The universal hill shape and ridge structure of entropy per baryon number near the QCD critical point

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QCD phase diagram

\[ T \sim 10^{12} \text{ K} \]

\[ \sim 10^{12} \text{ kg/cm}^3 \]

Quark-Gluon plasma (QGP)

Hadron

Color superconductivity

\[ \mu_B \]
QCD phase diagram

\[ T \approx 10^{12} \text{ K} \]

Quark-Gluon plasma (QGP)
- Experimental consequences?
- How to reach?
- Upward or downward? etc.

\[ \mu_B \approx 10^{12} \text{ kg/cm}^3 \]

Hadron

Color superconductivity
Setup

• Ideal hydrodynamics: 0th-order approximation in heavy-ion collisions

• Entropy and baryon number conservation in boost invariance:

\[ s\tau = s_{\text{ini}}\tau_{\text{ini}}, \quad n\tau = n_{\text{ini}}\tau_{\text{ini}} \rightarrow \hat{s} \equiv \frac{s}{n} = \frac{s_{\text{ini}}}{n_{\text{ini}}} \]

→ Time evolution = \( \hat{s} \) contours on the \((\mu, T)\) plane

• The universality of critical phenomena

→ EOS of QCD \( \approx \) 3D Ising mode

→ Universal properties on the \( \hat{s} \) contour
Ridge and the hill-shape

M. Pradeep, M. Stephanov, NS, and H. Yee (in preparation)

\[ \hat{s}(\mu, T) = \text{const.} \]

\[ \hat{s}(\mu, T_{1st}(\mu)) \]

Inevitable non-monotonic hill shape:
- Critical degeneracy: \[ \hat{s} \approx \pm (T - T_c)^\beta \]
- Third law of thermodynamics: \[ \hat{s}(T = 0) = 0 \]
Outline

1. Mapping 3d-Ising to QCD
2. $\hat{s}$ along the first-order line
3. $\hat{s}$ contours
4. Summary
Mapping \((h, r) \leftrightarrow (\mu, T)\)

- Pressure \(P(h, r)\) with relevant parameters \((h, r)\) given by RG

- Near the QCD critical point:

  \[
  \frac{\mu - \mu_c}{T_c} = - w(r \rho \cos \alpha_1 + h \cos \alpha_2)
  \]

  \[
  \frac{T - T_c}{T_c} = w(r \rho \sin \alpha_1 + h \sin \alpha_2)
  \]

  \(\rightarrow h(\mu, T), r(\mu, T)\)

- \(\alpha_2\) : Shrink/squeeze ratio

Entropy/baryon near the critical point

- Thermodynamic quantities:
  \[ s \equiv \partial_T p + s_0 = m \partial_T h + \sigma \partial_T r + s_0 \] (Chain rule)
  \[ n \equiv \partial_\mu p + n_0 = m \partial_\mu h + \sigma \partial_T r + n_0 \]

- Relevant parameter derivatives:
  - Order parameter: \( m = \partial_h p \)
  - Ising entropy: \( \sigma = \partial_r p \)


- Entropy/baryon number:
  \[ \hat{s} \simeq \hat{s}_0 + (\partial_m \hat{s})_0 m + (\partial_\sigma \hat{s})_0 \sigma \]

Map → (h, r) → m, σ → \( \hat{s} \)

Universal EOS
Along the first-order line

M. Pradeep, M. Stephanov, NS, and H. Yee (in preparation)

- Set $h = \pm 0$, $r \leq 0$

\[
\hat{s}_1^{1st} = \hat{s}_0 + \lambda (\partial_m \hat{s})_0 (-r)^\beta + (\partial_\sigma \hat{s})_0 C(-r)^{1-\alpha} \quad (\lambda = \pm 1)
\]

Competition between $m \propto (-r)^\beta$ and $\sigma \propto (-r)^{1-\alpha}$

(\(\beta = 0.326\), \(\alpha = 0.11\))

→ non-monotonic behavior

- The critical double: \((-r_{db})^{1-\alpha-\beta} = \left| \frac{(\partial_m \hat{s})_0}{C(\partial_\sigma \hat{s})_0} \right| \propto \left| \frac{\cos \alpha_1 - \hat{s}_0 \sin \alpha_1}{\cos \alpha_2 - \hat{s}_0 \sin \alpha_2} \right|

- Non-monotonic hill side: \(\lambda = \lambda_{hill}^{(r,h)} = \text{sgn} (\partial_m \hat{s})_0 \begin{pmatrix} (\partial_\sigma \hat{s})_0 > 0 \\ \text{for down concavity} \end{pmatrix} \)
(\(\mu, T\)) plane

M. Pradeep, M. Stephanov, NS, and H. Yee (in preparation)

- Solve \(h(\mu, T) = 0, \ r(\mu, T) = r_{db} \rightarrow (\mu, T) = (\mu_{db}, T_{db})\)

- Hillside index: \(\lambda_{\text{hill}}^{(\mu, T)} \equiv \begin{cases} 
1 & \text{above the boundary} \\
-1 & \text{below} 
\end{cases}\)

\[
\lambda_{\text{hill}}^{(r, h)} \text{sgn} (\alpha_2 - \alpha_1) = \text{sgn} \left( \cot \alpha_1 - \hat{s}_0 \right)
\]

Large region for plausible parameters: \(\hat{s}_0 \gg 1\)
Contours: parameter sets

- Default: \((\mu_c, T_c, s_0, n_0, w, \rho) = (350 \text{ MeV}, 143.2 \text{ MeV}, 21, 1, 1, 2)\)  
  Parotto et al. (2020)

- Vary slope angles \(\alpha_1\) (the first-order line) and \(\alpha_2\) (shrink/stretch rate):
  
  - Crossover from lattice QCD \(0 < \alpha_1 \ll 90^\circ\)
  
  - \(|\alpha_1 - \alpha_2|\) can be suppressed by \(m_q\)  
    M. Pradeep and M. Stephanov (2019)

- \(\alpha_2\) can be negative

<table>
<thead>
<tr>
<th>(\alpha_1)</th>
<th>(3.85^\circ (\cot \alpha_1 &lt; \hat{s}_0))</th>
<th>(1^\circ (\cot \alpha_1 &gt; \hat{s}_0))</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_2)</td>
<td>90°</td>
<td>13°</td>
</tr>
<tr>
<td></td>
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<td>13°</td>
</tr>
</tbody>
</table>
Contours \((r, h)\) plane

\(\alpha_1 = 3.85^\circ\) \((\cot \alpha_1 < \hat{s}_0)\)

- \(\hat{s}/\hat{s}_0\)

\(\alpha_2 = 90^\circ\)

\(\hat{s}/\hat{s}_0\)

\(\alpha_2 = 13^\circ\)

\(\hat{s}/\hat{s}_0\)

\(\alpha_2 = -5^\circ\)

\(\hat{s}/\hat{s}_0\)

- Mixing of \(m\) and \(\sigma\) ➞ The ridge(valley) lines of \(\sigma\) moved and curved
Contours \((\mu, T)\) plane

\[\alpha_1 = 3.85^\circ \quad (\cot \alpha_1 < \hat{s}_0)\]

- \(\alpha_2 = 90^\circ\)
- \(\hat{s}/\hat{s}_0\)
- \(\alpha_2 = 13^\circ\)
- \(\hat{s}/\hat{s}_0\)
- \(\alpha_2 = -5^\circ\)
- \(\hat{s}/\hat{s}_0\)

- Ridgeline and valley line are robust under the shrink and squeeze of \(\hat{s}(r, h)\)
- \(\lambda_{\text{hill}}^{(\mu, T)} = -1\)
Contours \((\mu, T)\) plane

\[ \alpha_1 = 1^\circ \quad (\cot \alpha_1 > \hat{s}_0) \]

\[ \alpha_2 = 90^\circ \]

\[ \alpha_2 = 13^\circ \]

\[ \alpha_2 = -13^\circ \]

- Narrow region for \(\lambda_{\text{hill}}^{(\mu,T)} = 1: 0 < \alpha_1 < \cot^{-1} \hat{s}_0 \approx \hat{s}_0^{-1}, (\hat{s}_0 \gg 1)\)
Slope formula

• The slope at an arbitral point on the contours:

\[
\left\{ \begin{aligned}
\left( \frac{\partial \mu T}{\partial \mu} \right)_{\hat{s}} &= -\frac{\partial \mu}{\partial T} \\
\partial X_{\hat{s}} &= (\partial m \hat{s})_0 \partial Xm + (\partial \sigma \hat{s})_0 \partial X\sigma
\end{aligned} \right.
\]

\[X = (\mu, T)\]

Susceptibilities

• Relative slope: \( \tan \Delta \theta_c \equiv \frac{\left( \frac{\partial \mu T}{\partial \mu} \right)_{\hat{s}} + \tan \alpha_1}{1 - \tan \alpha_1 \left( \frac{\partial \mu T}{\partial \mu} \right)_{\hat{s}}} \)

• Two branches along the first-order boundary

\((\alpha_1, \alpha_2) = (3.85^\circ, 90^\circ)\)
Classification of contours

M. Pradeep, M. Stephanov, NS, and H. Yee (in preparation)

\[ \lambda_{\text{hill}}^{(\mu,T)} = -1 \]

\[ \lambda_{\text{hill}}^{(\mu,T)} = 1 \]

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<tr>
<td>(I)</td>
<td>Crossover</td>
<td>QGP → QGP</td>
</tr>
<tr>
<td>(II)</td>
<td>Hadron → Hadron</td>
<td></td>
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<td>(III)</td>
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</table>

\( \hat{s} \)

\( \hat{s}_{\text{top}} \)

\( \hat{s}_0 \)
Summary

M. Pradeep, M. Stephanov, NS, and H. Yee (in preparation)

• Universal properties of $s/n$ near the QCD critical point based on the 3D Ising EOS
• Relevance to the heavy-ion collision trajectories within the ideal hydrodynamics
• Ridge structure and the hill shape along the first-order boundary
• Outlook:
  - Kinematics on the coexistence line
  - Connection to the freeze-out curve