r}} \cdot \langle |\delta \boldsymbol{u}(\boldsymbol{r})|^2 \delta \boldsymbol{u}(\boldsymbol{r}) \rangle - \text{Dissipation} + \text{Forcing}$$

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$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

Point-splitting KE budget (a form of scale decomposition) Assuming homogeneity

$$\partial_t \langle \boldsymbol{u}(x) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r})\rangle = \frac{1}{4}\nabla_{\boldsymbol{r}} \cdot \langle |\delta \boldsymbol{u}(\boldsymbol{r})|^2 \delta \boldsymbol{u}(\boldsymbol{r})\rangle - \text{Dissipation} + \text{Forcing}$$

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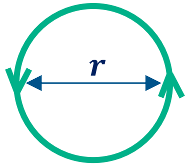
Increments:

$$\delta \boldsymbol{u}(\boldsymbol{r};\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$$



Galilean Invariant

Scaling Conditions $\quad \|\delta \mathbf{u}(\mathbf{r})\|_p \sim u_{rms} A_p (r/L)^{\sigma_p^u}, \qquad 0 < \sigma_p^u < 1,$

Kolmogorov's 4/5-th Law

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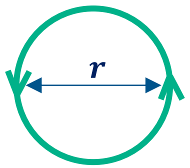
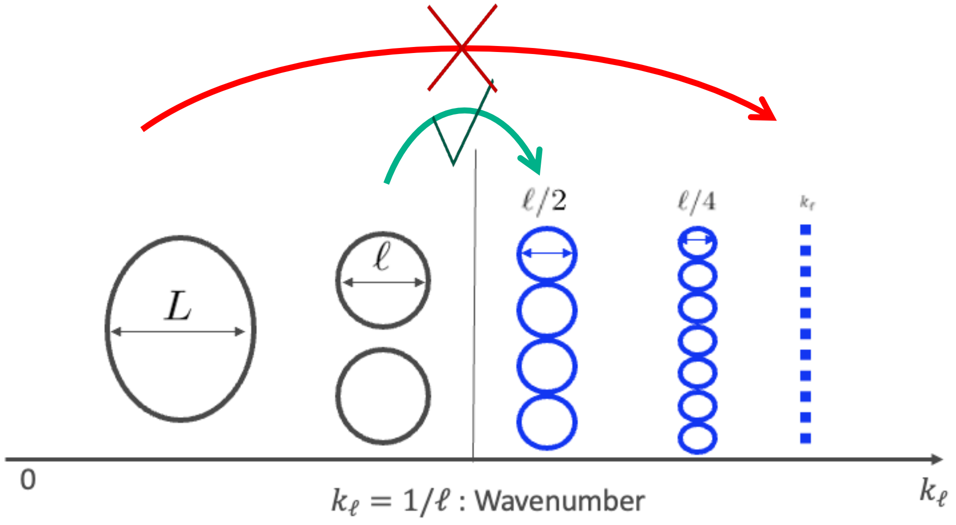
Increments:

$$\delta \boldsymbol{u}(\boldsymbol{r}; \boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$$

- Galilean Invariant
- Scaling Conditions
- Scale-local[1]





1. G. Eyink, 2005, Physica D.

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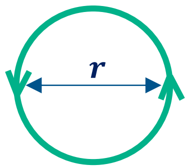
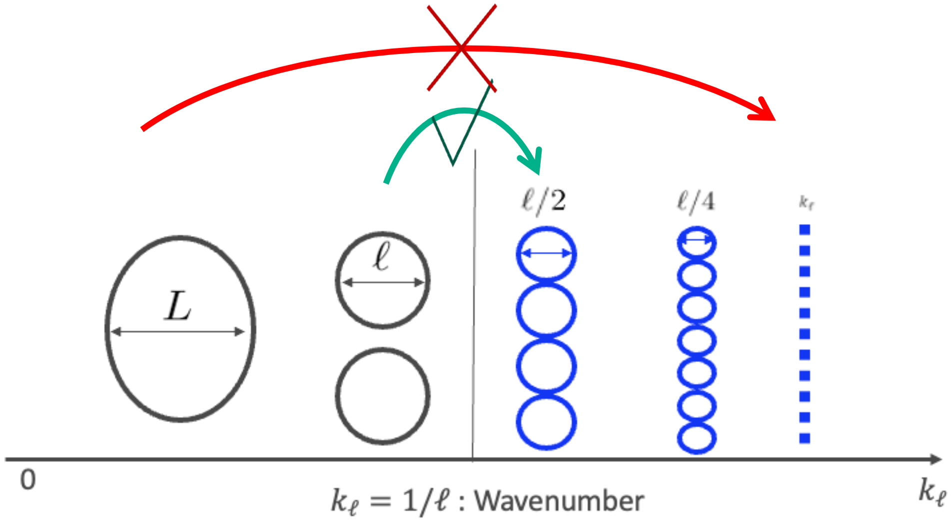
$$\partial_t \langle \boldsymbol{u}(x) \cdot \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) \rangle = \underbrace{\boxed{\frac{1}{4} \nabla_{\boldsymbol{r}} \cdot \langle |\delta \boldsymbol{u}(\boldsymbol{r})|^2 \delta \boldsymbol{u}(\boldsymbol{r}) \rangle}}_{\text{The Cascade Term}} - \text{Dissipation} + \text{Forcing}$$

Increments:

$$\delta \boldsymbol{u}(\boldsymbol{r};\boldsymbol{x}) = \boldsymbol{u}(\boldsymbol{x}+\boldsymbol{r}) - \boldsymbol{u}(\boldsymbol{x})$$

- Galilean Invariant
- Scaling Conditions
- Scale-local





Kolmogorov's 4/5-th Law

This is just conservation of kinetic energy at each scale, r!

Traditional derivation (e.g. Frisch's 1995 book)

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}$$

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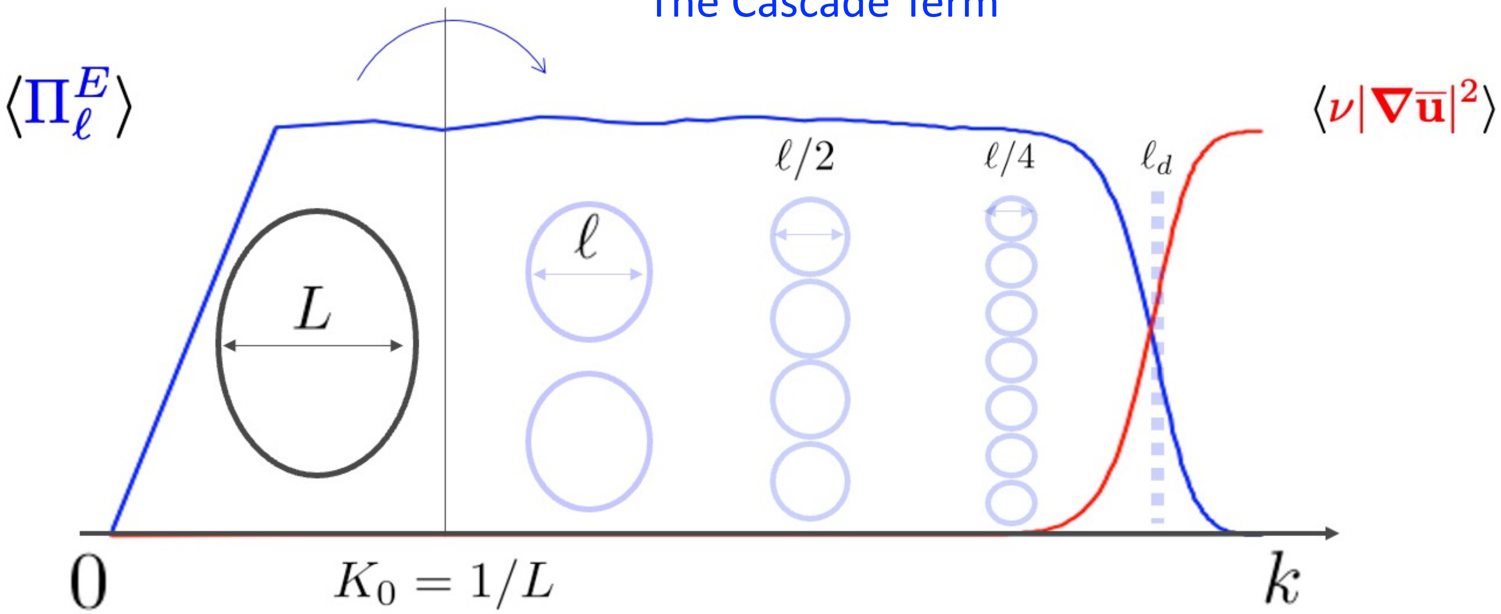
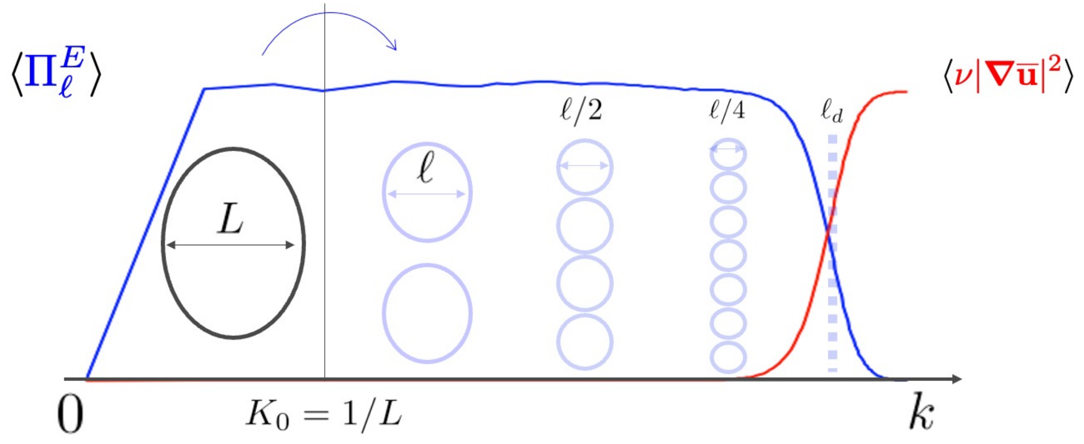
The Cascade Term



Kolmogorov's 4/5-th Law



For r in the inertial range:

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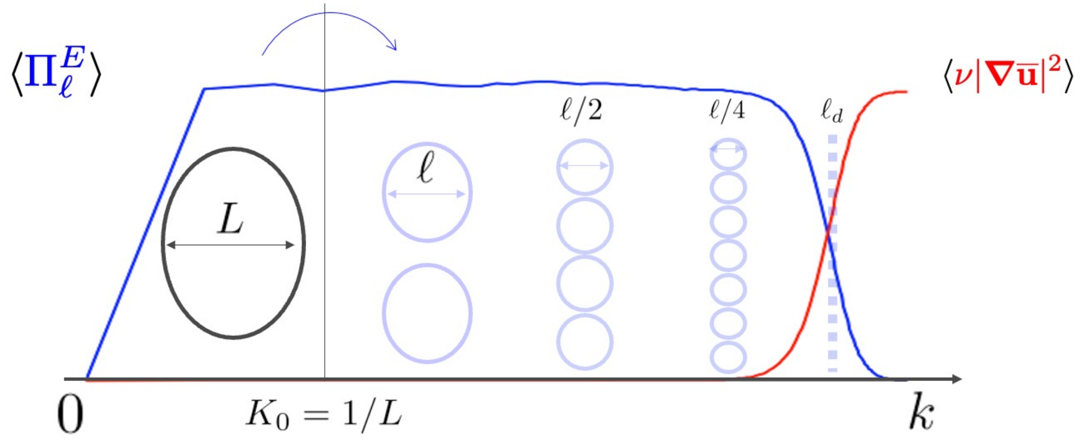
Kolmogorov's 4/5-th Law



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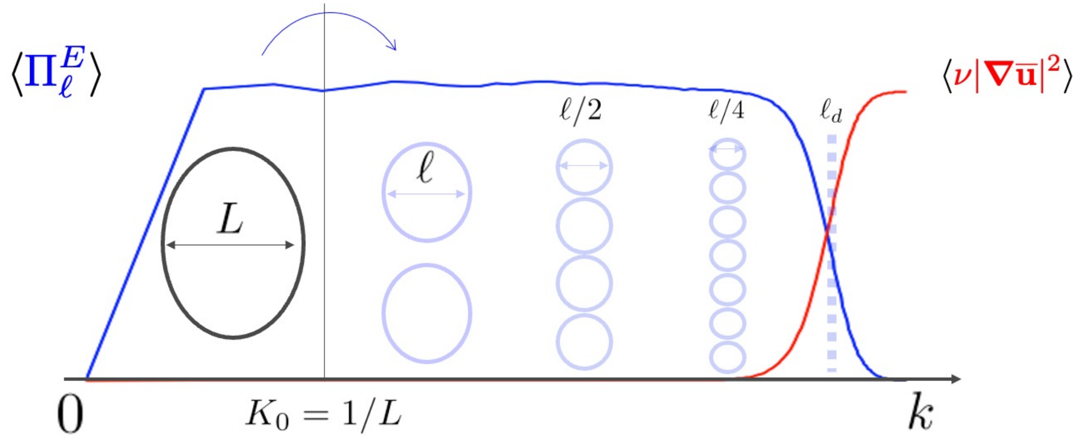
$$S_3(r) = \left\langle \left(\delta u_{\parallel}(r) \right)^3 \right\rangle = -\frac{4}{5} \varepsilon\, r$$

Assuming self-similarity, no intermittent event!

$$S_p(r) = \left\langle \left(\delta u_{\parallel}(r) \right)^p \right\rangle = C_p (\varepsilon\, r)^{p/3}$$

Kolmogorov’s 4/5-th Law



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$$S_p(r) = \left\langle \left(\delta u_{\parallel}(r) \right)^p \right\rangle = C_p (\varepsilon\, r)^{p/3}$$

Works fine for p $= 3 \pm 1$!

$$S_2(r) = \left\langle \left(\delta u_{\parallel}(r) \right)^2 \right\rangle \sim r^{2/3} \quad \Longleftrightarrow \quad E(k) \sim k^{-5/3}$$

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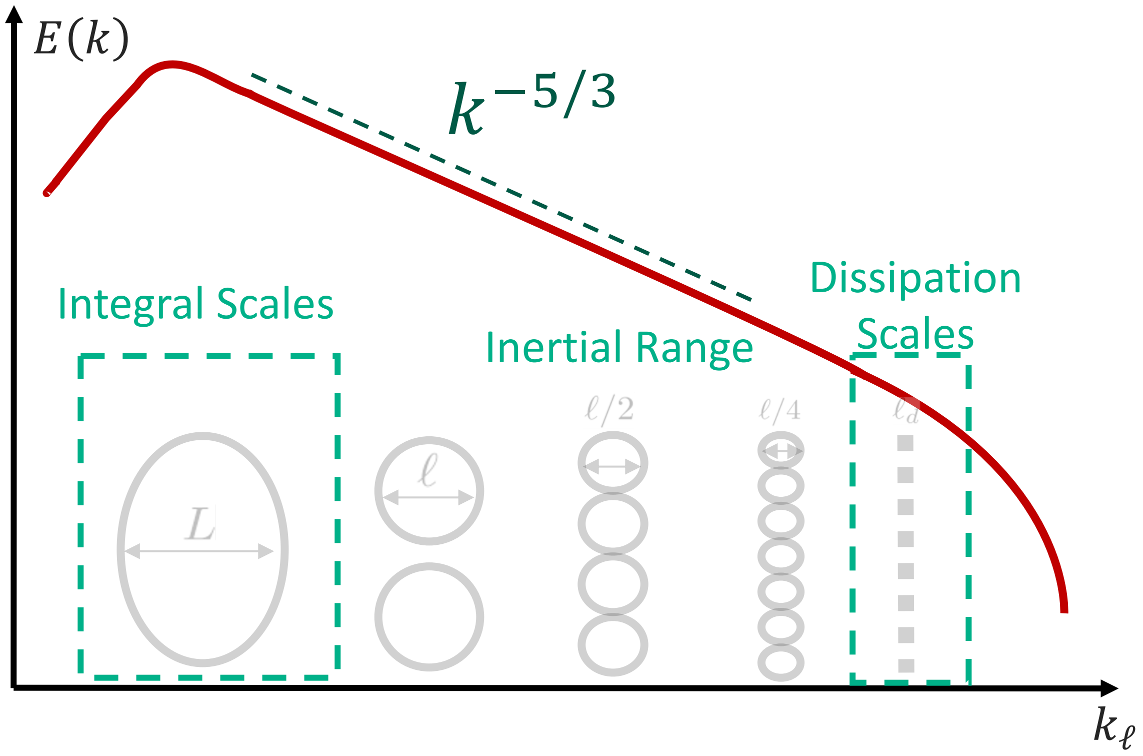
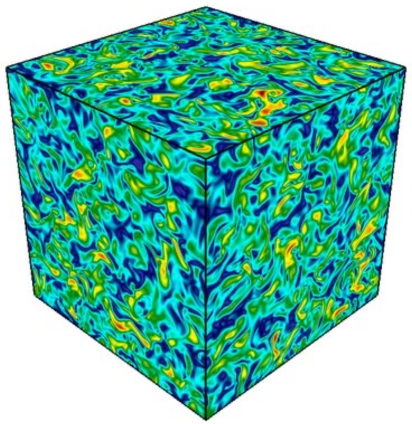
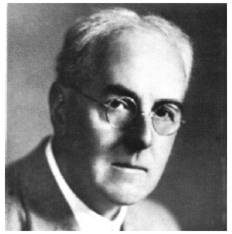
We have a relatively clear picture of the dynamics in **incompressible** turbulent systems.



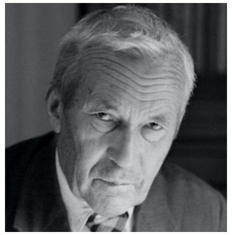


Decaying, isotropic, homogeneous turbulent system (Saadat, et al., 2021)

- Forcing and Boundary Conditions at integral scales.
- Viscous Dissipation at smallest scales.
- Inertial range



L. Richardson (1922)



A. Kolmogorov (1941)

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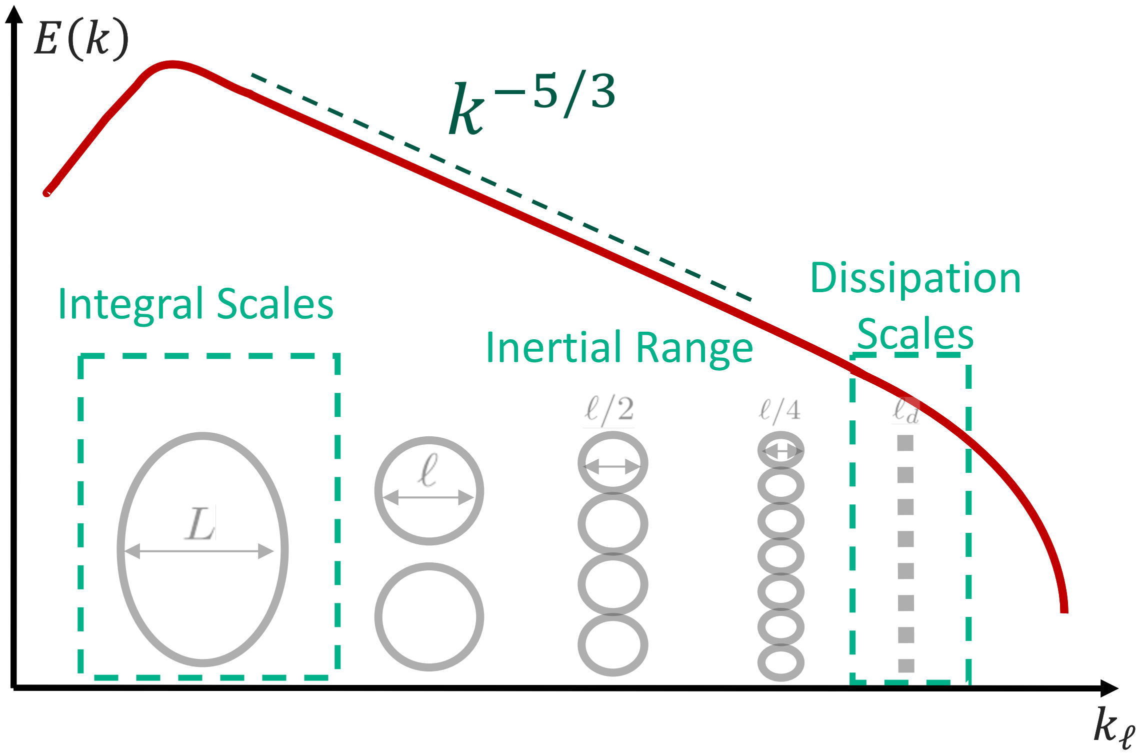
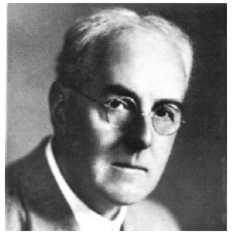


- **Scale-locality:**
  - Different modes exchange energy with neighbors mostly.
  - Justifies expectation of universal statistics.
  - Implicit assumption in LES Modeling.



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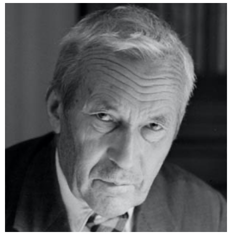


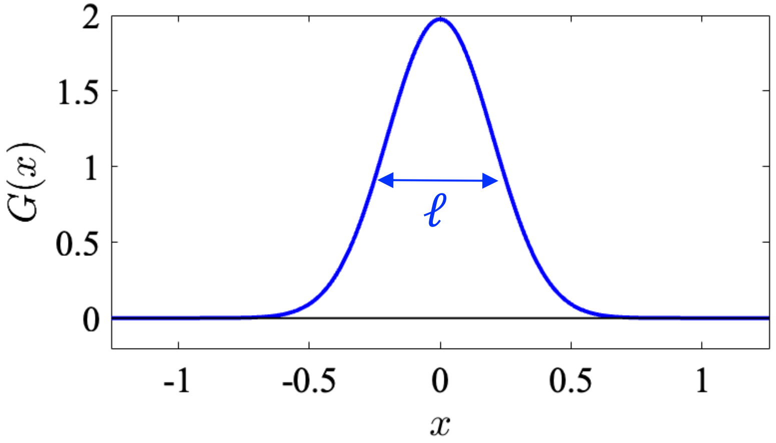
A. Kolmogorov (1941)

L. Onsager (1945)

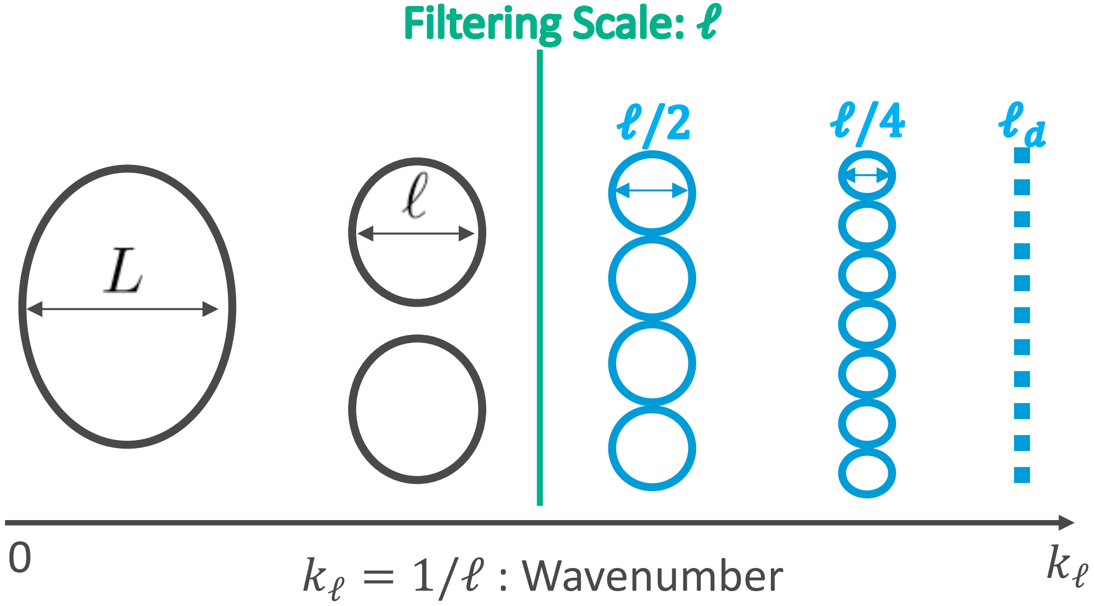
R. Kraichnan (1959)

G. Eyink (1994, 2005)

Method: Coarse-graining Approach



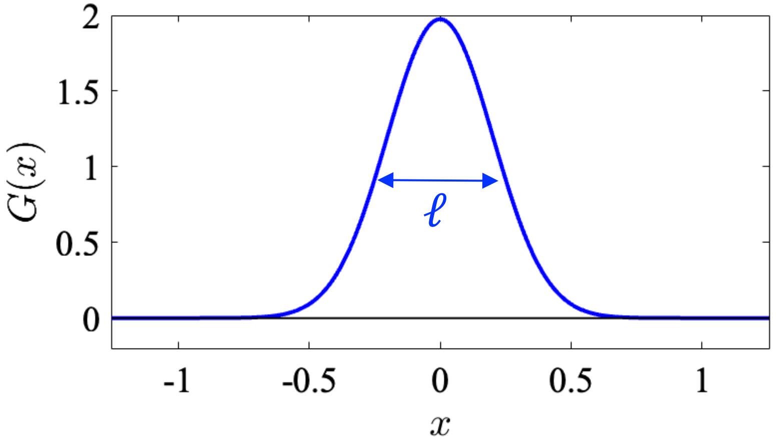
$$\overline{\boldsymbol{u}}_\ell(\boldsymbol{x}) = \int d\boldsymbol{r}\, \boldsymbol{u}(\boldsymbol{r}) G_\ell(\boldsymbol{x}+\boldsymbol{r})$$



$G_\ell(\boldsymbol{r})$ should be:

- **Homogeneous,**
- **Normalized,**
- **Spatially localized.**

Method: Coarse-graining Approach



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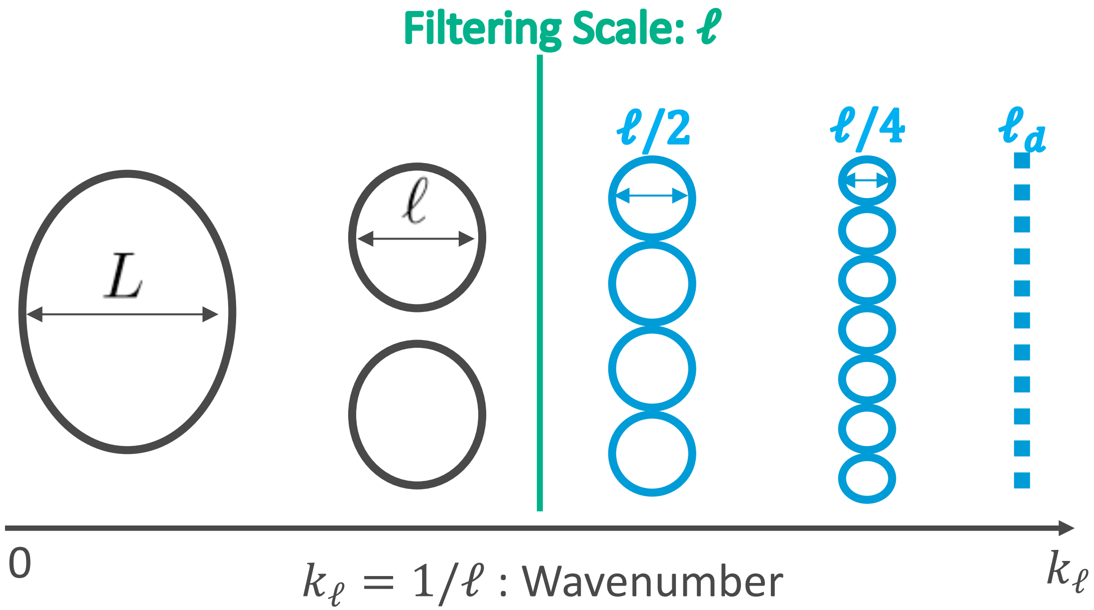


Coarse Cons. of Momentum:

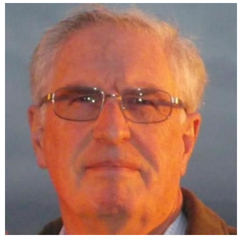
$$\partial_t \bar{u} + \bar{u} \cdot \boldsymbol{\nabla} \bar{u} = -\boldsymbol{\nabla} \bar{p} - \nabla \cdot \tau_\ell + \mu \nabla^2 \bar{u} \qquad \nabla \cdot \bar{u} = 0$$

Sub-scale Stress:

$$\tau_\ell = \left(\overline{uu}_\ell - \bar{u}_\ell \bar{u}_\ell\right)$$

…

A. Leonard (1974)



M. Germano (1992)

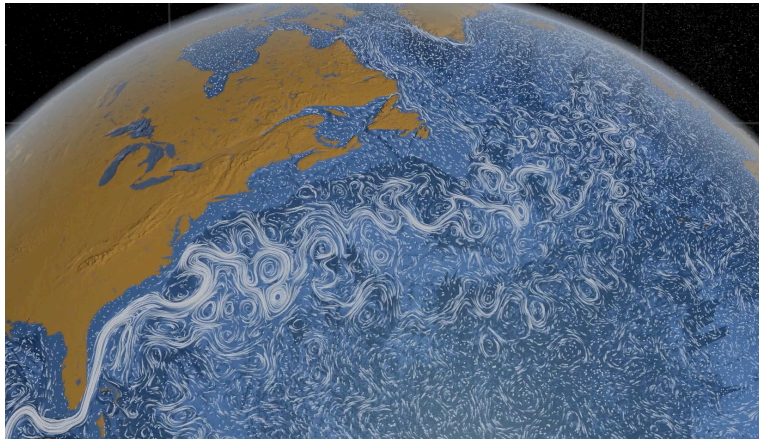
G. Eyink (1994)

C. Meneveau (1994)

H. Aluie (2011)

With the incompressible Turbulence Theory, we can explain:

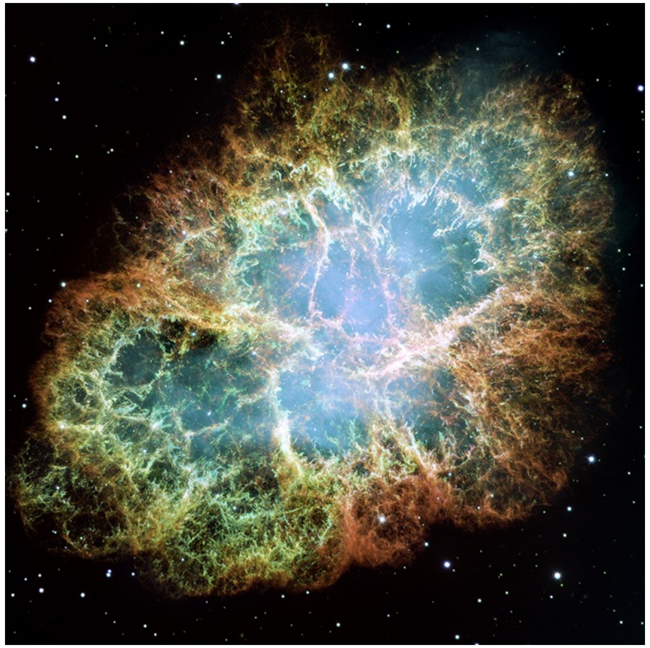
Ocean Dynamics



Terrestrial Flows

Atmospheric Dynamics

But compressible turbulence theory is essential to understand:

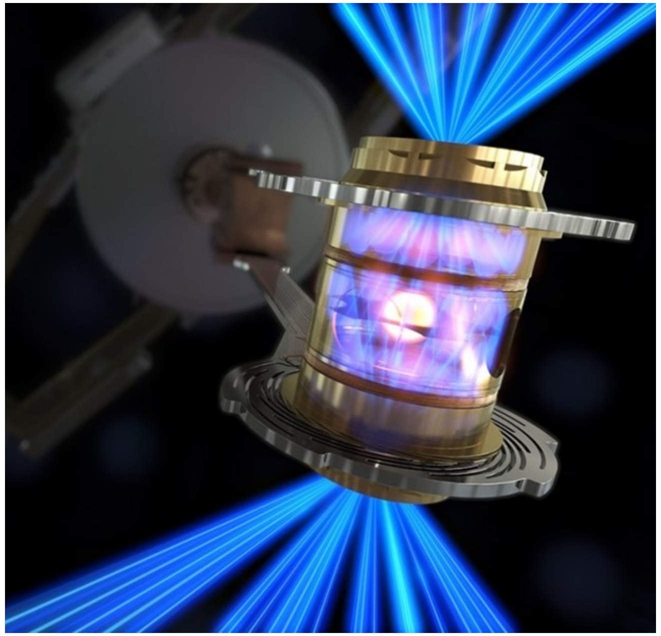


Crab Nebula, NASA

Supernovae Explosions

NASA

Supersonic Flight



NIF, Lawrence Livermore National Laboratory

Inertial Confinement Fusion

Energy Transfer Across Scales: Compressible

Mass Cons. $\partial_t \rho + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u}) = 0$

Momentum Cons. $\partial_t (\rho \boldsymbol{u}) + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{u} \boldsymbol{u}) = \boldsymbol{\nabla} P + \boldsymbol{\nabla} \cdot (2\eta \boldsymbol{S} \cdot \boldsymbol{u} + \zeta \theta \boldsymbol{u})$ + Equation of State

Total Energy Cons. $\partial_t \left(\frac{1}{2} \rho u^2 + e \right) + \boldsymbol{\nabla} \cdot \left[\left(e + P + \frac{1}{2} \rho u^2 \right) \boldsymbol{u} - \kappa \boldsymbol{\nabla} T - 2\eta \boldsymbol{S} \cdot \boldsymbol{u} - \zeta \theta \boldsymbol{u} \right] = 0$

Energy Transfer Across Scales: Compressible

Mass Cons. $\partial_t \rho + \nabla \cdot (\rho \boldsymbol{u}) = 0$

Momentum Cons. $\partial_t (\rho \boldsymbol{u}) + \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u}) = \nabla P + \nabla \cdot (2\eta \boldsymbol{S} \cdot \boldsymbol{u} + \zeta \theta \boldsymbol{u})$ + Equation of State

Total Energy Cons. $\partial_t \left(\frac{1}{2}\rho u^2 + e\right) + \nabla \cdot \left[\left(e + P + \frac{1}{2}\rho u^2\right)\boldsymbol{u} - \kappa \nabla T - 2\eta \boldsymbol{S} \cdot \boldsymbol{u} - \zeta \theta \boldsymbol{u}\right] = 0$

Kinetic Energy Budget

$$\partial_t \left(\frac{1}{2}\rho u^2\right) + \nabla \cdot J_{KE} = \underbrace{p(\nabla \cdot \boldsymbol{u})}_{\text{Pressure-Dilatation}} \underbrace{- 2\eta |\boldsymbol{S}|^2 - \zeta \theta^2}_{\text{Viscous Dissipation}}$$

Internal Energy Budget

$$\partial_t (e) + \nabla \cdot J_{IE} = \underbrace{-p(\nabla \cdot \boldsymbol{u})}_{\text{Pressure-Dilatation}} \underbrace{+ 2\eta |\boldsymbol{S}|^2 + \zeta \theta^2}_{\text{Viscous Dissipation}}$$

Modify the Coarse-graining Method

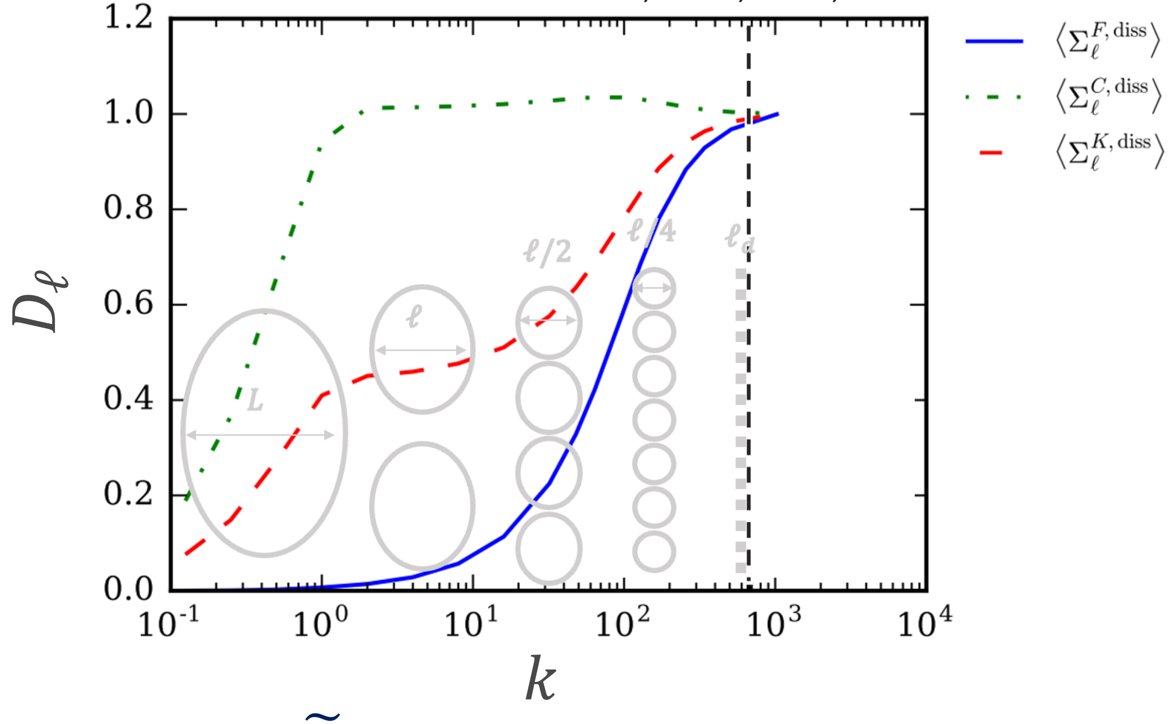
Favre Averaging (Favre, 1965)

$$\overline{\boldsymbol{u}}_\ell(\boldsymbol{x}) = \int d\boldsymbol{r}\,\boldsymbol{u}(\boldsymbol{r})G_\ell(\boldsymbol{x}+\boldsymbol{r})$$

$$\widetilde{\boldsymbol{u}}_\ell(\boldsymbol{x}) \equiv \frac{\overline{\rho\boldsymbol{u}_\ell}}{\overline{\rho}_\ell}$$

Why Favre Averaging?
The Scale-decomposition must satisfy the Inviscid Criterion [Aluie Physica D (2013)].

Zhao, Aluie, 2019, PRF

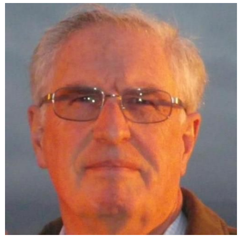


Coarse Cons. of Momentum:

$$\partial_t(\bar{\rho}_\ell\widetilde{\boldsymbol{u}}_\ell) + \nabla\cdot(\bar{\rho}_\ell\widetilde{\boldsymbol{u}}_\ell\widetilde{\boldsymbol{u}}_\ell) = -\nabla\bar{P}_\ell - \nabla\cdot(\bar{\rho}\tilde{\tau}_\ell) + \mu\nabla\cdot\bar{\sigma}_\ell + \bar{\rho}_\ell\widetilde{\boldsymbol{F}} \qquad \bar{\rho}\tilde{\tau}_\ell = \bar{\rho}_\ell(\widetilde{\boldsymbol{u}\boldsymbol{u}}_\ell - \widetilde{\boldsymbol{u}}_\ell\widetilde{\boldsymbol{u}}_\ell)$$

…

A. Leonard (1974)



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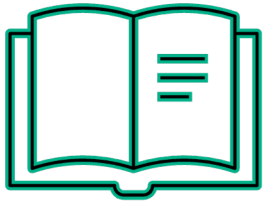
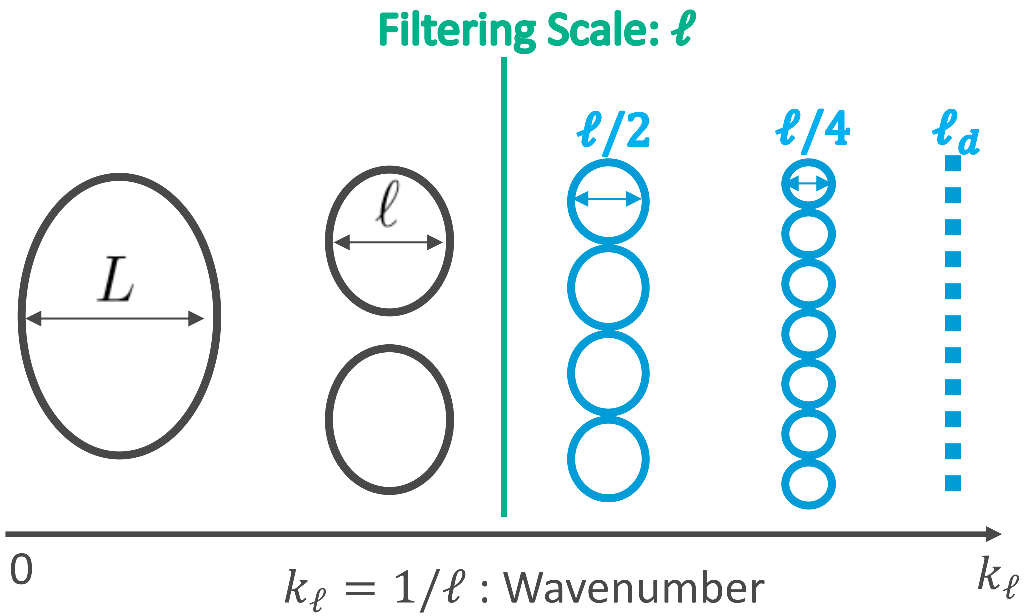


My PhD Work (2022-Present)

Coarse-grained KE Budget

$$\bar{f}_\ell(\boldsymbol{x}) = \int d\boldsymbol{r}\, G_\ell(\boldsymbol{r}) \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r})$$

$$\tilde{f}_\ell(\boldsymbol{x}) \equiv \frac{\overline{\rho f_\ell}}{\overline{\rho_\ell}}$$



$$\partial_t \left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2} \right) + \boldsymbol{\nabla} \cdot \boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p, \boldsymbol{\nabla} \cdot \boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla} \cdot u_\ell}) - D_\ell$$

Coarse-grained KE Budget

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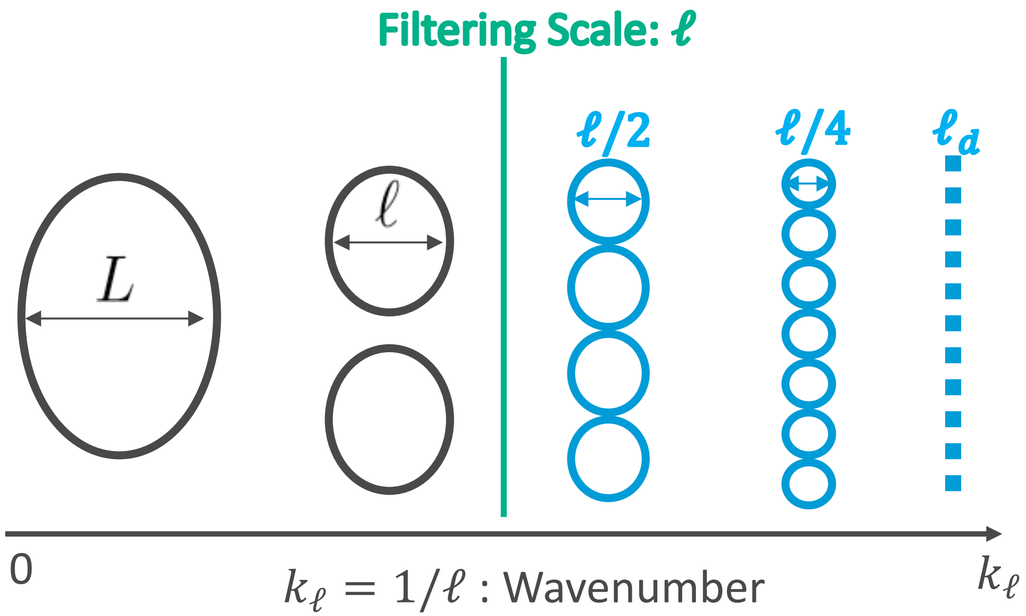


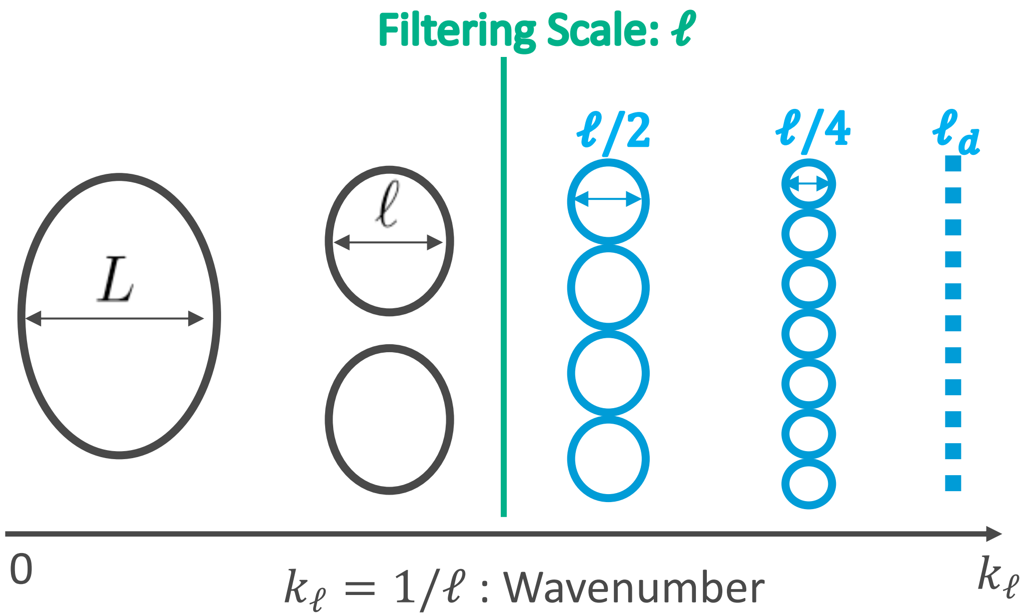
Temporal Change
of Coarse KE

$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \nabla \cdot J_\ell = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p, \nabla \cdot \boldsymbol{u}) - (-\overline{P\nabla \cdot u_\ell}) - D_\ell$$

Coarse-grained KE Budget

$$\bar{f}_\ell(\boldsymbol{x}) = \int d\boldsymbol{r}\, G_\ell(\boldsymbol{r}) \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r})$$

$$\tilde{f}_\ell(\boldsymbol{x}) \equiv \frac{\overline{\rho f_\ell}}{\overline{\rho_\ell}}$$



$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \boldsymbol{\nabla}\cdot\boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p, \boldsymbol{\nabla}\cdot\boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla}\cdot u_\ell}) - D_\ell$$

Spatial Transport

Coarse-grained KE Budget

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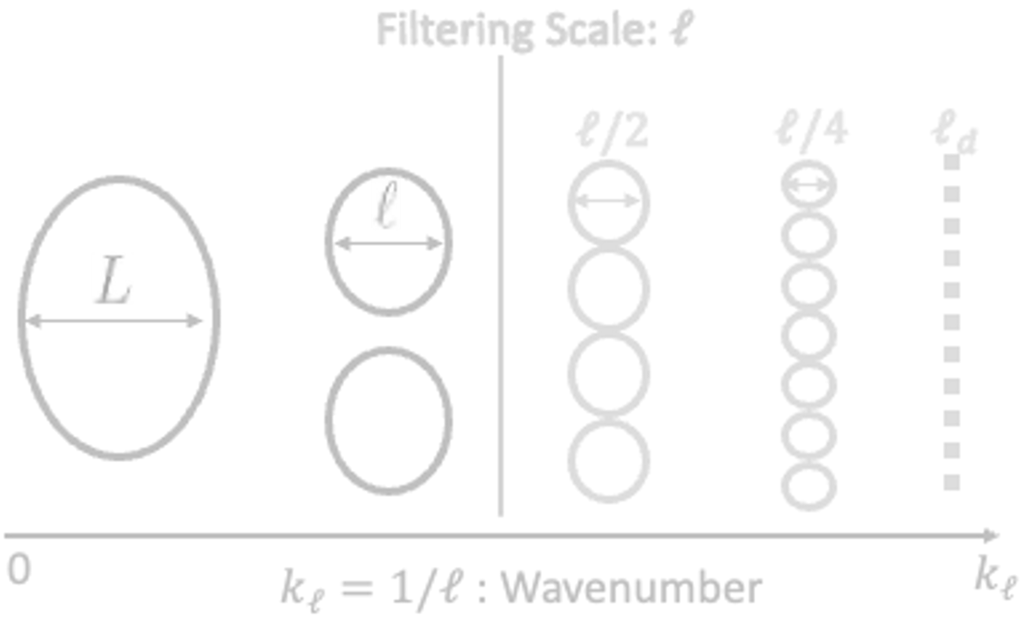

No transfer in scale.
Zero on average for spatially homogeneous flows.

$$\partial_t \left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2} \right) + \cancel{\boldsymbol{\nabla} \cdot \boldsymbol{J}_\ell} = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p, \boldsymbol{\nabla} \cdot \boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla} \cdot u_\ell}) - D_\ell$$

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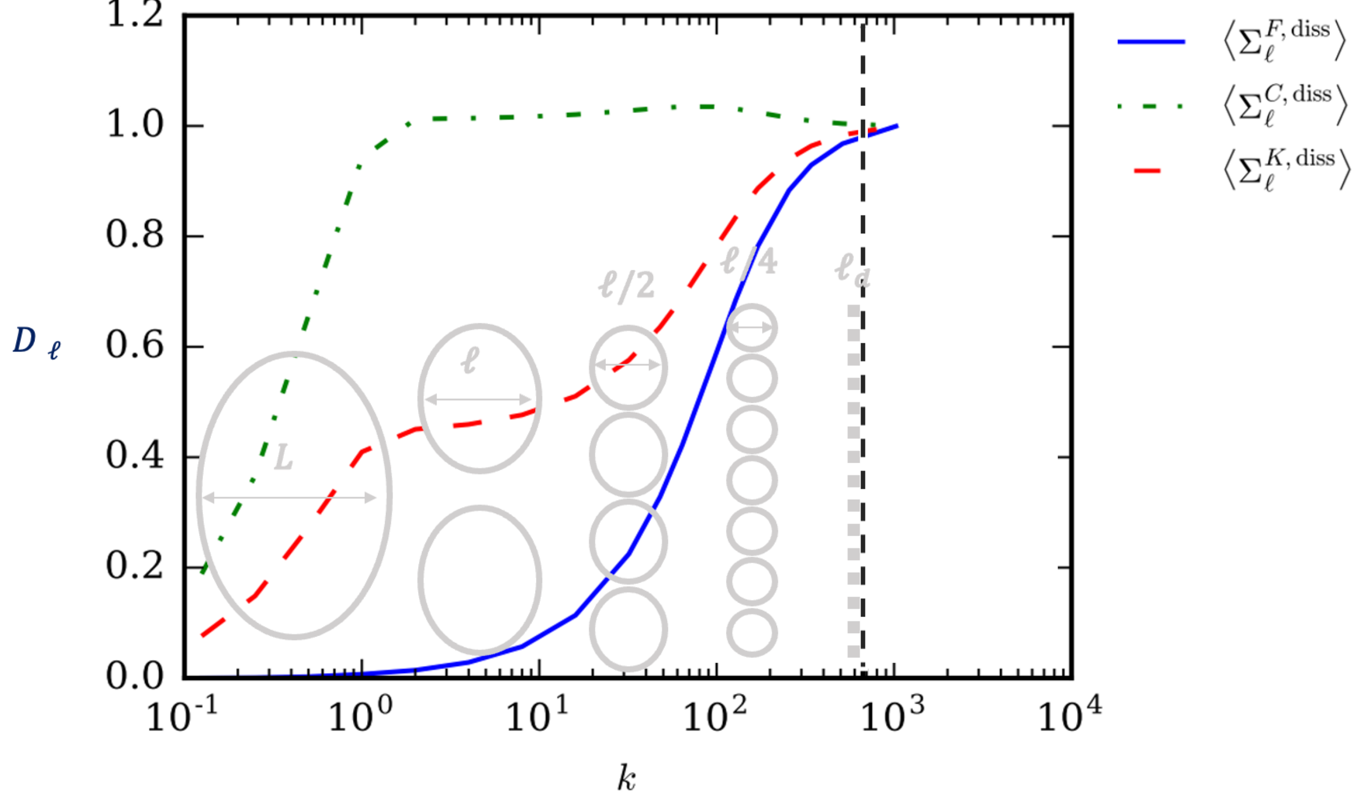
Zhao, Aluie, 2019, PRF



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Viscous Dissipation

Plays role at Smallest Scales[1]

1. Aluie, 2013, Physica D

Coarse-grained KE Budget

Filtering Scale: ℓ

$$\bar{f}_\ell(\boldsymbol{x}) = \int d\boldsymbol{r}\, G_\ell(\boldsymbol{r})\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r})$$

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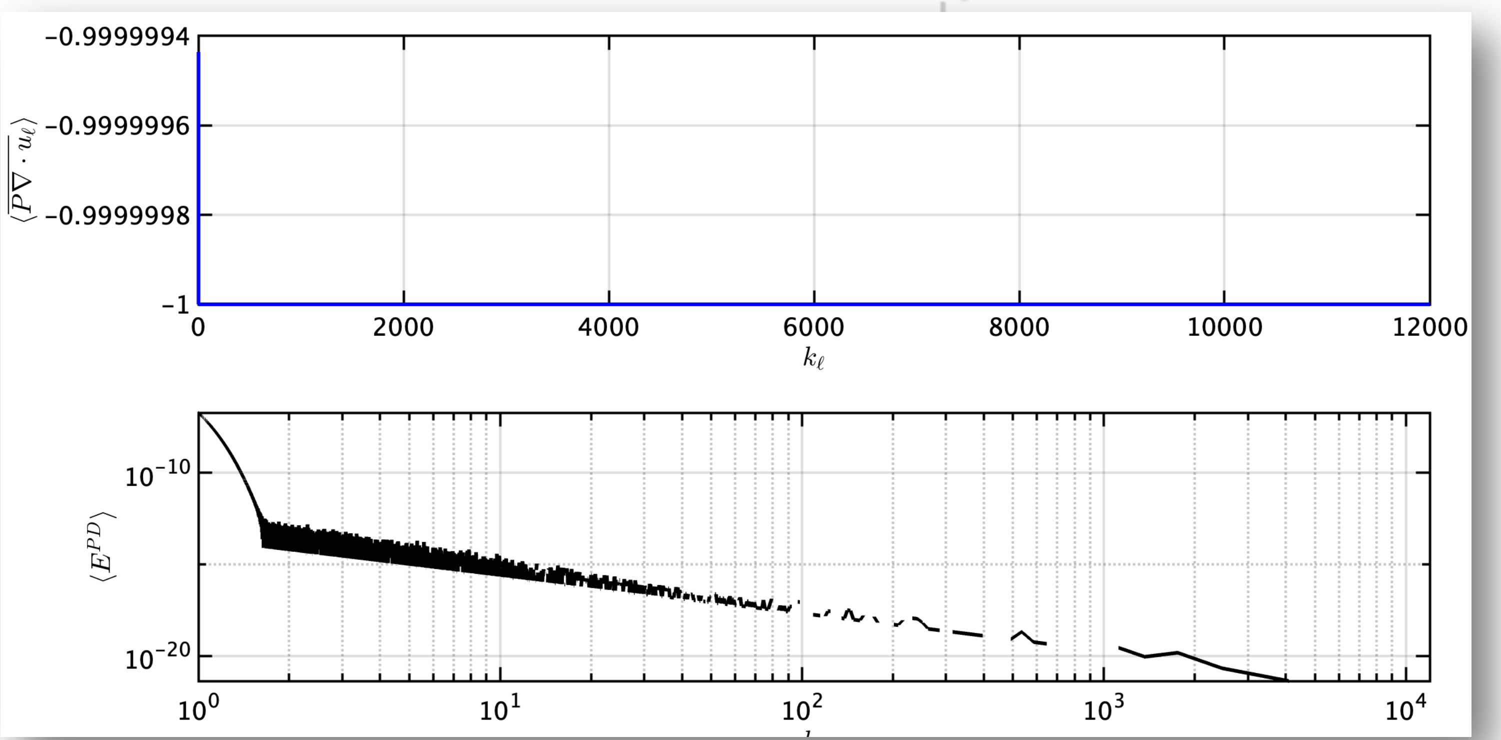

Acts as a large-scale forcing

$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \nabla\cdot \boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p, \nabla\cdot \boldsymbol{u}) - \cancel{(-\overline{P\nabla\cdot u_\ell})} - D_\ell$$

Pressure-Dilatation

Coarse-grained KE Budget

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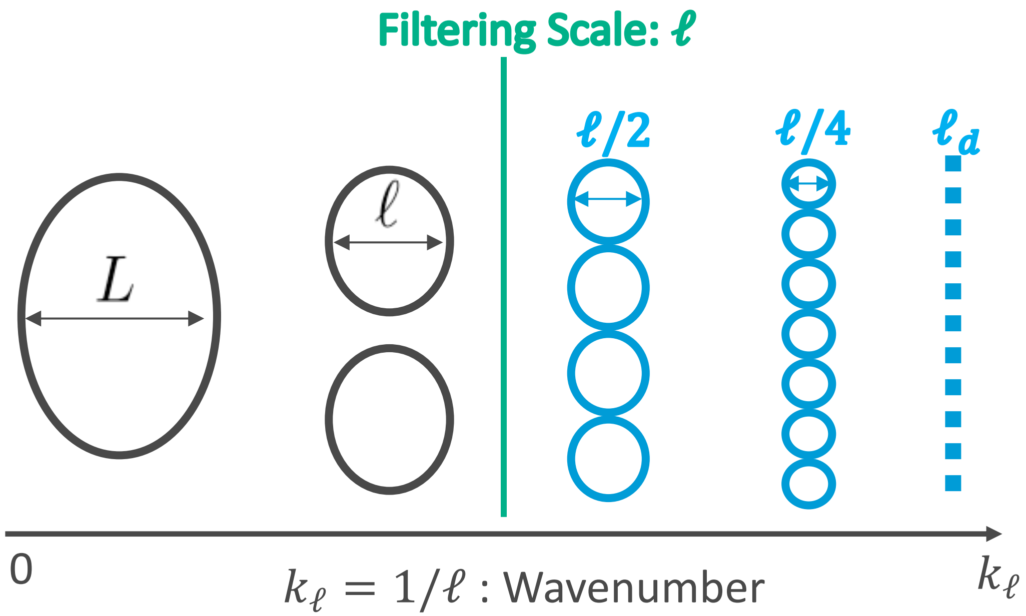

$$\Lambda_\ell(\mathrm{x}) = \frac{1}{\bar{\rho}_\ell}\partial_x \bar{p}_\ell \left[\overline{(\rho u)}_\ell - \bar{\rho}_\ell \bar{u}_\ell\right]$$

Baropycnal Work

$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \boldsymbol{\nabla}\cdot \boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \bar{\tau}(p, \boldsymbol{\nabla}\cdot\boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla}\cdot u_\ell}) - D_\ell$$

$$\Pi_\ell(\mathrm{x}) = -\partial_x \bar{u}_\ell \left[\overline{(u^2)}_\ell - \bar{u}_\ell^2\right]$$

Deformation Work

$$\bar{\tau}_\ell(\mathrm{x}) = \overline{(p\, \nabla\cdot u)}_\ell - \bar{p}_\ell \overline{\nabla\cdot u_\ell}$$

Pressure-Dilatation Defect

Coarse-grained KE Budget

$$\bar{f}_\ell(\boldsymbol{x}) = \int d\boldsymbol{r}\, G_\ell(\boldsymbol{r})\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r})$$

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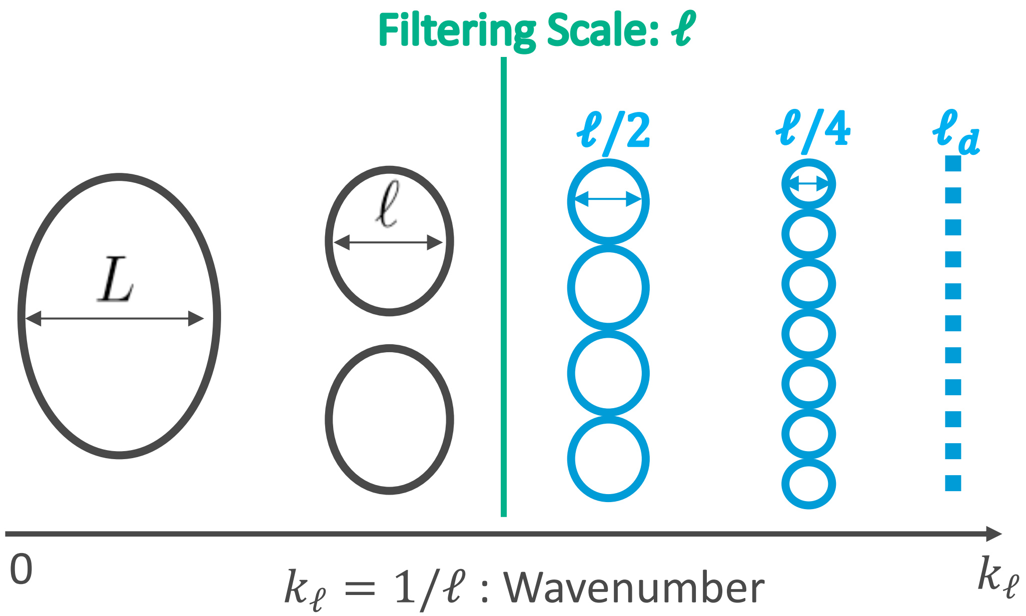
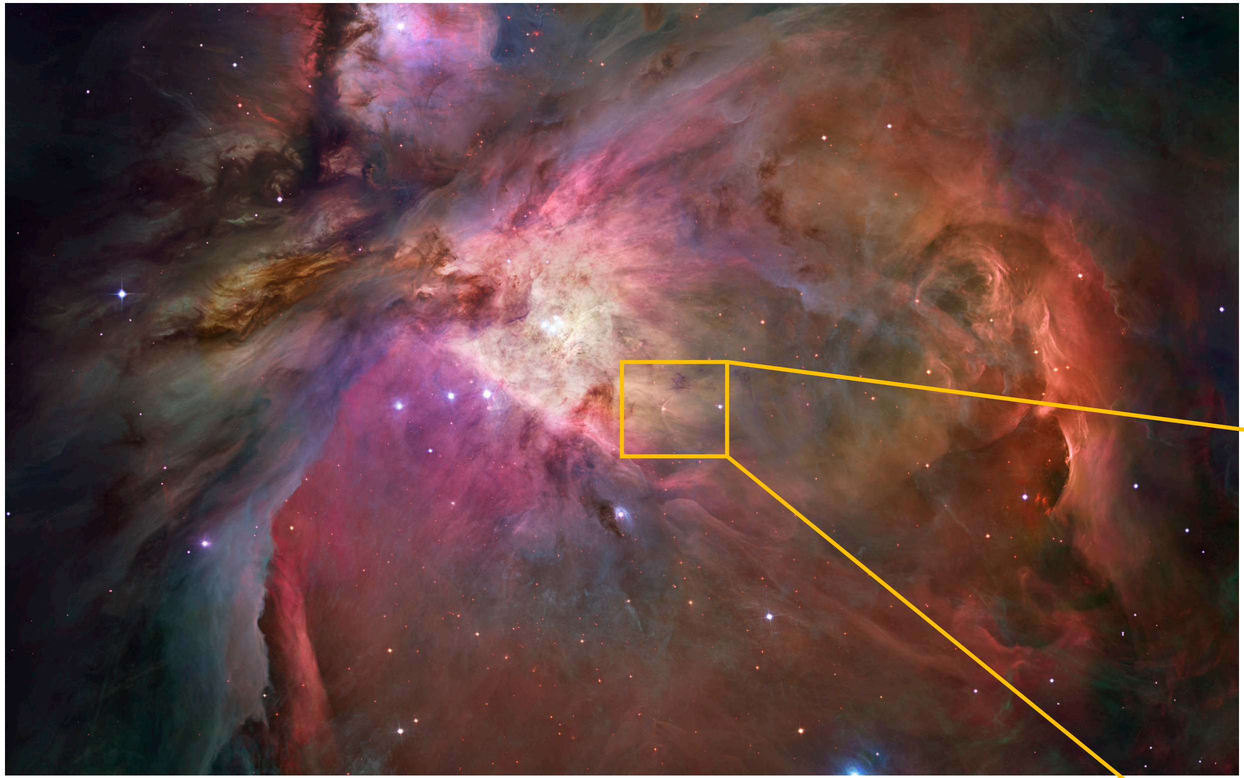


$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \boldsymbol{\nabla}\cdot \boldsymbol{J}_\ell = \boxed{-\Pi_\ell - \Lambda_\ell - \bar{\tau}(p, \boldsymbol{\nabla}\cdot\boldsymbol{u})} - (-\overline{P\boldsymbol{\nabla}\cdot u_\ell}) - D_\ell$$

Sub-scale Flux Mechanisms

There is an unrefined intuition about shocks …



Messier 42: Orion Nebula [Source: NASA]

… some portion of the energy at a given scale must be directly dissipated via shocks, rather than cascading conservatively through intermediate scales until dissipation scales is reached. [McKee & Ostriker (2007)]

Bow shock near young star, LL Ori
[Source: NASA]

1. McKee & Ostriker, 2007, Ann. Rev. Fluid. Mech.
2. Ferrand et al., 2020, ApJ.

Not just in the astrophysics community, …

Lake Superior, US [Source: UCAR]

… spectrally nonlocal pathway for downscale energy transfer that is phenomenologically distinct from traditional concepts of turbulent cascades and can contribute substantially to total kinetic energy dissipation.
[Samelson & Skyllingstad (2016)]

Kraichnan and Locality

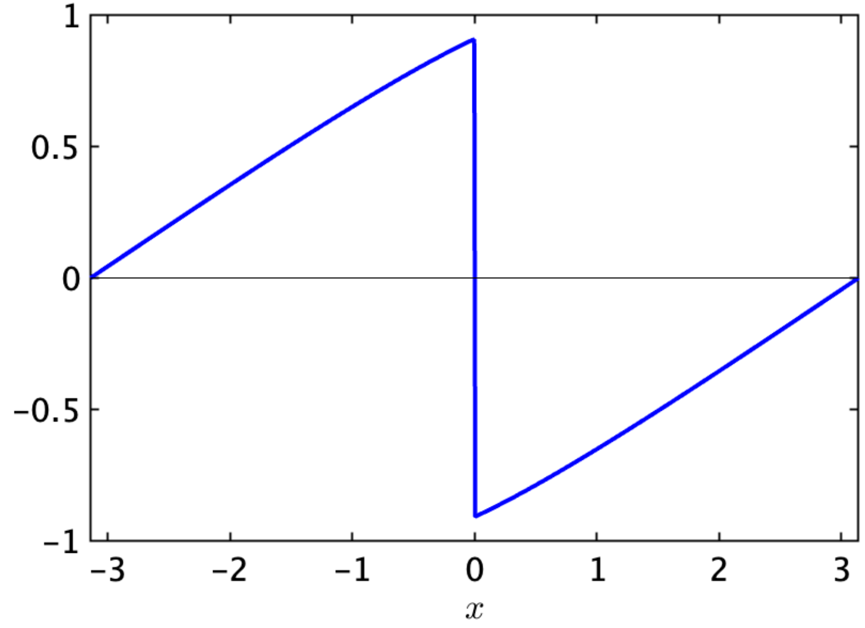
Kraichnan (1974):

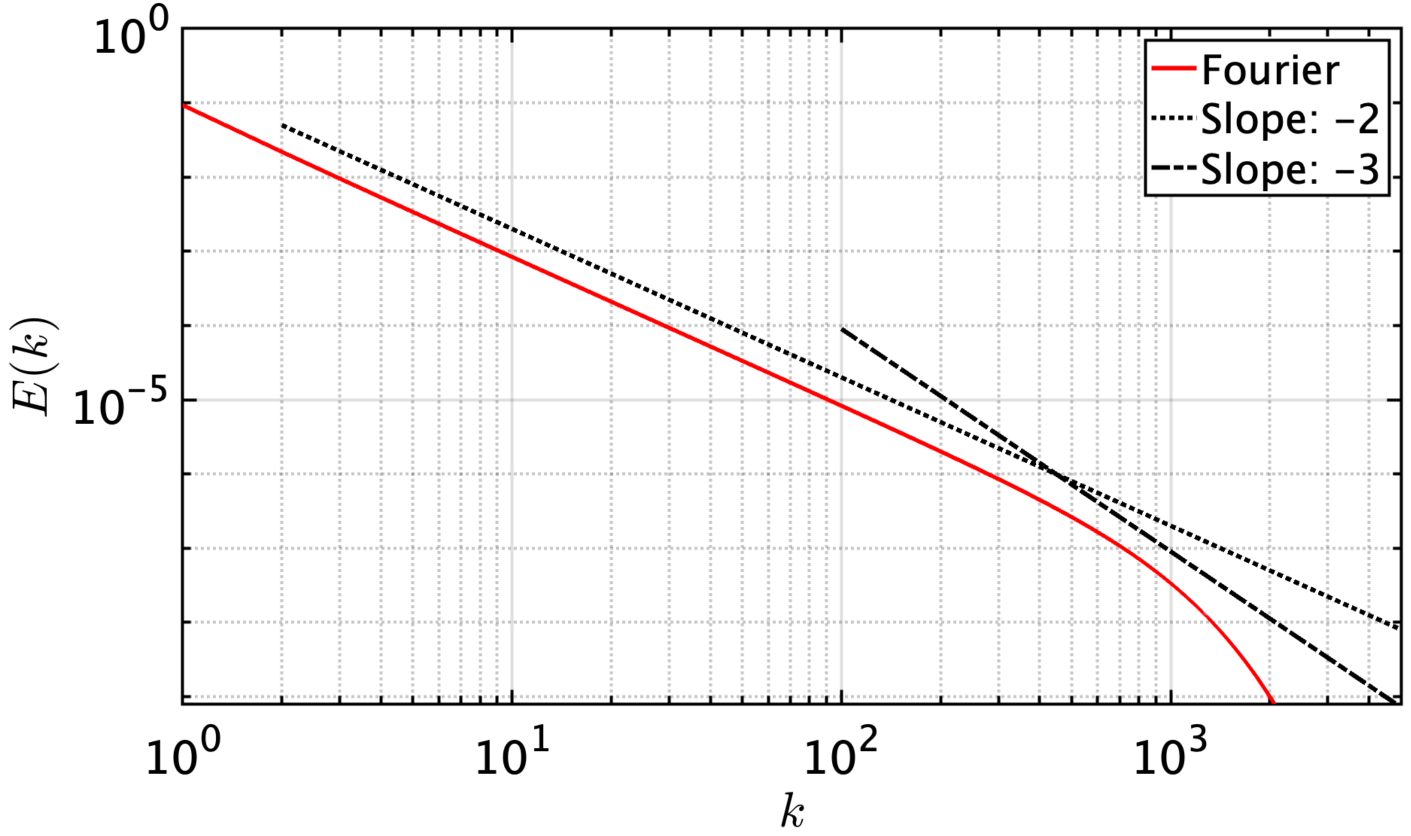
A generally unappreciated point is that the inertial-range energy cascade is local in wavenumber, even when the inertial-range spectrum is the spectral tail of coherent discontinuous structures,… .

Take Burgers Shock — A Clean Toy Model

$\partial_t u + u\partial_x u = \nu\partial_{xx},$

$u(x,0) = -sin\left(\frac{2\pi}{L}x\right).$





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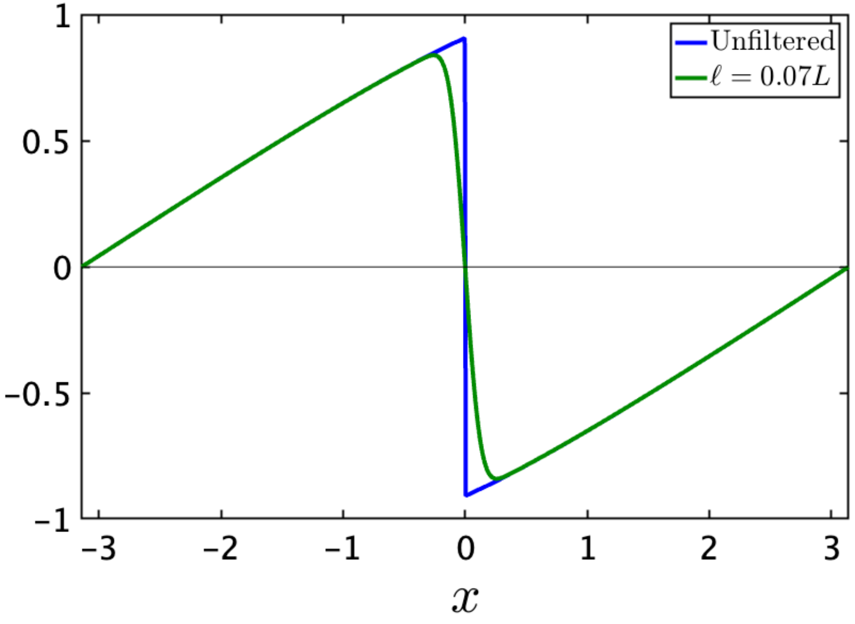

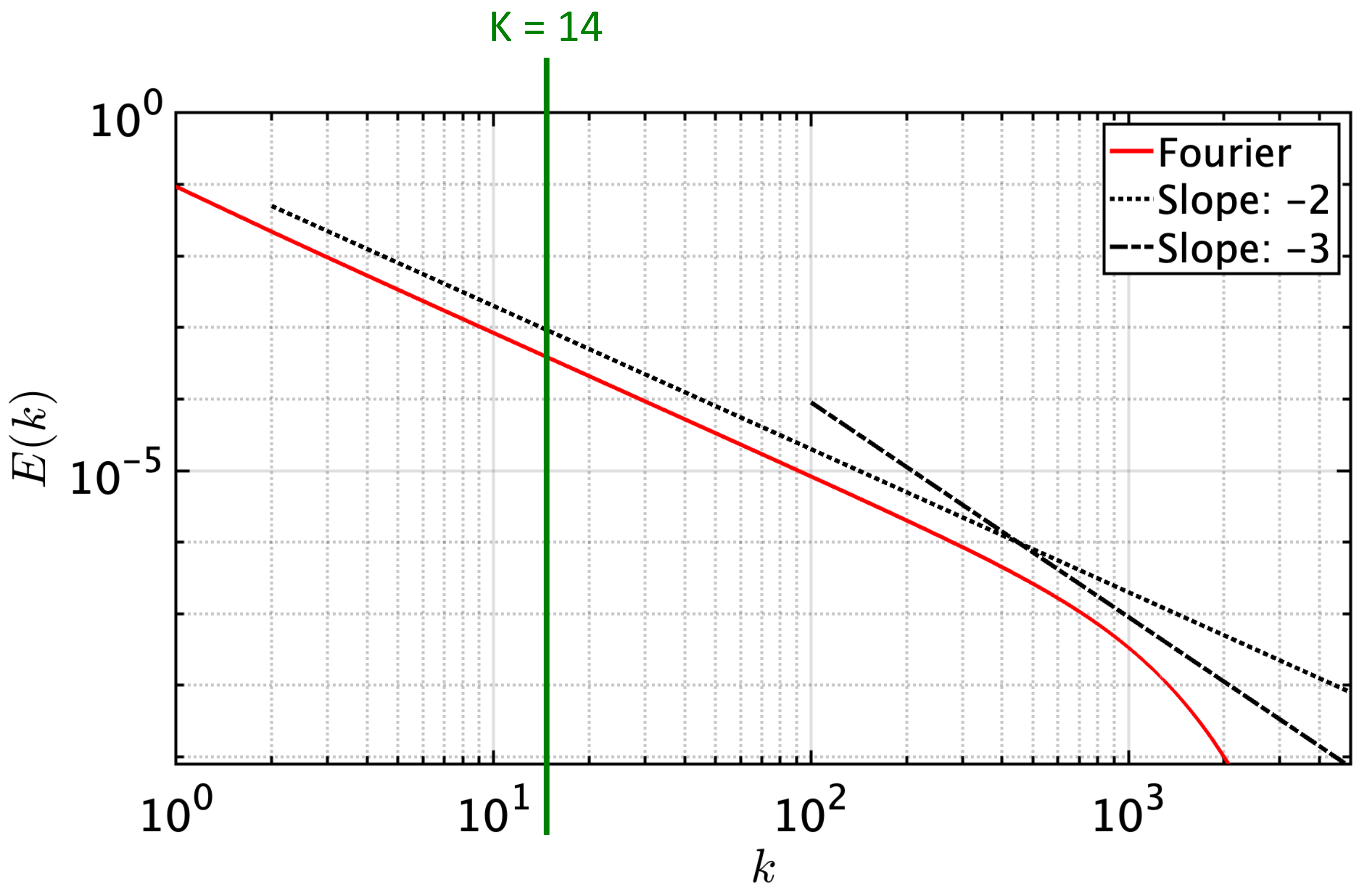

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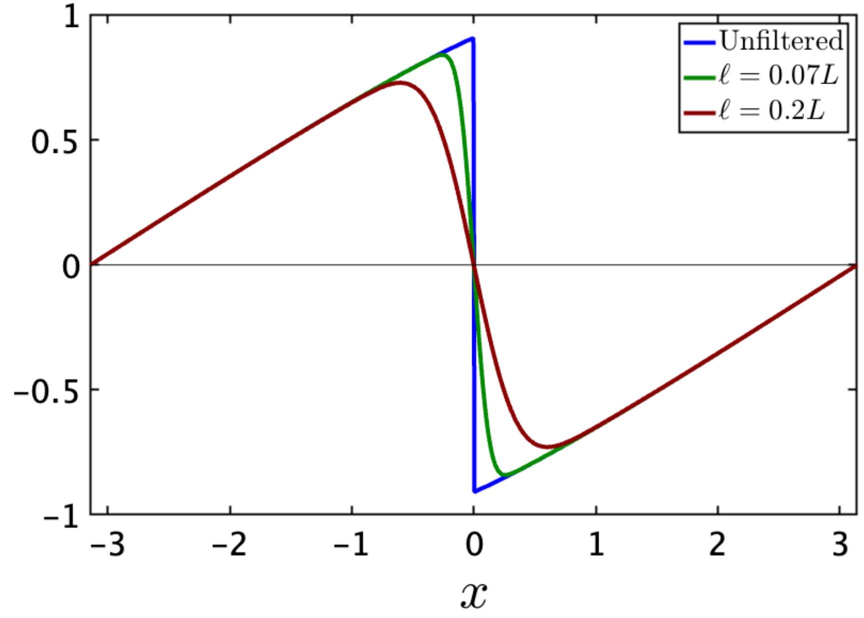

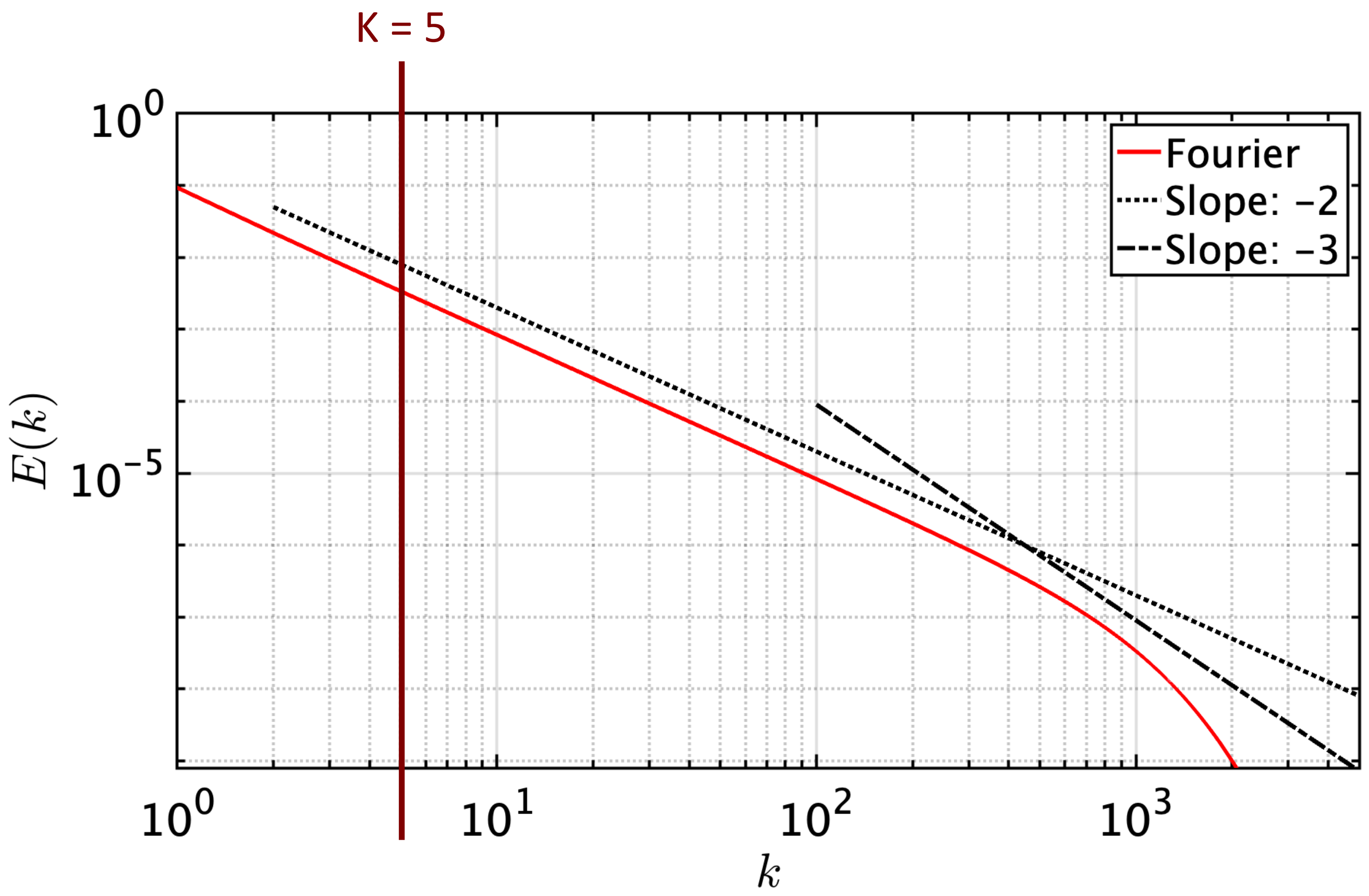

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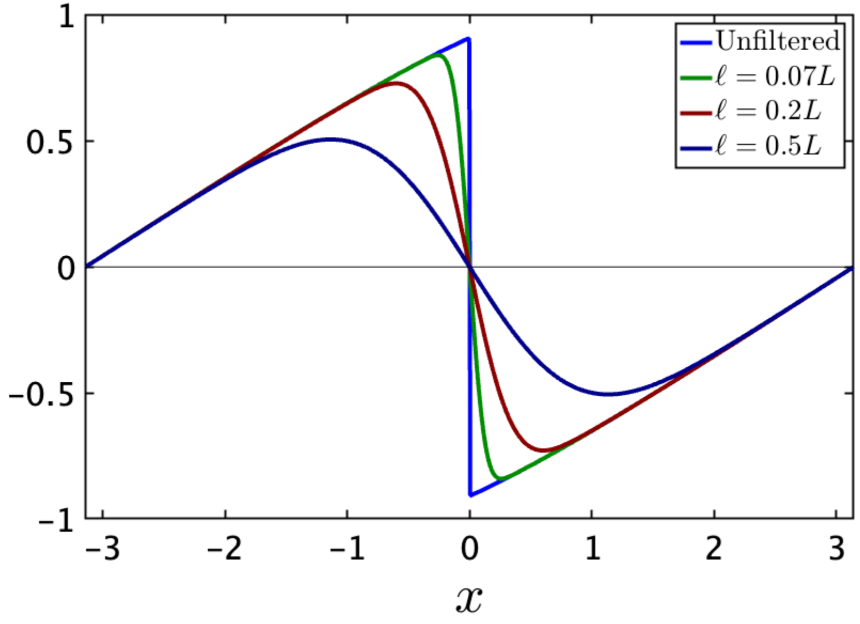

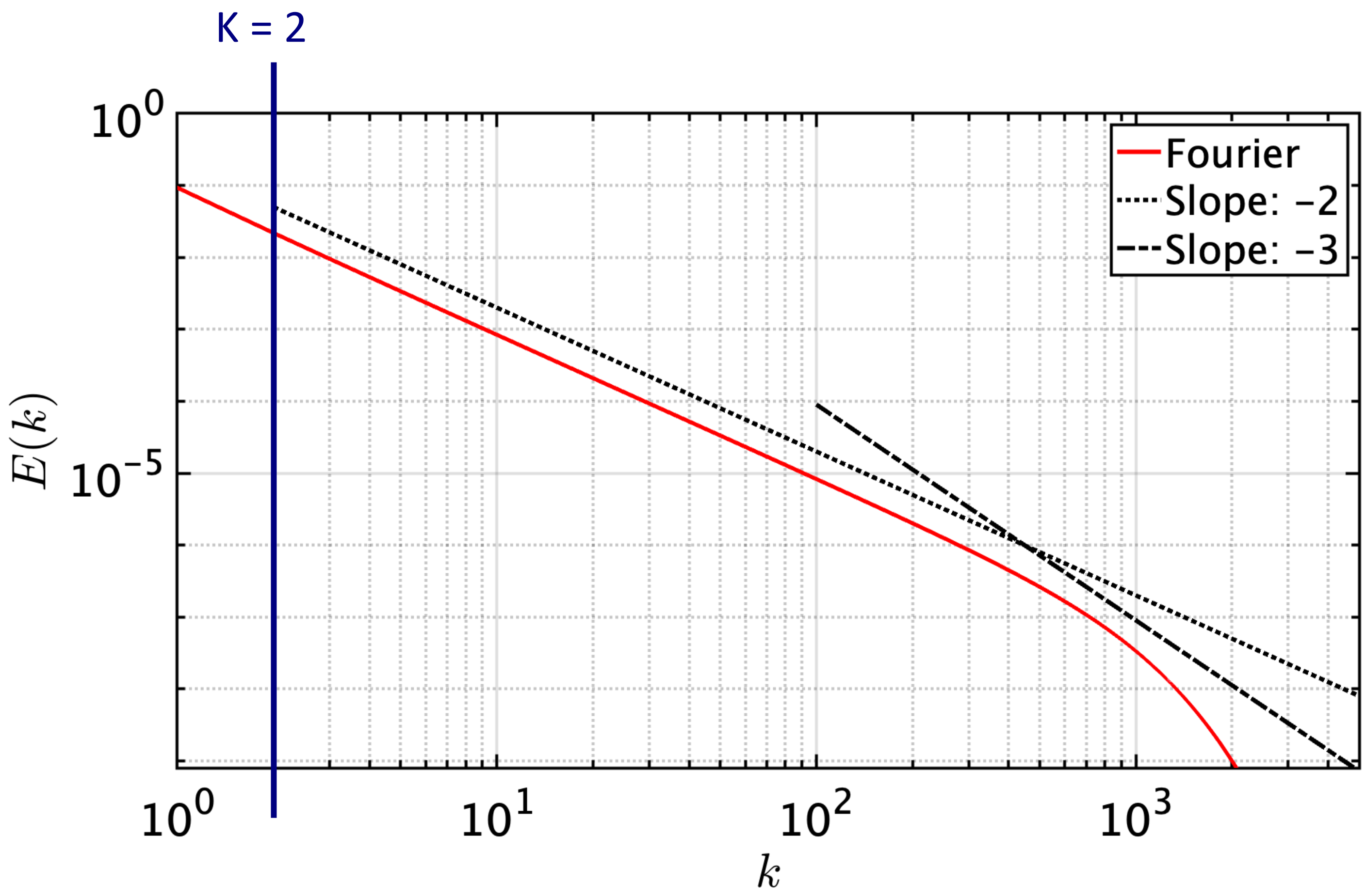

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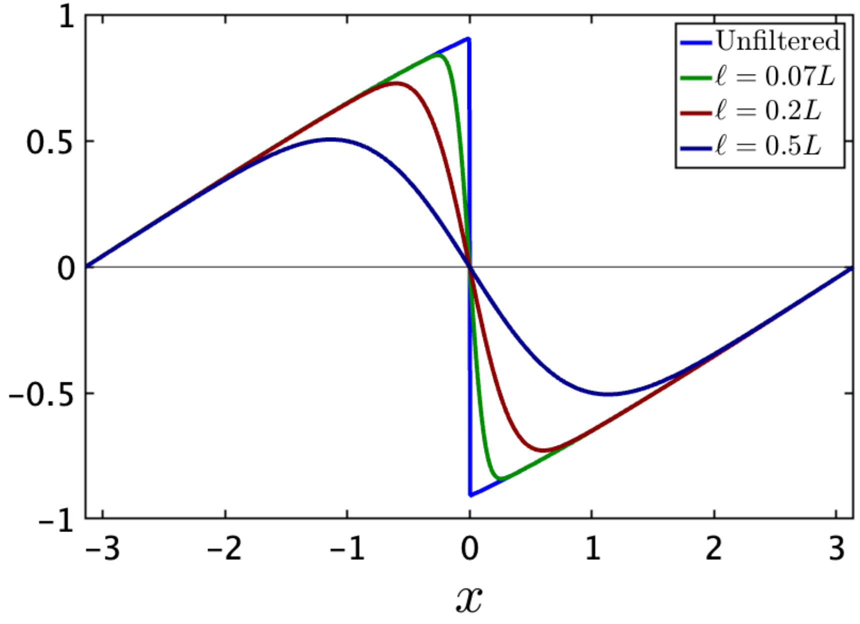

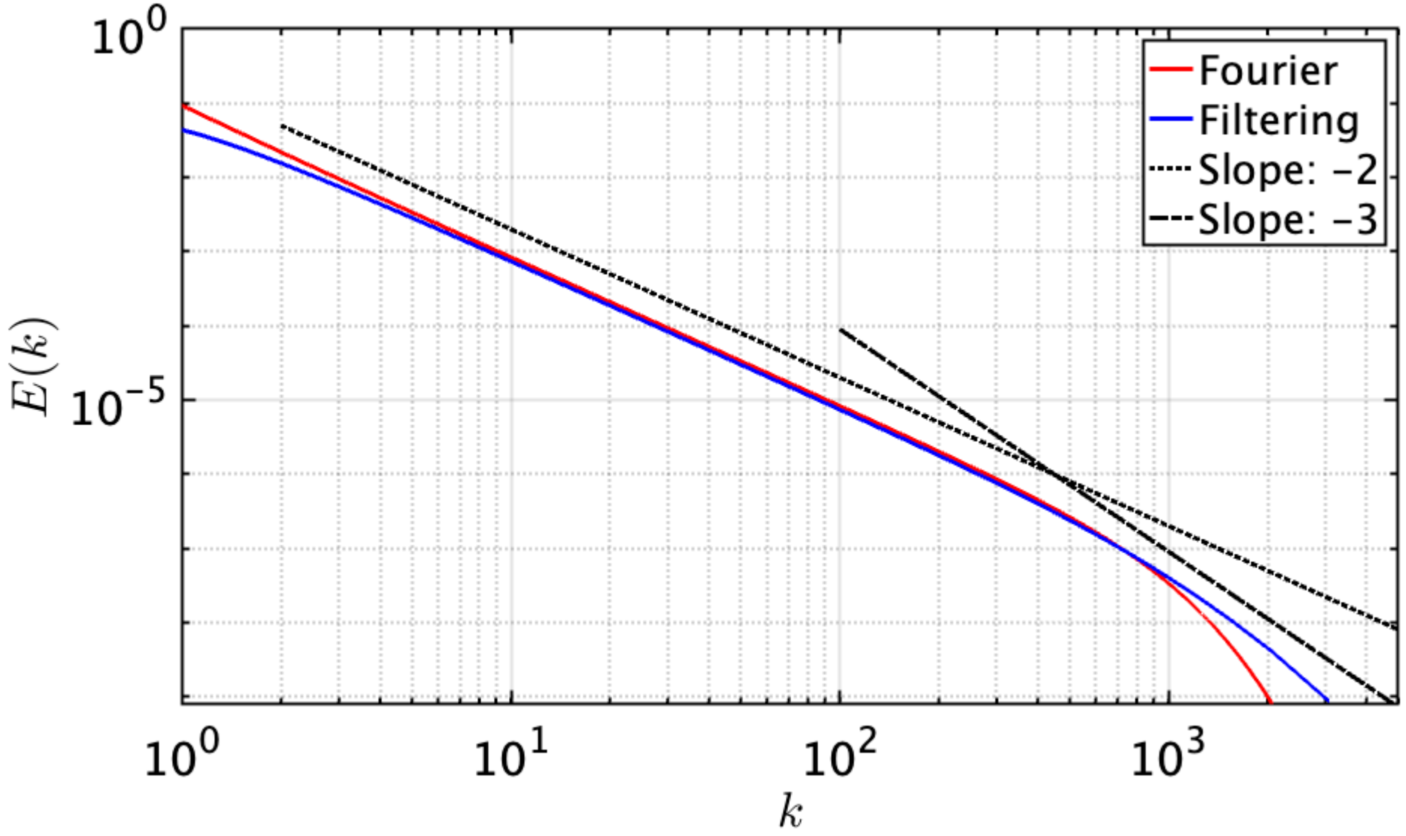

Even in 1D, shock is a multiscale structure!

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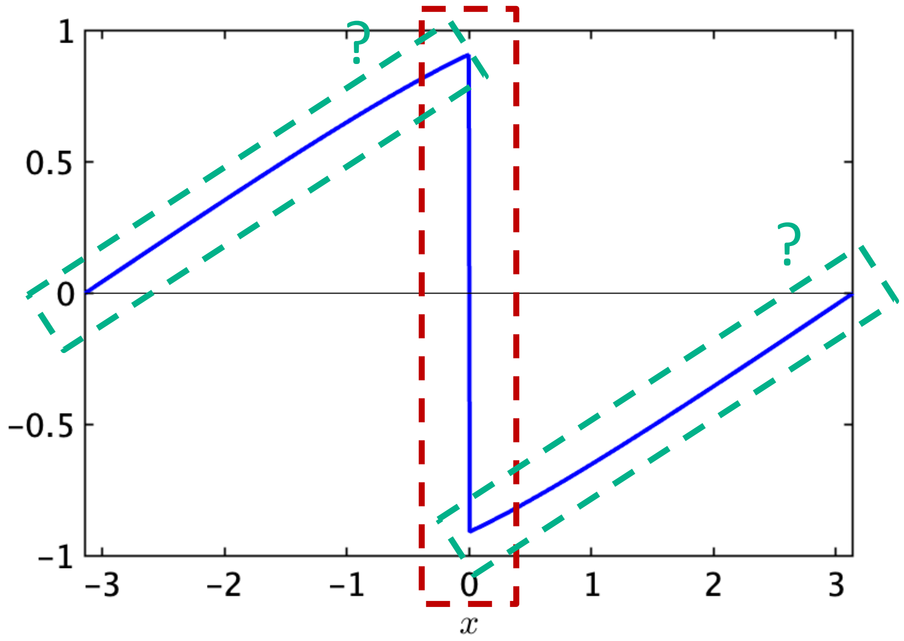

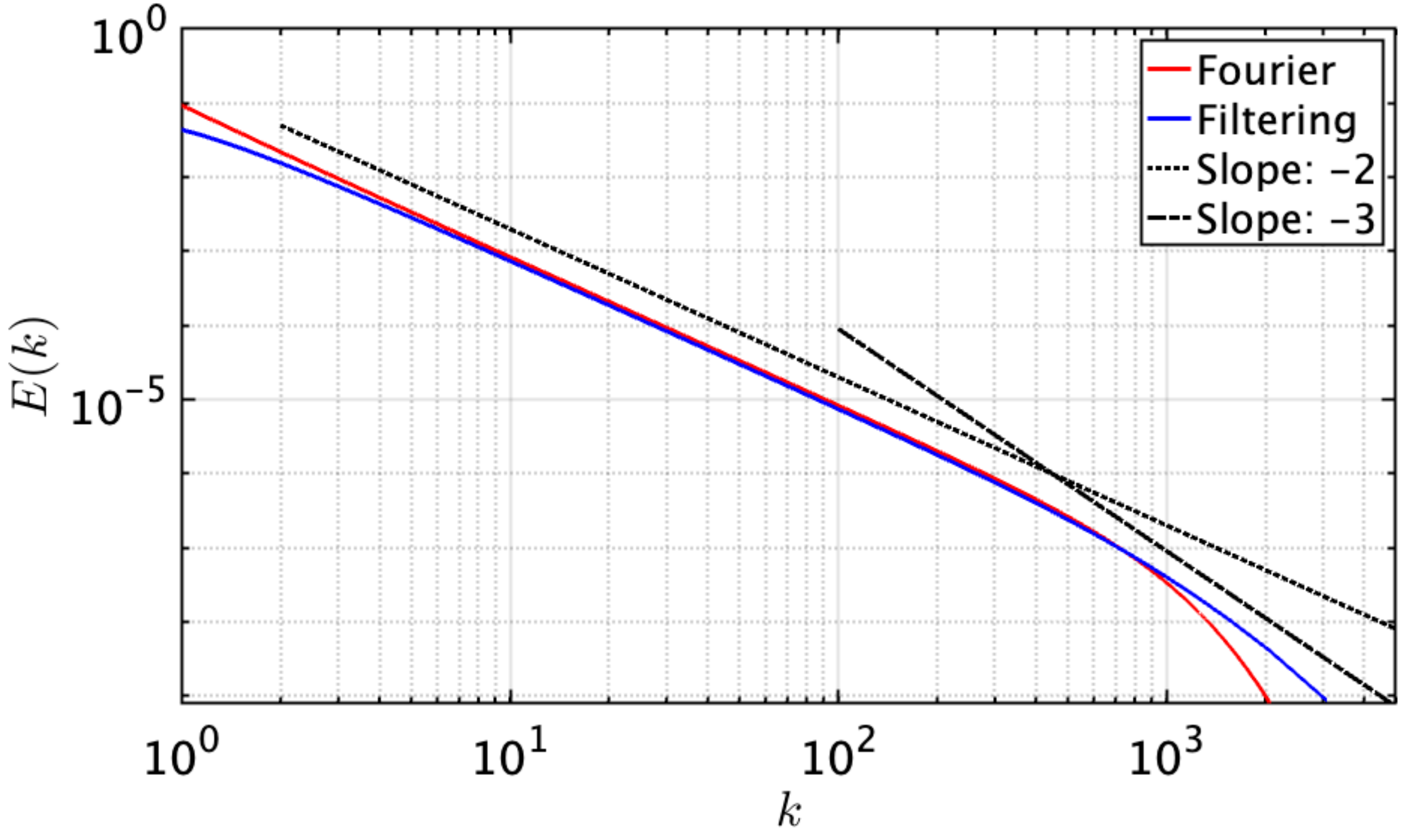

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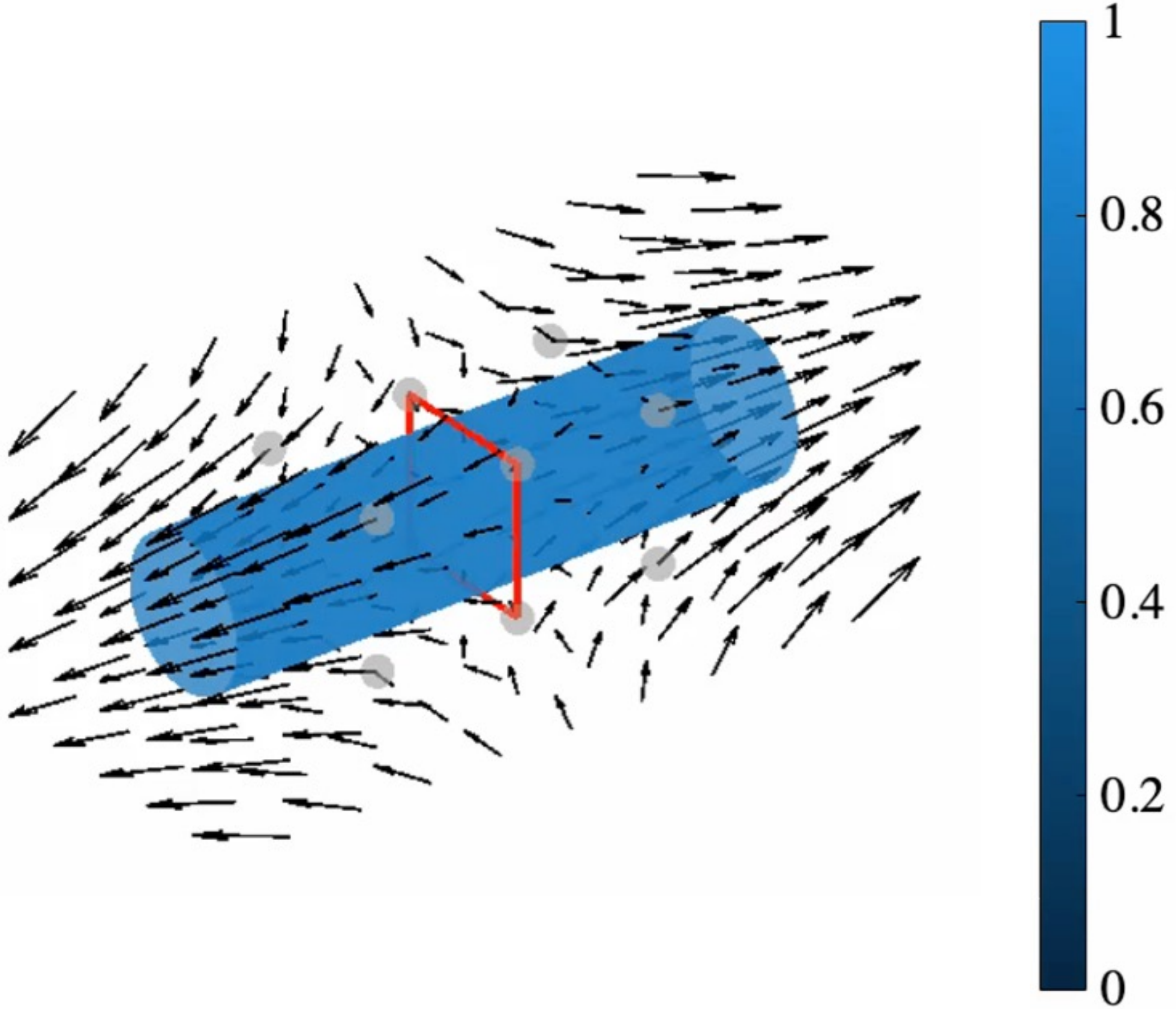
Cascade in Burgers Shock

$$\partial_t u + u\partial_x u = \nu\partial_{xx},$$

$$\partial_t \frac{|\bar{u}_\ell|^2}{2} = -\Pi_\ell - D_\ell,$$

$$\Pi_\ell(\mathrm{x}) = -\partial_x \bar{u}_\ell\left[\overline{(u^2)}_\ell - \bar{u}_\ell^2\right]$$

The mechanism that represents vortex stretching or energy cascade.



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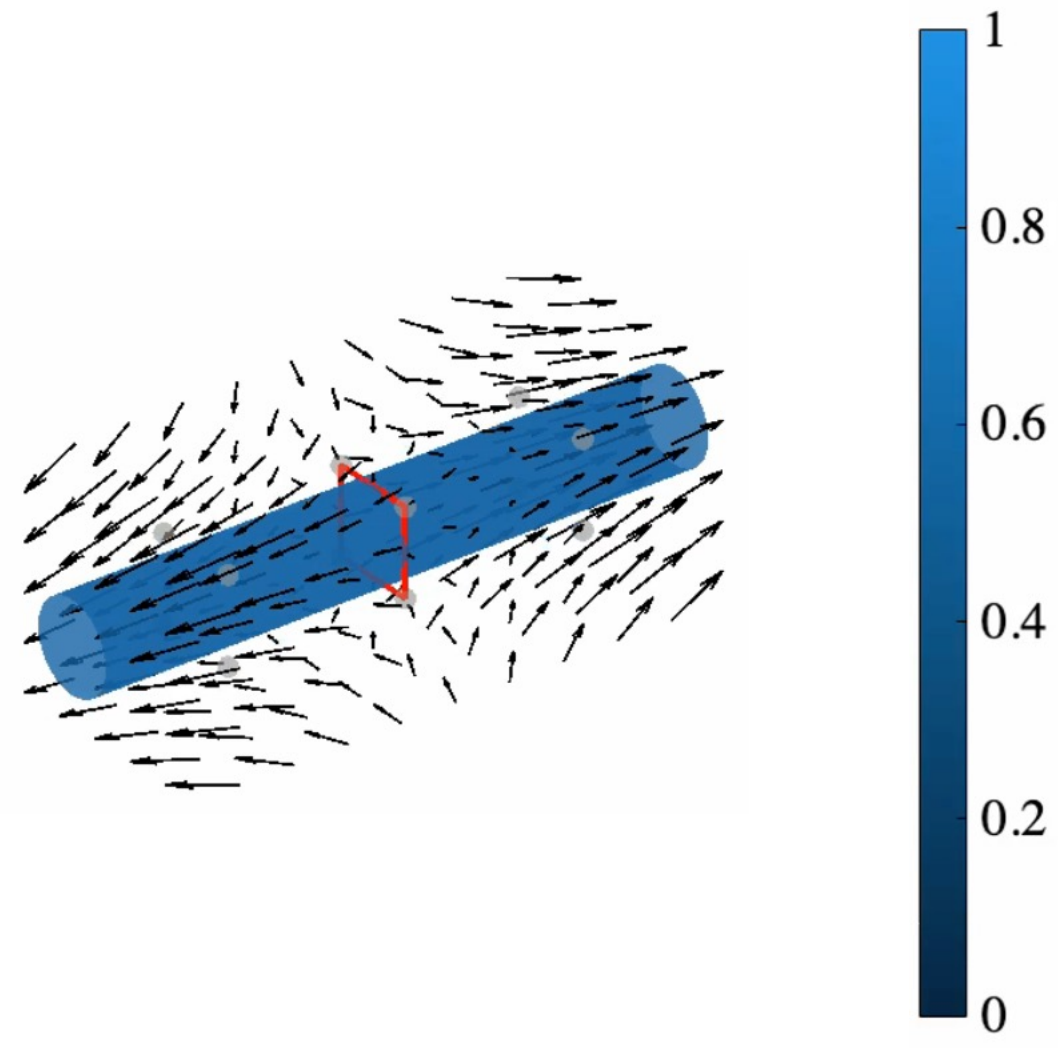

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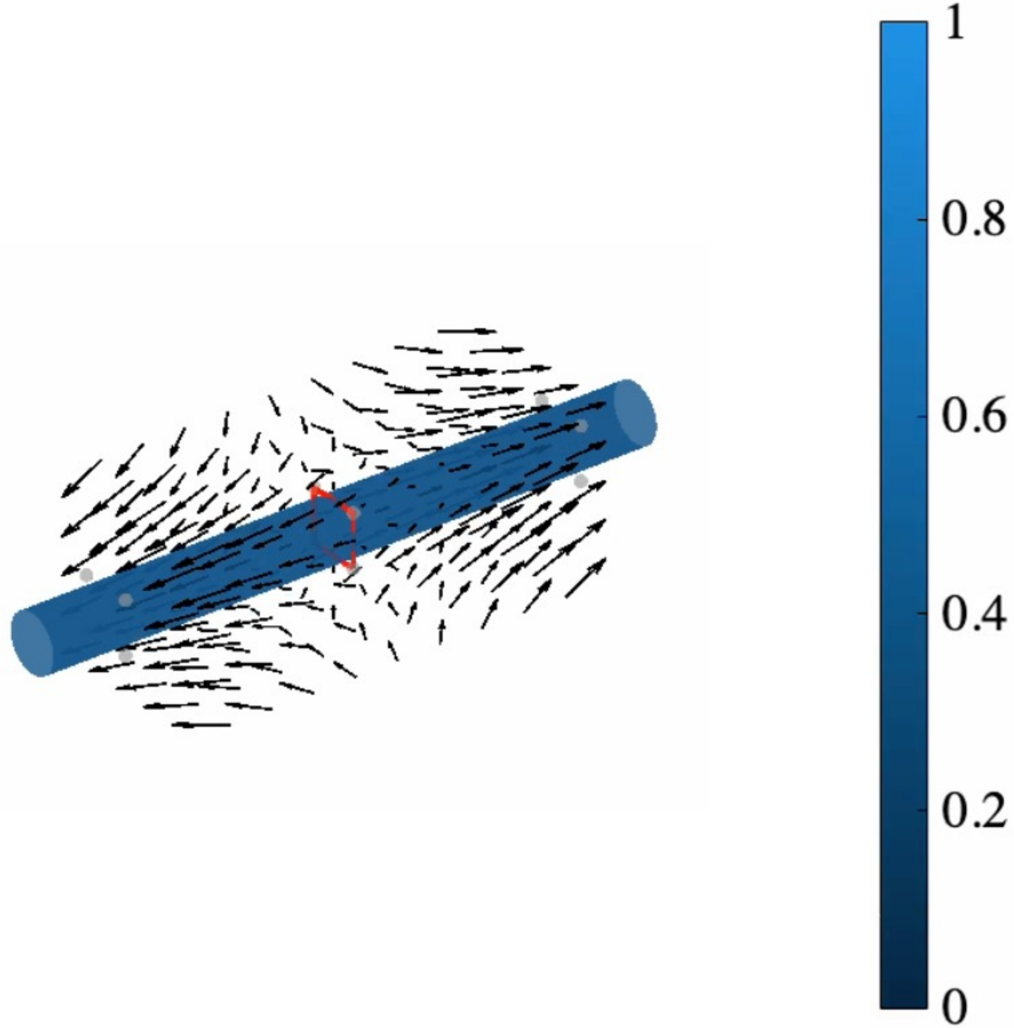

Cascade in Burgers Shock

$\partial_t u + u\partial_x u = \nu\partial_{xx},$

For $\ell = 0.15L$ $\longrightarrow$ $\bar{u}_\ell(x) = \int dr\, u(r) G_\ell\,(x+r)$

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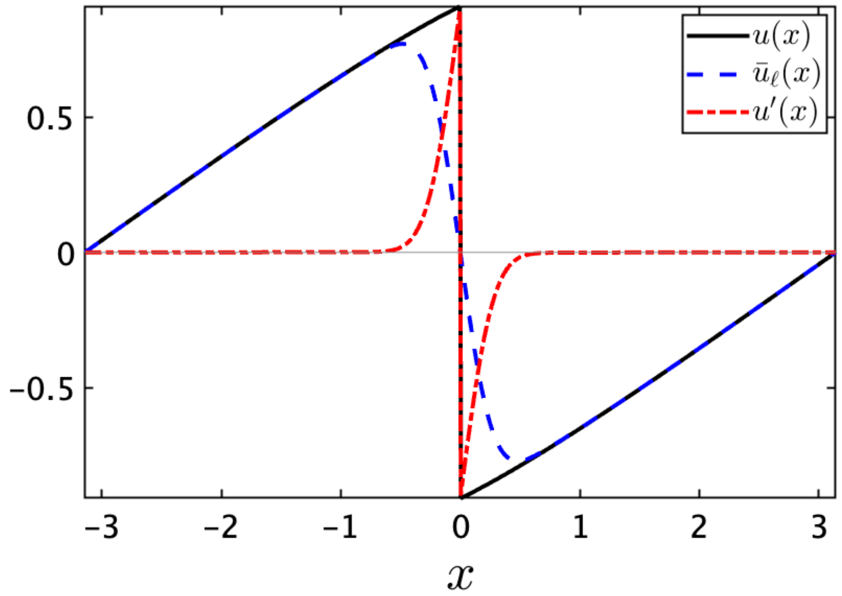

Deformation Work in Burgers Shock: $\boldsymbol{\Pi_\ell(x)}$

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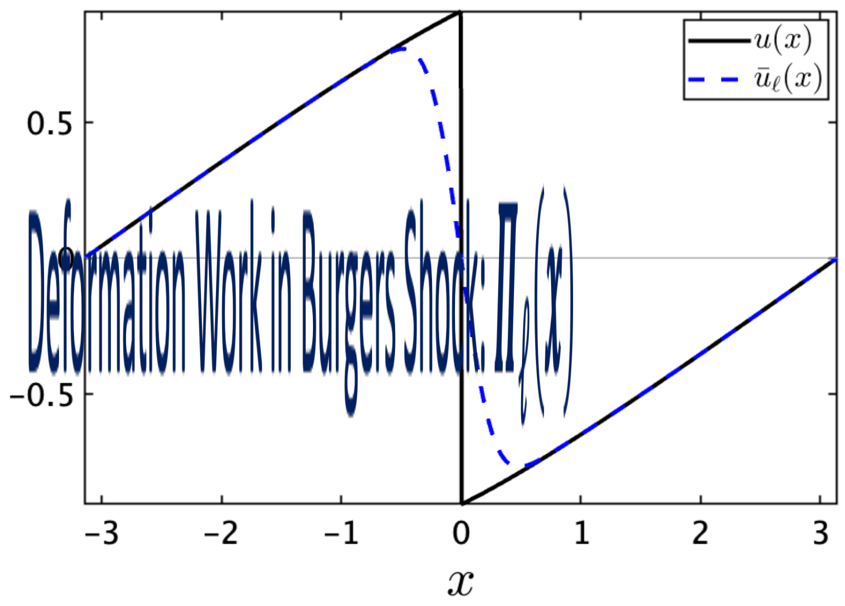

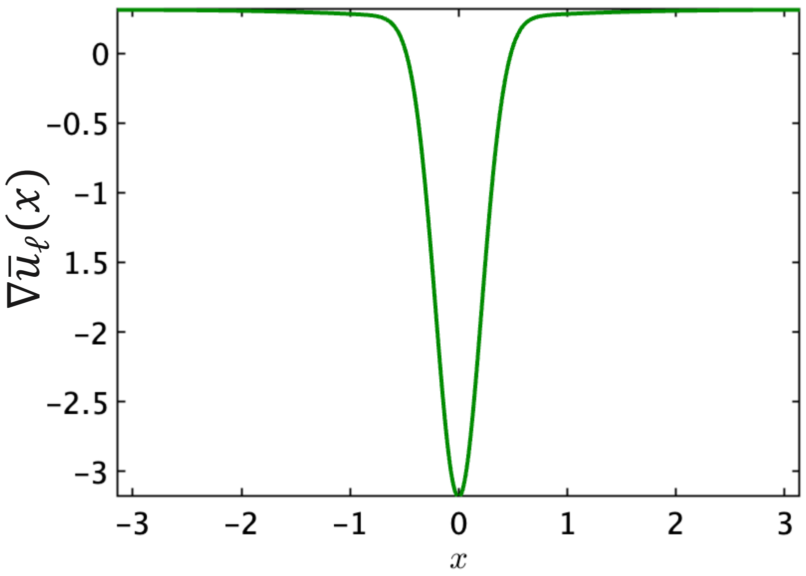

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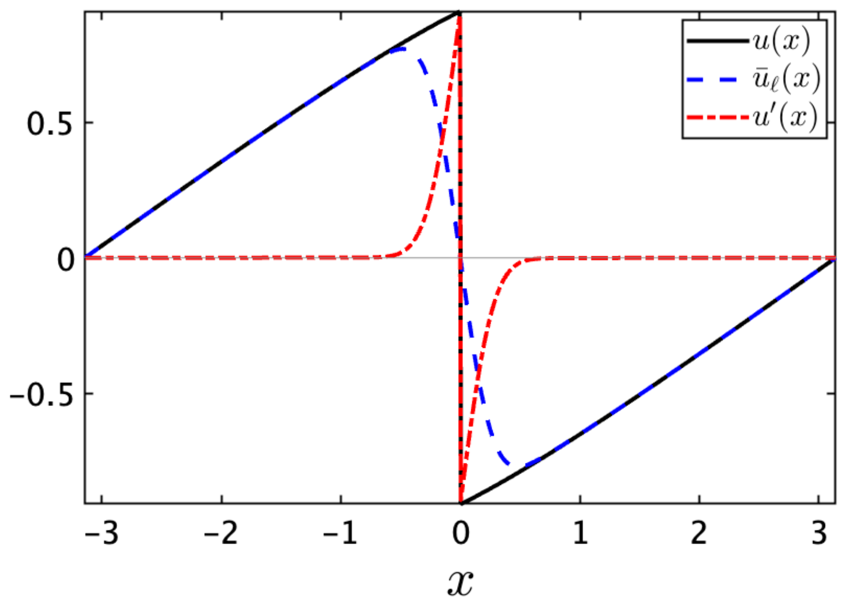

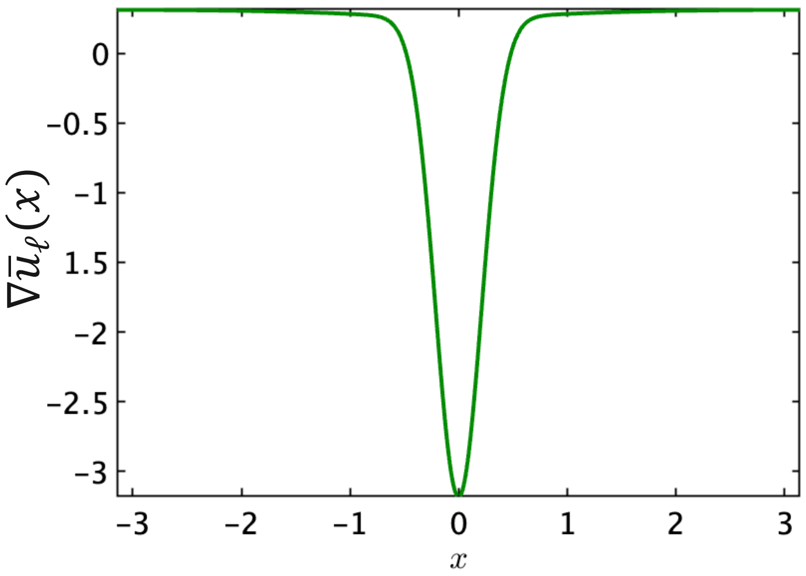

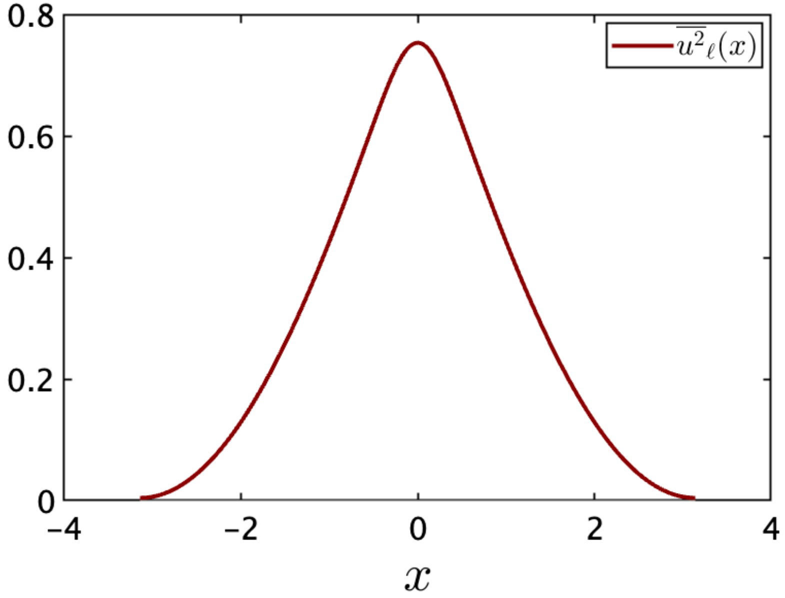

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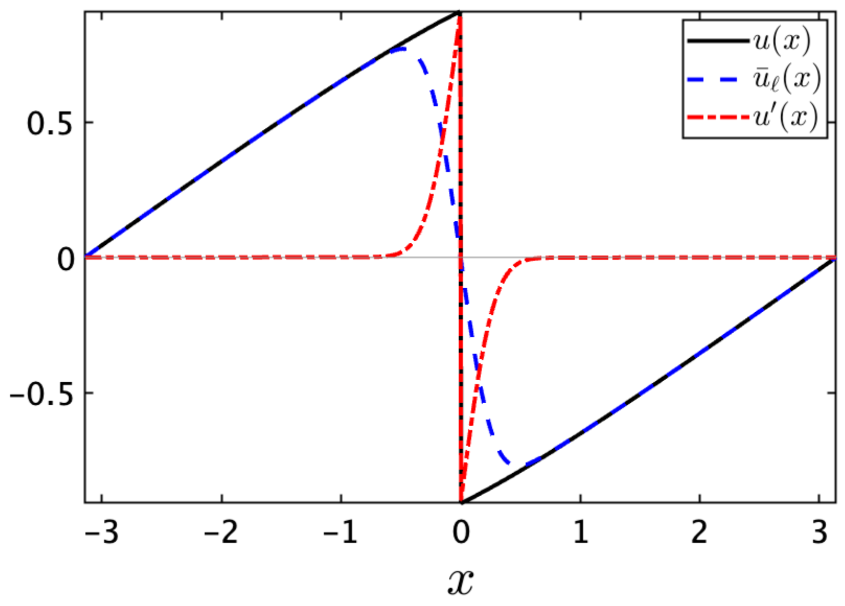

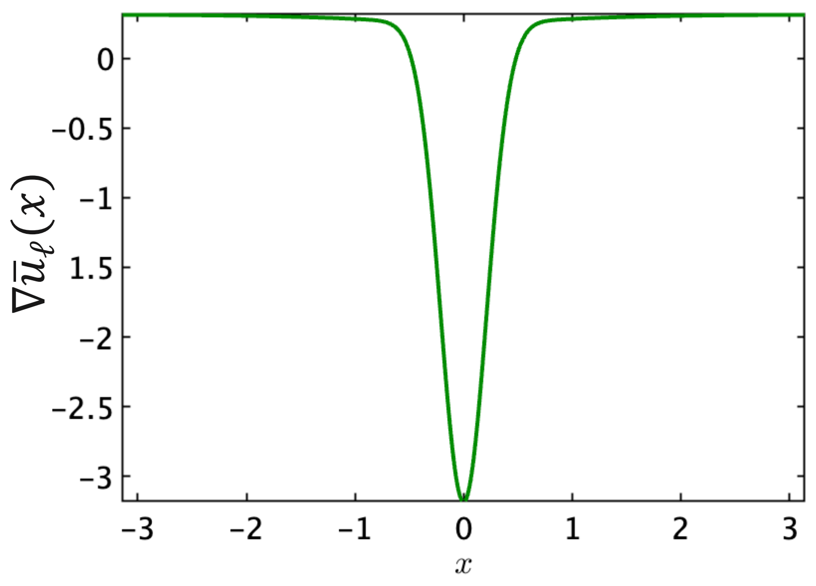

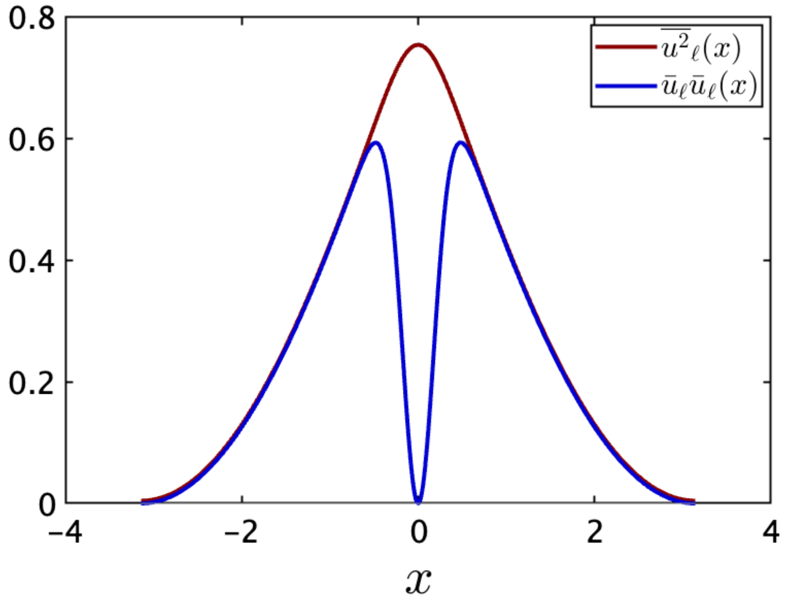

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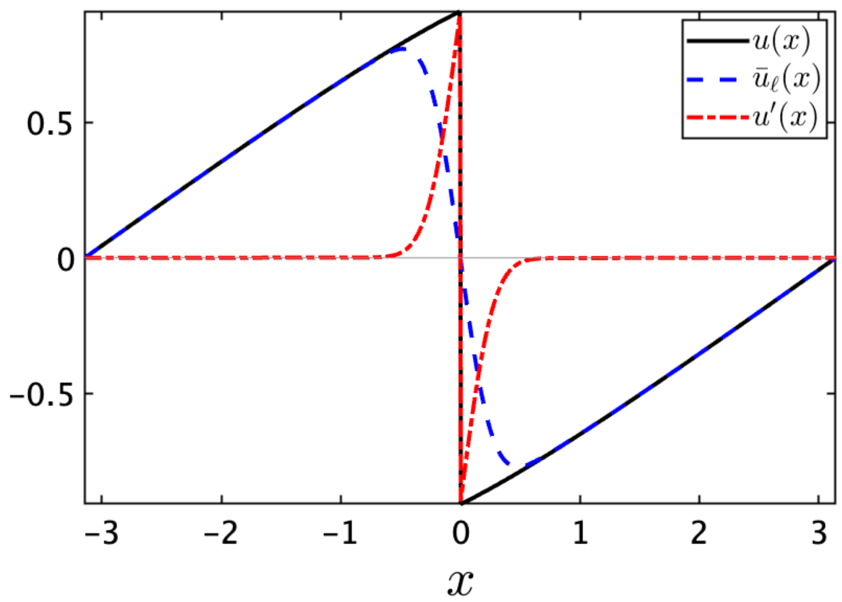
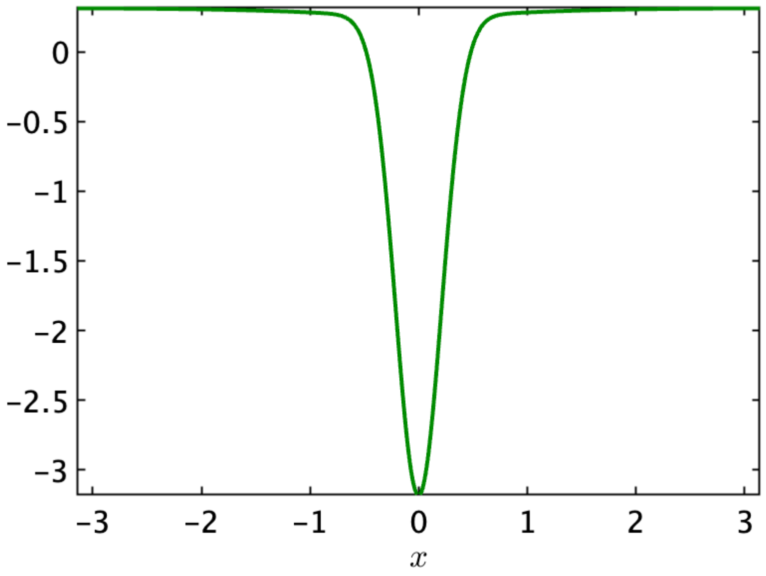
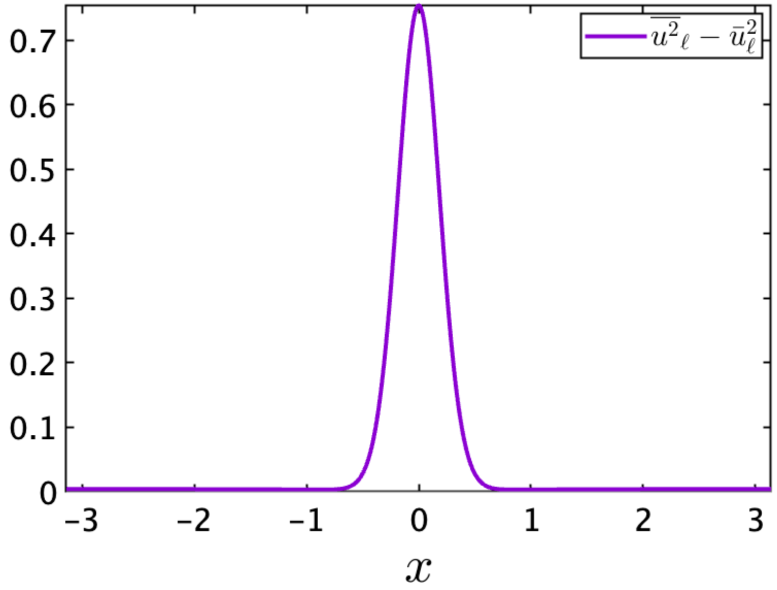


$\nabla \bar{u}_\ell(x)$



$\overline{u^2}_\ell - \bar{u}_\ell^2$



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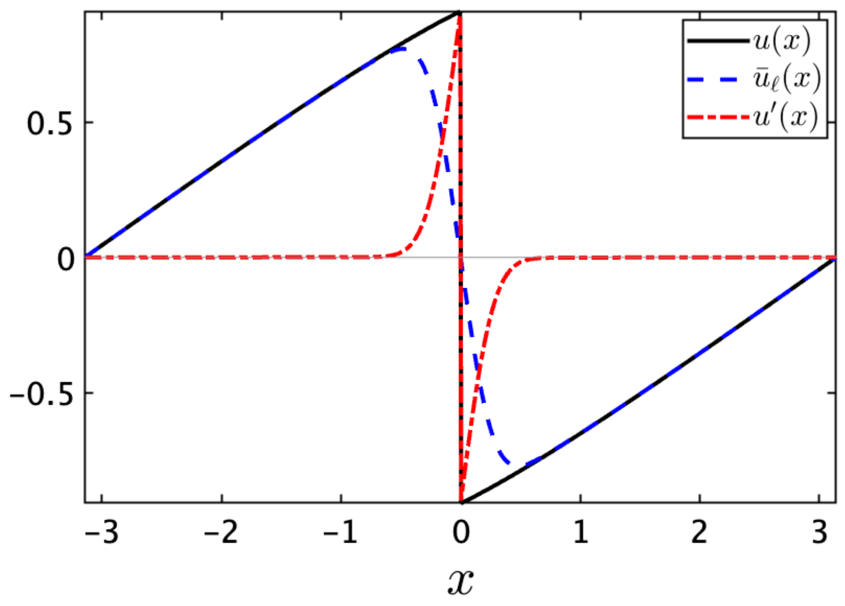

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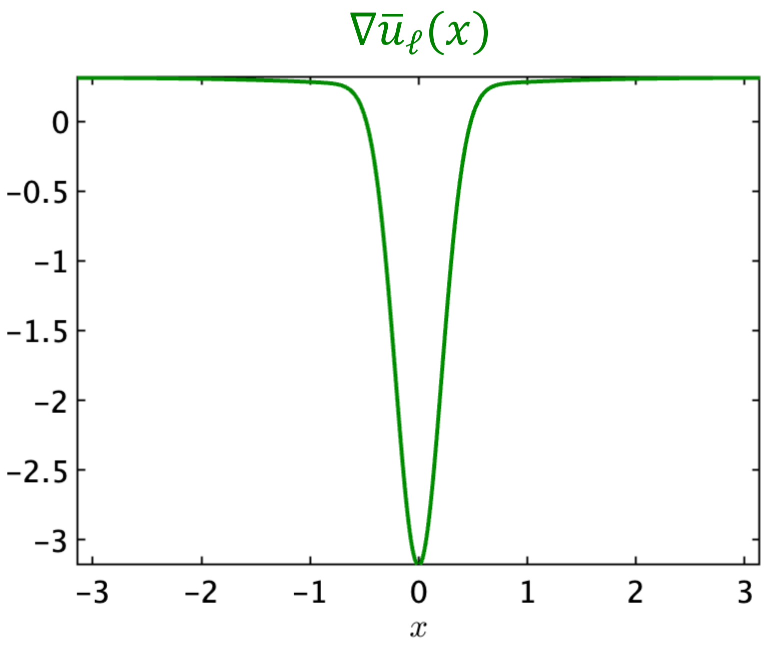
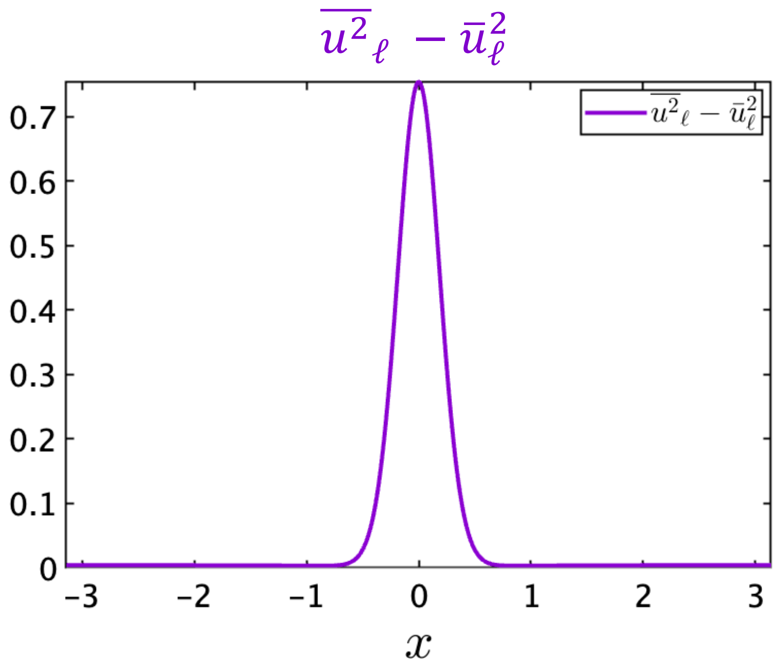
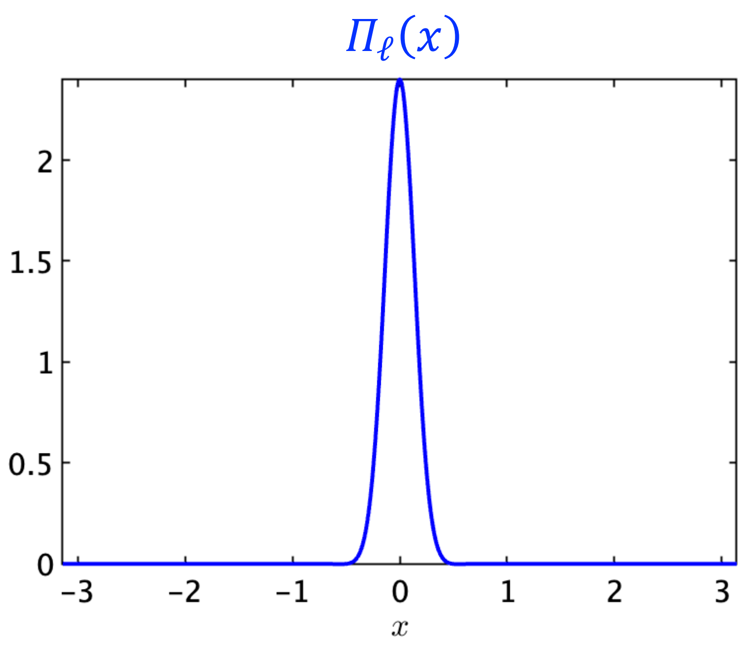
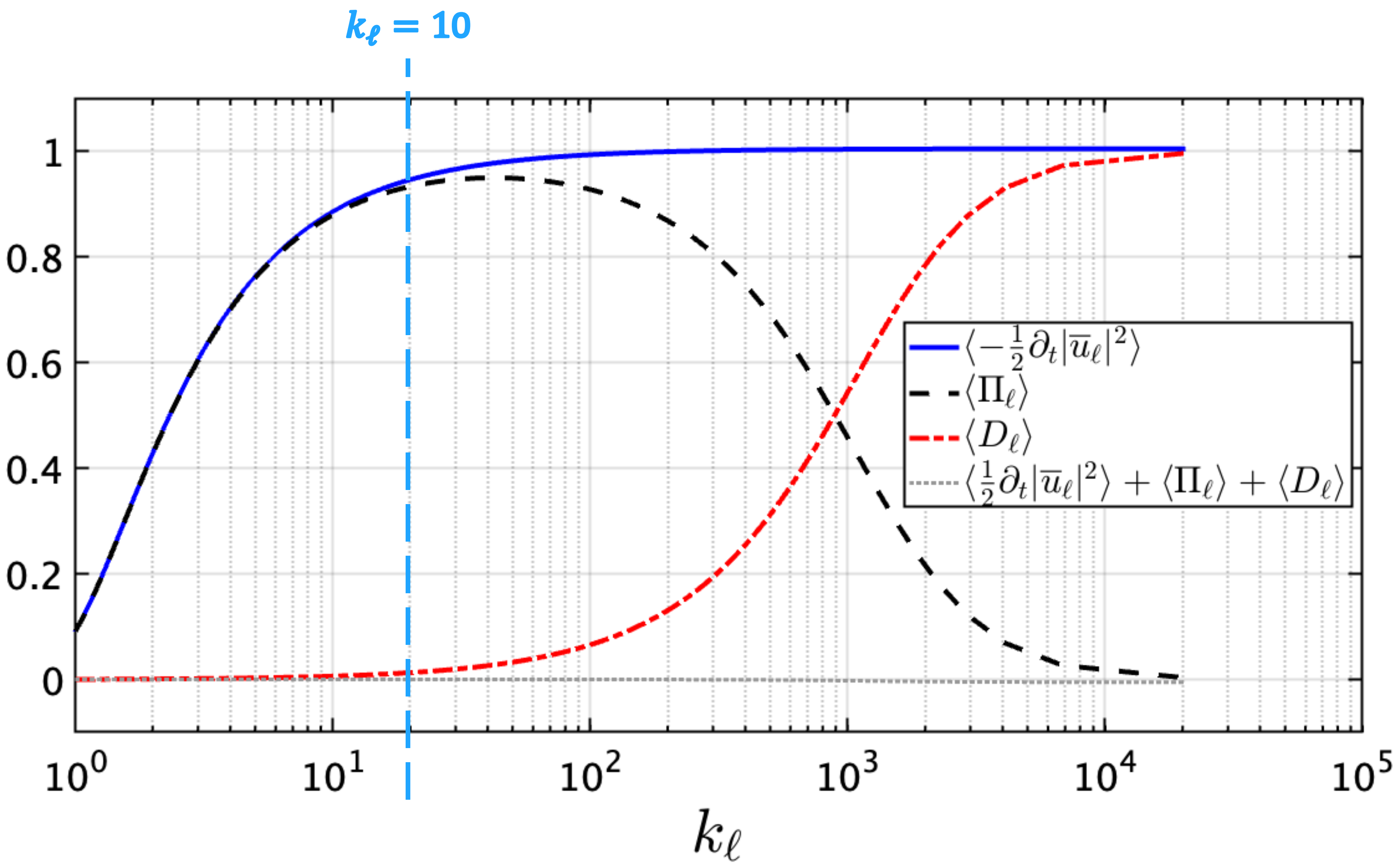






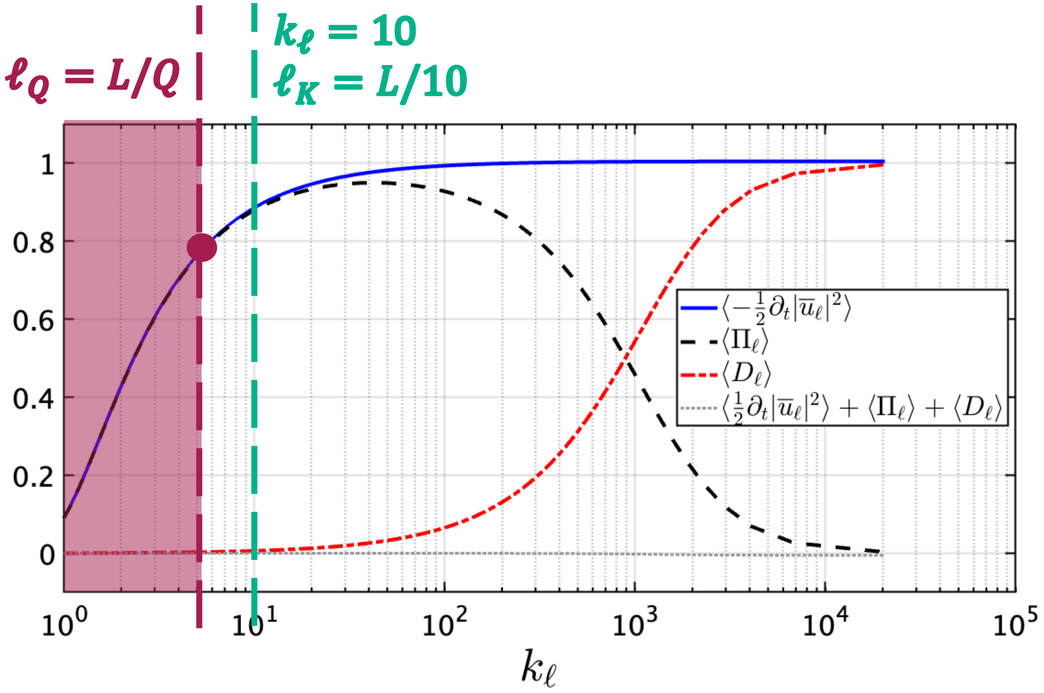
Scale Locality of the cascade in Burgers Shock: $\Pi_\ell(x)$

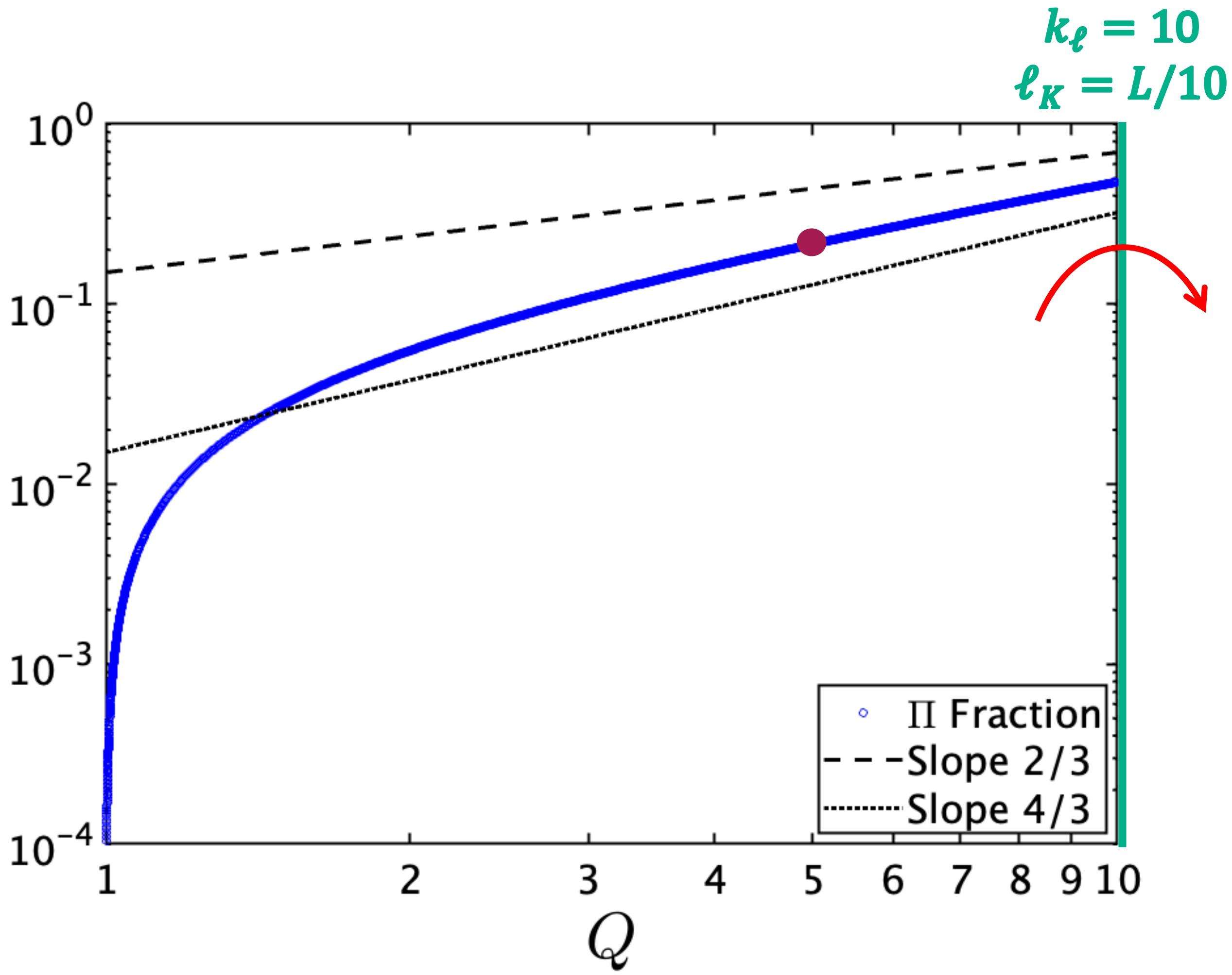


Scale Locality of the cascade in Burgers Shock: $\Pi_\ell(x)$

$$\left\| \left\langle \Pi_{\ell_K}\left(\bar{u}_{\ell_Q}, u, u\right)\right\rangle / \langle \Pi_{\ell_K}(u,u,u)\rangle \right\| \to 0 \qquad \text{As } Q \to 0$$

Deformation work is infrared local.

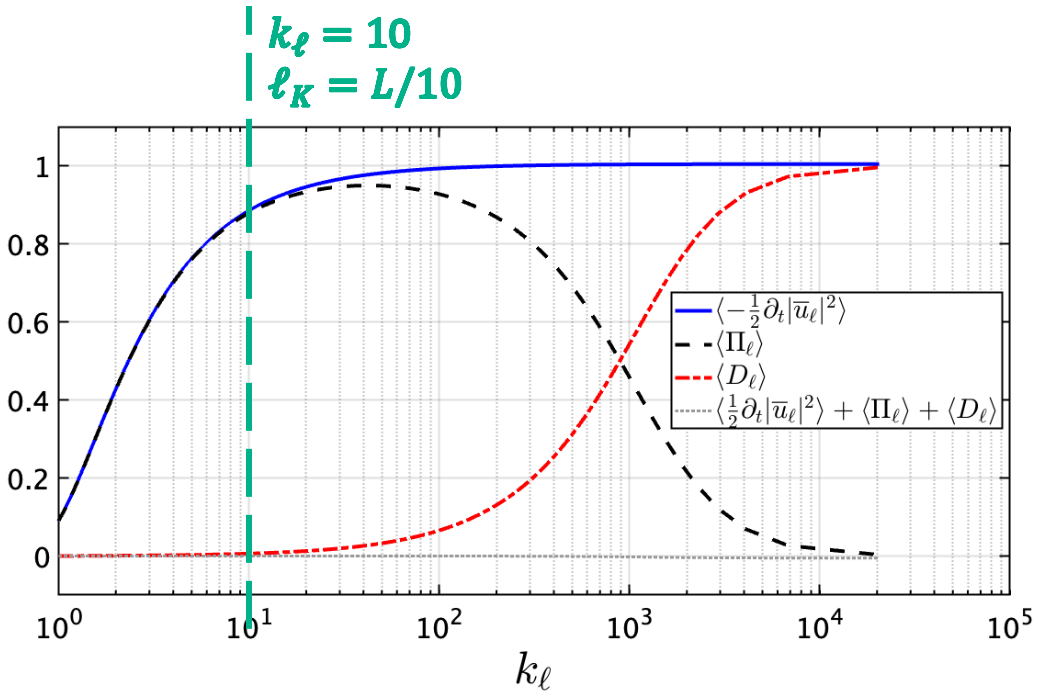


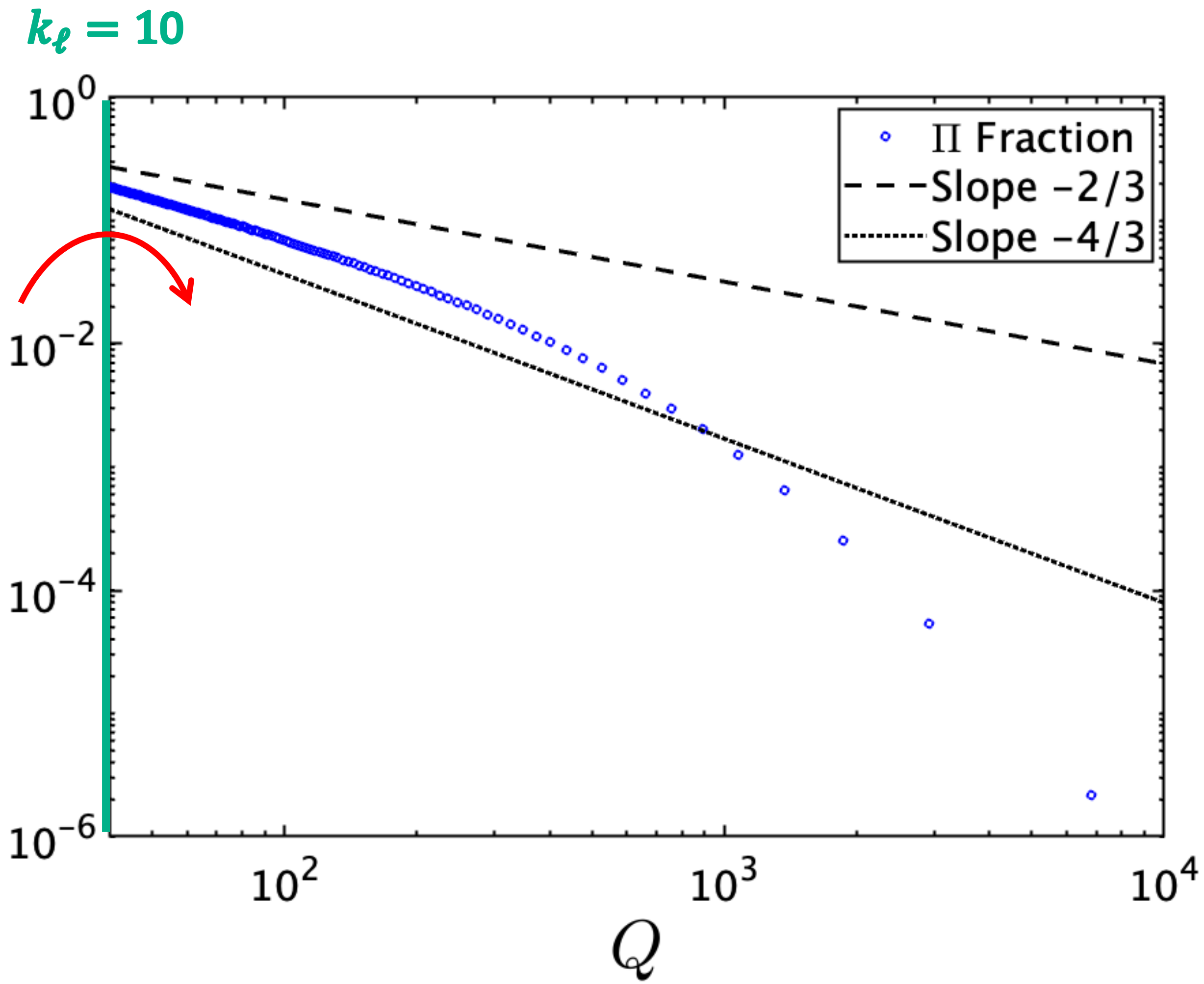


Scale Locality of the cascade in Burgers Shock: $\Pi_\ell(x)$

$\|\langle \Pi_{\ell_K}(u,u',u)\rangle / \langle \Pi_{\ell_K}(u,u,u)\rangle \| \to 0$ As $Q \to \infty$

Deformation work is ultraviolet local.

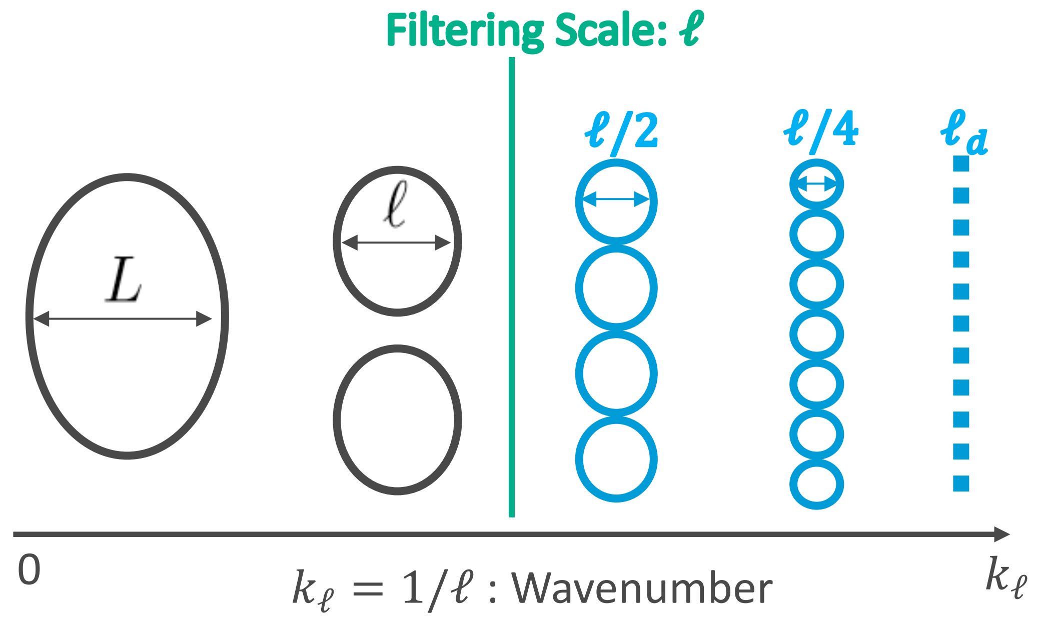




Coarse-grained KE Budget

$$\bar{f}_\ell(\boldsymbol{x}) = \int d\boldsymbol{r}\, G_\ell(\boldsymbol{r}) \boldsymbol{f}(\boldsymbol{x}+\boldsymbol{r})$$

$$\tilde{f}_\ell(\boldsymbol{x}) \equiv \frac{\overline{\rho f_\ell}}{\overline{\rho_\ell}}$$



$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \nabla\cdot J_\ell = \boxed{-\Pi_\ell} - \Lambda_\ell - \overline{\tau}(p, \boldsymbol{\nabla}\cdot\boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla}\cdot u_\ell}) - D_\ell$$

√ Deformation Work is scale-local.

Coarse-grained KE Budget

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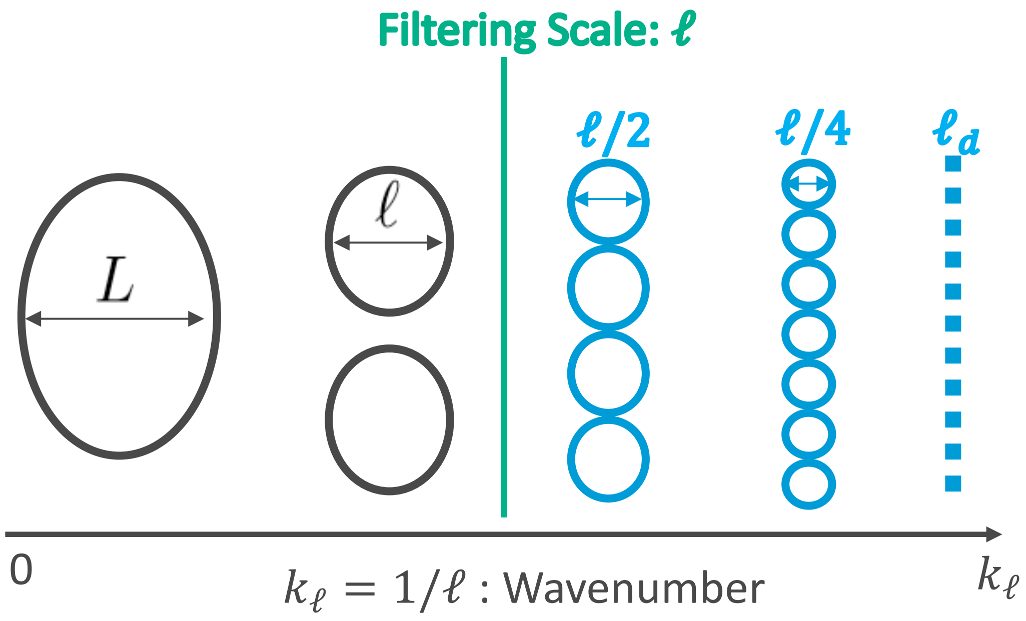


$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \nabla\cdot J_\ell = -\Pi_\ell \boxed{-\Lambda_\ell} - \overline{\tau}(p, \boldsymbol{\nabla}\cdot\boldsymbol{u}) - (-\overline{P\nabla\cdot u_\ell}) - D_\ell$$

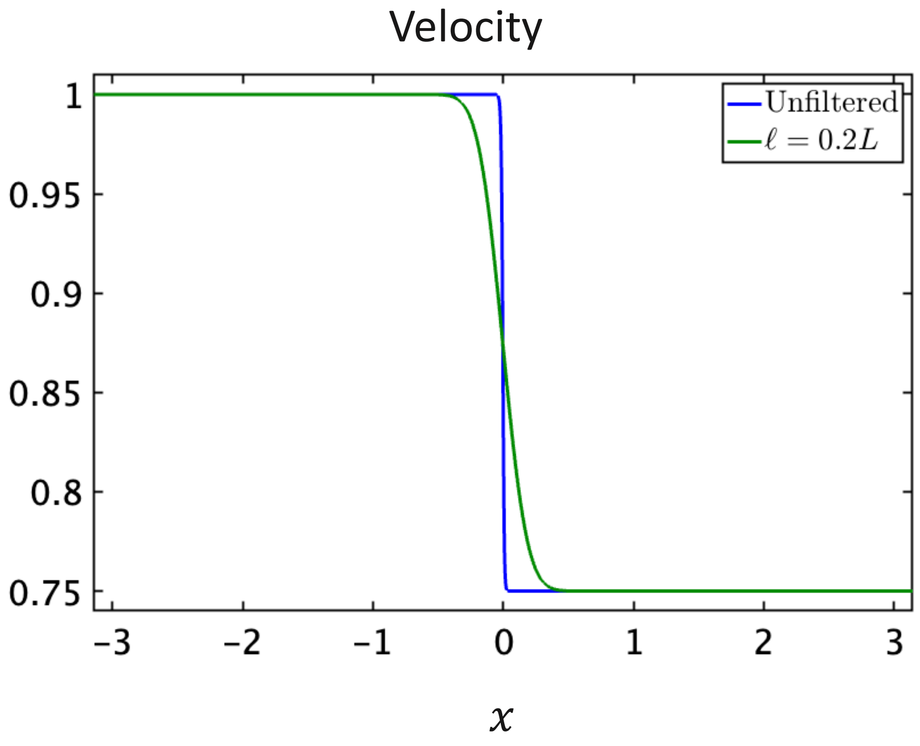
Baropycnal Work

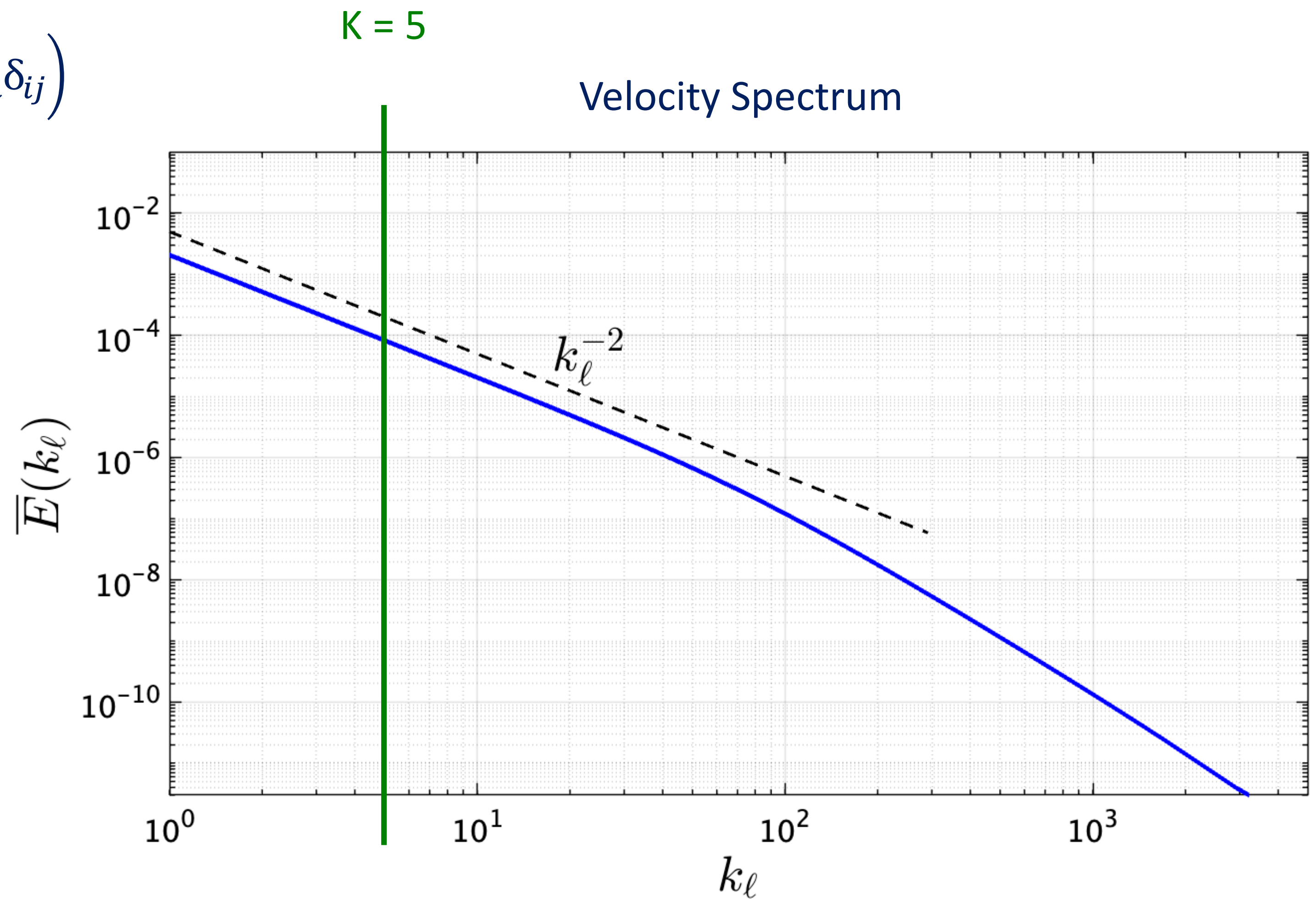
Normal Shock: A Real Compressible System

Analytical Solutions[1] for 1D, Normal Shock, Ma = 1.7

$$\partial_t \rho + \partial_x(\rho u_x) = 0,$$

$$\partial_t(\rho u) + \partial_x(\rho u_x u_x) = -\partial_x P + \mu \partial_x \left(\partial_x u_x + \frac{1}{3} \partial_m u_m \delta_{ij} \right)$$





1. Johnson, 2014, J. Fluid. Mech.

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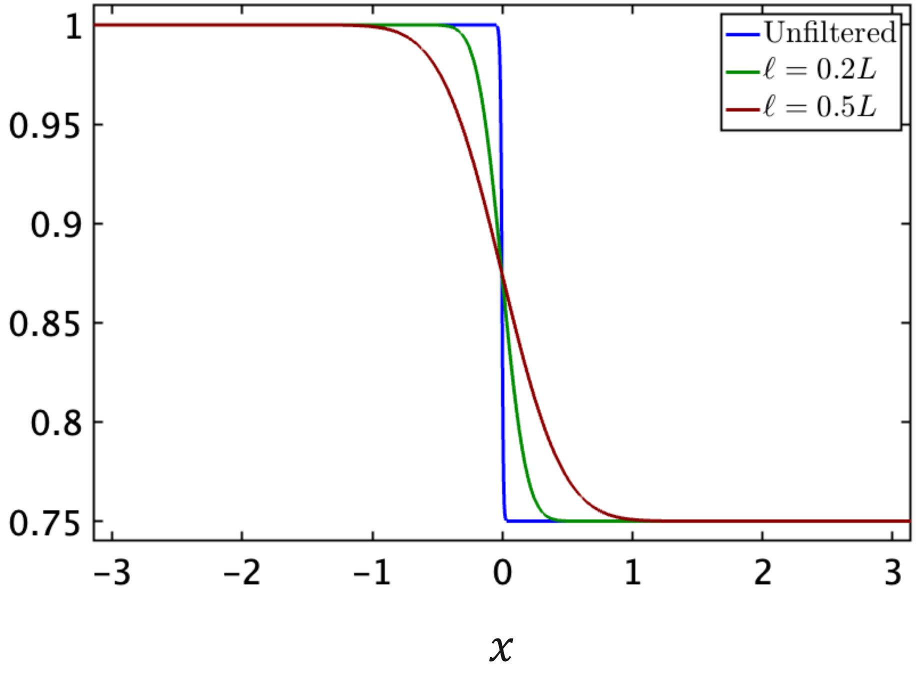
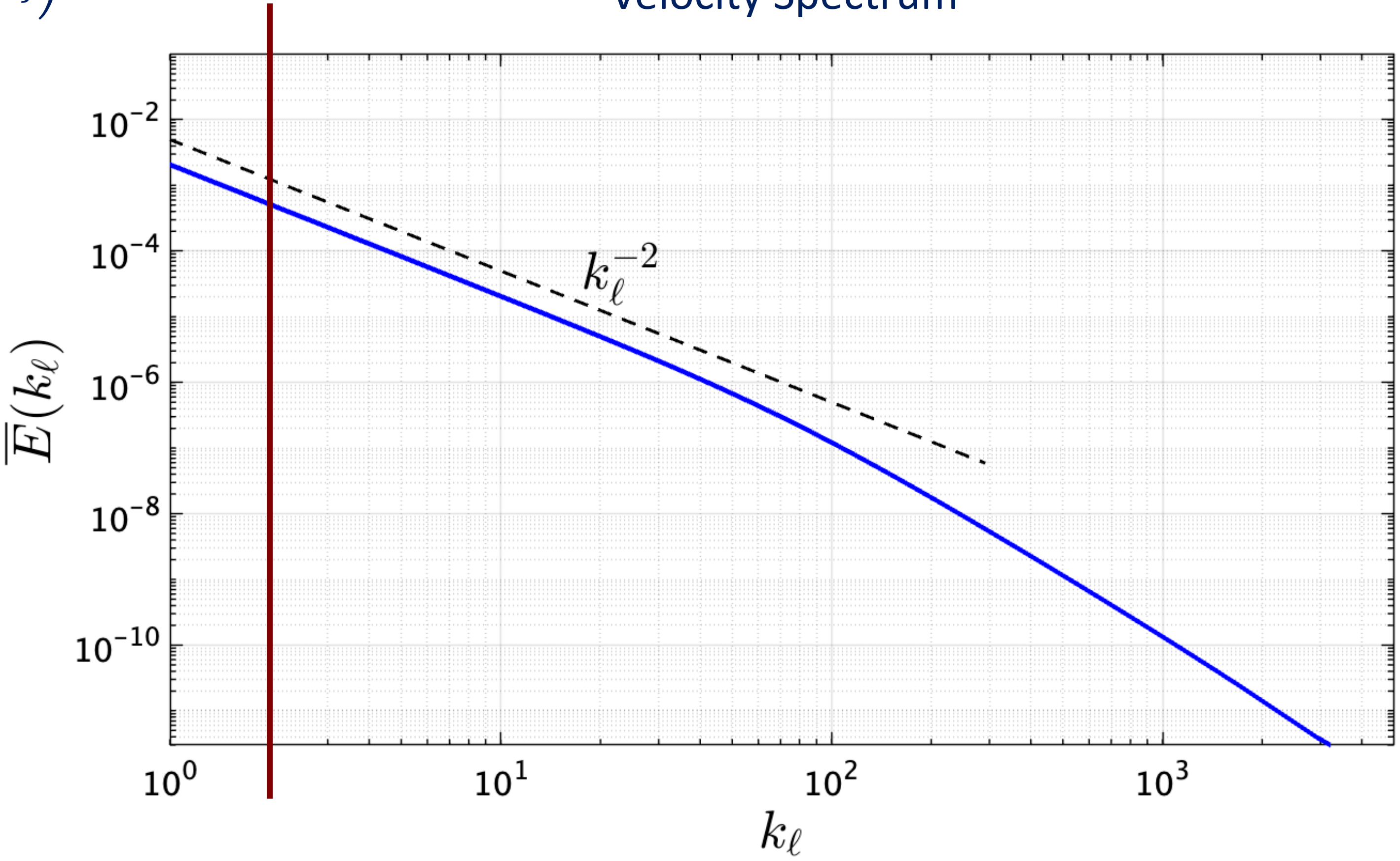
Velocity



K = 2

Velocity Spectrum



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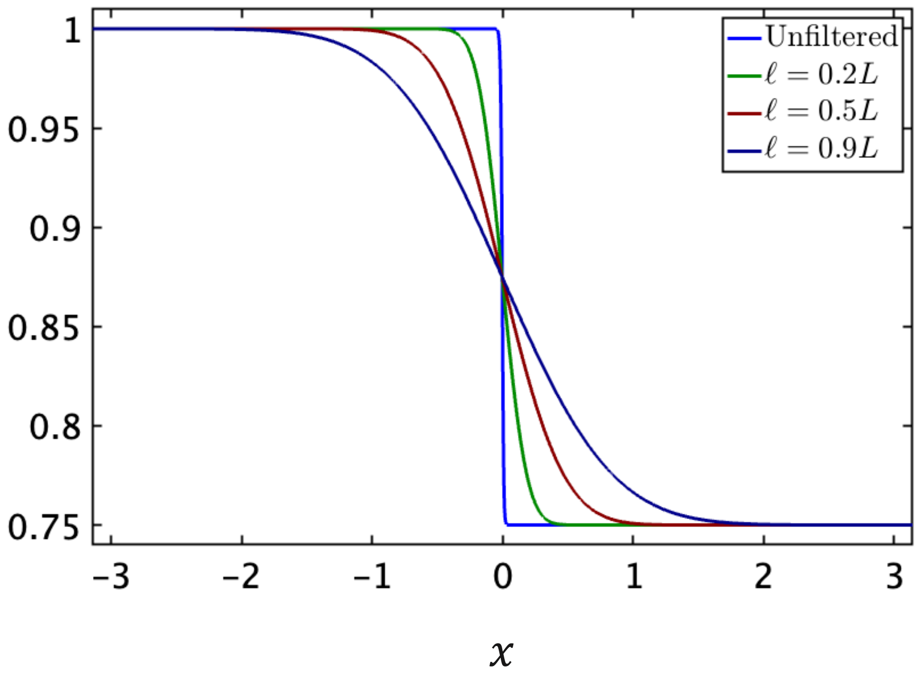
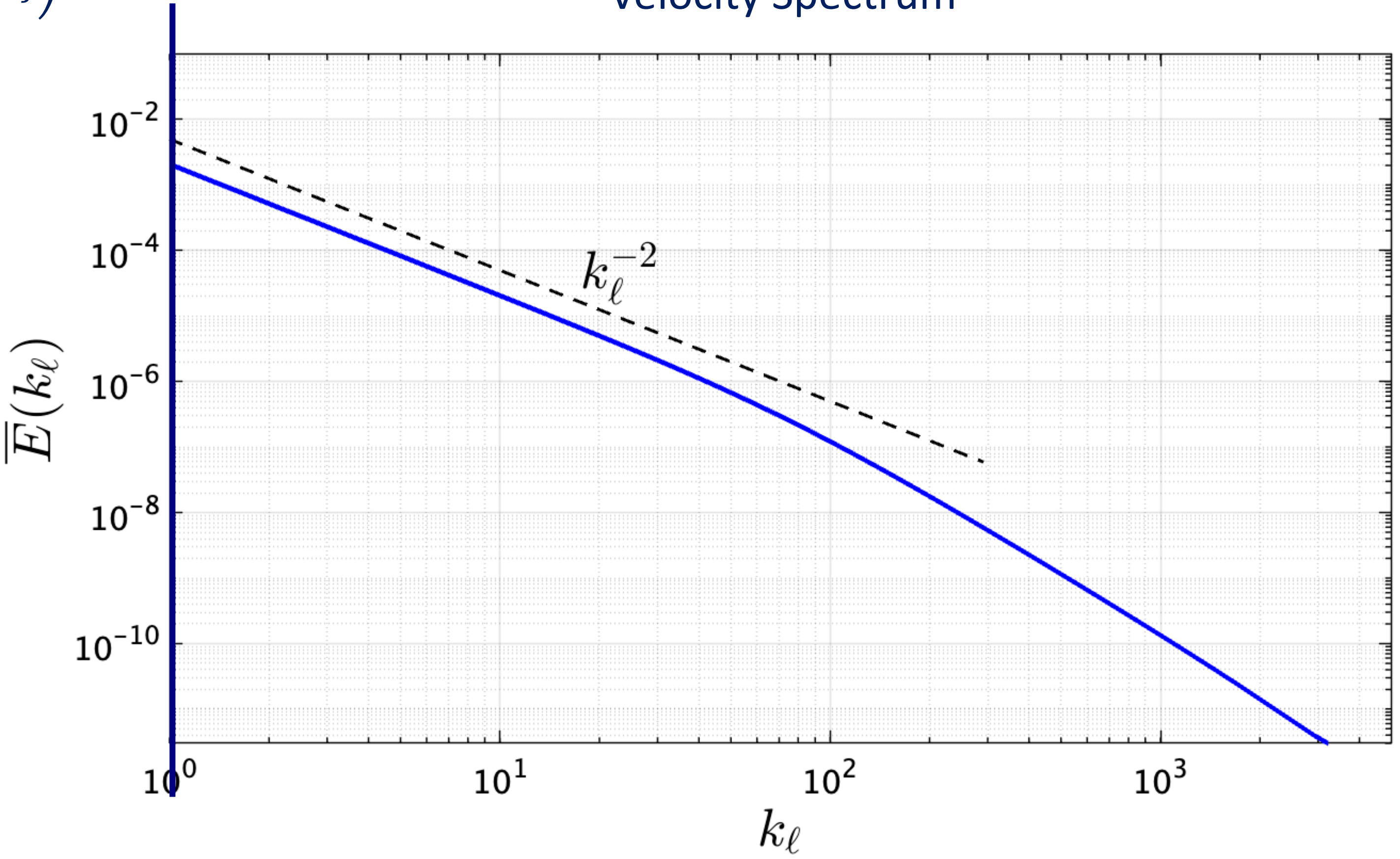
A normal shock is a multiscale structure, even in 1D!

K = 1

Velocity



Velocity Spectrum

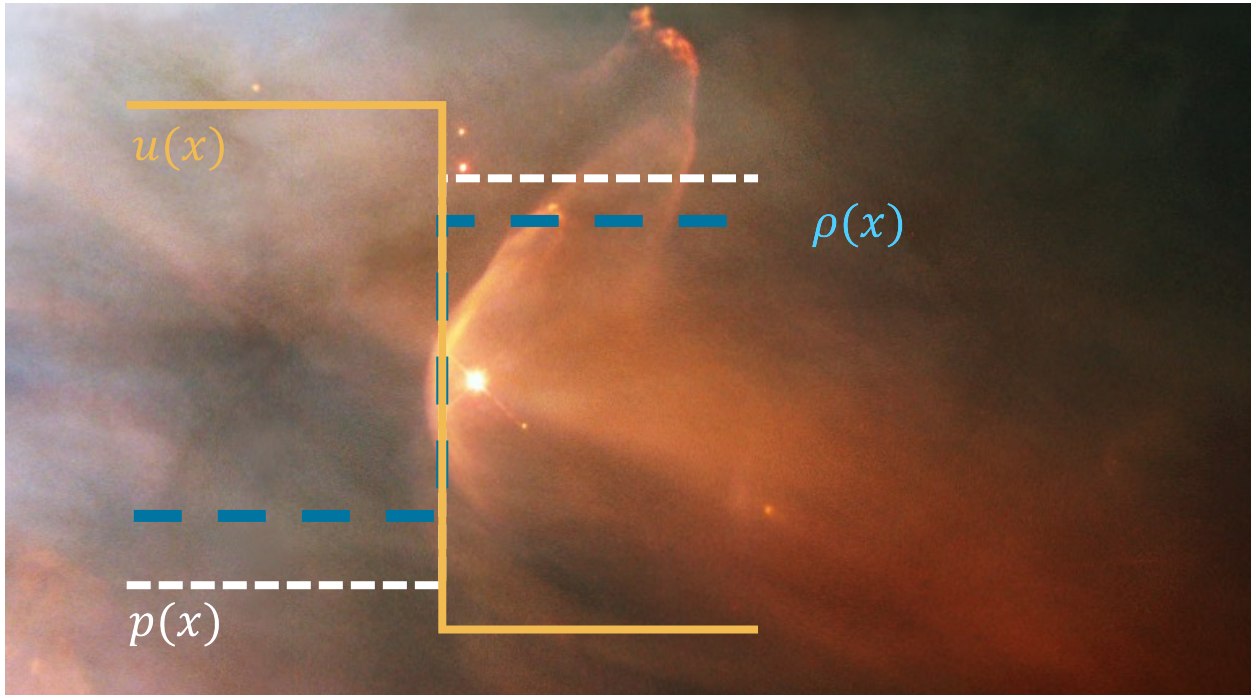


1. Johnson, 2014, J. Fluid. Mech.

Baropycnal Work in Normal Shock: $\boldsymbol{\Lambda}_\ell(\boldsymbol{x})$

$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \boldsymbol{\nabla}\cdot\boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p, \boldsymbol{\nabla}\cdot\boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla}\cdot u_\ell}) - D_\ell$$

$$\Lambda_\ell(\mathrm{x}) = \frac{1}{\bar{\rho}_\ell}\partial_x \bar{p}_\ell\left[\overline{(\rho u)}_\ell - \bar{\rho}_\ell \bar{u}_\ell\right]$$



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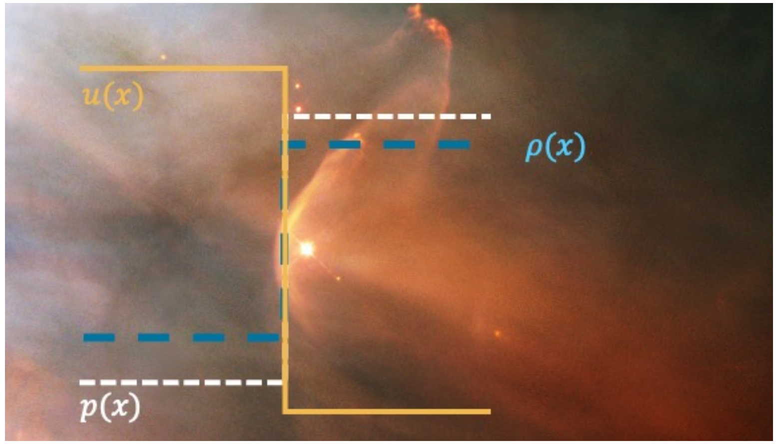
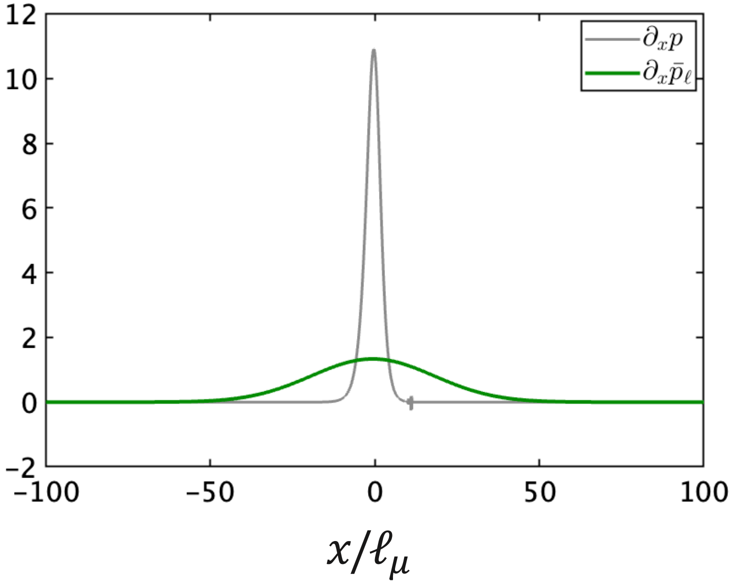
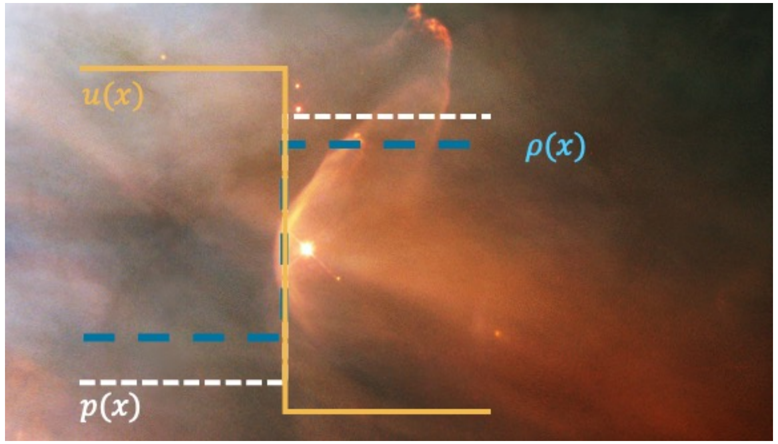
For $\ell = 0.15L$



$\partial_x\bar{p}_\ell$



Baropycnal Work in Normal Shock: $\boldsymbol{\Lambda_\ell(x)}$



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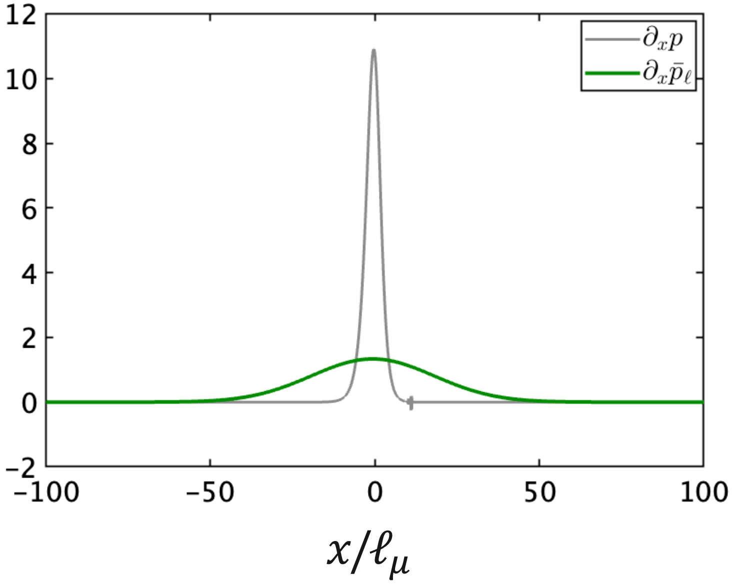
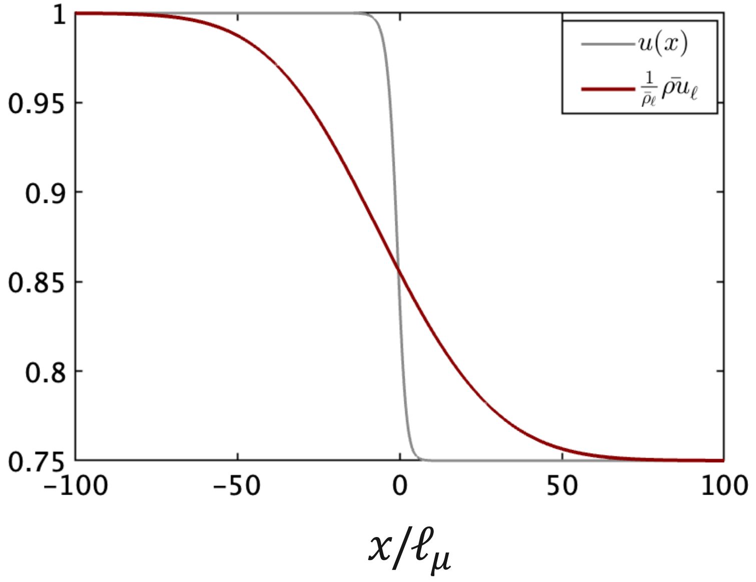
For $\ell = 0.15L$

$\partial_x\bar{p}_\ell$

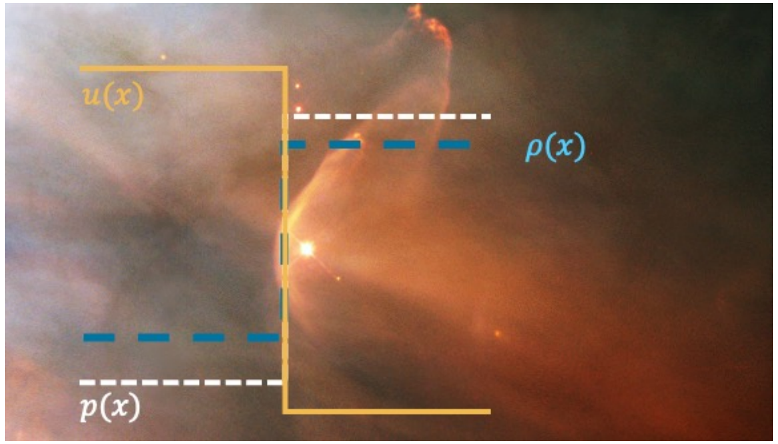




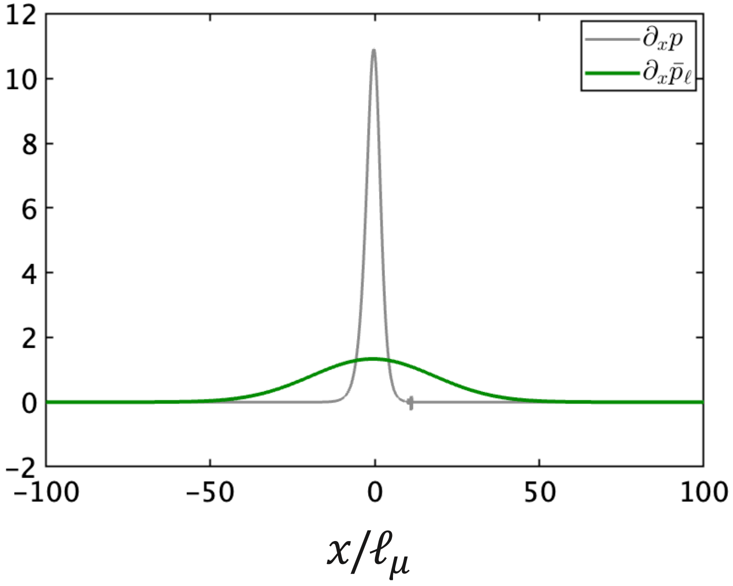
Baropycnal Work in Normal Shock: $\boldsymbol{\Lambda}_\ell(\boldsymbol{x})$

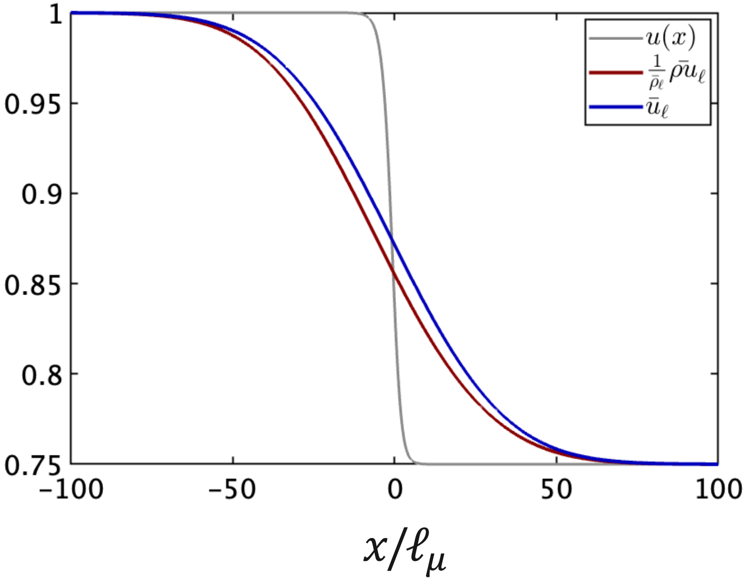
$$\Lambda_\ell(\mathrm{x}) = \frac{1}{\bar{\rho}_\ell}\partial_x \bar{p}_\ell\left[\overline{(\rho u)}_\ell \; - \bar{\rho}_\ell \bar{u}_\ell\right]$$

For $\ell = 0.15L$



$\partial_x \bar{p}_\ell$





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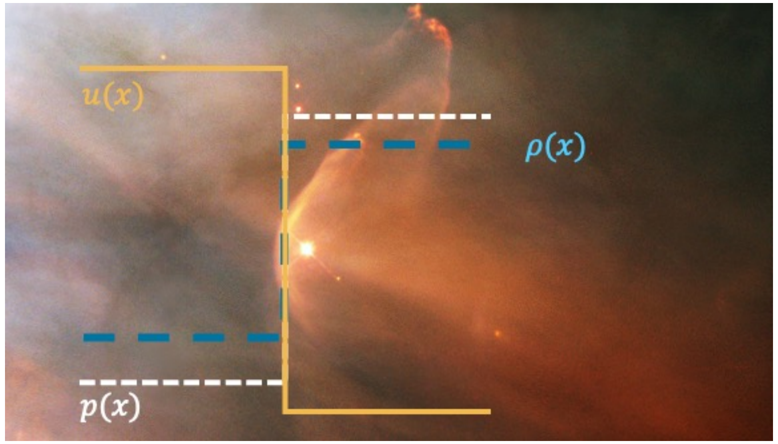

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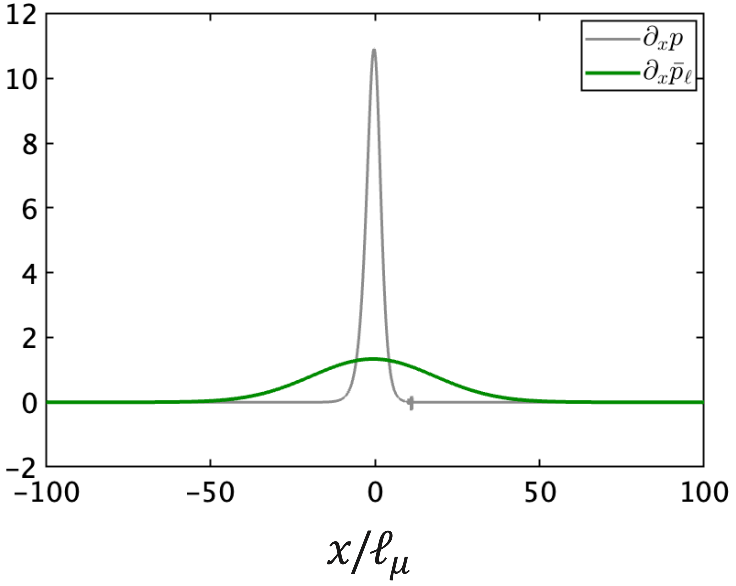
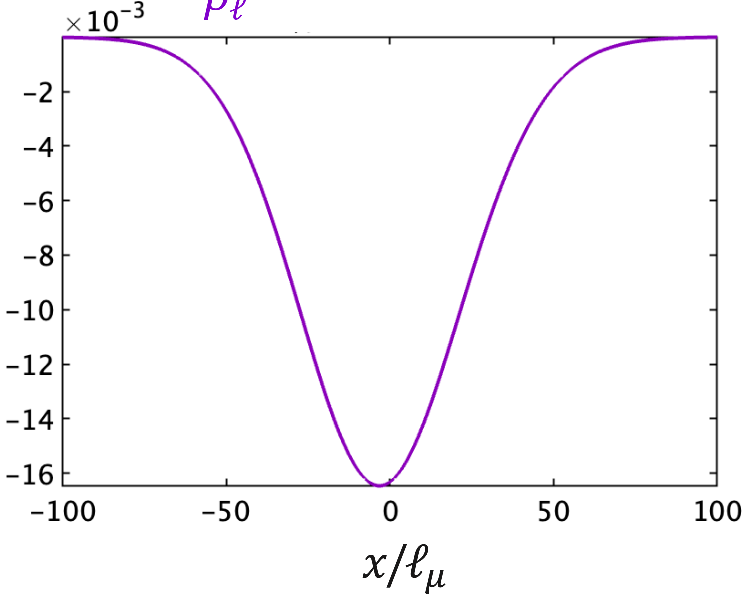
For $\ell = 0.15L$

$\partial_x\bar{p}_\ell$



$\frac{1}{\bar{\rho}_\ell}\left[\overline{(\rho u)}_\ell - \bar{\rho}_\ell\bar{u}_\ell\right]$



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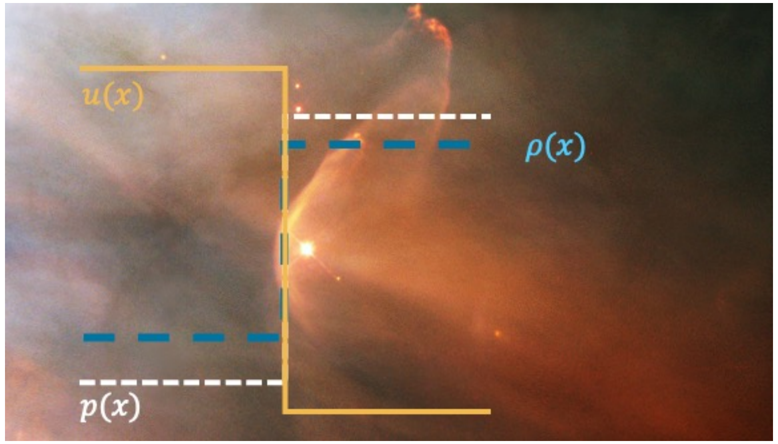
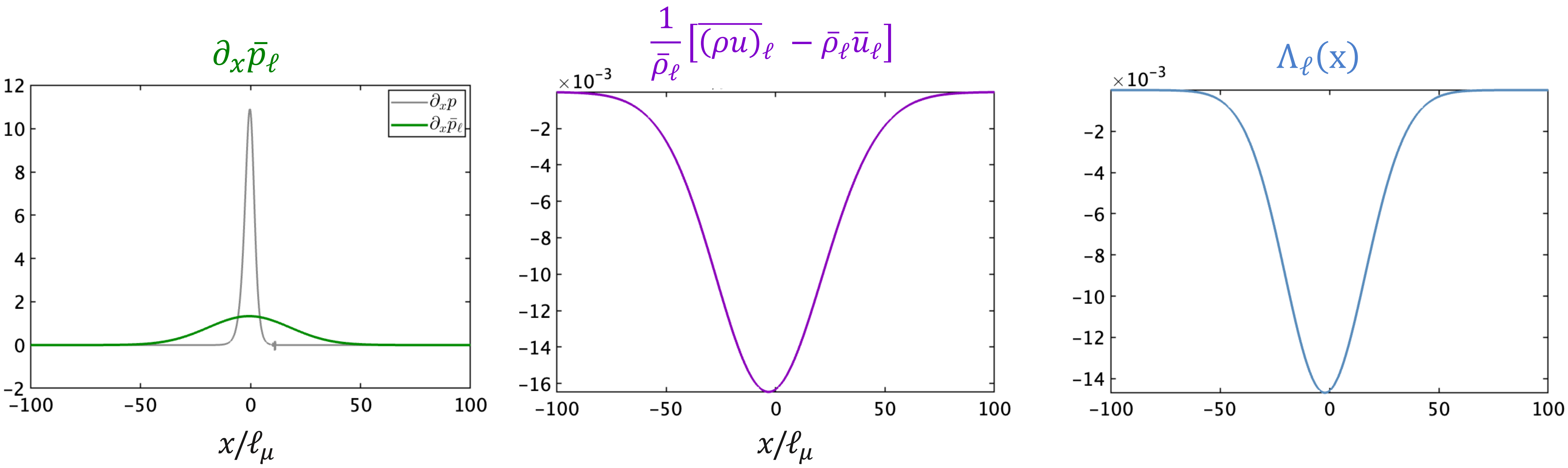
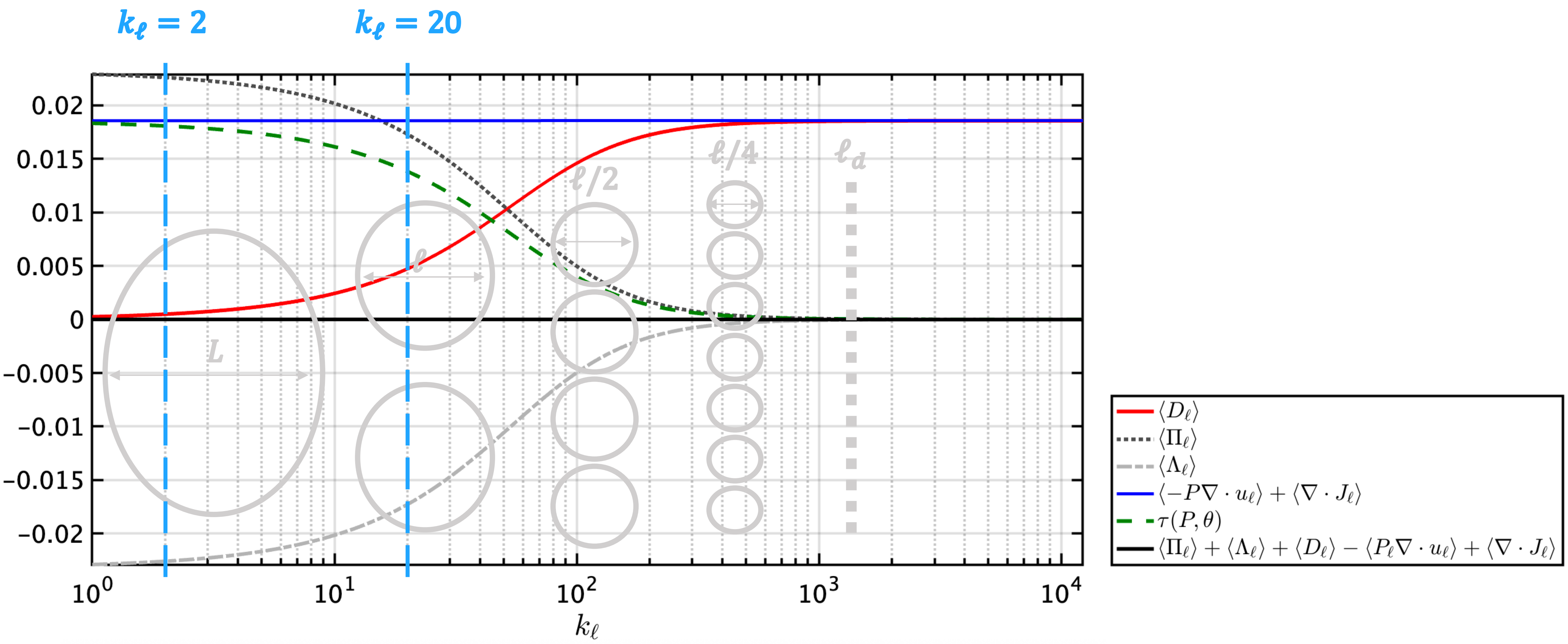




Coarse-grained KE Budget For Normal Shock

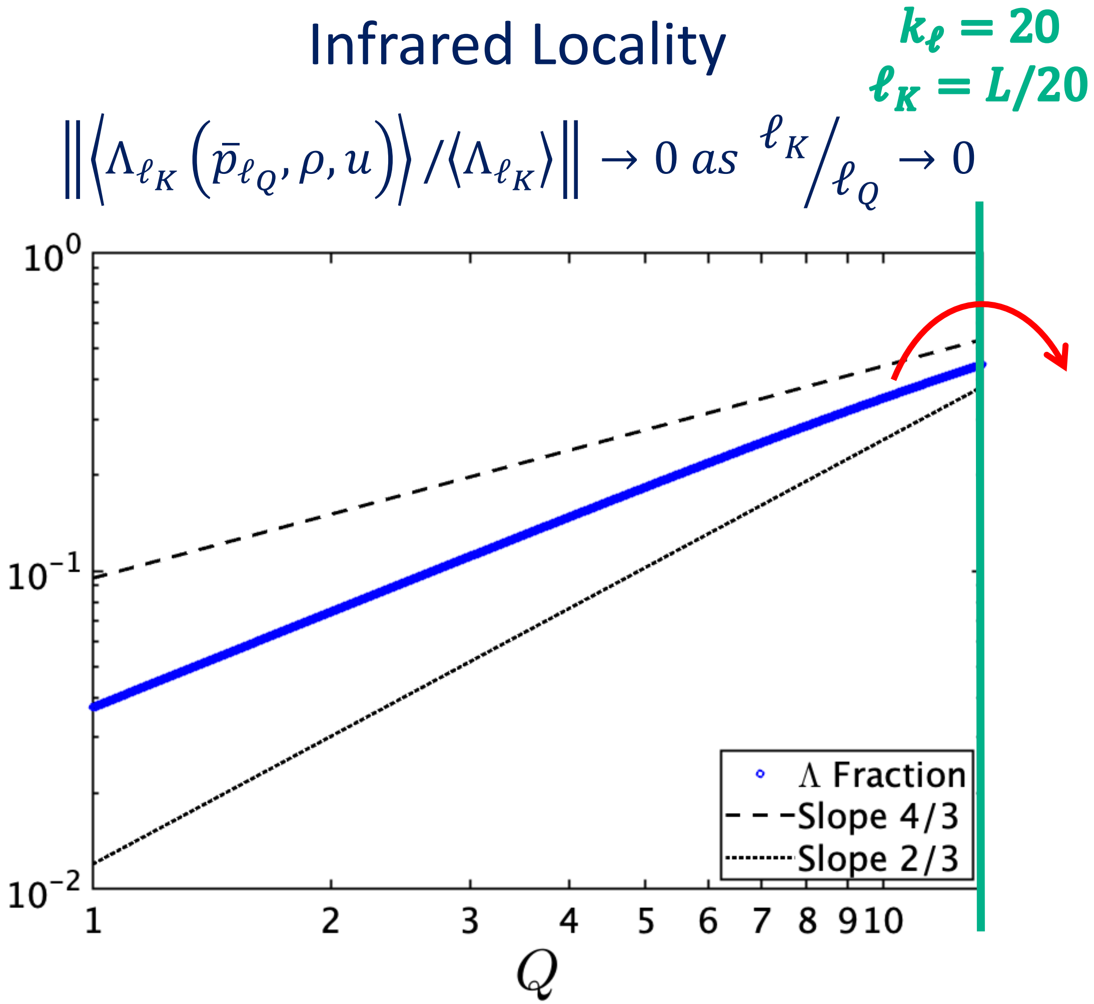
$$\nabla \cdot \boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \bar{\tau}_\ell(p, \nabla \cdot \boldsymbol{u}) - (-\overline{P\nabla \cdot u_\ell}) \ - D_\ell$$



Scale Locality of the Sub-scale Flux: $\Lambda_\ell(x)$

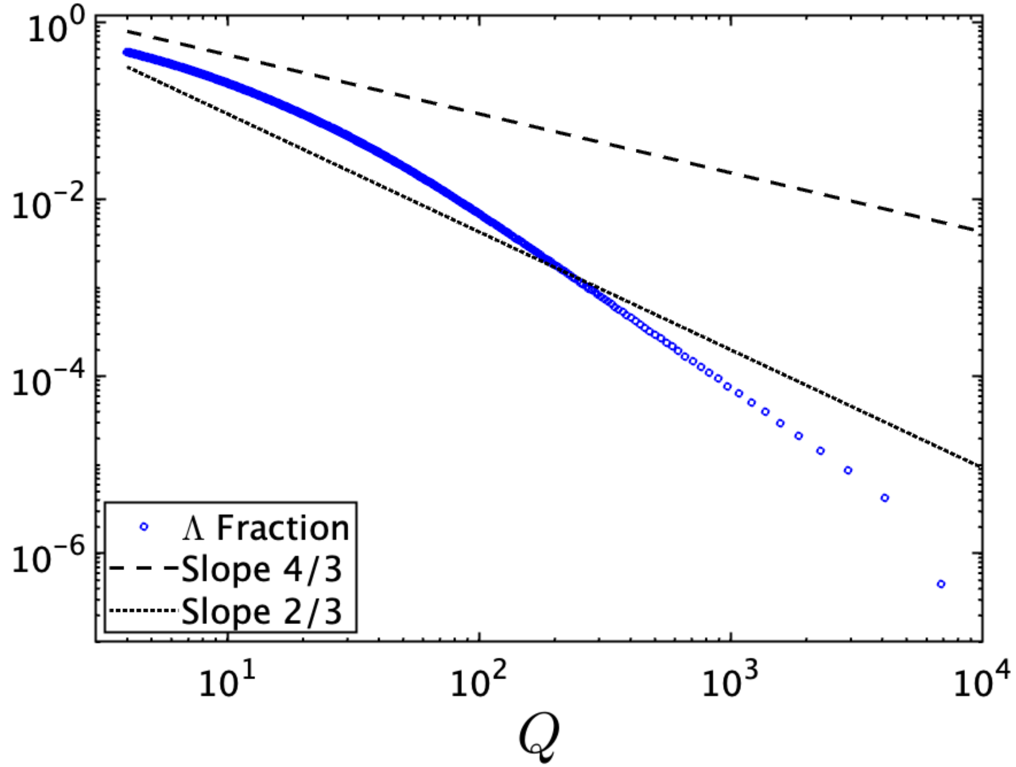
Infrared Locality

$$\left\| \left\langle \Lambda_{\ell_K}\left(\bar{p}_{\ell_Q}, \rho, u\right)\right\rangle / \langle \Lambda_{\ell_K} \rangle \right\| \to 0 \ as \ {}^{\ell_K}\!/_{\ell_Q} \to 0$$



Ultraviolet Locality

$$\| \langle \Lambda_{\ell_K}(P, \rho', u) \rangle / \langle \Lambda_{\ell_K} \rangle \| \to 0 \ as \ {}^{\ell_Q}\!/_{\ell_K} \to 0$$



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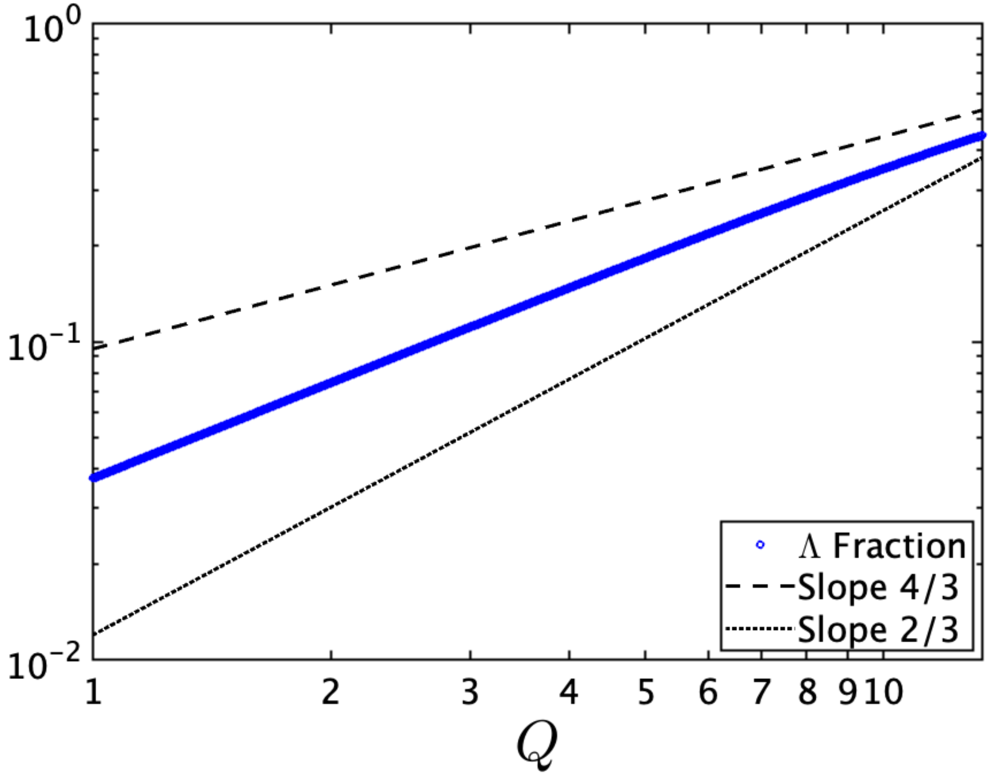
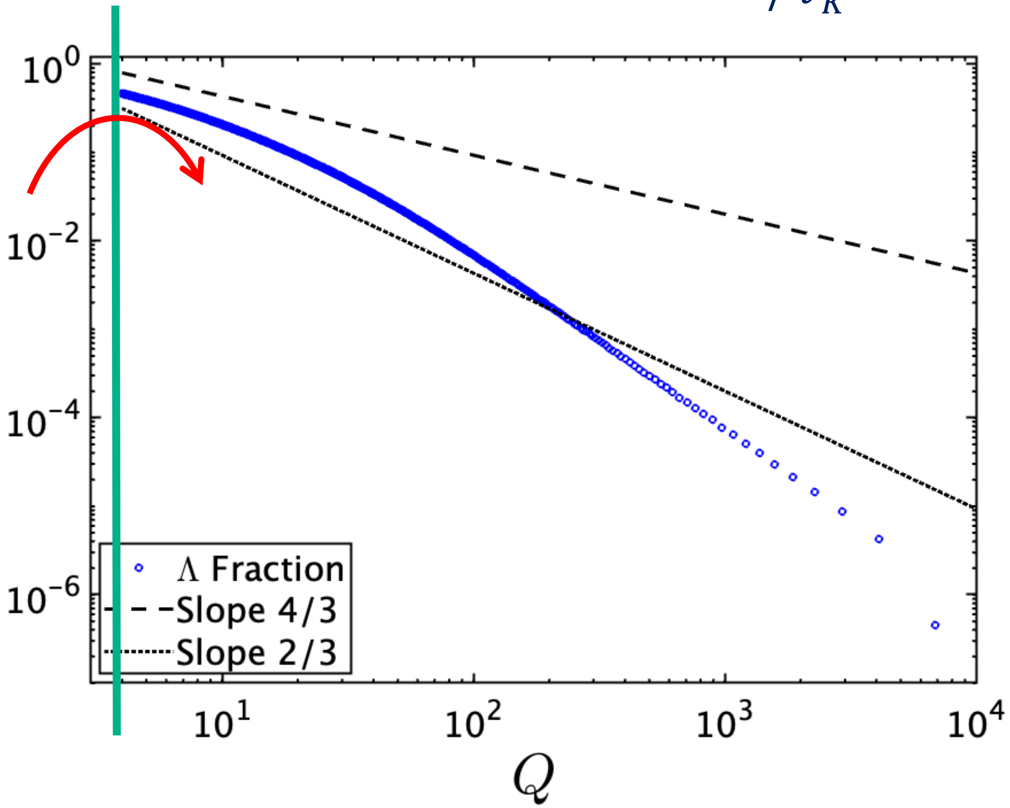


Ultraviolet Locality

$\boldsymbol{k_\ell = 2}$

$\boldsymbol{\ell_K = L/2}$

$$\left\| \langle \Lambda_{\ell_K}(P, \rho', u)\rangle / \langle \Lambda_{\ell_K}\rangle \right\| \to 0 \ as \ {}^{\ell_Q}/_{\ell_K} \to 0$$



Coarse-grained KE Budget For Normal Shock

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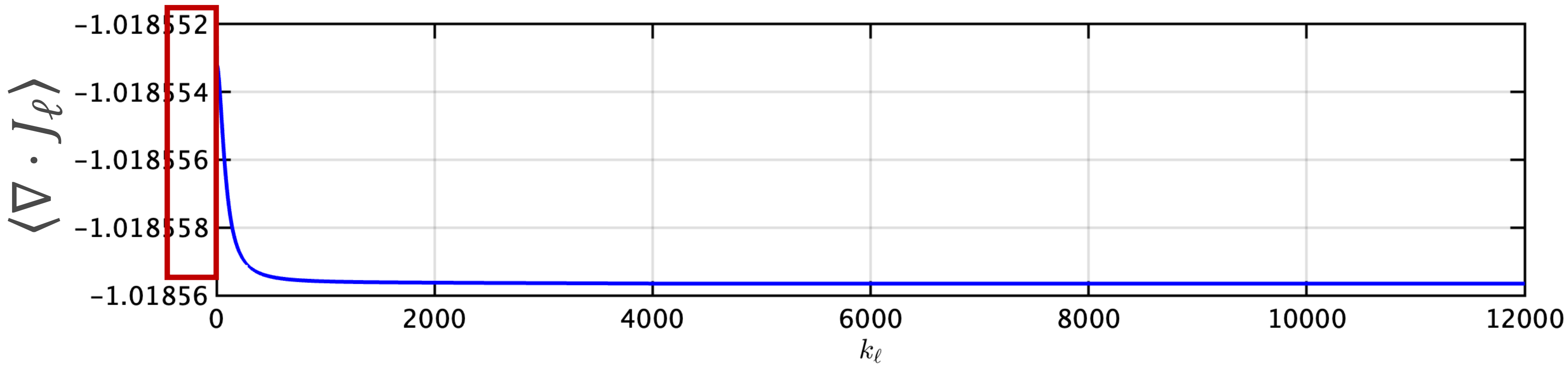

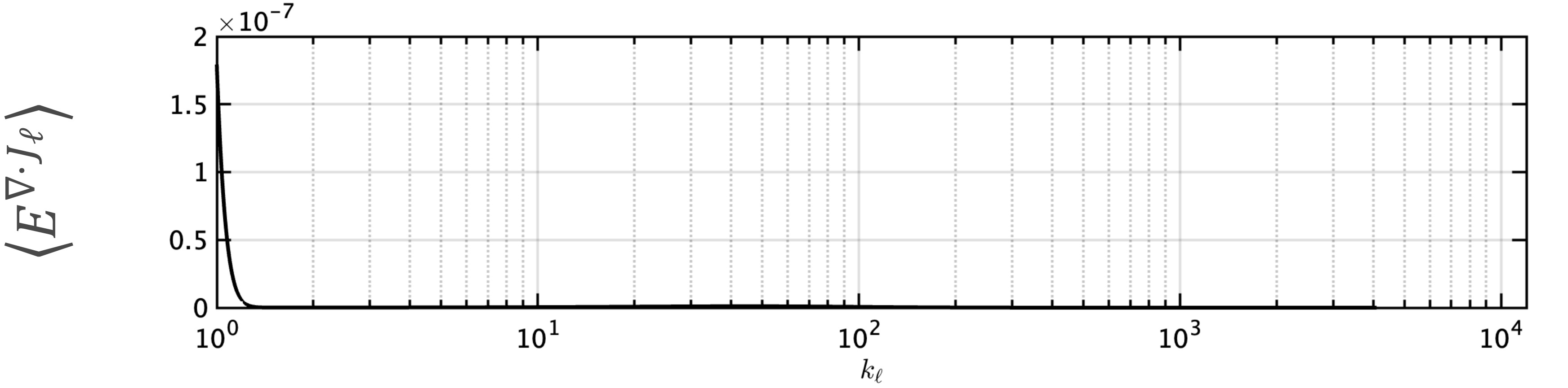

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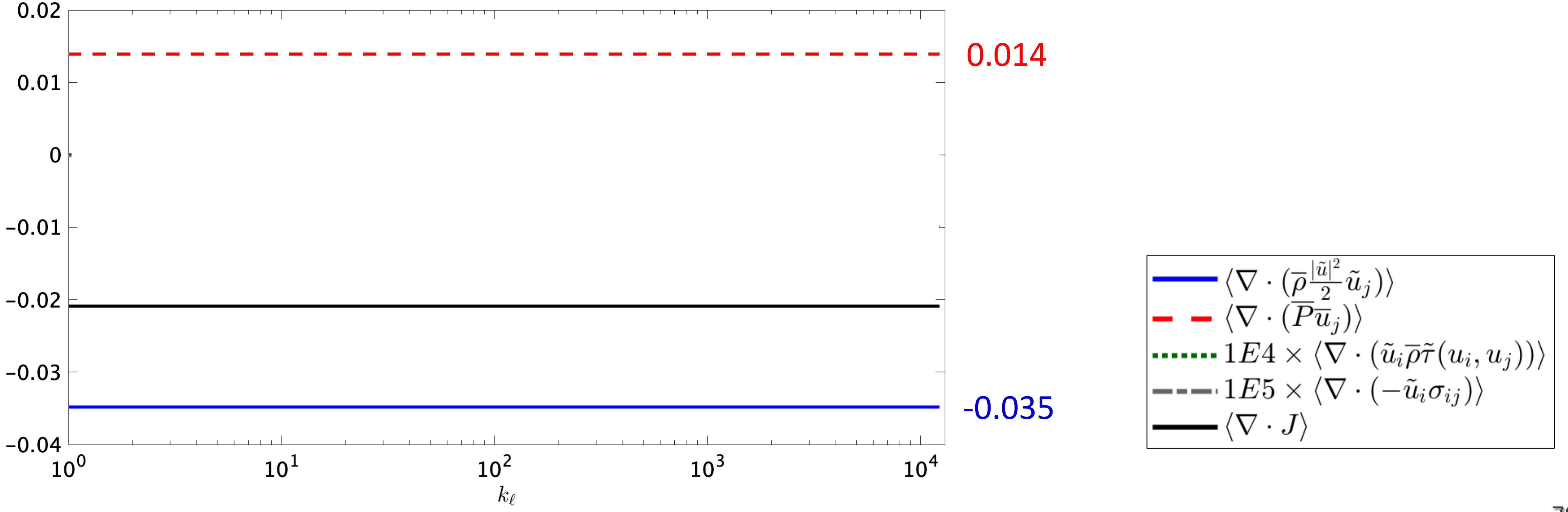

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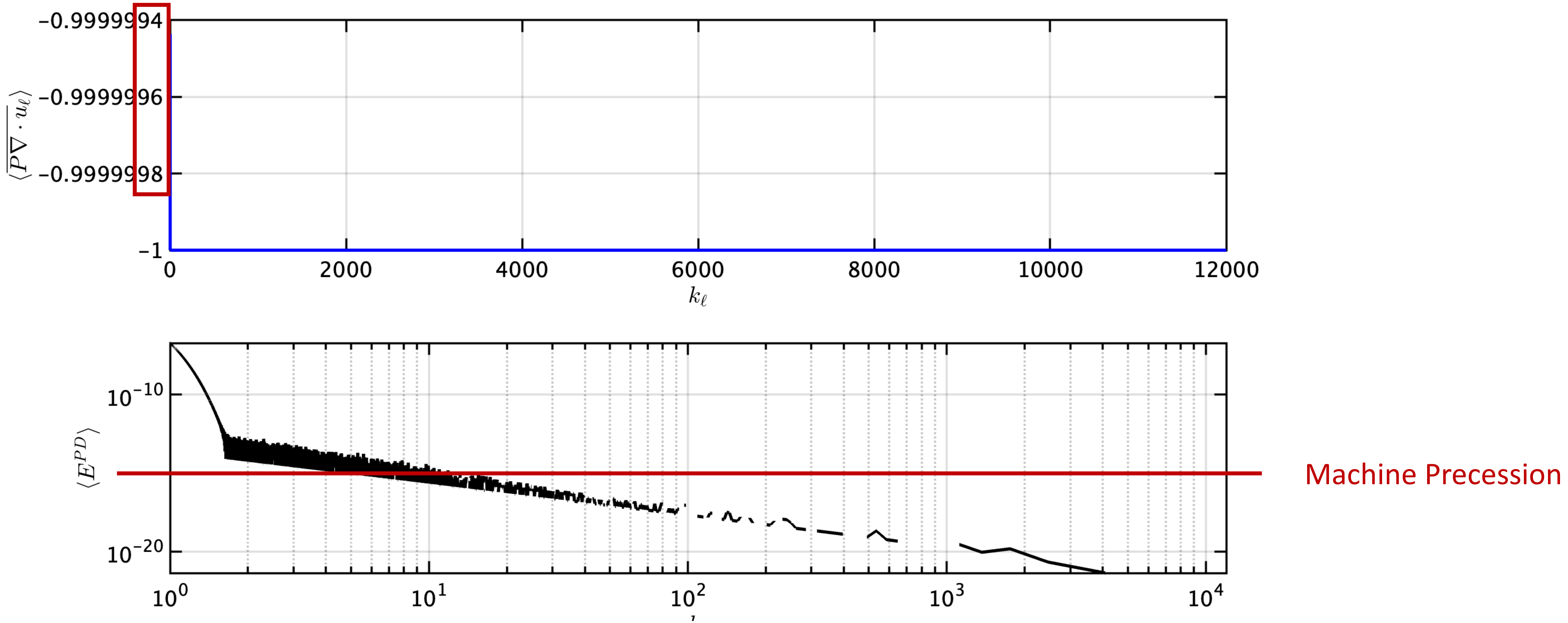

Baropycnal Work in Rayleigh-Taylor: $\boldsymbol{\Lambda}_\ell(\boldsymbol{x})$

$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \boldsymbol{\nabla}\cdot\boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p\,,\boldsymbol{\nabla}\cdot\boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla}\cdot u_\ell}) - D_\ell$$

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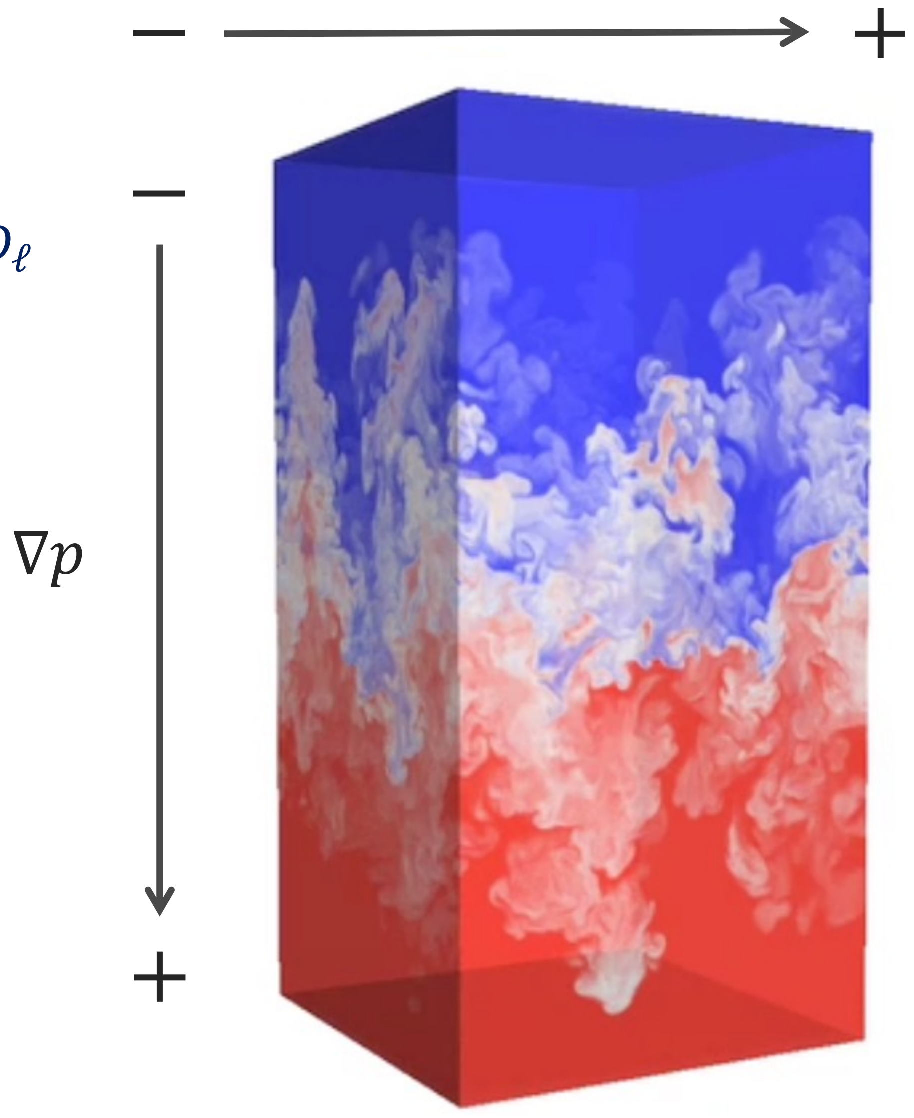

G. Boffetta

Baropycnal Work in Rayleigh-Taylor: $\boldsymbol{\Lambda_\ell(x)}$

$$\partial_t\left(\bar{\rho}_\ell \frac{|\tilde{u}_\ell|^2}{2}\right) + \boldsymbol{\nabla}\cdot\boldsymbol{J}_\ell = -\Pi_\ell - \Lambda_\ell - \overline{\tau}(p\,,\boldsymbol{\nabla}\cdot\boldsymbol{u}) - (-\overline{P\boldsymbol{\nabla}\cdot u_\ell}) - D_\ell$$

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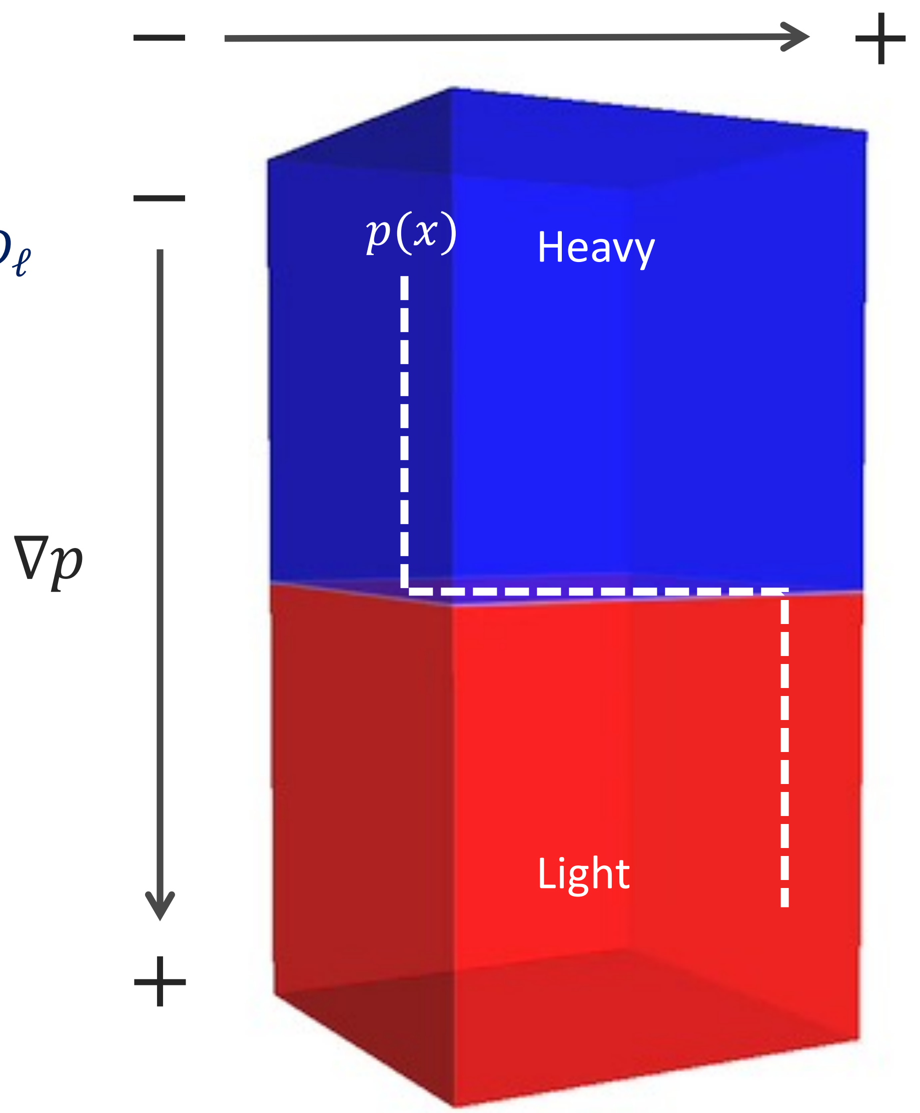
$+$



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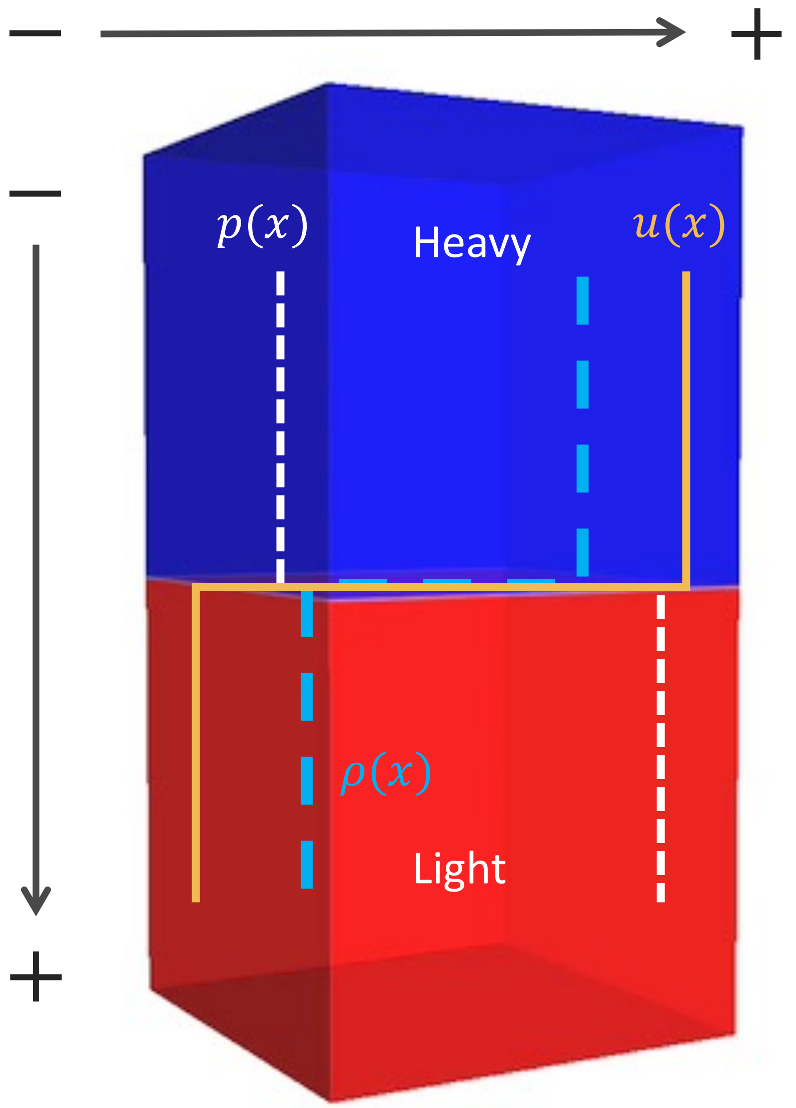
$$\Lambda_\ell(\mathrm{x})=\frac{1}{\bar{\rho}_\ell}\partial_x\bar{p}_\ell\boxed{\left[\overline{(\rho u)}_\ell-\bar{\rho}_\ell\bar{u}_\ell\right]}$$

$+$

∇p



G. Boffetta

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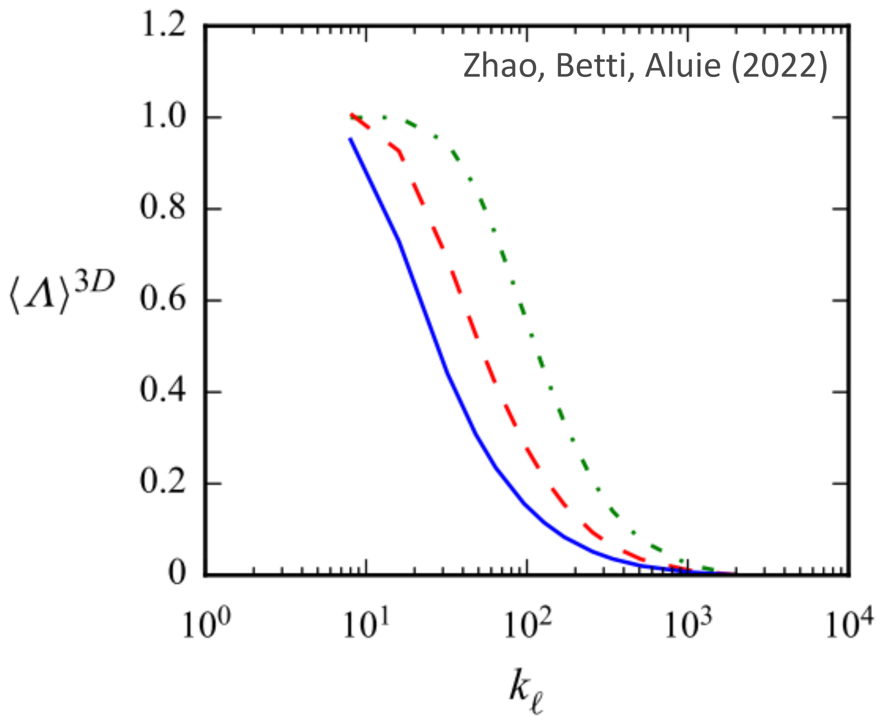
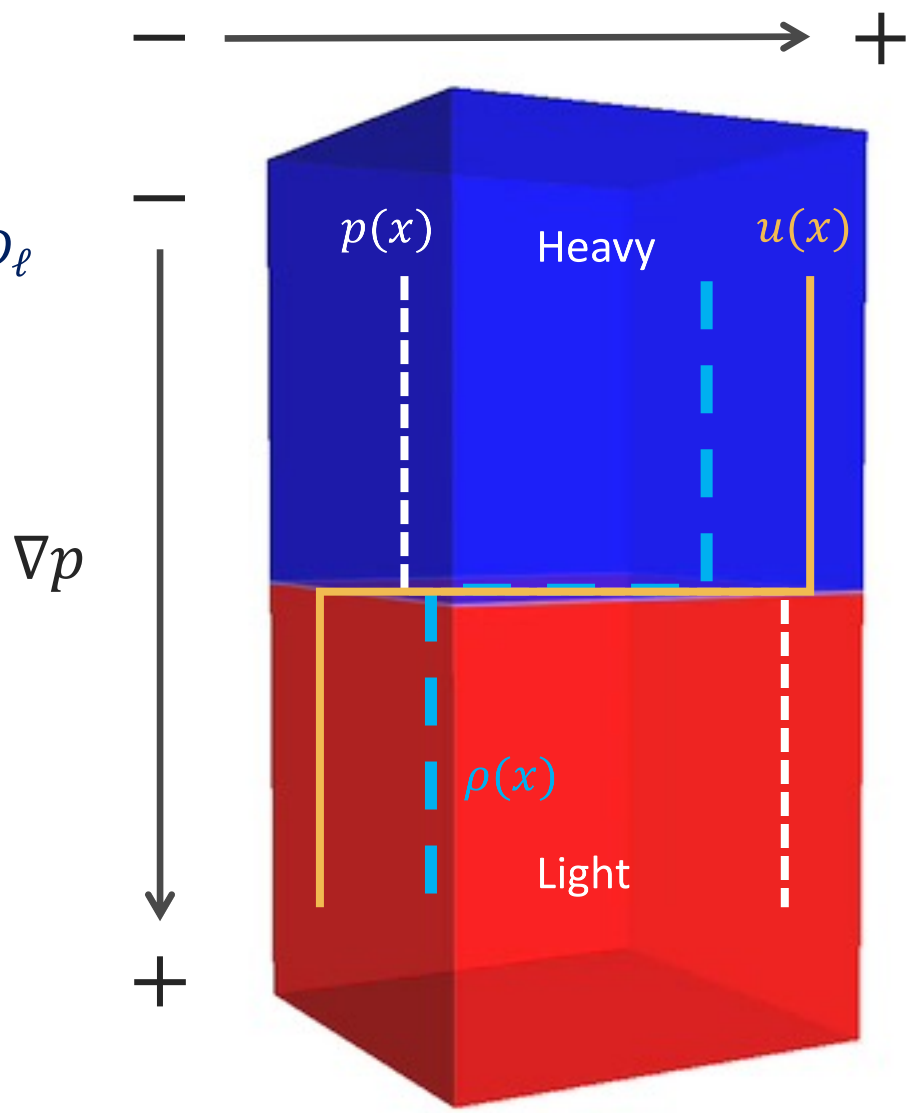
$$\boxed{\Lambda_\ell(\mathrm{x})=\frac{1}{\bar{\rho}_\ell}\partial_x\bar{p}_\ell\left[\overline{(\rho u)}_\ell-\bar{\rho}_\ell\bar{u}_\ell\right]}$$

+





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