# Theoretical Developments in Lattice and Continuum TMDs


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**MIT**


**Parton Distributions and Nucleon Structure**
**INT workshop**
**University of Washington, Seattle**
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Massachusetts Institute of Technology

U.S. DEPARTMENT OF ENERGY | Office of Science

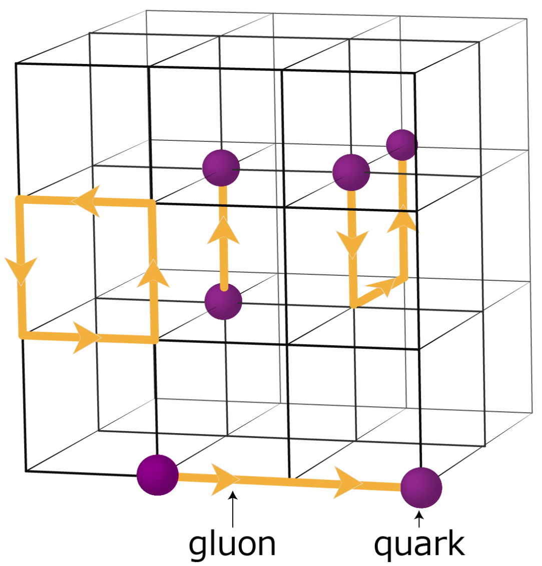
TMD Collaboration

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

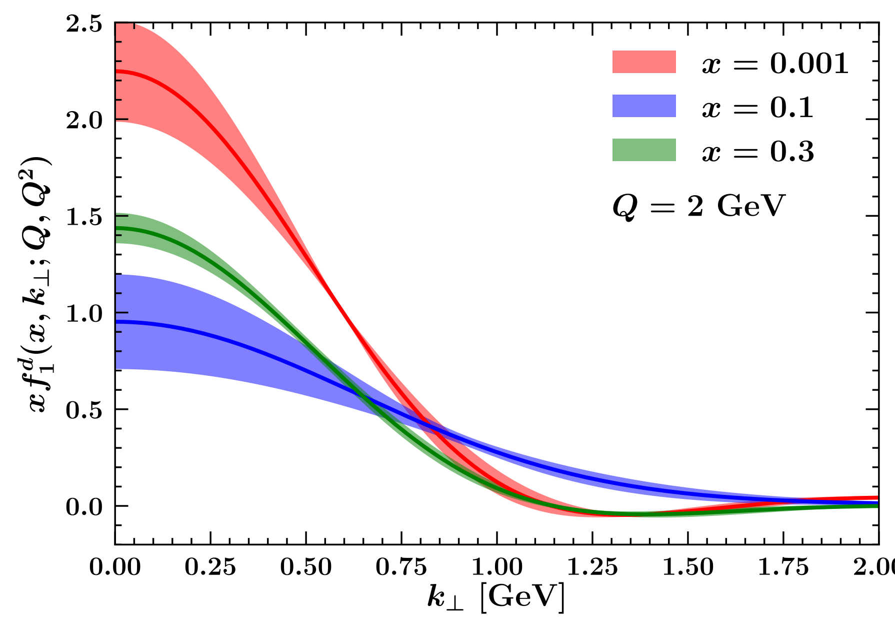
# Goal

**Proof of Factorization connecting**
**Quasi-TMDs (Lattice) and Collins-TMDs (cross section)**

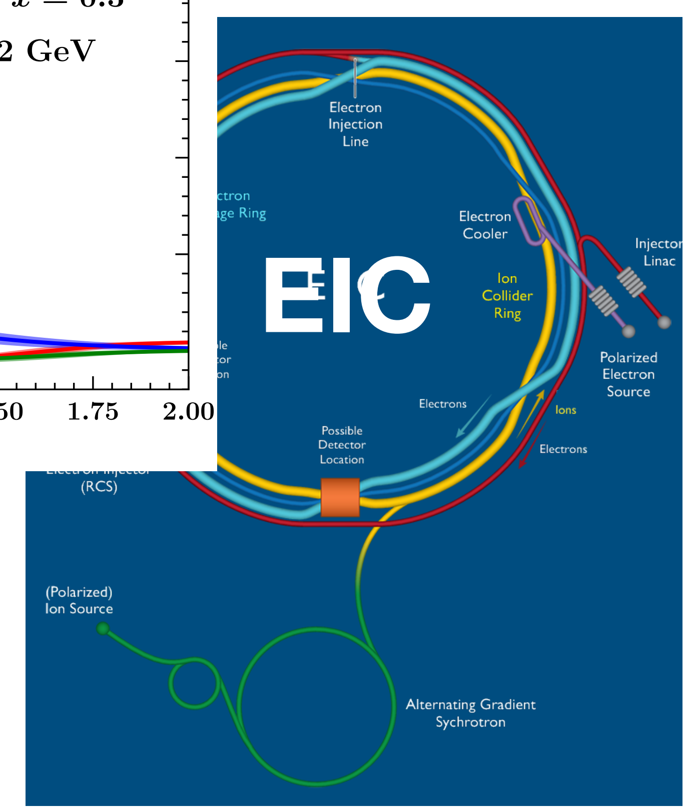
$$\lim_{\tilde\eta\to\infty} \tilde f_{i/h}(x,\vec b_T,\mu,\zeta,x\tilde P^z,\tilde\eta) = C_i\big(x\tilde P^z,\mu\big)\, f^C_{i/h}(x,\vec b_T,\mu,\zeta) + \ldots$$








Bacchetta et.al. (1912.07550)



## Goal

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

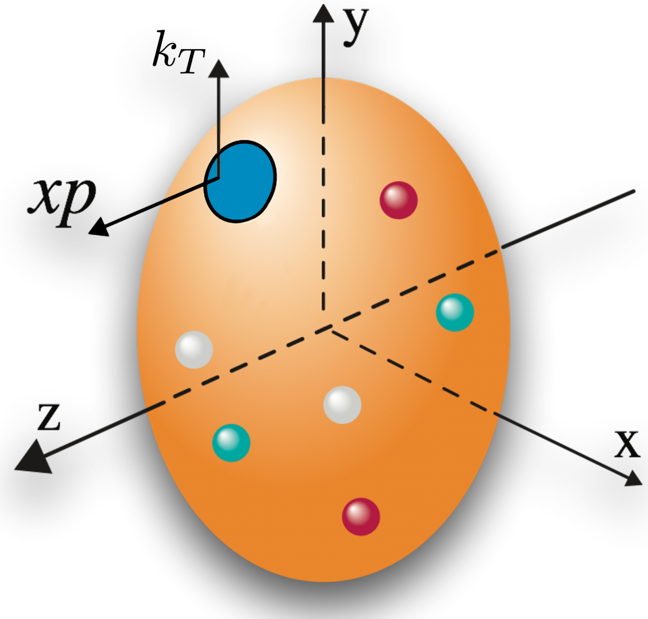
**Proof of Factorization connecting**
**Quasi-TMDs (Lattice) and Collins-TMDs (Continuum)**

$$\lim_{\tilde\eta\to\infty} \tilde f_{i/h}(x, \vec b_T, \mu, \zeta, x\tilde P^z, \tilde\eta) = C_i\big(x\tilde P^z, \mu\big)\, f^C_{i/h}(x, \vec b_T, \mu, \zeta) + \ldots$$

## Outline

- Introduction
- Setup a General Framework
- Proof
- Implications

# TMDs

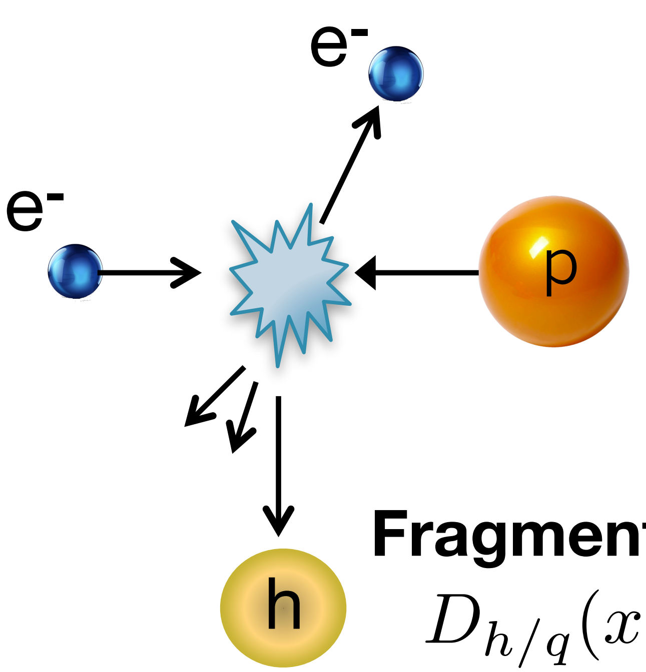



$$f_{q/P}(x, k_T, \mu, \zeta)$$

longitudinal & Transverse

## Semi-Inclusive DIS

$$\sigma \sim f_{q/P}(x, k_T) D_{h/q}(x, k_T)$$

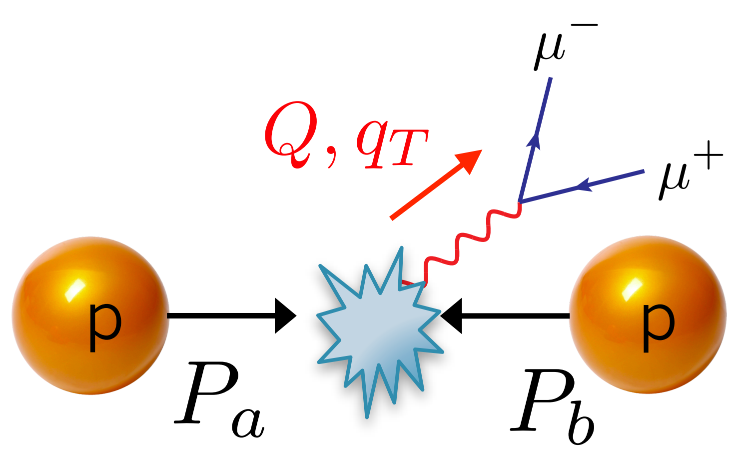



**Fragmentation**

$D_{h/q}(x, k_T)$

## Drell-Yan

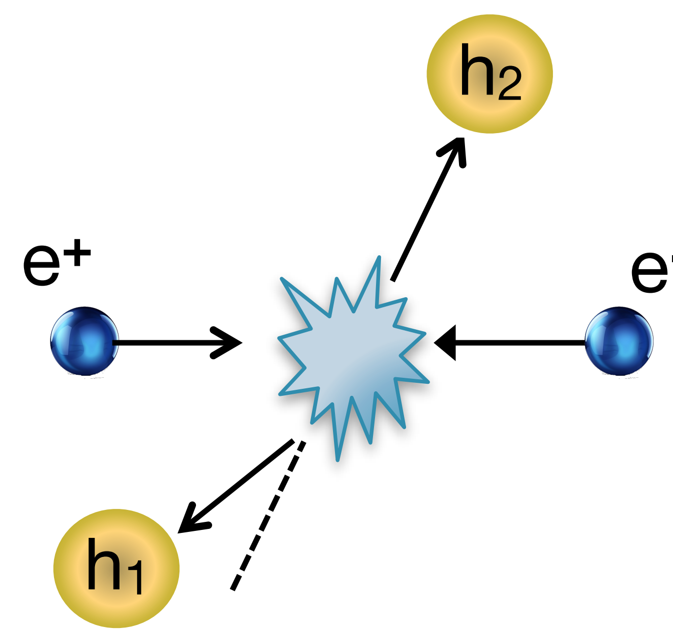
$$\sigma \sim f_{q/P}(x, k_T) f_{q/P}(x, k_T)$$




$$q_T \ll Q$$

## Dihadron in e+e-

$$\sigma \sim D_{h_1/q}(x, k_T) D_{h_2/q}(x, k_T)$$



# TMD Factorization (Drell Yan)

**CSS (Collins, Soper, Sterman)**

**SCET (Soft Collinear Effective Theory)**

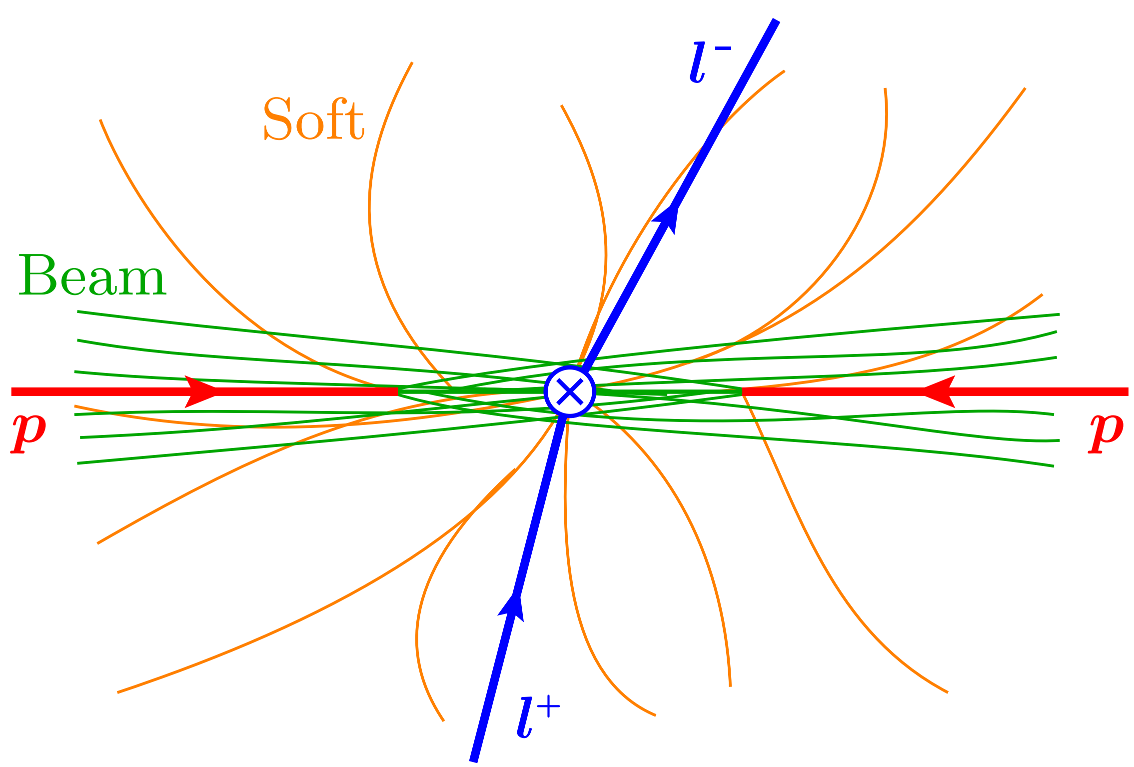
$$\frac{d\sigma}{dQ\,dY\,dq_T^2} = H(Q,\mu) \int d^2\vec{b}_T\, e^{i\vec{q}_T\cdot\vec{b}_T}\, f_q(x_a,\vec{b}_T,\mu,\zeta_a)\, f_q(x_b,\vec{b}_T,\mu,\zeta_b) \Big[1+\mathcal{O}\Big(\frac{q_T^2}{Q^2}\Big)\Big]$$

**Hard virtual corrections**

**TMDs**

$$\zeta_a\zeta_b = Q^4$$

$$f_q(x,\vec{b}_T,\mu,\zeta) \sim Z_{\text{uv}}\, B_q/\sqrt{S_q}$$



# TMD Definitions

Beam Function

Soft factor

$$f_q(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z_{\mathrm{uv}}(\epsilon, \mu, \zeta) \lim_{\tau\to 0} B_q(x, \vec{b}_T, \epsilon, \tau, \zeta) \;/\sqrt{S_q(b_T, \epsilon, \tau)}$$

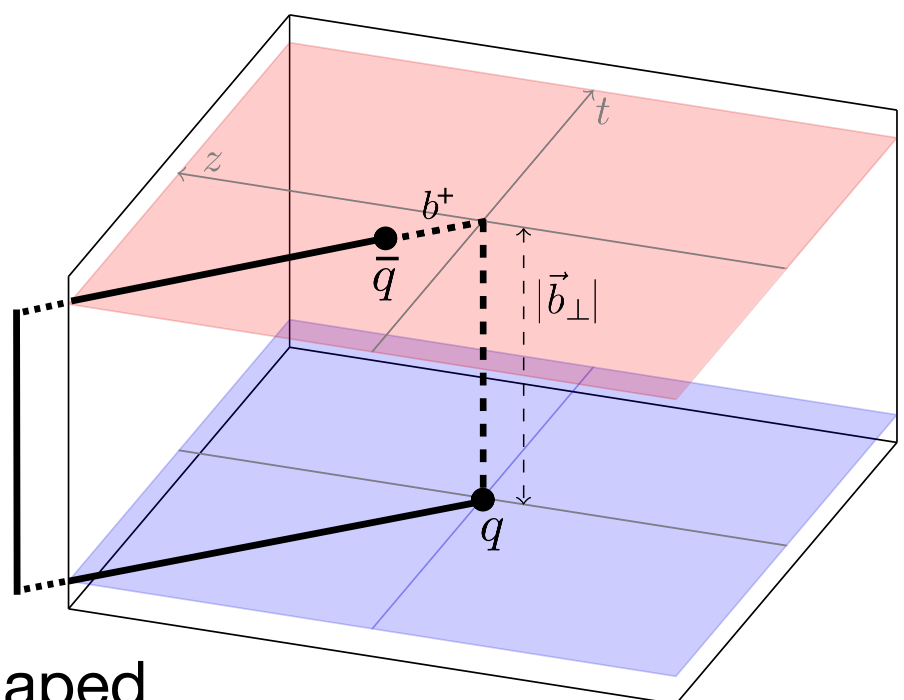
**UV limit**

**rapidity limit**

$$B_{q/p} = \mathrm{FT}_{b^+\to x}\,\Omega_{q/p}$$

$$\Omega_{q/p} = \langle p|O_B|p\rangle$$

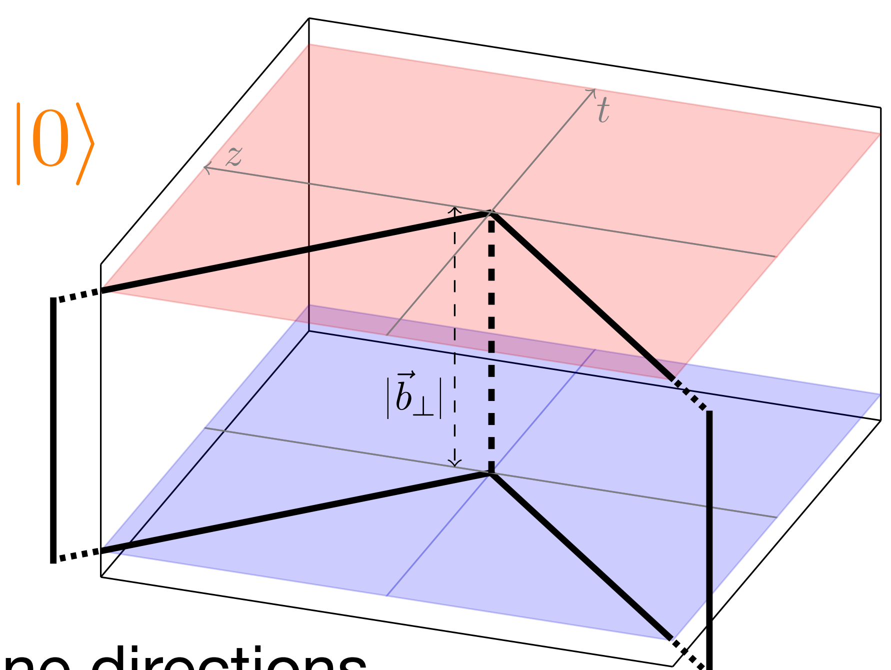
$O_B$ :




staple shaped Wilson lines

$$S_q = \langle 0|O_S|0\rangle$$

$O_S$ :




two light-cone directions

# Evolution

**Sum large logarithms:** $\ln(Q^2 b_T^2) \sim \ln \frac{Q^2}{q_T^2}$

$$f_q(x, \vec{b}_T, \mu, \zeta) = \exp\left[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_\mu^q(\mu', \zeta_0)\right] \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln \frac{\zeta}{\zeta_0}\right] f_q(x, \vec{b}_T, \mu_0, \zeta_0)$$

**CS kernel** **boundary data**

$\mu$ = renormalization scale

$\zeta$ = Collins-Soper parameter $= 2\left(xP^+ e^{-y_n}\right)^2$

**Nonperturbative contributions in both**

$b_T^{-1} \sim \Lambda_{\mathrm{QCD}}$

**Targets for Lattice Calculations**

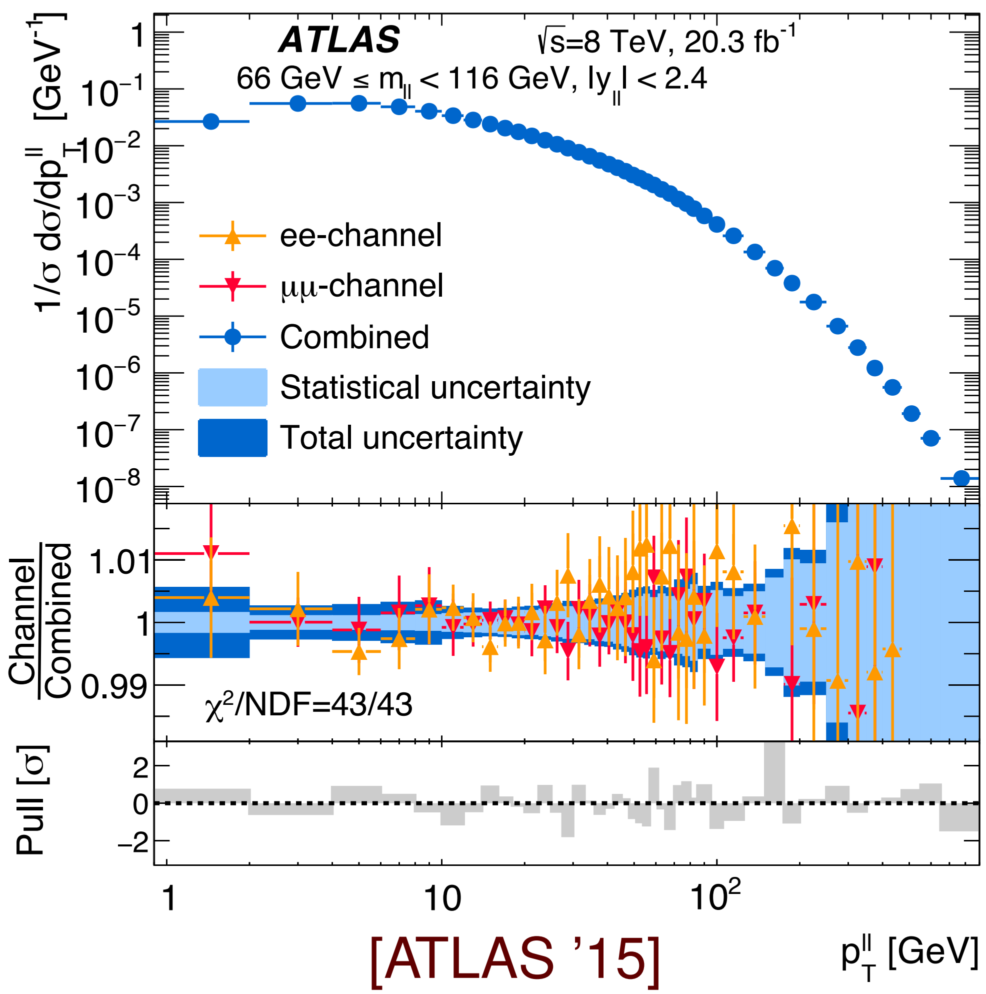
**Scales:**

in lattice calculation or model: $\mu_0, \sqrt{\zeta_0} \sim \mathrm{GeV}$

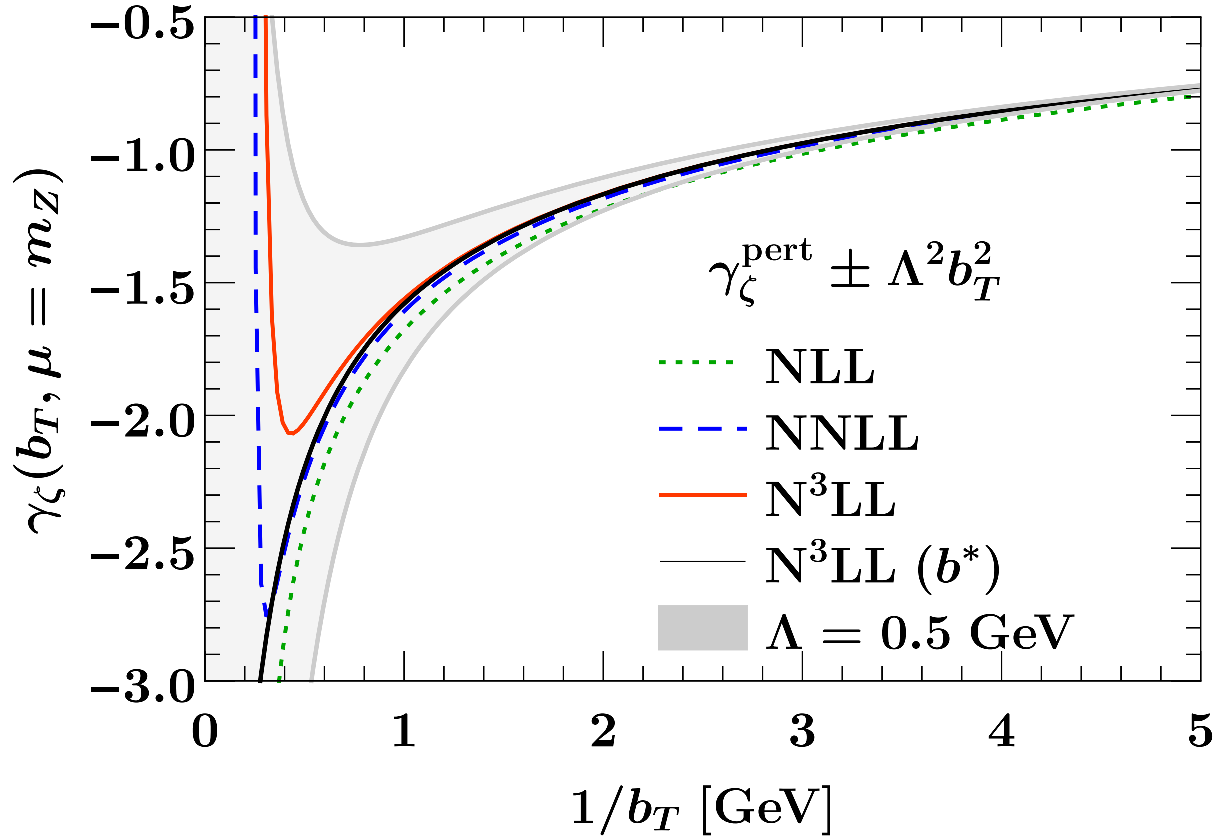
in cross-section: $\mu, \sqrt{\zeta} \sim Q$

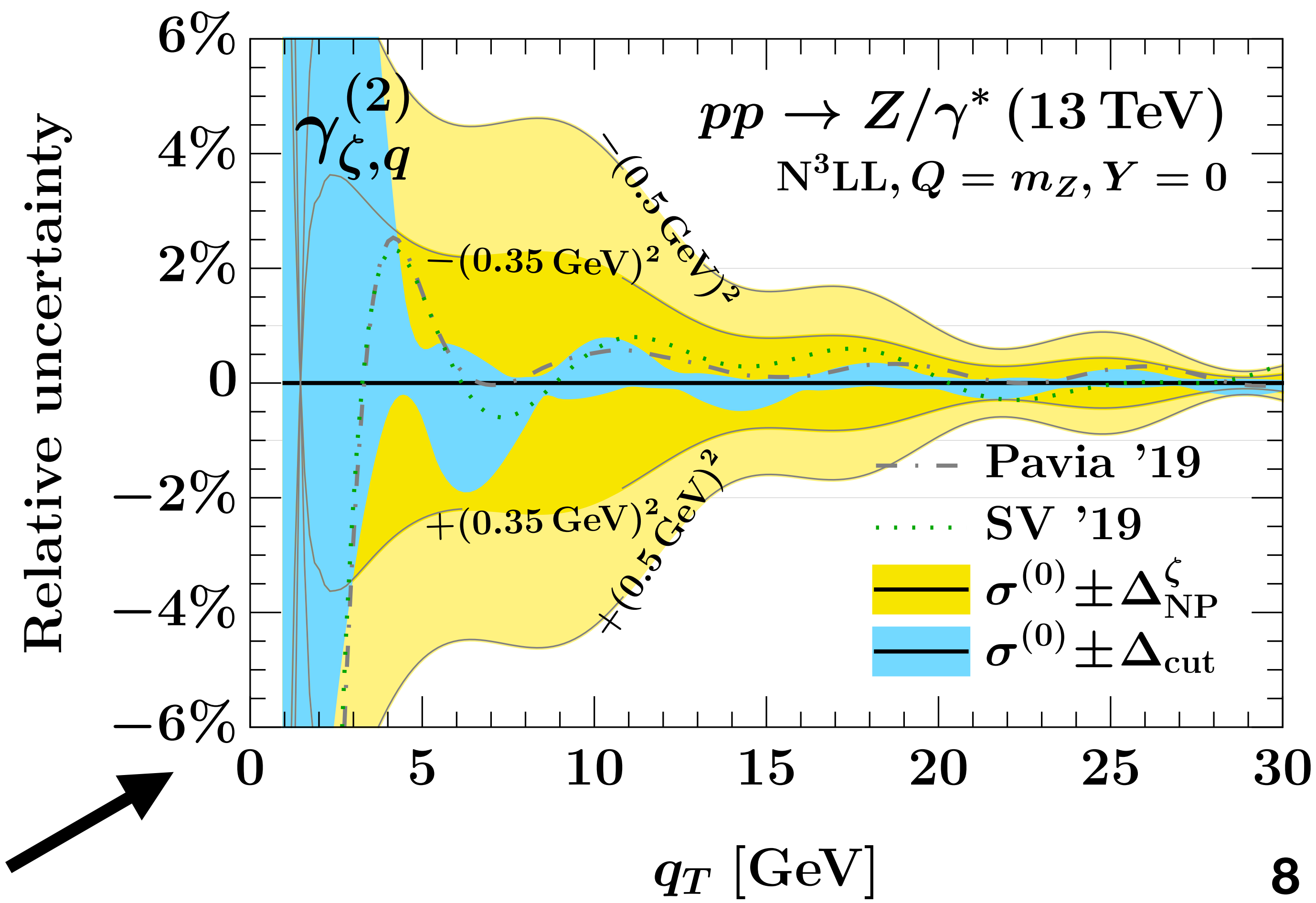
# Nonperturbative Contributions to DY Cross Sections

## Collins-Soper kernel




[ATLAS '15]

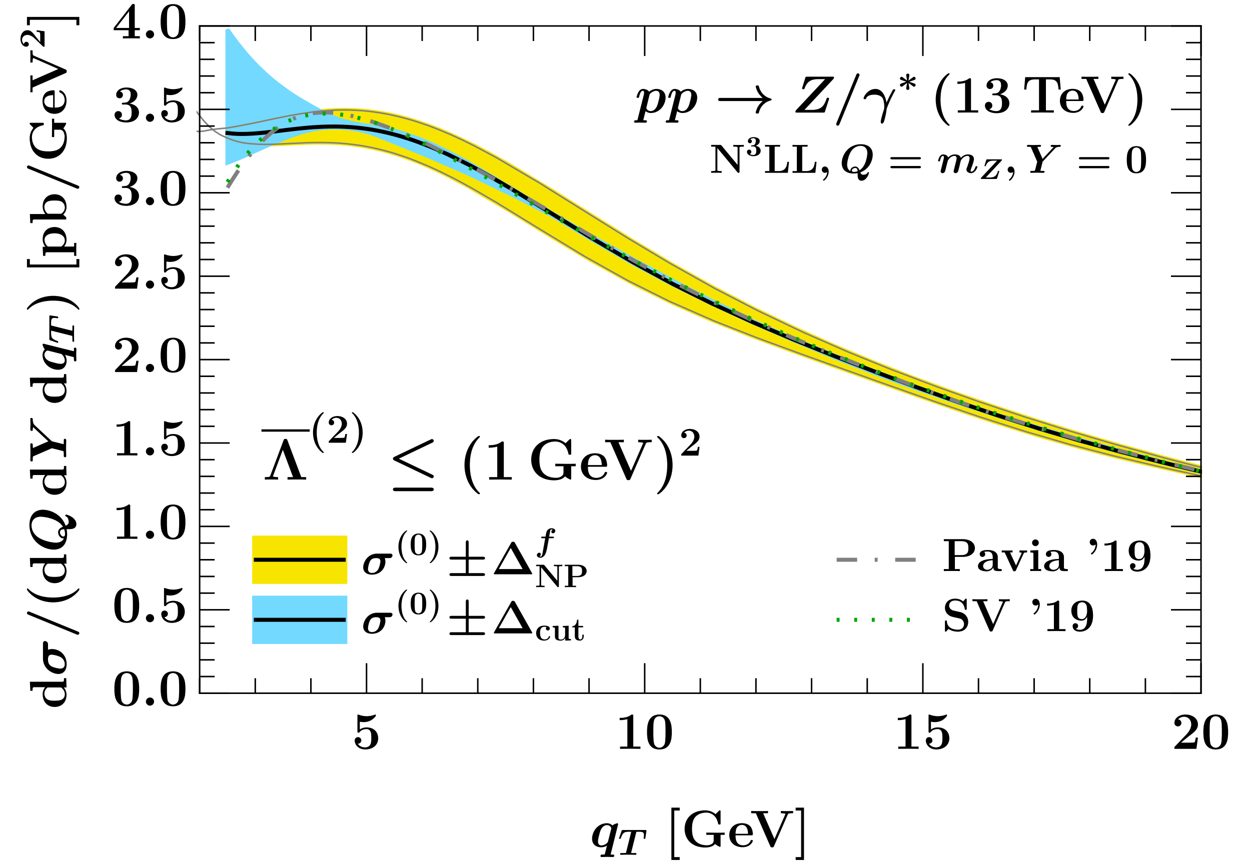


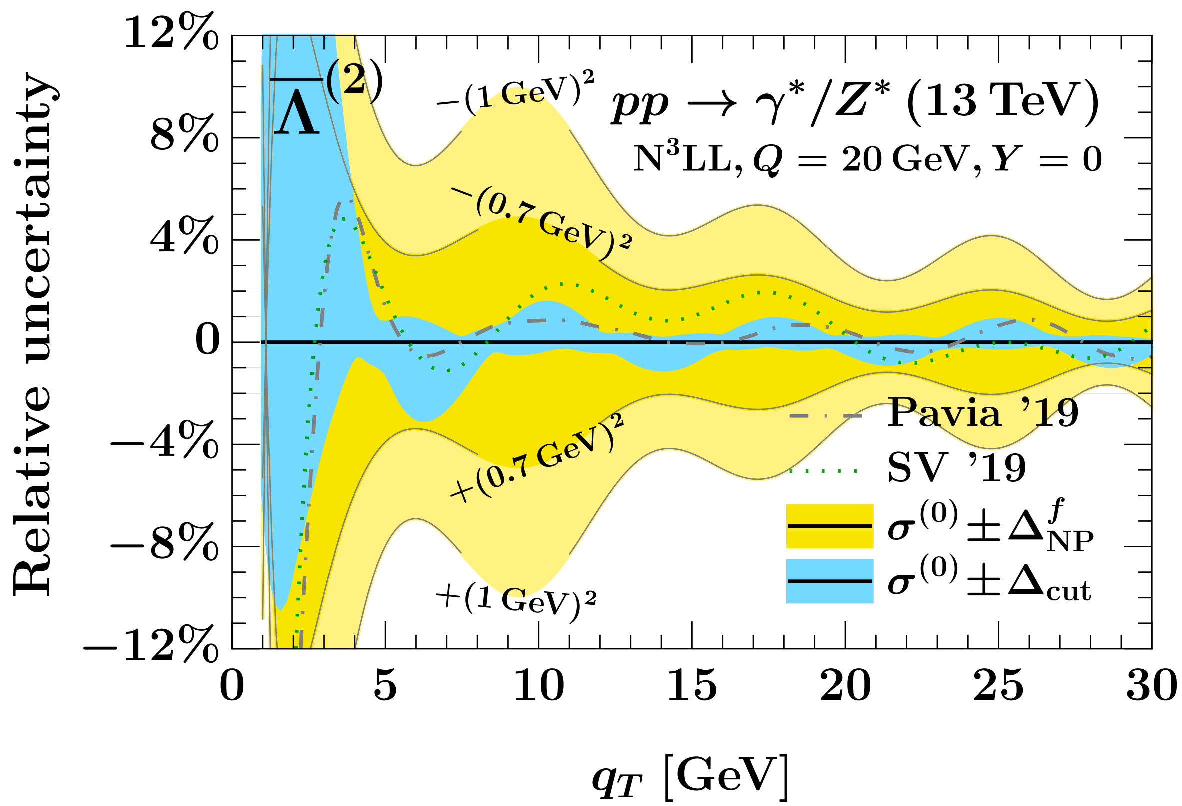



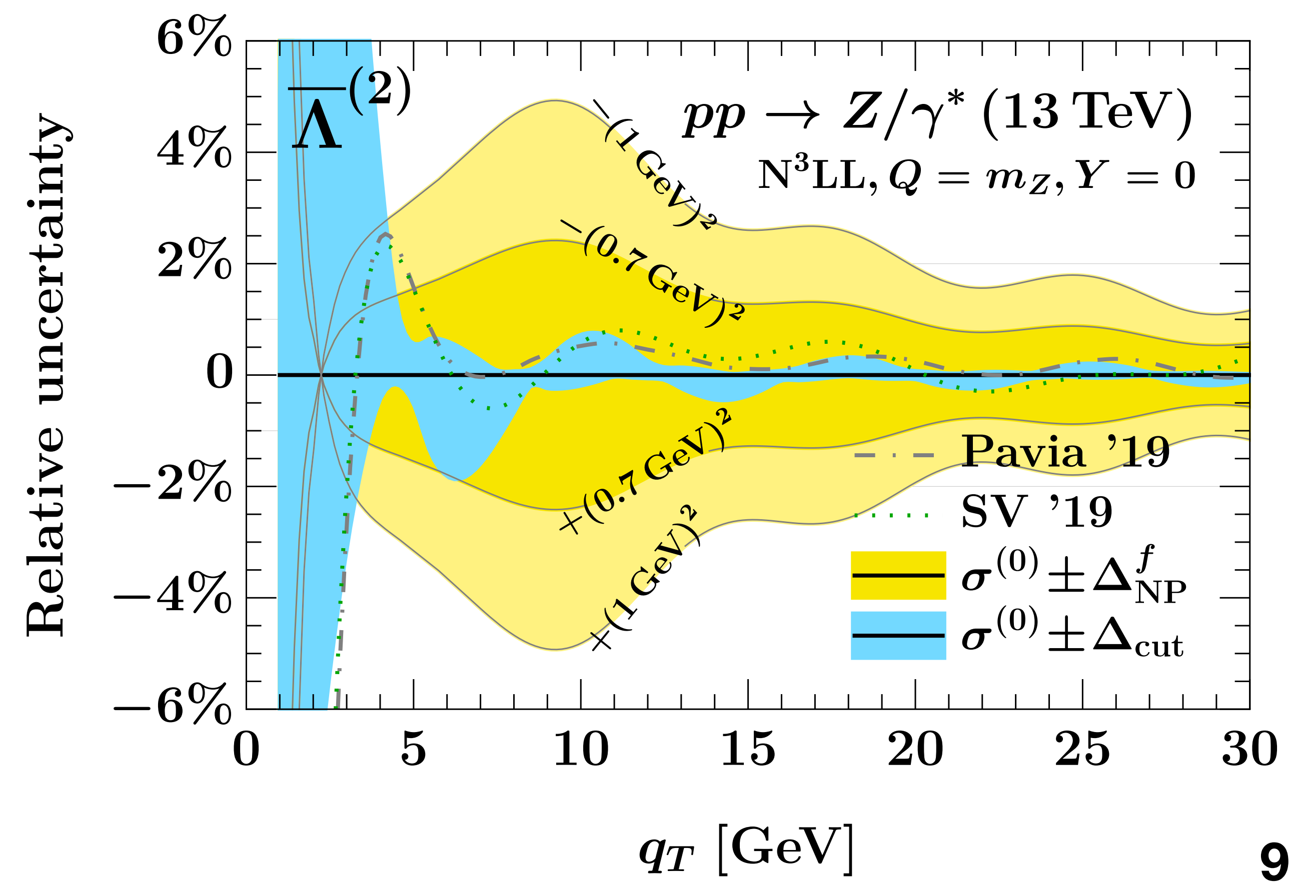

Ebert, Michel, IS, Sun (arXiv:2201.07237)
[without need for b* scheme]

# Nonperturbative Contributions to DY Cross Sections

## Boundary TMD PDF

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[without need for b* scheme]

# Large Momentum EFT:  Quasi-PDFs

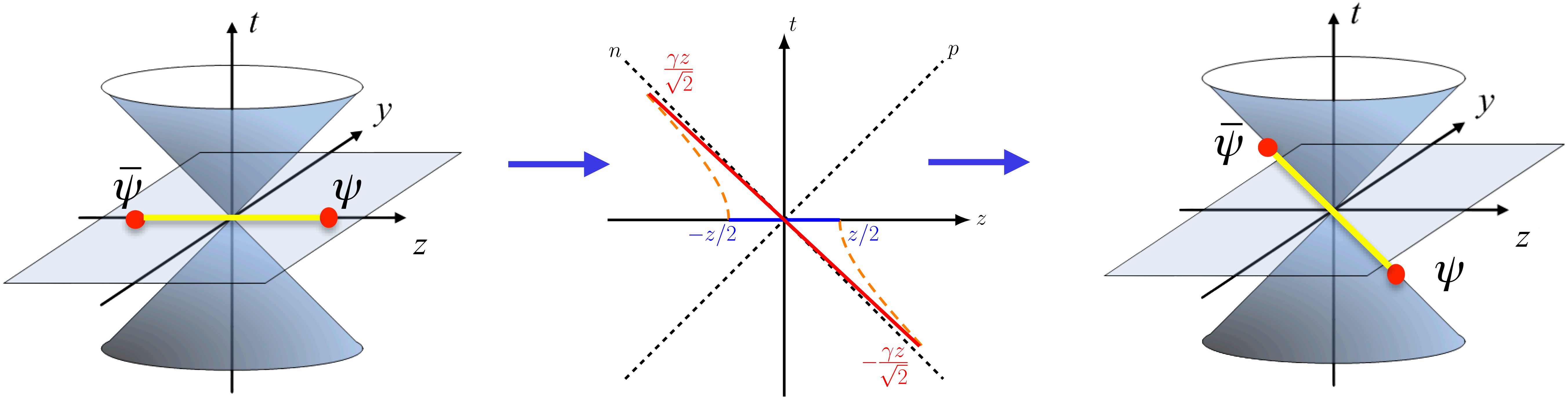
Xiangdong Ji
2013

$$\Lambda_{\rm QCD} \ll P^z \quad (\text{finite large } P^z)$$

$t = 0,\ \ z \neq 0$

Related by Lorentz boost

$z + ct = 0,\ \ z - ct \neq 0$




**quasi-PDF computable with Lattice QCD**

**Perturbative matching coefficient**

**PDF**

$$\tilde f_i(x, P^z, \tilde\mu) = \int_{-1}^{1} \frac{dy}{|y|}\, C_{ij}\left(\frac{x}{y}, \frac{\tilde\mu}{P^z}, \frac{\mu}{yP^z}\right) f_j(y,\mu) + \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}\right)$$

**Power corrections**

**quasi-PDF and PDF:  same IR physics**

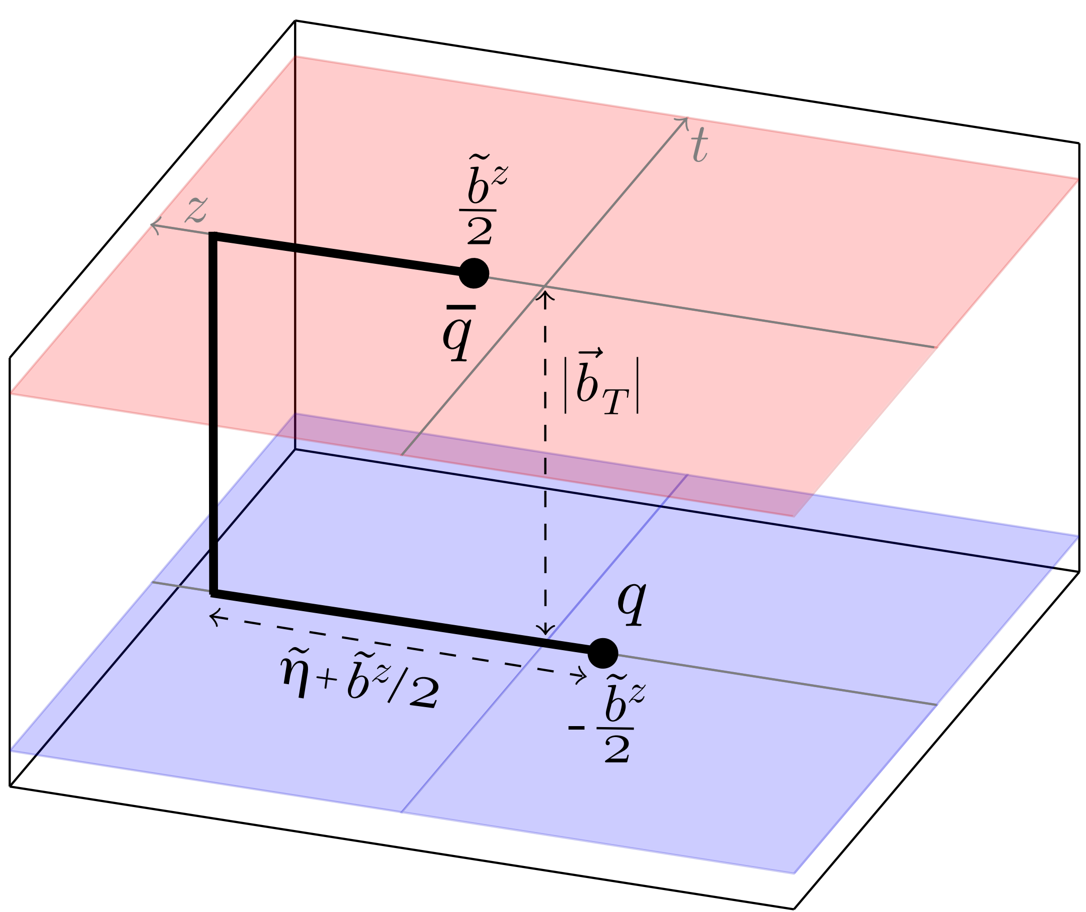
# Requirements for a useful Lattice-TMD

Lattice:

1. Equal time correlators $\tilde{B}$
2. Finite length staples $\tilde{\eta}$ = finite

Physics:

1. Same IR physics as physical TMDs
2. Cancel linear divergences (need soft factor $\tilde{S}$ )
3. Relation to physical TMDs: $\tilde{f} \to f$




$$\tilde{f} \sim Z_{\mathrm{UV}} \tilde{B} / \sqrt{\tilde{S}}$$

<u>Two approaches:</u>

- Lorentz Invariant method (MHENS TMDs)
- quasi-TMDs

# Literature:

- **MHENS:Musch, Hägler, Engelhardt, Negele, Schäfer ('10, '11, '15)**

  **Pioneered Lattice studies of TMDs, exploit Lorentz Invariance ratios to cancel soft, focus on moments ( $b^z \to 0$ )**

- **Ji, Sun, Xiong, Yuan ('14); Ji, Link, Yuan, Zhang, Zhao ('18)**

  **Quasi TMDs, propose factorization ($\eta = \infty$), calculate C**

- **Ebert, Stewart, Zhao ('18, '19, '19)**

  **Propose factorization (finite $\eta$ ) and CS kernel method,**
  **IR tests, calculate C**

- **Ji, Liu, Liu ('19, '19)**

  **Proposal for diagrammatic proof of factorization**
  **Proposed proper quasi-soft factor & indirect lattice method**

- **Vladimirov, Schäfer ('20)**

  **Factorization analysis**

# Unifying Correlators

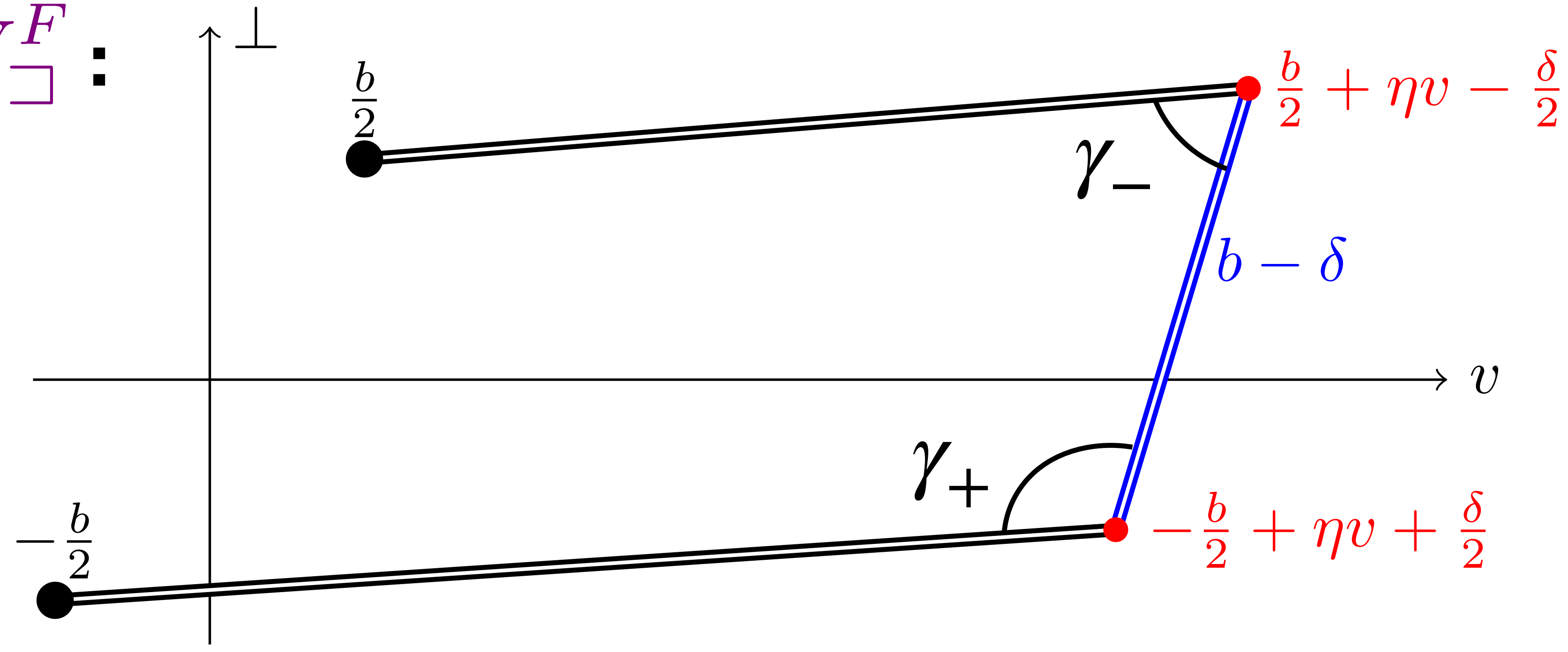
Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

Make clear connection between lattice (MHENS, Quasi) and physical (Collins, JMY, …) schemes

**Introduce a universal Beam Function:**

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) = \Big\langle h(P) \Big| \bar{q}_i\Big(\frac{b}{2}\Big) \frac{\Gamma}{2} W^F_{\sqsupset}(b, \eta v, \delta) q_i\Big(-\frac{b}{2}\Big) \Big| h(P) \Big\rangle$$

$W^F_{\sqsupset}$:




**MHENS:** $\delta = 0$

**Quasi:** $b - \delta = b_T$

**Collins:** **also works**

**Path Length:**

$$L_{\sqsupset} = |\eta v - \delta/2| + |\eta v + \delta/2| + |b - \delta|$$

**Cusp angles:**

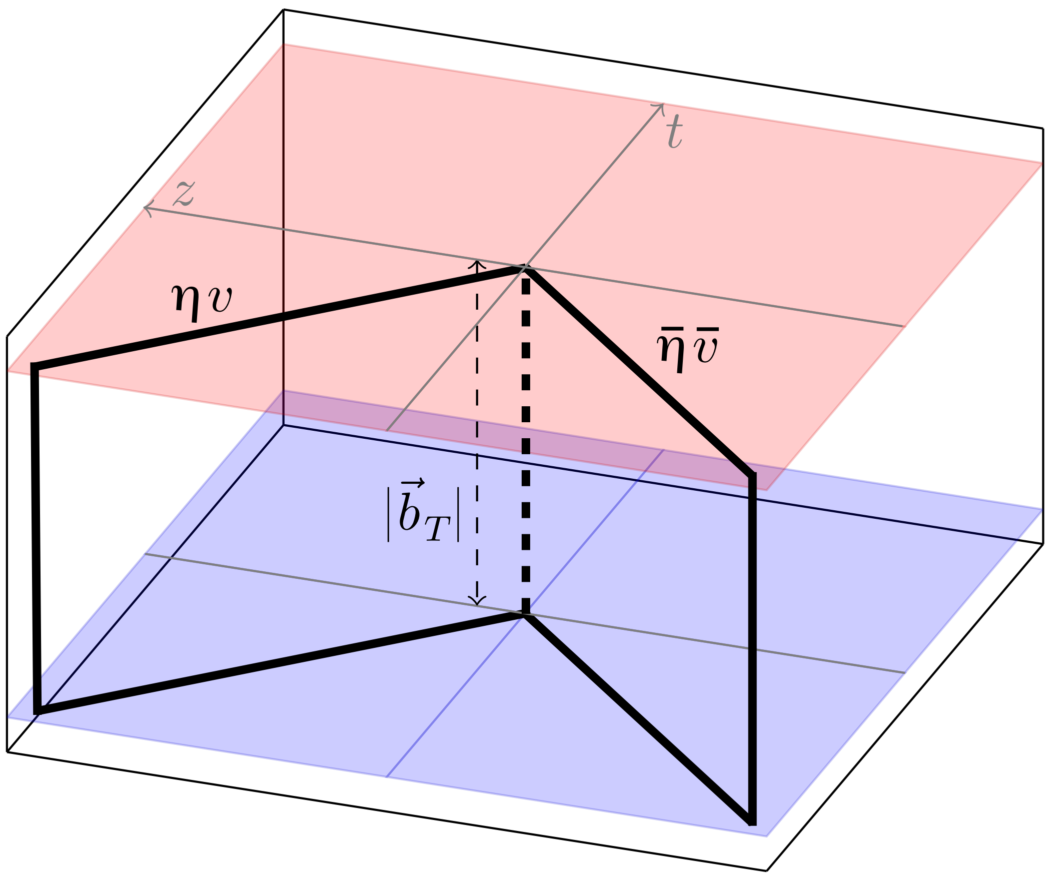
$$\cosh \gamma_{\pm} = \frac{(\eta v \pm \delta/2)\cdot(b-\delta)}{|\eta v \pm \delta/2||b-\delta|}$$

**These matter for Renormalization**

# General Soft Function (finite $\eta$)

$$S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v}) = \frac{1}{d_R} \Big\langle 0 \Big| \mathrm{Tr}\Big[ S^R_{\gg}(b, \eta v, \bar{\eta}\bar{v}) \Big] \Big| 0 \Big\rangle$$

$S^R_{\gg}$:




Path Length:

$$L_{\gg} = 2|\bar{\eta}\bar{v}| + 2|\eta v| + 2|b|$$

- Needed to match IR structure (Ebert, IS, Zhao '19)
- Needed to cancel $\eta$ linear div. in $\Omega$

- **No direct method to calculate on the Lattice.**
- **Indirect method exists to calculate proper quasi-Soft Fn (Ji, Liu, Liu '19)**

# Can parameterize $\Omega$ with 10 Lorentz Invariants:

$$P^2\,, \quad b^2\,, \quad \eta^2 v^2\,, \quad P\cdot b\,, \quad \frac{P\cdot(\eta v)}{\sqrt{P^2|(\eta v)^2|}}\,, \quad \frac{b\cdot(\eta v)}{\sqrt{|b^2(\eta v)^2|}}\,,$$

$$\frac{\delta^2}{b^2}\,, \quad \frac{b\cdot\delta}{b^2}\,, \quad \frac{P\cdot\delta}{P\cdot b}\,, \quad \frac{\delta\cdot(\eta v)}{b\cdot(\eta v)}\,.$$

**6 like Musch et.al.**

**+4 that fix scheme category**

# Choices for various TMDs:

**Continuum TMDs** **Lattice TMDs**

| | Collins / LR | JMY | Quasi | MHENS |
|---|---|---|---|---|
| $b^\mu$ | $(0, b^-, b_\perp)$ | $(0, b^-, b_\perp)$ | $(0, b_T^x, b_T^y, \tilde{b}^z)$ | $(0, b_T^x, b_T^y, \tilde{b}^z)$ |
| $v^\mu$ | $(-e^{2y_B}, 1, 0_\perp)$ | $(v^- e^{2y'_B}, v^-, 0_\perp)$ | $(0, 0, 0, -1)$ | $(0, v^x, v^y, v^z)$ |
| $\delta^\mu$ | $(0, b^-, 0_\perp)$ | $(0, b^-, 0_\perp)$ | $(0, 0, 0, \tilde{b}^z)$ | $(0, 0, 0_\perp)$ |
| $P^\mu$ | $\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{-y_P}, 0_\perp)$ | $\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{-y_P}, 0_\perp)$ | $m_h(\cosh y_{\tilde{P}}, 0, 0, \sinh y_{\tilde{P}})$ | $m_h\left(\cosh y_P, \frac{P^x}{m_h}, \frac{P^y}{m_h}, \sinh y_P\right)$ |

# TMDs differ by how 3 limits (UV, large rapidity, large $\eta$) are taken:

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

| | TMD | Beam function | Soft function |
|---|---|---|---|
| Collins | $\lim_{\epsilon\to 0} Z^R_{\mathrm{UV}} \lim_{y_B\to-\infty} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$ | $S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$ |
| LR | $\lim_{-y_B\gg 1} \lim_{\epsilon\to 0} Z^R_{\mathrm{UV}} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$ | $S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$ |
| JMY | $\lim_{\frac{v^-}{v^+}\gg 1} \lim_{\epsilon\to 0} Z^R_{\mathrm{UV}} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \mu, -\infty v, b^- n_b\right]$ | $S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$ |
| Quasi | $\lim_{a\to 0} Z_{\mathrm{UV}} \frac{B_{i/h}}{\sqrt{\tilde{S}^R}}$ | $\Omega^{[\gamma^{0,z}]}_{q/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$ | $S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$ |
| MHENS | | $\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$ | |

# TMDs differ by how 3 limits (UV, large rapidity, large $\eta$) are taken:

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

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| LR | $\lim\limits_{-y_B\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\rm UV} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$ | $S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$ |
| JMY | $\lim\limits_{\frac{v^-}{v^+}\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\rm UV} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \mu, -\infty v, b^- n_b\right]$ | $S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$ |
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| MHENS | | $\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$ | |

**Collins:**

$$f^C_{i/h}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z^R_{\rm uv}(\epsilon, \mu, \zeta) \lim_{y_B\to-\infty} \frac{B^C_{i/h}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S^R_C(b_T, \epsilon, 2y_n, 2y_B)}}$$

$$B^C_{q_i/h}(x, \vec{b}_T, \epsilon, y_P - y_B) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)}\, \Omega^{[\gamma^+]}_{q_i/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$$

$$n^\mu_B(y_B) = (-e^{2y_B}, 1, 0_\perp)$$

# TMDs differ by how 3 limits (UV, large rapidity, large $\eta$) are taken:

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| Quasi | $\lim\limits_{a\to 0} Z_{\rm UV} \frac{B_{i/h}}{\sqrt{\tilde S^R}}$ | $\Omega_{q/h}^{[\gamma^{0,z}]}(\tilde b, \tilde P, a, \tilde\eta\hat z, \tilde b^z \hat z)$ | $S^R\left[b_\perp, a, -\tilde\eta \frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde\eta \frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$ |
| MHENS | | $\Omega_{q/h}^{[\Gamma]}(b, P, a, \eta v, 0)$ | |

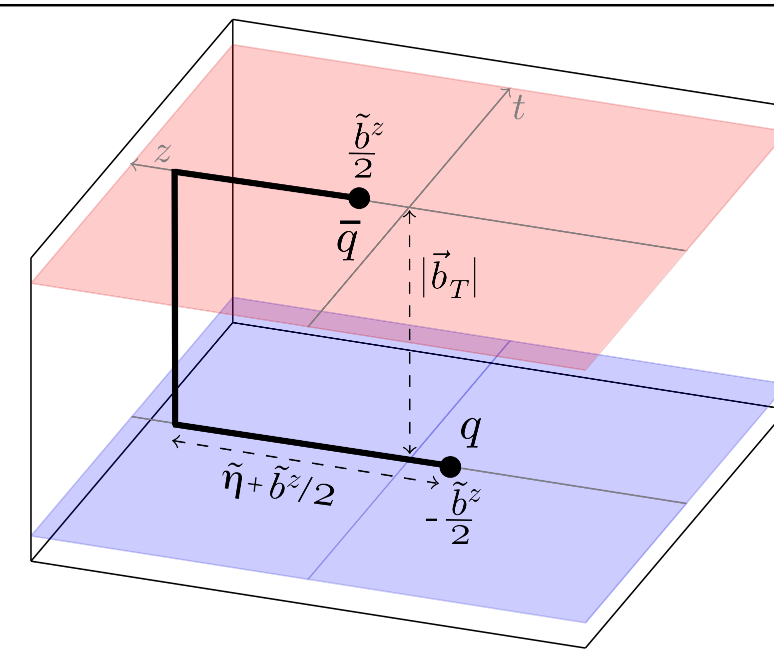
## Quasi Beam:

$$\tilde B_{i/h}^{[\tilde\Gamma]}(x, \vec b_T, a, \tilde\eta, x\tilde P^z)$$

$$= \int \frac{d\tilde b^z}{2\pi}\, e^{i\tilde b^z (x\tilde P^z)}\, \Omega_{q_i/h}^{[\tilde\Gamma]}(\tilde b, \tilde P, a, \tilde\eta\hat z, \tilde b^z \hat z)$$

a = lattice spacing (UV regulator)

**equal time:**



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| Collins | $\lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \lim\limits_{y_B\to-\infty} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$ | $S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$ |
| LR | $\lim\limits_{-y_B\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$ | $S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$ |
| JMY | $\lim\limits_{\frac{v^-}{v^+}\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega^{[\gamma^+]}_{q/h}\left[b, P, \mu, -\infty v, b^- n_b\right]$ | $S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$ |
| Quasi | $\lim\limits_{a\to 0} Z_{\mathrm{UV}} \frac{B_{i/h}}{\sqrt{\tilde{S}^R}}$ | $\Omega^{[\gamma^{0,z}]}_{q/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$ | $S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$ |
| MHENS | | $\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$ | |

# Quasi TMD:

$$\tilde{f}^{[\tilde{\Gamma}]}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z) = \lim_{\substack{\tilde{\eta}\to\infty \\ a\to 0}} Z'_{\mathrm{uv}}(\mu, \tilde{\mu}) Z_{\mathrm{uv}}(a, \tilde{\mu}, y_n - y_B) \frac{\tilde{B}^{[\tilde{\Gamma}]}_{i/h}(x, \vec{b}_T, a, \tilde{\eta}, x\tilde{P}^z)}{\sqrt{S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_B(y_B)}{\lvert n_B(y_B)\rvert}\right]}}$$

Finite $\eta$ Collins soft function

(In ratio: limit $\tilde{\eta} \to \infty$ exists)

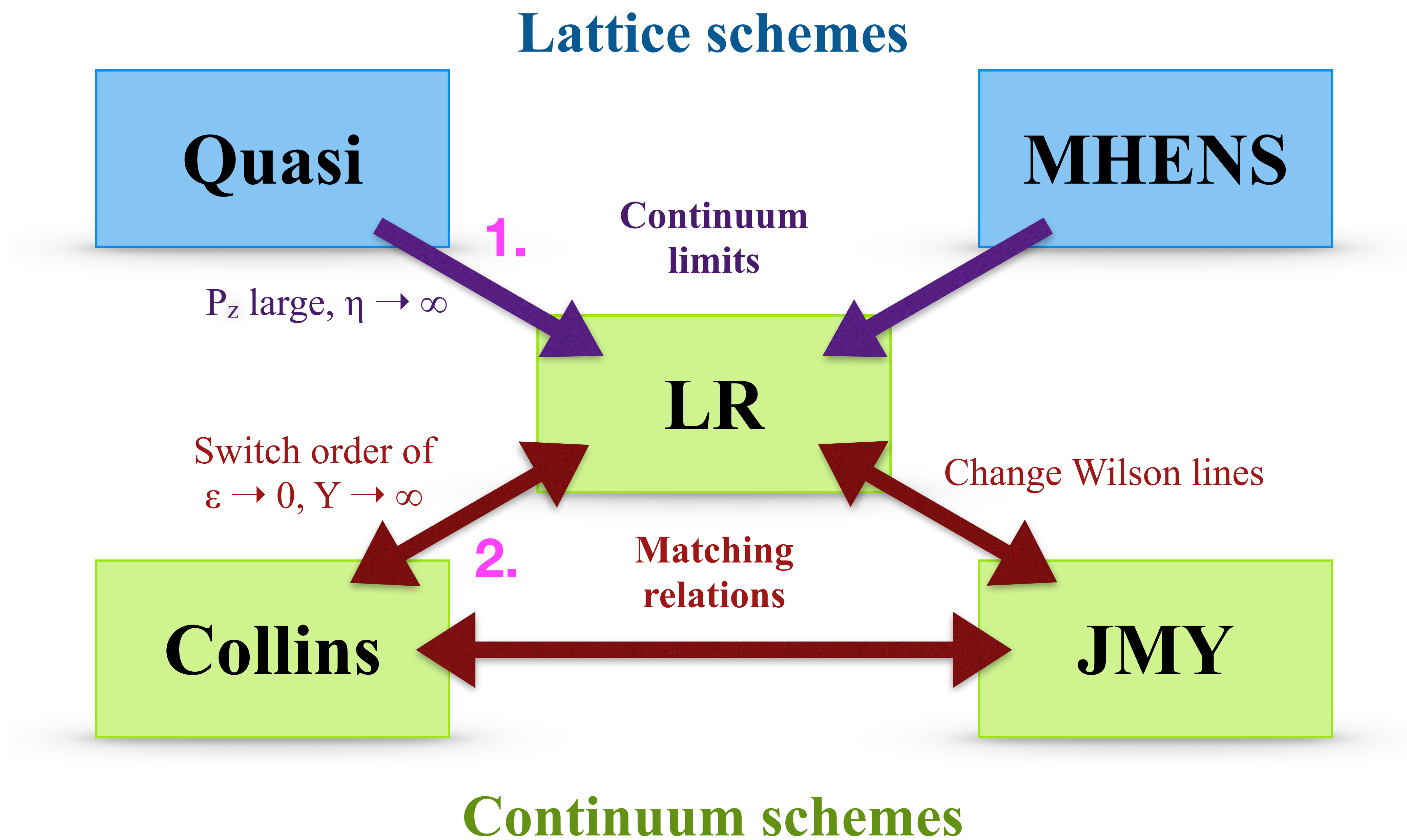
# TMDs differ by how 3 limits (UV, large rapidity, large $\eta$) are taken:

$$\Omega_{q/h}^{[\Gamma]}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

| | TMD | Beam function | Soft function |
|---|---|---|---|
| Collins | $\lim_{\epsilon \to 0} Z_{\text{UV}}^R \lim_{y_B \to -\infty} \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega_{q/h}^{[\gamma^+]}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$ | $S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$ |
| LR | $\lim_{-y_B \gg 1} \lim_{\epsilon \to 0} Z_{\text{UV}}^R \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega_{q/h}^{[\gamma^+]}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$ | $S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$ |
| JMY | $\lim_{\frac{v^-}{v^+} \gg 1} \lim_{\epsilon \to 0} Z_{\text{UV}}^R \frac{\Omega_{i/h}}{\sqrt{S^R}}$ | $\Omega_{q/h}^{[\gamma^+]}\left[b, P, \mu, -\infty v, b^- n_b\right]$ | $S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$ |
| Quasi | $\lim_{a \to 0} Z_{\text{UV}} \frac{B_{i/h}}{\sqrt{\tilde{S}^R}}$ | $\Omega_{q/h}^{[\gamma^{0,z}]}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z \hat{z})$ | $S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$ |
| MHENS | | $\Omega_{q/h}^{[\Gamma]}(b, P, a, \eta v, 0)$ | |

**LR scheme:** **new,**
**differs from Collins only by order of (UV & rapidity) limits,**
**useful for our proof**

## Proof




**Steps:**

1. **Quasi** → **LR**: related by large rapidity ($P^z \gg \Lambda_{\mathrm{QCD}}$)
   IF we properly map variables,
   take $|\eta| \to \infty$
2. **LR → Collins:** UV ren. & non-trivial Matching coefficient

# Step 1

| | Collins / LR | Quasi | MHENS |
|---|---|---|---|
| $b^2$ | $-b_T^2$ | $-b_T^2 - (\tilde{b}^z)^2$ | $-b_T^2 - (\tilde{b}^z)^2$ |
| $(\eta v)^2$ | $-2\eta^2 e^{2y_B}$ | $-\tilde{\eta}^2$ | $-\eta^2 \vec{v}^{\,2}$ |
| $P \cdot b$ | $\frac{m_h}{\sqrt{2}} b^- e^{y_P}$ | $-m_h \tilde{b}^z \sinh y_{\tilde{P}}$ | $m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$ |
| $\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$ | $-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$ | $\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$ | $\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$ |
| $\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$ | $\sinh(y_P - y_B) \operatorname{sgn}(\eta)$ | $\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)$ | $\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$ |
| $\frac{\delta^2}{b^2}$ | $0$ | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{b \cdot \delta}{b^2}$ | $0$ | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{P \cdot \delta}{P \cdot b}$ | $1$ | $1$ | $0$ |
| $\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$ | $1$ | $1$ | $0$ |
| $P^2$ | $m_h^2$ | $m_h^2$ | $m_h^2$ |

# Step 1

| | Collins / LR | Quasi | MHENS |
|---|---|---|---|
| $b^2$ | $-b_T^2$ | $-b_T^2 - (\tilde{b}^z)^2$ | $-b_T^2 - (\tilde{b}^z)^2$ |
| $(\eta v)^2$ | $-2\eta^2 e^{2y_B}$ | $-\tilde{\eta}^2$ | $-\eta^2 \vec{v}^{\,2}$ |
| $P \cdot b$ | $\frac{m_h}{\sqrt{2}} b^- e^{y_P}$ | $-m_h \tilde{b}^z \sinh y_{\tilde{P}}$ | $m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$ |
| $\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$ | $-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$ | $\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$ | $\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$ |
| $\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$ | $\boxed{\sinh(y_P - y_B) \operatorname{sgn}(\eta)}$ | $\boxed{\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)}$ | $\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$ |
| | | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| | | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{P \cdot \delta}{P \cdot b}$ | $1$ | $1$ | $0$ |
| $\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$ | $1$ | $1$ | $0$ |
| $P^2$ | $m_h^2$ | $m_h^2$ | $m_h^2$ |

$$\sinh(y_P - y_B) = \sinh y_{\tilde{P}}$$

$$\Rightarrow y_{\tilde{P}} = y_P - y_B$$

# Step 1

| | Collins / LR | Quasi | MHENS |
|---|---|---|---|
| $b^2$ | $-b_T^2$ | $-b_T^2 - (\tilde{b}^z)^2$ | $-b_T^2 - (\tilde{b}^z)^2$ |
| $(\eta v)^2$ | $-2\eta^2 e^{2y_B}$ | $-\tilde{\eta}^2$ | $-\eta^2 \vec{v}^{\,2}$ |
| $P \cdot b$ | $\frac{m_h}{\sqrt{2}} b^- e^{y_P}$ | $-m_h \tilde{b}^z \sinh y_{\tilde{P}}$ | $m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$ |
| | | | $\frac{\cdots b_T^y v^y + \tilde{b}^z v^z}{\cdots \sqrt{b_T^2 + (\tilde{b}^z)^2}}$ |
| | | | $\frac{\cdots + m_h v^z \sinh y_P}{\cdots \sqrt{m_h^2 + P_x^2 + P_y^2}}$ |
| | | | $0$ |
| $\frac{b \cdot \delta}{b^2}$ | $0$ | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{P \cdot \delta}{P \cdot b}$ | $1$ | $1$ | $0$ |
| $\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$ | $1$ | $1$ | $0$ |
| $P^2$ | $m_h^2$ | $m_h^2$ | $m_h^2$ |

$y_P\,,\, b^-$ finite **Boost quasi by $y_B = y_P - y_{\tilde{P}}$**

$\tilde{P}^z = m_h \sinh y_{\tilde{P}} \gg \Lambda_{\rm QCD}$ $(y_{\tilde{P}} \to \infty, y_B \to -\infty)$

$$-m_h \tilde{b}^z \sinh y_{\tilde{P}} = m_h \sqrt{2} e^{y_B} b^- \sinh(y_P - y_B) \xrightarrow{y_B \to -\infty} \frac{m_h}{\sqrt{2}} b^- e^{y_P}$$

# Step 1

| | Collins / LR | Quasi | MHENS |
|---|---|---|---|
| $b^2$ | $-b_T^2$ | $-b_T^2-(\tilde b^z)^2$ | $-b_T^2-(\tilde b^z)^2$ |
| $(\eta v)^2$ | $-2\eta^2 e^{2y_B}$ | $-\tilde\eta^2$ | [illegible] |
| $P\cdot b$ | $\frac{m_h}{\sqrt2}b^- e^{y_P}$ | $-m_h\tilde b^z \sinh y_{\tilde P}$ | $m_h \sinh y_P \tilde b^z + P^x b_T^x + P^y b_T^y$ |
| $\frac{b\cdot(\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$ | $-\frac{b^- e^{y_B}}{\sqrt2\, b_T}\,\mathrm{sgn}(\eta)$ | $\frac{\tilde b^z}{\sqrt{(\tilde b^z)^2+b_T^2}}\,\mathrm{sgn}(\eta)$ | [illegible] |
| $\frac{P\cdot(\eta v)}{\sqrt{P^2\lvert\eta v\rvert^2}}$ | $\sinh(y_P-y_B)\,\mathrm{sgn}(\eta)$ | $\sinh y_{\tilde P}\,\mathrm{sgn}(\eta)$ | [illegible] |
| $\frac{\delta^2}{b^2}$ | $0$ | $\frac{(\tilde b^z)^2}{b_T^2+(\tilde b^z)^2}$ | $0$ |
| $\frac{b\cdot\delta}{b^2}$ | $0$ | $\frac{(\tilde b^z)^2}{b_T^2+(\tilde b^z)^2}$ | $0$ |
| $\frac{P\cdot\delta}{P\cdot b}$ | $1$ | $1$ | $0$ |
| $\frac{\delta\cdot(\eta v)}{b\cdot(\eta v)}$ | $1$ | $1$ | $0$ |
| $P^2$ | $m_h^2$ | $m_h^2$ | $m_h^2$ |

$$\tilde\eta = \sqrt{2}\,e^{y_B}\,\eta$$

$$\frac{\tilde b^z}{b_T} = \frac{\sqrt{2}\,b^- e^{y_B}}{b_T} \xrightarrow{y_B\to-\infty} 0$$

**Also** $\tilde\zeta = \zeta$

# Step 1 Quasi $\to$ LR

$$\tilde{f}_{i/h}^{[\tilde{\Gamma}]}(x,\vec{b}_T,\mu,\tilde{\zeta},x\tilde{P}^z) = \lim_{\substack{\tilde{\eta}\to\infty \\ a\to 0}} Z'_{\rm uv}(\mu,\tilde{\mu}) Z_{\rm uv}(a,\tilde{\mu},y_n-y_B)\, \frac{\tilde{B}_{i/h}^{[\tilde{\Gamma}]}(x,\vec{b}_T,a,\tilde{\eta},x\tilde{P}^z)}{\sqrt{S^R\Big[b_\perp,a,-\tilde{\eta}\frac{n_A(y_A)}{|n_A(y_A)|},-\tilde{\eta}\frac{n_B(y_B)}{|n_B(y_B)|}\Big]}}$$

**Quasi and LR have same UV renormalization**

**Quasi and LR have same $\tilde{\eta}\to\infty$ limit**

**Thus Quasi = LR after expansion**

## Step 2 Quasi=LR → Collins

**LR and Collins differ by order of $y_B \to -\infty$ and $\epsilon \to 0$ limits**

**LaMET: this induces a matching coefficient**

$$\tilde{f}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z)$$

$$= C_q(x\tilde{P}^z, \mu) f_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}) + \mathcal{O}(\tilde{y}_P^k e^{-\tilde{y}_P})$$

$$= C_q(x\tilde{P}^z, \mu) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln\frac{\tilde{\zeta}}{\zeta}\right] f_{q_i/h}(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}(\tilde{y}_P^k e^{-\tilde{y}_P})$$

standard CS evolution

- Note: if we were satisfied relating Quasi to LR then there would be no $C_i$

# Result

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

**Same steps work for any spin structure & for gluon TMDs**

$$\tilde{f}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = C_i\big(x\tilde{P}^z, \mu\big) \exp\left[\frac{1}{2}\gamma_\zeta^i(\mu, b_T) \ln\frac{\tilde{\zeta}}{\zeta}\right] f^C_{i/h}(x, \vec{b}_T, \mu, \zeta)$$
$$\times \left\{1 + \mathcal{O}\left[\frac{b_T}{\tilde{\eta}}, \frac{1}{\tilde{x}P^z\tilde{\eta}}, \frac{1}{(x\tilde{P}^z b_T)^2}, \frac{\Lambda^2_{\rm QCD}}{(x\tilde{P}^z)^2}\right]\right\}$$

- Direct diagrammatic proof exists at one-loop for all channels (see Stella's talk after lunch)
- All orders diagrammatic proof is in progress (Ji et.al.)

Note: The diagrammatic proof provides a cross-check on the LaMET matching between two Continuum Schemes (LR & Collins).

If performed for both finite and infinite $\tilde{\eta}$, then it can also confirm the equality of LR and Quasi.

# Implications

Quasi → Collins

$$\tilde{f}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = C_i\big(x\tilde{P}^z, \mu\big) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln\frac{\tilde{\zeta}}{\zeta}\right] f^C_{i/h}(x, \vec{b}_T, \mu, \zeta)$$

- **Extract CS Kernel from ratios of quasi-TMDs** (Ebert, IS, Zhao '18)
- **No mixing of flavors, quarks and gluons, or spin structures (except perhaps by lattice-fermion discretization)**
- **Ratios can be calculated in x-space**

Ebert, Schindler, IS, Zhao '20
Vladimirov, Schafer '20
Ji, Liu, Schaefer, Yuan '20

$$\lim_{\tilde{\eta}\to\infty} \frac{\tilde{B}^{[\tilde{\Gamma}_1]}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z)}{\tilde{B}^{[\tilde{\Gamma}_2]}_{q_j/h'}(x, \vec{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z)} = \frac{f^{[\Gamma_1]}_{q_i/h}(x, \vec{b}_T, \mu, \zeta)}{f^{[\Gamma_2]}_{q_j/h'}(x, \vec{b}_T, \mu, \zeta)}$$

- **NLL calculation** $\quad C_q\big(x\tilde{P}^z, \mu\big)^{\mathrm{NLL}} = \exp\Big[-2K^q_\Gamma\big(2x\tilde{P}^z, \mu\big) - K^q_\gamma\big(2x\tilde{P}^z, \mu\big)\Big]$

$$K^q_\Gamma(\mu_0, \mu) = -\frac{\Gamma^q_0}{4\beta_0^2}\left\{\frac{4\pi}{\alpha_s(\mu_0)}\Big(1 - \frac{1}{r} - \ln r\Big) + \Big(\frac{\Gamma^q_1}{\Gamma^q_0} - \frac{\beta_1}{\beta_0}\Big)(1 - r + \ln r) + \frac{\beta_1}{2\beta_0}\ln^2 r\right\} \quad , \quad r = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}$$

$$K^q_\gamma(\mu_0, \mu) = -\frac{\gamma^q_{C0}}{2\beta_0}\ln r$$

# MHENS $\to$ Collins

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

| | Collins / LR | Quasi | MHENS |
|---|---|---|---|
| $b^2$ | $-b_T^2$ | $-b_T^2 - (\tilde{b}^z)^2$ | $-b_T^2 - (\tilde{b}^z)^2$ |
| $(\eta v)^2$ | $-2\eta^2 e^{2y_B}$ | $-\tilde{\eta}^2$ | $-\eta^2 \vec{v}^{\,2}$ |
| $P \cdot b$ | $\frac{m_h}{\sqrt{2}} b^- e^{y_P}$ | $-m_h \tilde{b}^z \sinh y_{\tilde{P}}$ | $m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$ |
| $\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$ | $-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \,\mathrm{sgn}(\eta)$ | $\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \,\mathrm{sgn}(\eta)$ | $\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$ |
| $\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$ | $\sinh(y_P - y_B)\,\mathrm{sgn}(\eta)$ | $\sinh y_{\tilde{P}}\,\mathrm{sgn}(\eta)$ | $\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$ |
| $\frac{\delta^2}{b^2}$ | $0$ | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{b \cdot \delta}{b^2}$ | $0$ | $\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$ | $0$ |
| $\frac{P \cdot \delta}{P \cdot b}$ | $1$ | $1$ | $0$ |
| $\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$ | $1$ | $1$ | $0$ |
| $P^2$ | $m_h^2$ | $m_h^2$ | $m_h^2$ |

?

# MHENS → Collins

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

- $P \cdot b = 0$ **case, our proof applies**

  **MHENS equivalent to Quasi (same soft fn, renormalization, …)**

  **This case was focus of Musch, Hägler, Engelhardt, Negele, Schäfer**

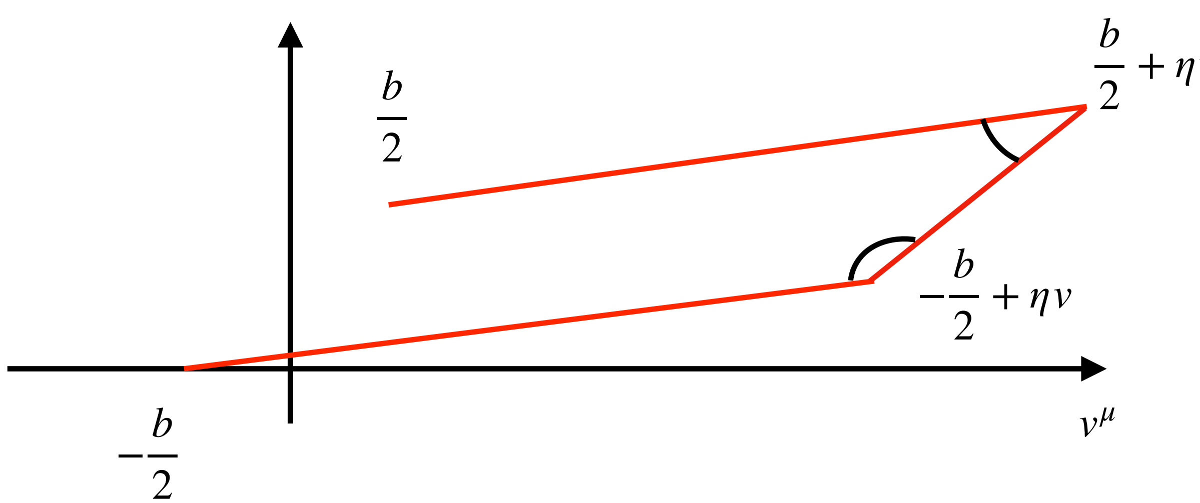
- $P \cdot b \neq 0$ **case ( $x$ dependence )**

Additional challenges

- $b^z$ - dependent renormalization

  Linear: $\propto \left(2\,|\eta v| + \sqrt{\tilde{b}_z^2 + b_T^2}\right)/a$

  Cusp: $\propto \left[3 - \frac{2\tilde{b}^z}{b_T}\tan^{-1}\frac{b_T}{\tilde{b}^z}\right]\ \ln(a)$




- $b^z$ - dependent soft function?

With proper lattice renormalization, Lorentz Inv. compensation, and construction of a suitable soft function, could connect MHENS to LR scheme (thus to Collins).

# Conclusion

**Quasi-TMD → Collins-TMD**

**Our proof enables rigorous lattice studies**

$$\tilde{f}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = C_i\big(x\tilde{P}^z, \mu\big) \, \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln \frac{\tilde{\zeta}}{\zeta}\right] f_{i/h}^C(x, \vec{b}_T, \mu, \zeta)$$

Lattice Targets:

(see Artur's talk)

- Non-perturbative CS Kernel
- Info on Spin-dependent TMDPDFs (in ratios)
- Info about 3D structure, $x$ and $b_T$ (in ratios)
- proton vs. pion TMD PDFs (in ratios)
- flavor dependence of TMD PDFs (in ratios)
- soft function for TMDs
- TMD PDF with $x$ and $b_T$ (normalization)
- Gluon TMD PDFs