Correlations between the Neutron Star M-R Relation and EoSs of Dense Matter

Sun, B., & Lattimer, J. M. 2025, The Astrophysical Journal, 984, 30

Boyang Sun
Stony Brook University
09/12/2025

INT Workshop
Seattle, USA

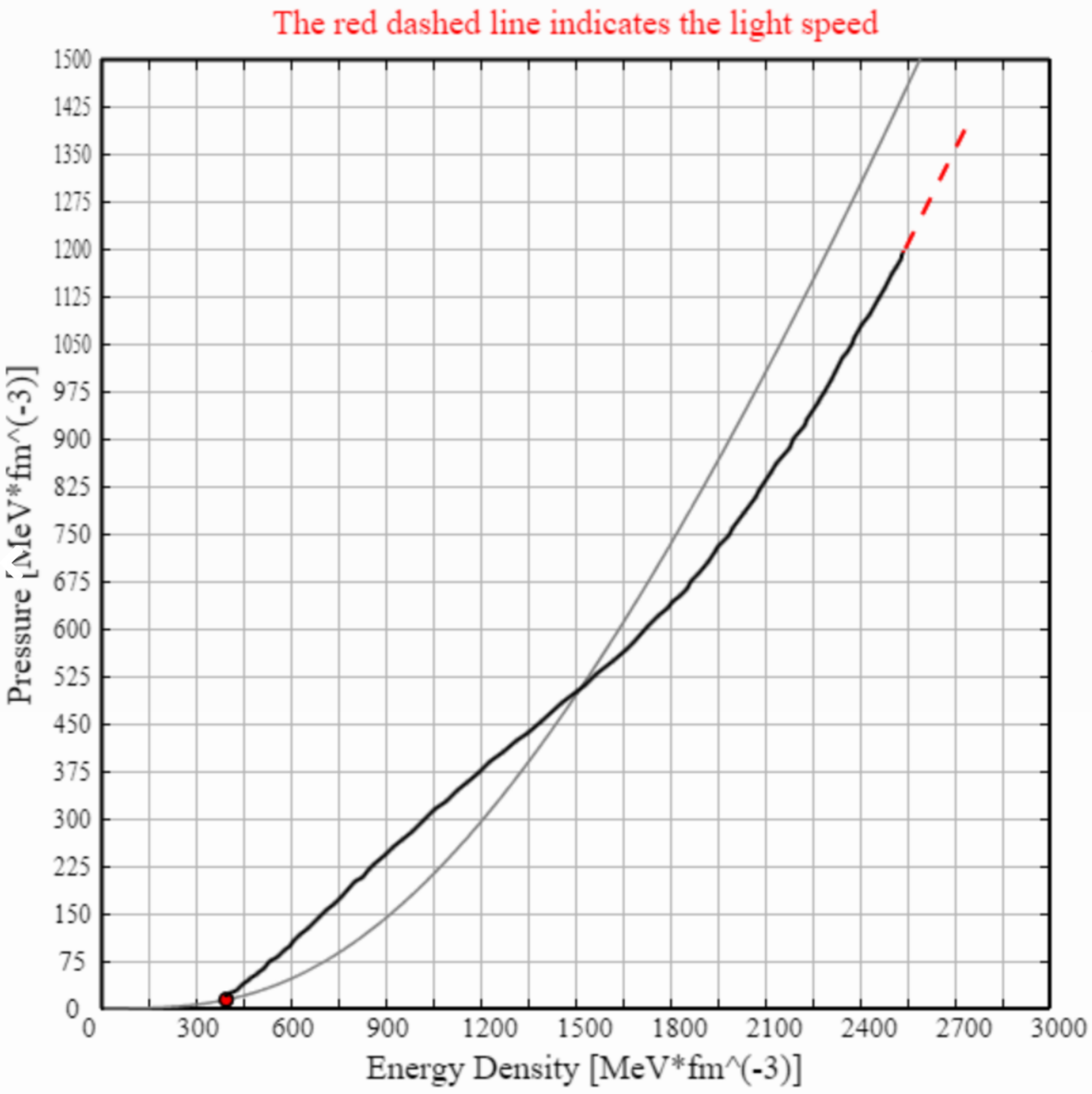
Motivation

Observables of neutron stars: Mass (M), Radius (R), Tidal deformability (Λ), Moment of Inertia (I)

Non-observables of neutron stars: Pressure (P), Energy density (Ɛ), Chemical potential (μ), Number density (n), etc. at certain radius.

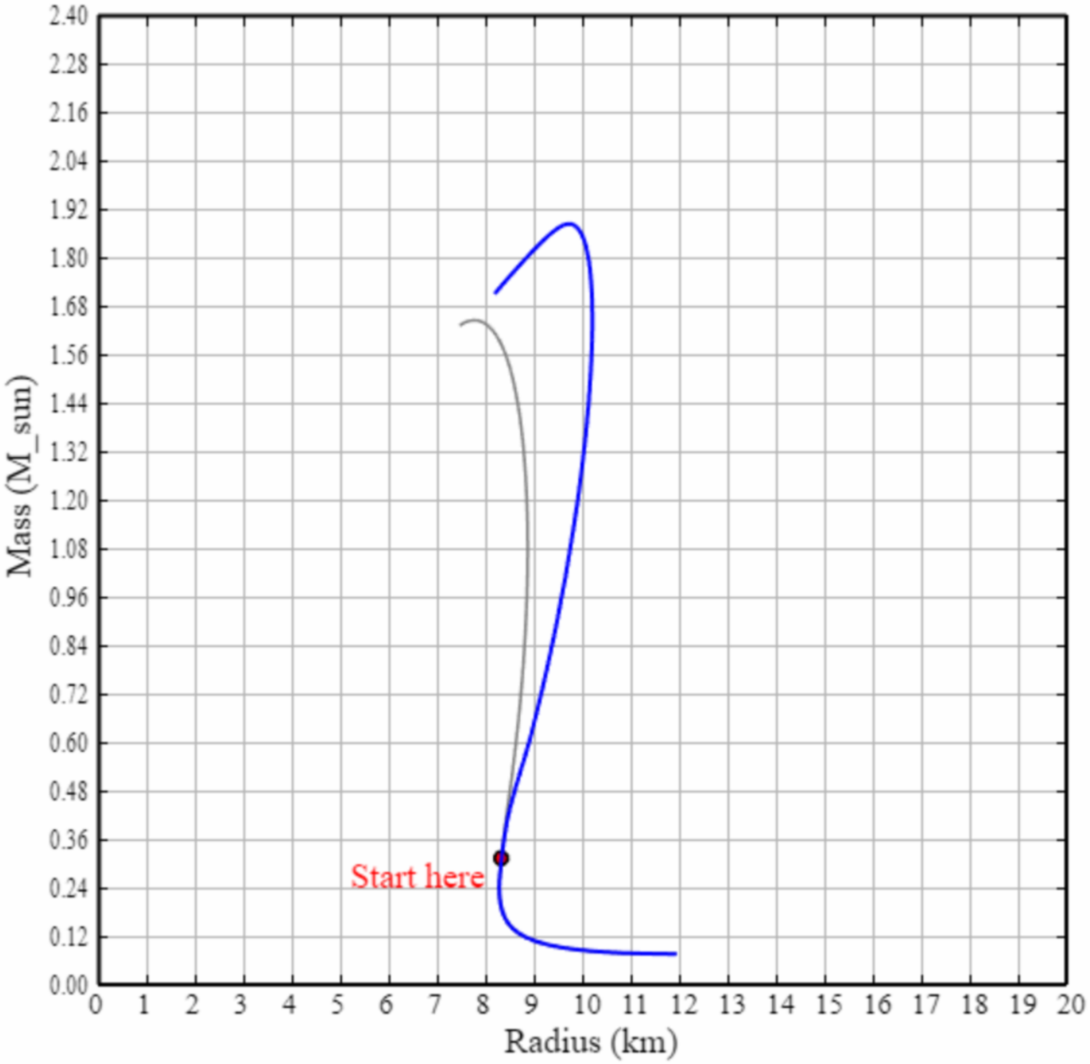
One of the most famous universal relations is the **I-Love-Q relation** which relates three observables, I, Love number and Quadrupole.

Is it possible to construct correlations between observables and non-observables, so that we can explore EOS with astronomical observations?



Solving the TOV equation

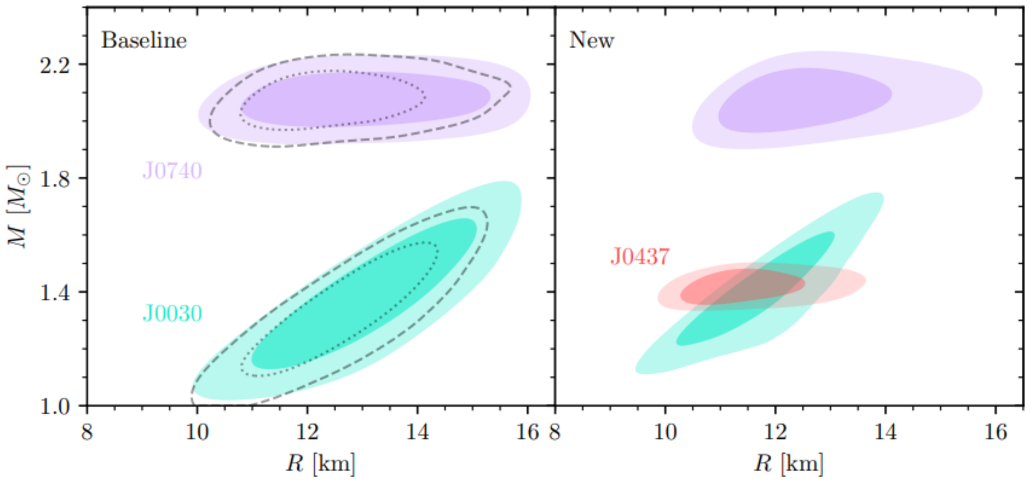
???

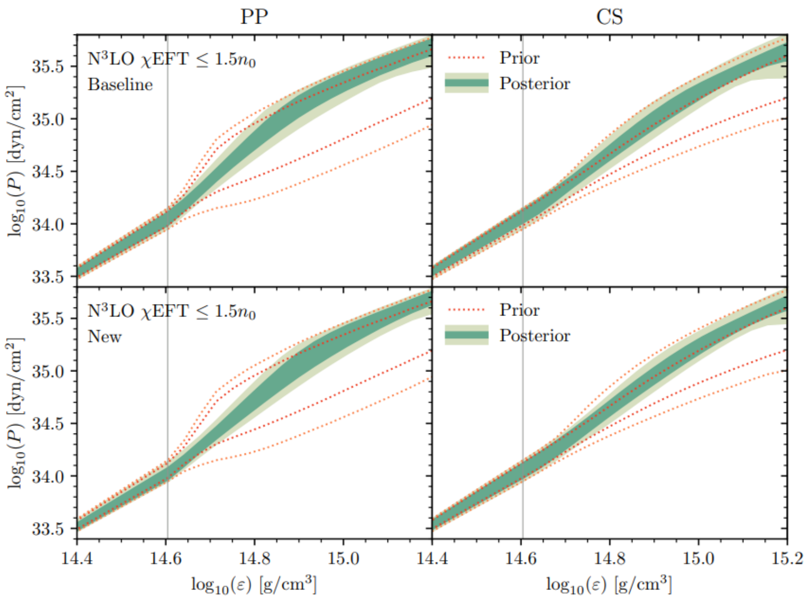


Previous Works

Constrain EoS from Bayesian approaches

Ex: Rutherford et al. *Constraining the Dense Matter Equation of State with New NICER Mass–Radius Measurements and New Chiral Effective Field Theory Inputs* (2024)





Inverting the entire M-R curve

We developed a method fitting with maximum mass and two radii at certain fractional mass points:

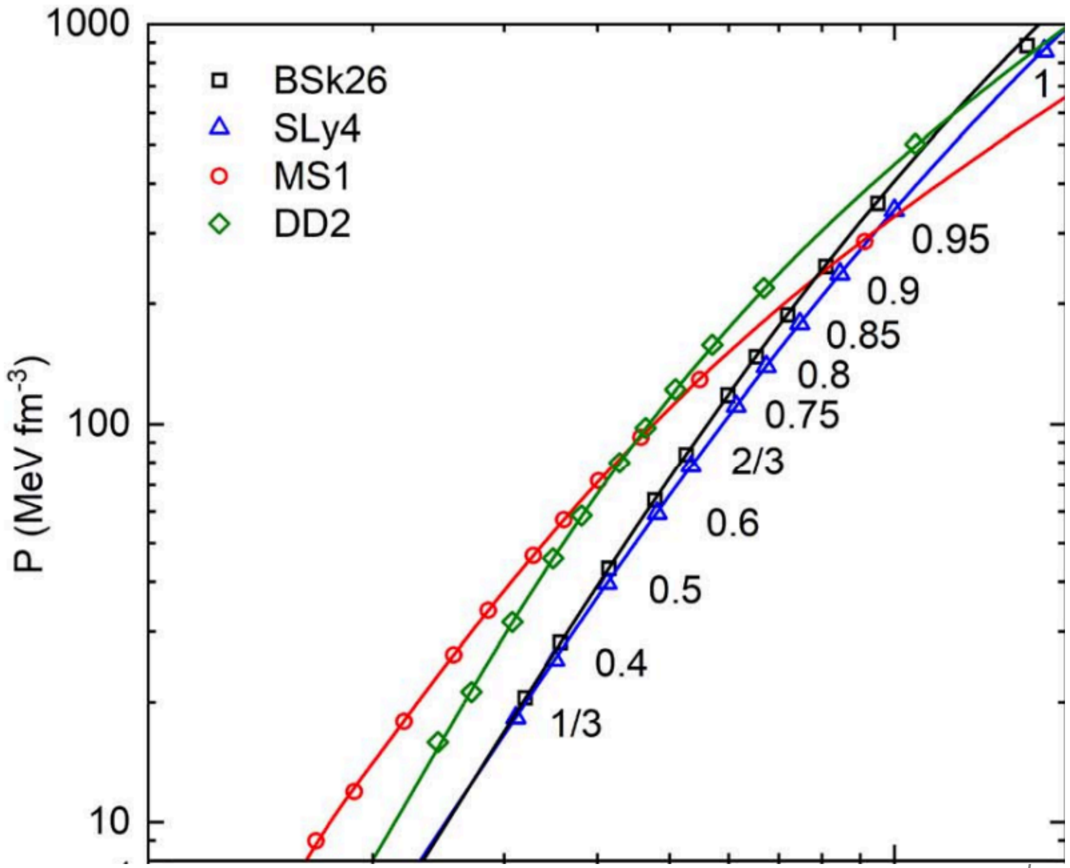
$$G_f = a_{Gf} \left(\frac{M_{max}}{M_\odot}\right)^{b_{Gf}} \left(\frac{R_g}{10\ \mathrm{km}}\right)^{c_{Gf}} \left(\frac{R_h}{10\ \mathrm{km}}\right)^{d_{Gf}}$$

where G ∈ [Ɛ, P, c_s / c , μ, n],

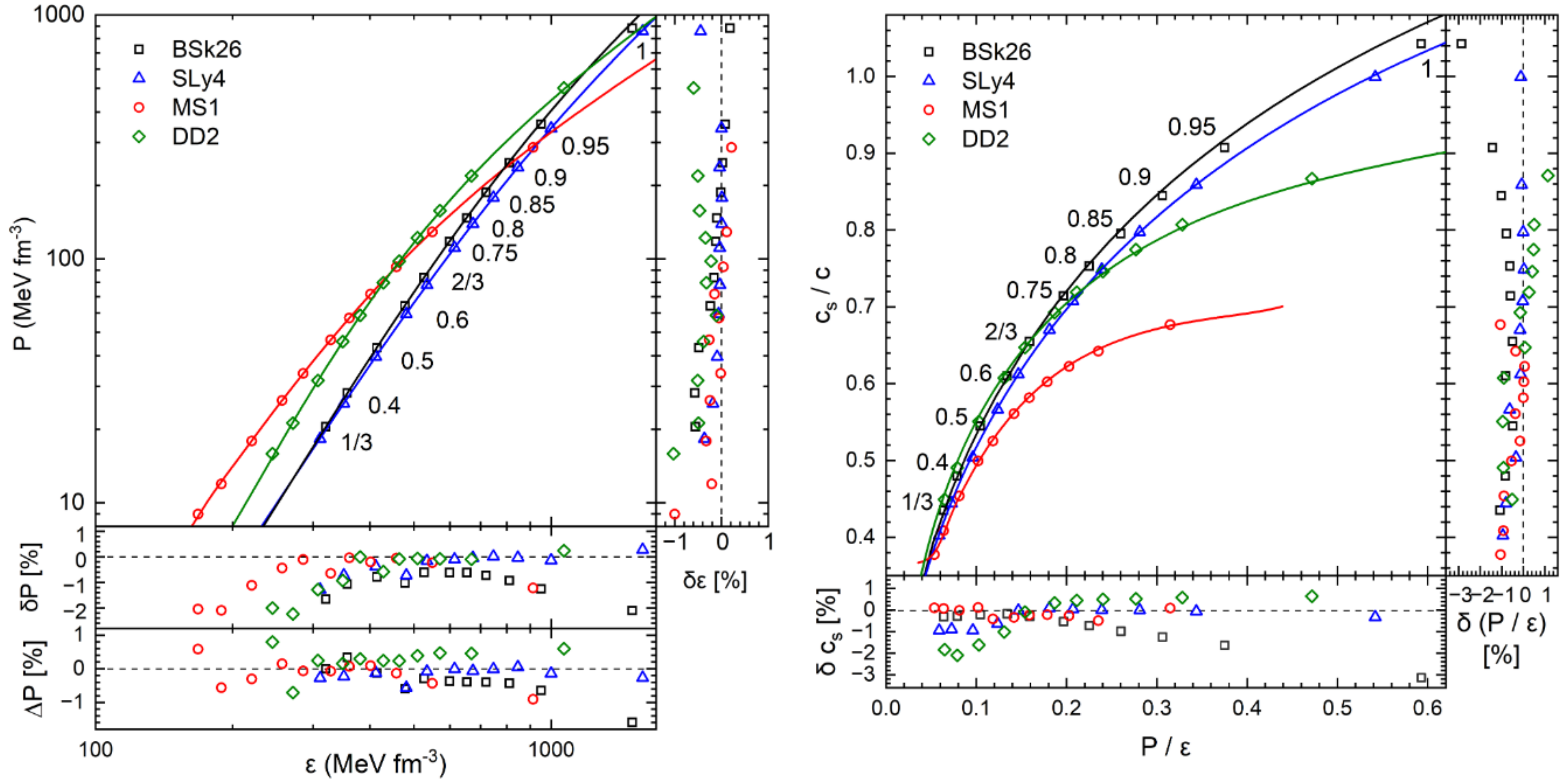
f is the grid we chose, in terms of M_{max}

g and h indicate which fraction in the grid

The fitting accuracy (RMS error) is about 0.5% for both Ɛ and P



Inverting the entire M-R curve



Inverting the entire M-R curve

1, How did we obtain these fitting parameters?

The parameter set (a_G, b_G, c_G, d_G) and (R_h, R_f) was fitted from our Database, which compiled 255 Skyrme-like, 270 RMF and 13 Gogny-like forces at zero temperature, no phase transition

Boyang Sun, Saketh Bhattiprolu, and James M. Lattimer. Compiled properties of nucleonic matter and nuclear and neutron star models from nonrelativistic and relativistic interactions (2024)
https://doi.org/10.1103/PhysRevC.109.055801

Contents of our Database

1, **Interaction parameters** of 268 non-relativistic models & 270 relativistic models

TABLE I. Parameters of Skyrme models. The unit for t_0 is MeV fm^3. t_1 and t_2 are both in MeV fm^5, t_{3i} is in MeV fm$^{3+3\alpha_i}$, and t_4 and t_5 are in MeV fm$^{5+3\delta}$ and MeV fm$^{5+3\gamma}$, respectively. Other parameters are dimensionless.

Model	t_0 / x_0	t_1 / x_1	t_2 / x_2	t_{31} / x_{31}	t_{32} / x_{32}	t_{33} / x_{33}
BSk1 [31]	−1830.45	262.97	−296.446	13444.7		
	0.599988	−0.5	−0.5	0.823074		
BSk2 [32]	−1790.62	260.996	−147.167	13215.1		
	0.498986	−0.08975	0.224411	0.515675		
BSk2′ [32]	−1792.71	259.053	−146.768	13267.9		
	0.498612	−0.08976	0.242854	0.509818		
BSk3 [33]	−1755.1297	233.262	−135.284	13543.2		
	0.476585	−0.03257	0.470393	0.422501		
BSk4 [34]	−1776.9376	306.884	−105.67	12302.1		
	0.542594	−0.53517	0.494738	0.759028		

TABLE V. Parameters of RMF models. Masses, A, α_1, and α_2 are in MeV. Other quantities are dimensionless.

Model	M_n / A	m_σ / B	m_ω / C	m_ρ / α_1	m_δ / α_1'	g_σ / α_2	g_ω / α_2'	g_ρ / α_3'	g_δ
Linear finite-range models (type 1)									
H1 [96]	939	550	783	770		11.079	13.806	5.258	
L1 [97]	938	550	783			10.3	12.6		
L2 [97]	938	546.94	780			11.397	14.248		
L3 [97]	938	492.26	780			10.692	14.87		
LBF [45]	938	615	1008	763		12.334	17.619	10.378	
LHS [97]	938.9	520	783	770		10.481	13.814	8.0849	

2, **Saturation properties for nuclei** (energy per particle, incompressibility, skewness of symmetric matter at saturation density [E_0, K_0, Q_0]

&

symmetry energy parameters [$J_{1,2}$, $L_{1,2}$, $K_{sym1,2}$, $Q_{sym1,2}$])

Contents of our Database

3, **Nuclear structure properties** (surface tension for symmetric matter; surface tension symmetry parameter; surface energy; neutron skin thickness and dipole polarizability for ^{208}Pb & ^{48}Ca)

$$\sigma_o = \int_0^{n_0} \sqrt{2\mathcal{Q}n(E_{1/2}(n) - E_0)}dn \qquad \sigma_\delta = \frac{J_2}{\sqrt{2}} \int_0^{n_0} \sqrt{\frac{\mathcal{Q}n}{E_{1/2}(n) - E_0}} \left[\frac{J_2}{\mathcal{S}_2(n)} - 1\right] dn$$

$$E_S = 4\pi r_o^2 \sigma_o, \qquad S_S = 4\pi r_o^2 \sigma_\delta$$

$$\alpha_D^{\rm h} = \frac{1}{20}\frac{AR^2}{J_2}\left(1 + \frac{5S_S}{3J_2A^{1/3}}\right),$$

$$R_n - R_p = \frac{2r_o}{3}\left(1 + \frac{S_S}{J_2A^{1/3}}\right)^{-1}\left[I\frac{S_S}{J_2} - \frac{3Ze^2}{140r_oJ_2}\left(1 + \frac{10S_S}{3J_2A^{1/3}}\right)\right]$$

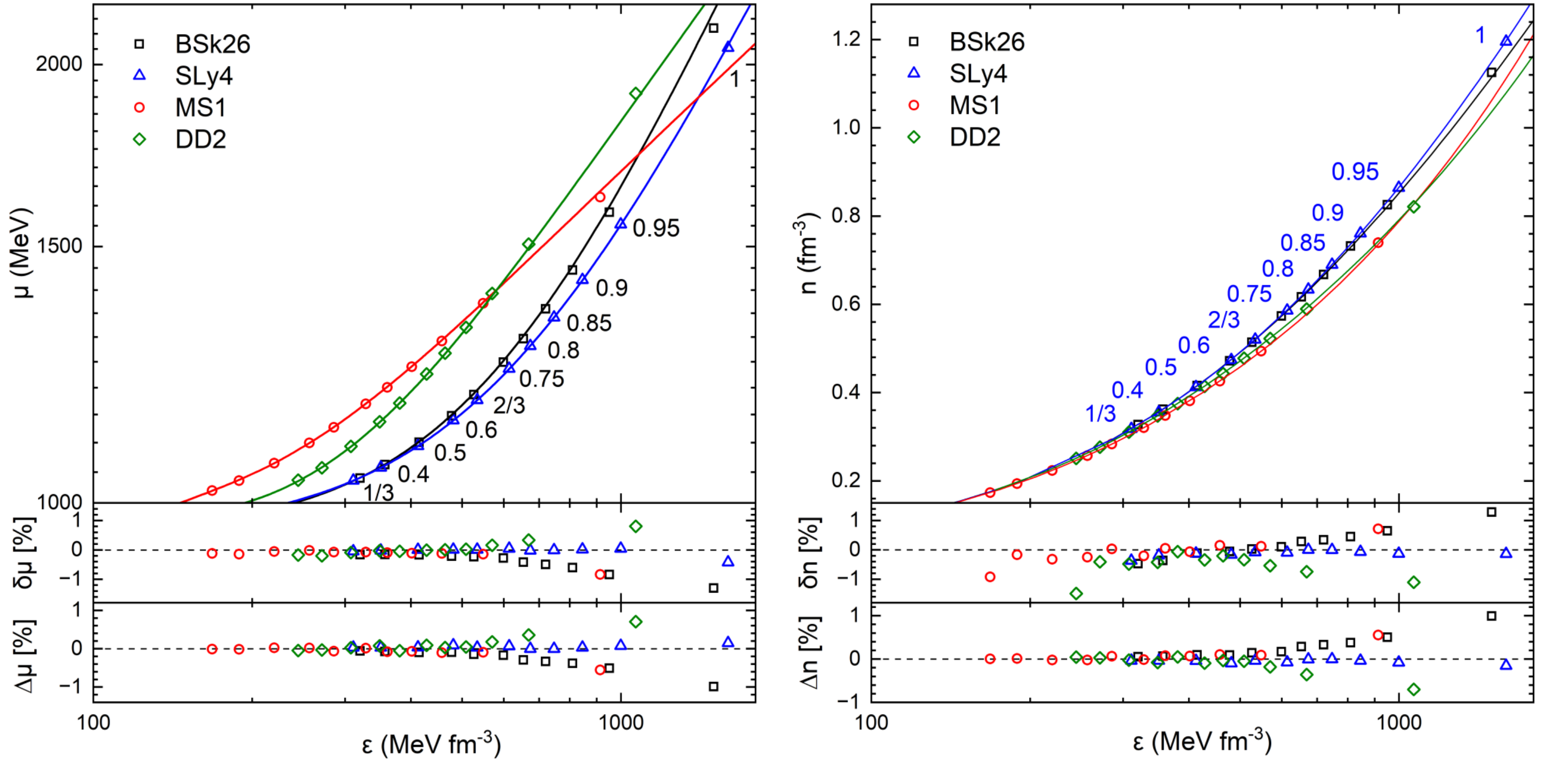
$$\alpha_D^{\rm DM} = \frac{\pi e^2}{90}\frac{AR^2}{J_2}\left(1 + \frac{5S_S}{3J_2A^{1/3}}\right)$$

4, **Neutron star properties** (radius, tidal deformability, moment of inertia, binding energy at 1.2, 1.4, 1.6 solar mass & maximum mass)

Inverting the entire M-R curve

2, Does this relation apply to chemical potential μ and number density n?

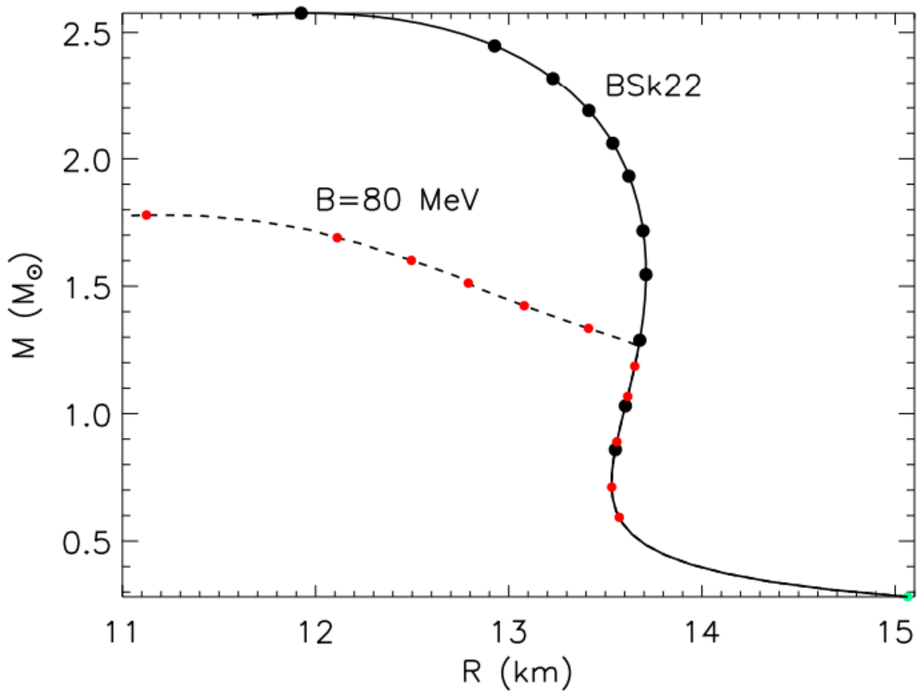
Yes, The RMS errors are even lower than 0.5%

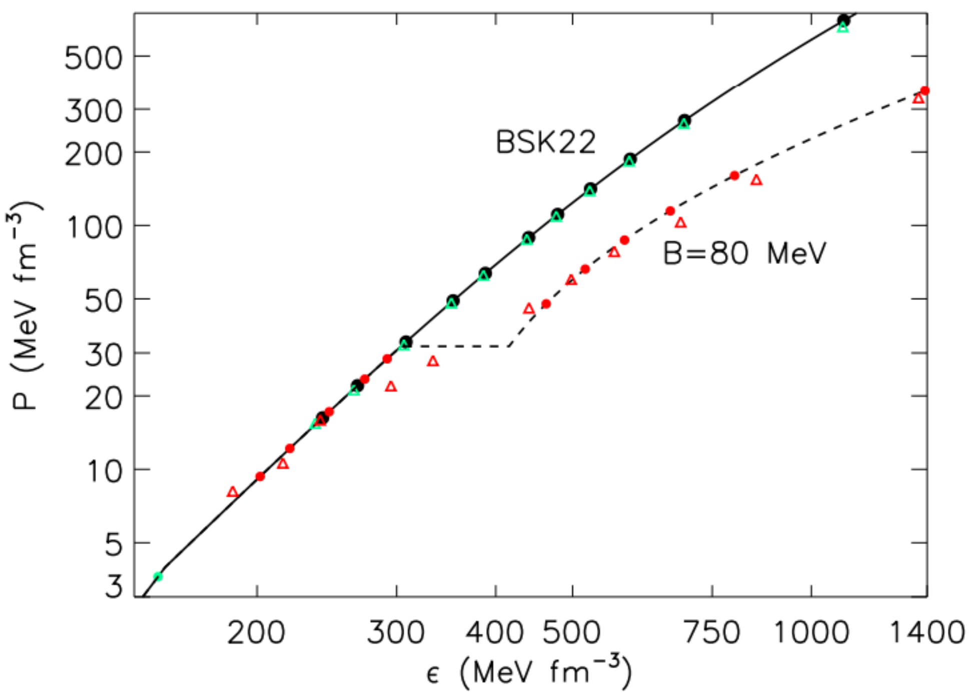


EoS with phase transition

3, What if there's 1st order phase transition?

We chose a hadronic EoS BSk22 combined with MIT bag model for quarks





Inverting the entire M-R curve

Why does this formula work?

Ref: *Cai, B.-J., Li, B.-A., & Zhang, Z. 2023, ApJ, 952, 147*

Cai, B.-J., Li, B.-A., & Zhang, Z. 2023, PhRvD, 108, 103041

$$\frac{d\hat{P}}{d\hat{r}} = -\frac{(\hat{\mathcal{E}} + \hat{P})(\hat{m} + \hat{r}^3\hat{P})}{\hat{r}(\hat{r} - 2\hat{m})}, \qquad \frac{d\hat{m}}{d\hat{r}} = \hat{\mathcal{E}}\hat{r}^2.$$

Expanding quantities in terms of radius:

$$\hat{\mathcal{E}} = \sum_i a_i\hat{r}^i, \qquad \hat{P} = \sum_i b_i\hat{r}^i, \qquad \hat{m} = \sum_i c_i\hat{r}^i,$$

The non-vanishing terms:

$$a_0 = 1, \qquad a_2 = -\frac{1}{\hat{R}^2}, \qquad b_0 = \hat{P}_c,$$

$$b_2 = -\frac{1 + 4\hat{P}_c + 3\hat{P}_c^2}{6}, \qquad c_3 = \frac{a_0}{3}, \qquad c_5 = \frac{a_2}{5},$$

Some results:

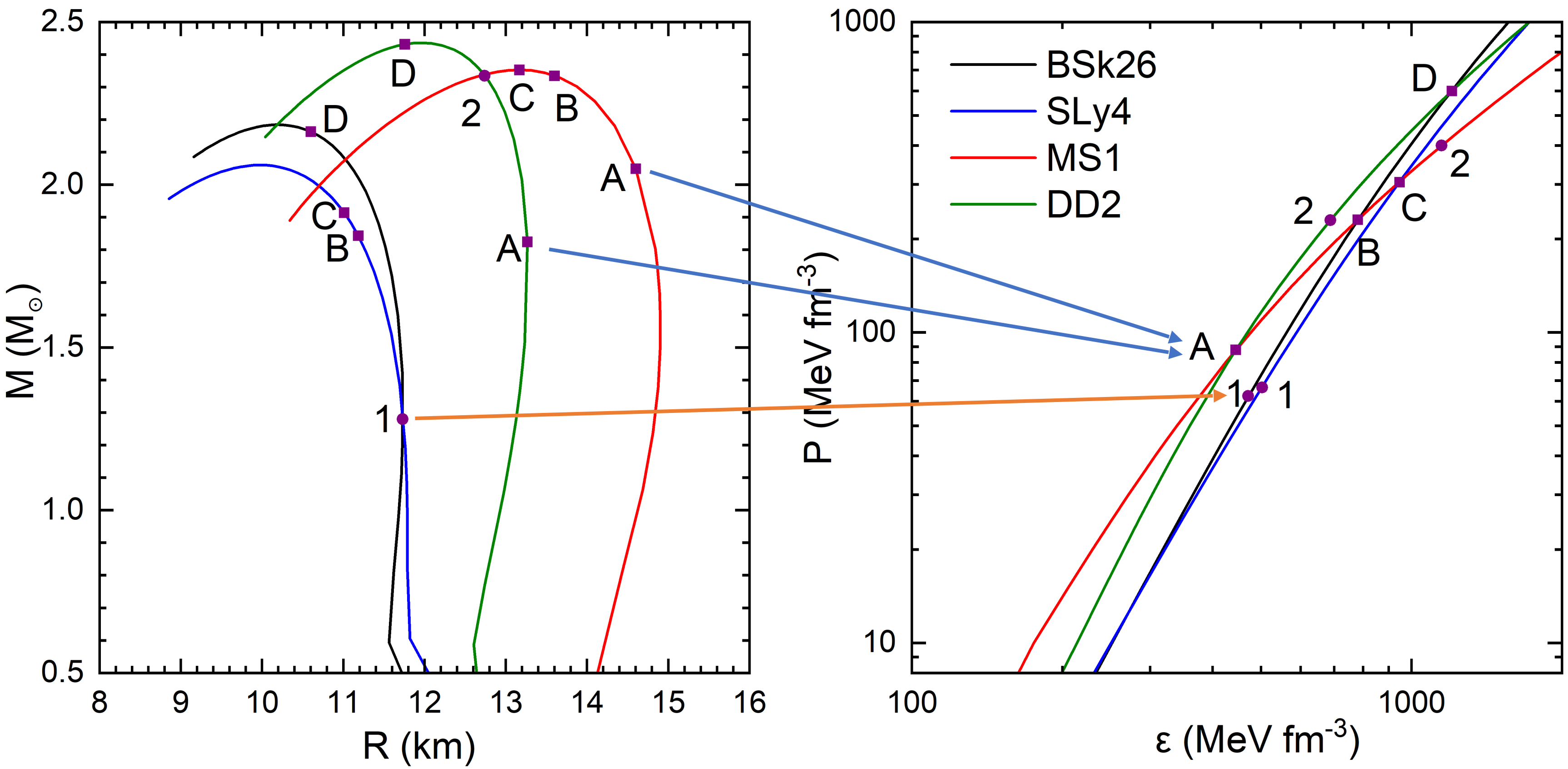
$$\hat{M}_{max} = \frac{GM_{max}}{c^2}\sqrt{\frac{4\pi G\mathcal{E}_{max}}{c^4}} \simeq \frac{2\hat{R}^3_{max}}{15},$$

$$\hat{R}_{max} = R_{max}\sqrt{\frac{4\pi G\mathcal{E}_{max}}{c^4}} \simeq \sqrt{\frac{6\hat{P}_{max}}{1 + 4\hat{P}_{max} + 3\hat{P}^2_{max}}} \equiv \sqrt{6\phi_{max}},$$

$$c^2_{s,max} = \left(\frac{d\mathcal{E}}{dP}\right)_{max} = \frac{b_2}{a_2} = \hat{P}_{max},$$

Inverting arbitrary points

Main issue: the inversion is not unique



Inverting arbitrary points

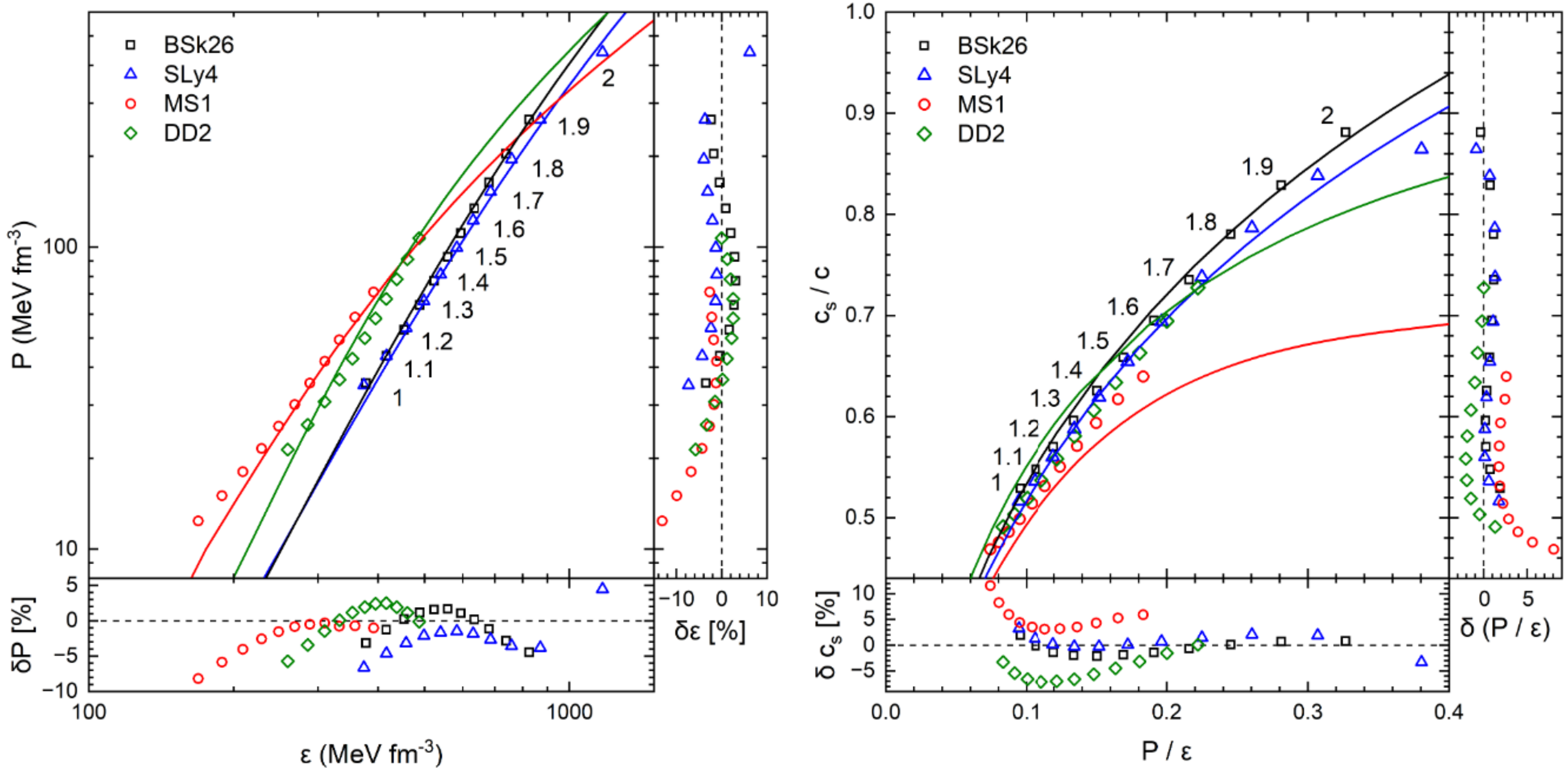
To improve the accuracy, more information from the M-R relation has to be incorporated into the fits.

One possibility is to utilize the inverse slope dR/dM at an (M, R) point:

$$\ln G = a_G + b_G \ln\left(\frac{M}{M_\odot}\right) + c_G \ln\left(\frac{R}{\text{km}}\right) + d_G \left[\ln\left(\frac{M}{M_\odot}\right)\right]^2 + e_G \ln\left(\frac{M}{M_\odot}\right) \ln\left(\frac{R}{\text{km}}\right) + f_G \left[\ln\left(\frac{R}{\text{km}}\right)\right]^2 + g_G \left(\frac{dR}{dM}\right)\left(\frac{M_\odot}{\text{km}}\right).$$

in which $M_j \in [2.0, 1.9, 1.8, 1.7, 1.6, 1.5, 1.4, 1.3]\ M_\odot$

Inverting arbitrary points



The RMS errors are around 3.5% and 4.0% for Ɛ and P, respectively (6.5% and 7.0% for Ɛ and P without the utilization of dR/dM)

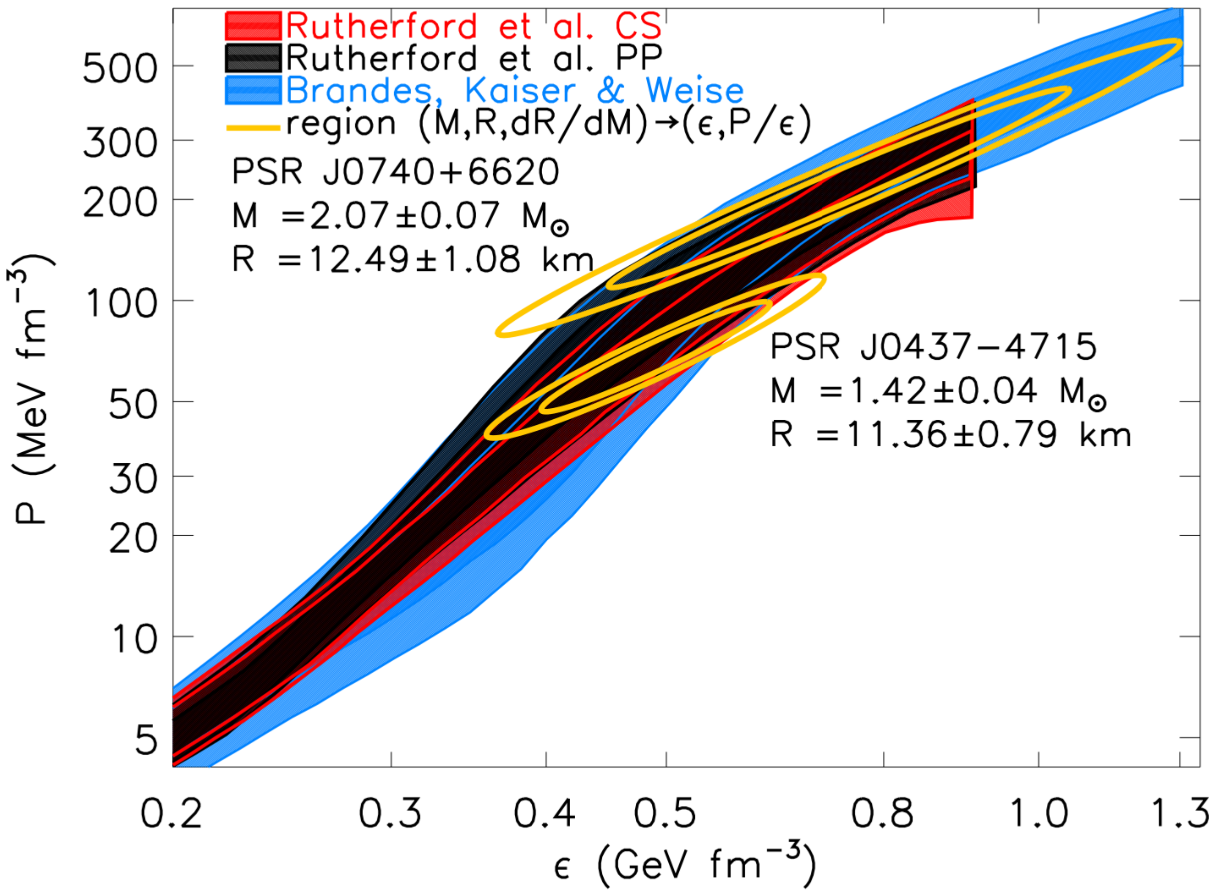
Fitting with astronomical observation

We compared our results of predicted P- Ɛ regions from NICER data with those from constrains:

Rutherford et al. Constraining the Dense Matter Equation of State with New NICER Mass–Radius Measurements and New Chiral Effective Field Theory Inputs (2024)

Brandes, Kaiser and Weise Brandes et al. Bayesian inference of multi-messenger astrophysical data: Joint and coherent inference of gravitational waves and kilonovae (2023)

Fitting with astronomical observation



Our approach is consistent with the results of Brandes et al. (2023) and Rutherford et. al. (2024) at both low energy and high energy regions.

Summary

Paper

- Previous works
  - Bayesian method
  - Linblom’s direct inversion
  - Cai’s method
  - Ofengeim’s method

  Section I, II

- Power-law fits
  - Inverse entire M-R curve

    $$G_f = a_{Gf}\left(\frac{M_{max}}{M_\odot}\right)^{b_{Gf}}\left(\frac{R_g}{10\ \mathrm{km}}\right)^{c_{Gf}}\left(\frac{R_h}{10\ \mathrm{km}}\right)^{d_{Gf}}$$

    Section III

  - Inverse arbitrary (M, R) points

    $$\ln G = a_G + b_G \ln\left(\frac{M}{M_\odot}\right) + c_G \ln\left(\frac{R}{\mathrm{km}}\right) + d_G\left[\ln\left(\frac{M}{M_\odot}\right)\right]^2 + e_G \ln\left(\frac{M}{M_\odot}\right)\ln\left(\frac{R}{\mathrm{km}}\right) + f_G\left[\ln\left(\frac{R}{\mathrm{km}}\right)\right]^2 + g_G\left(\frac{dR}{dM}\right)\left(\frac{M_\odot}{\mathrm{km}}\right).$$

    Section IV

- Applications
  - Astronomical observations
  - Visualization Software — Zenodo

  Section V

Applications

1, Inferring the entire EOS band with Bayesian method instead of just two regions

Applications

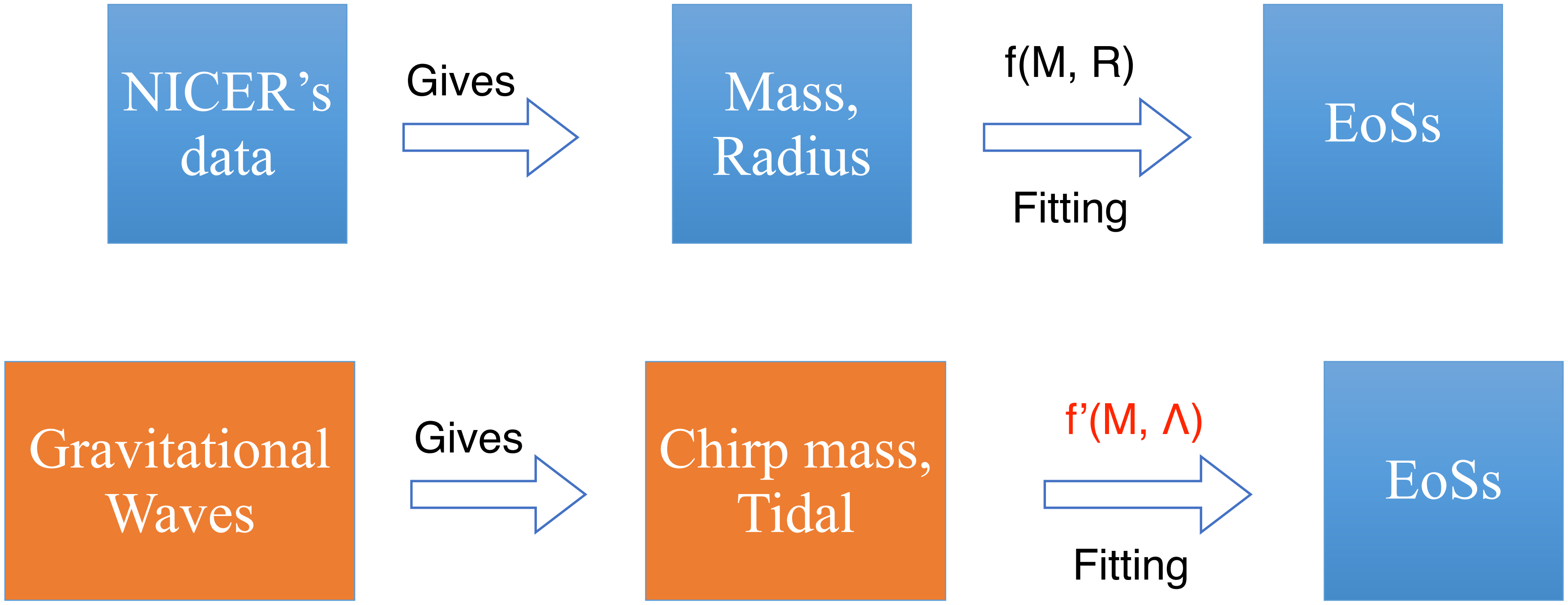
2, Combine our formula with other computing methods, ie. gradient descent / machine learning

Our fitting formula: Pretty fast but not that accurate (point to point)

gradient descent / machine learning: Can be more accurate but require lots of computing resources

Applications

3, Use mass and tidal deformability Λ for inversion



THANK YOU