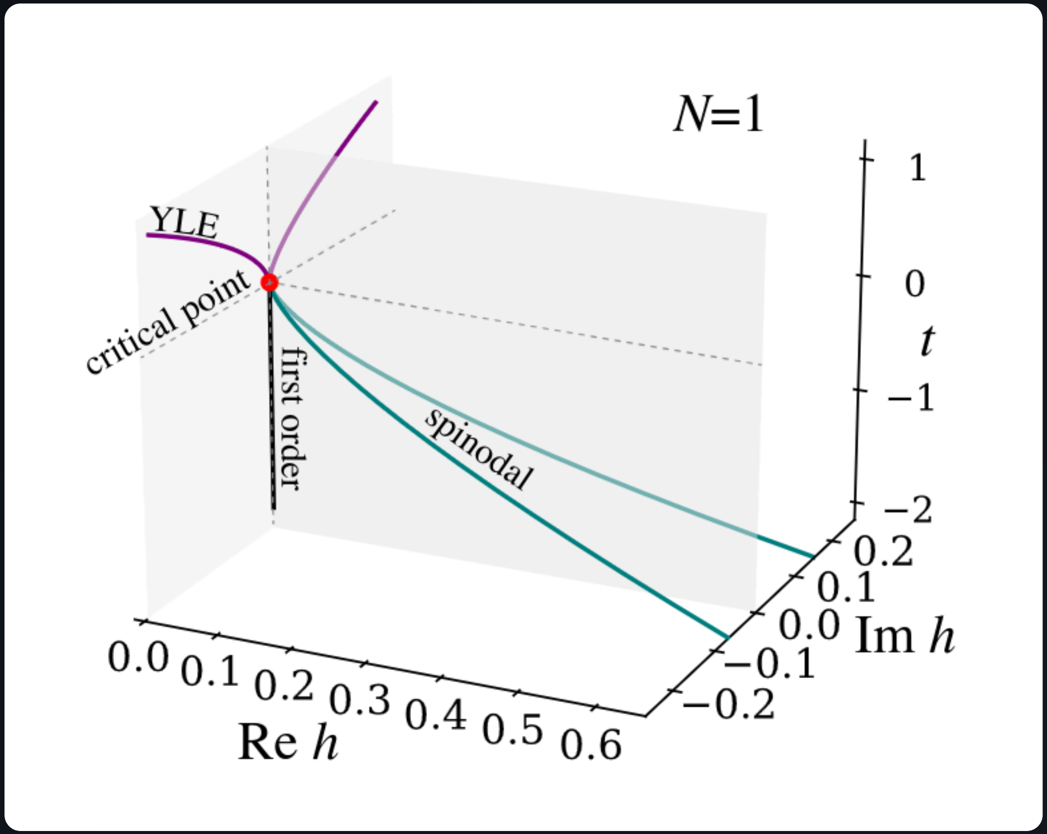
Yang-Lee Edge singularities, Lee-Yang Zeroes, Lattice results

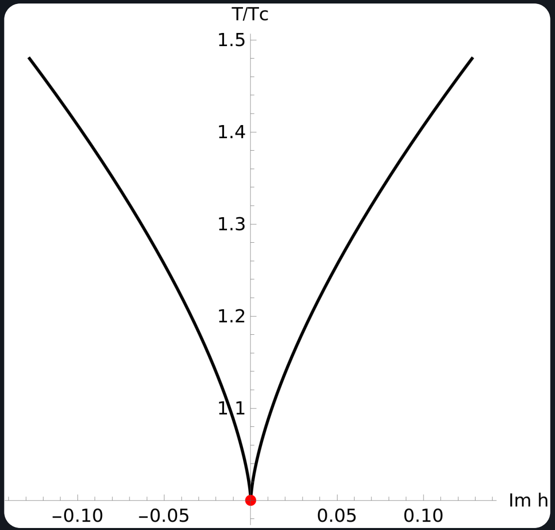
YLE and CP

Three dimensional Ising model for illustration:



- Every CP has related YLEs
- YLEs are at complex values of thermodynamic parameters
- When two YLEs pinch real axis ⇝ critical point

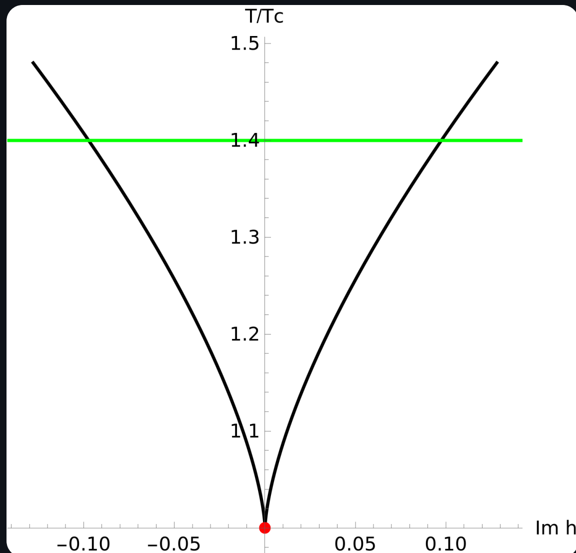
Above T_c:



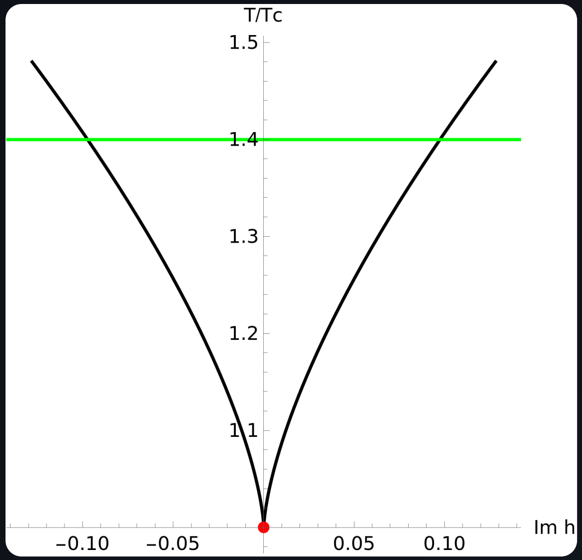
Yang-Lee Edge as CP

- At YLE, the correlation length is infinite
- YLE is characterized by a single independent critical exponent
- The susceptibility, χ_2, χ_4 diverge

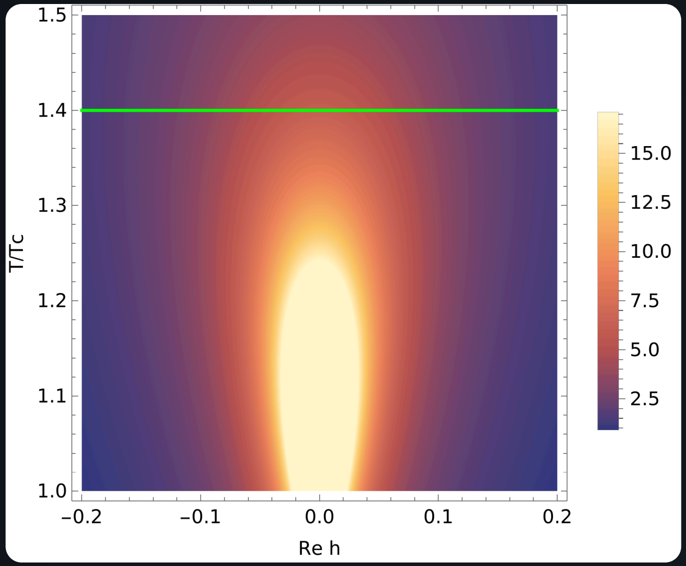
Phase diagram

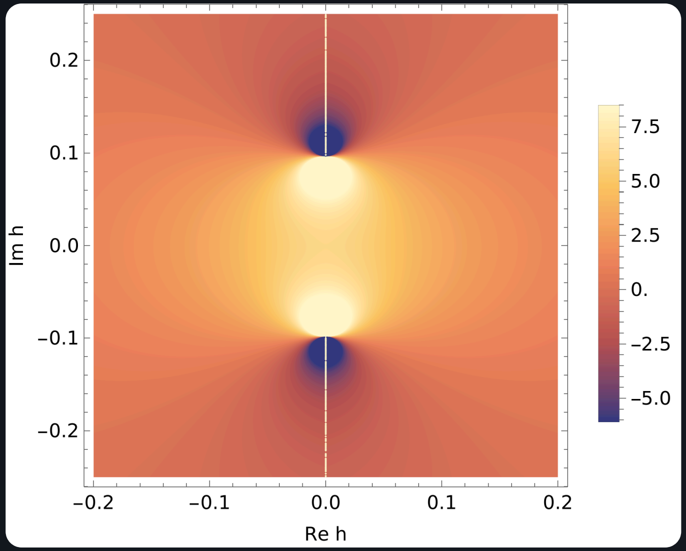


Phase diagram

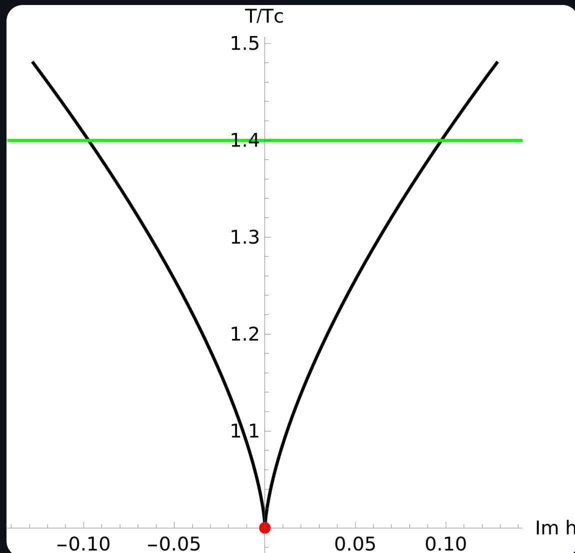


χ_2

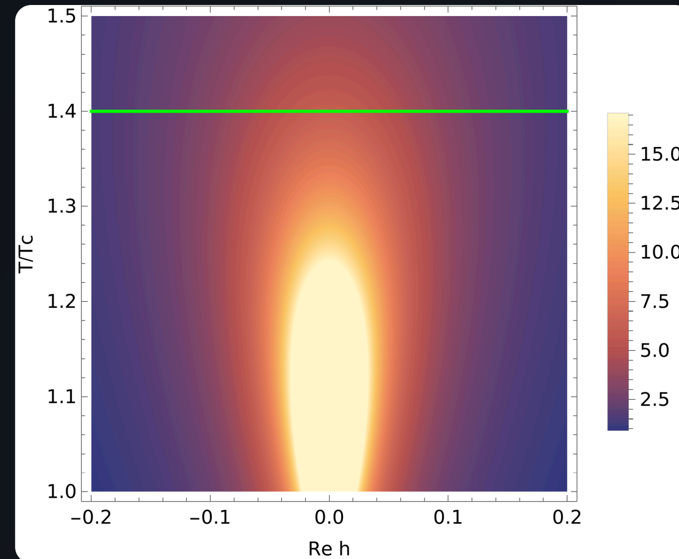


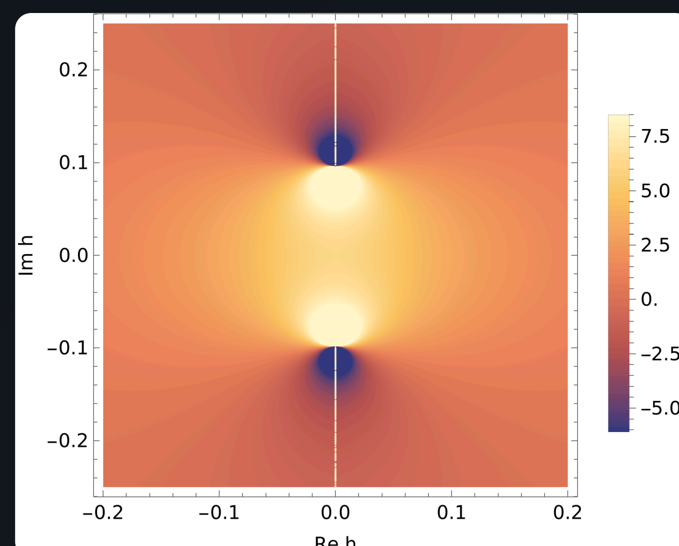


Phase diagram

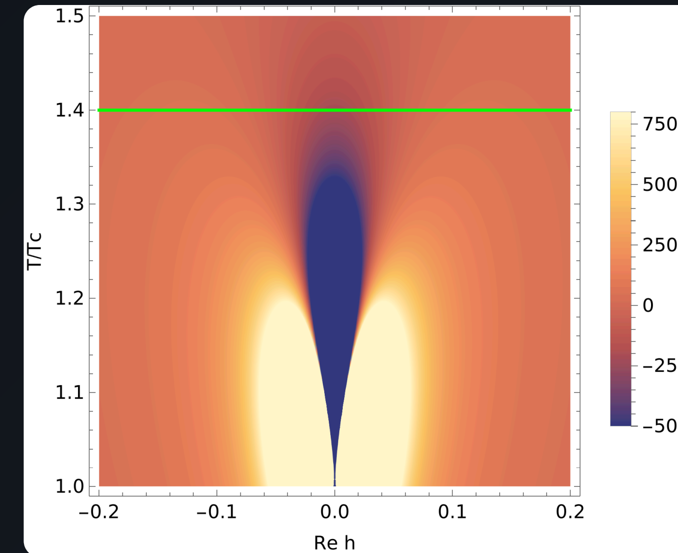


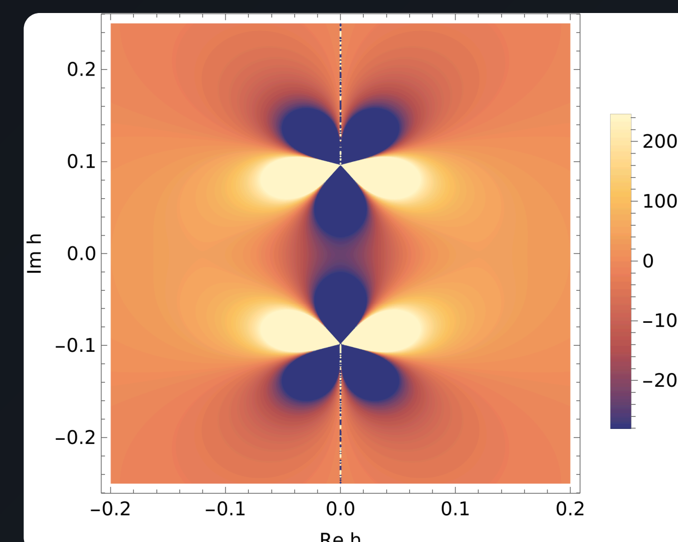
χ_2



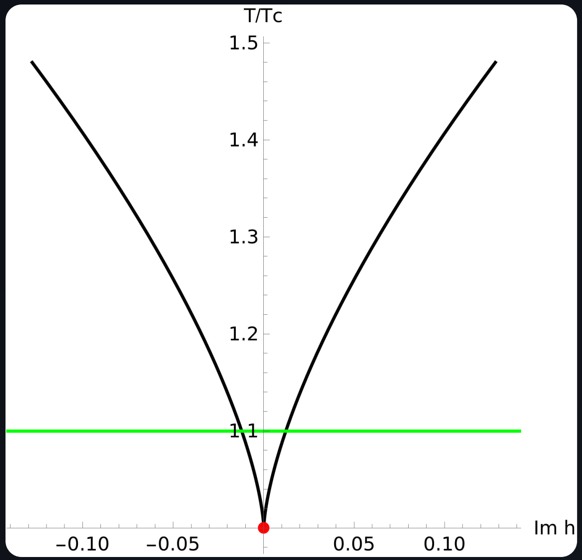


χ_4

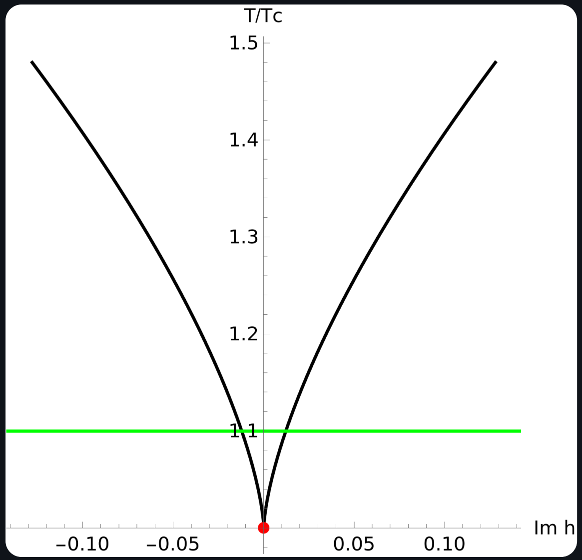




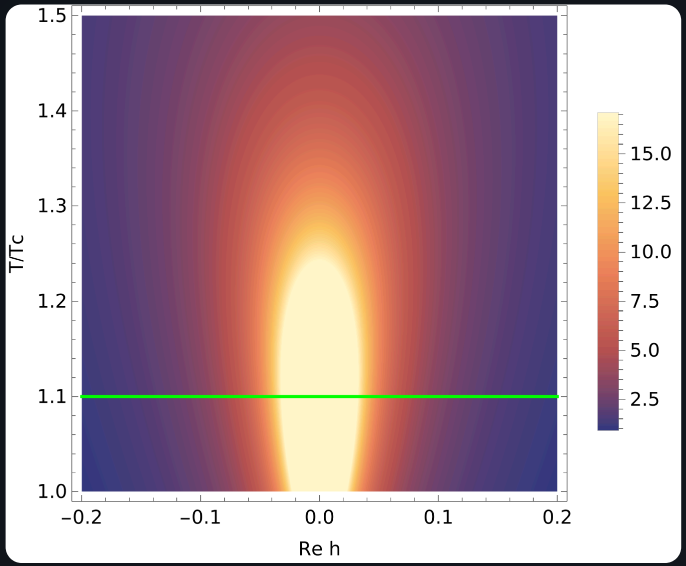
Phase diagram

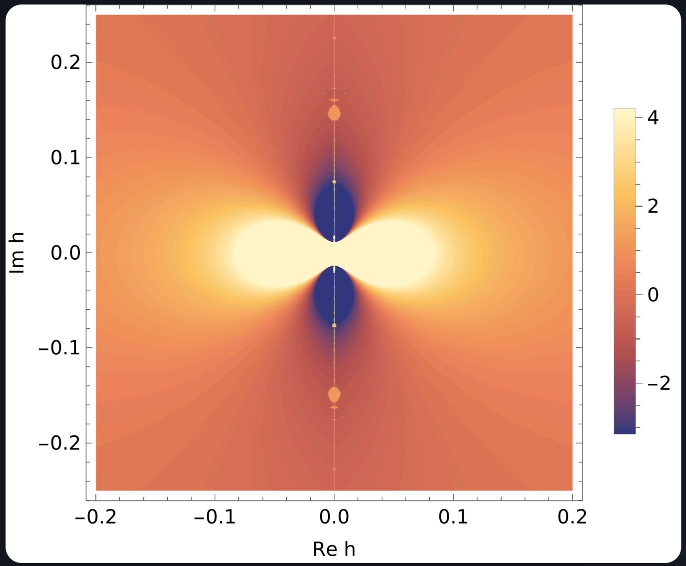


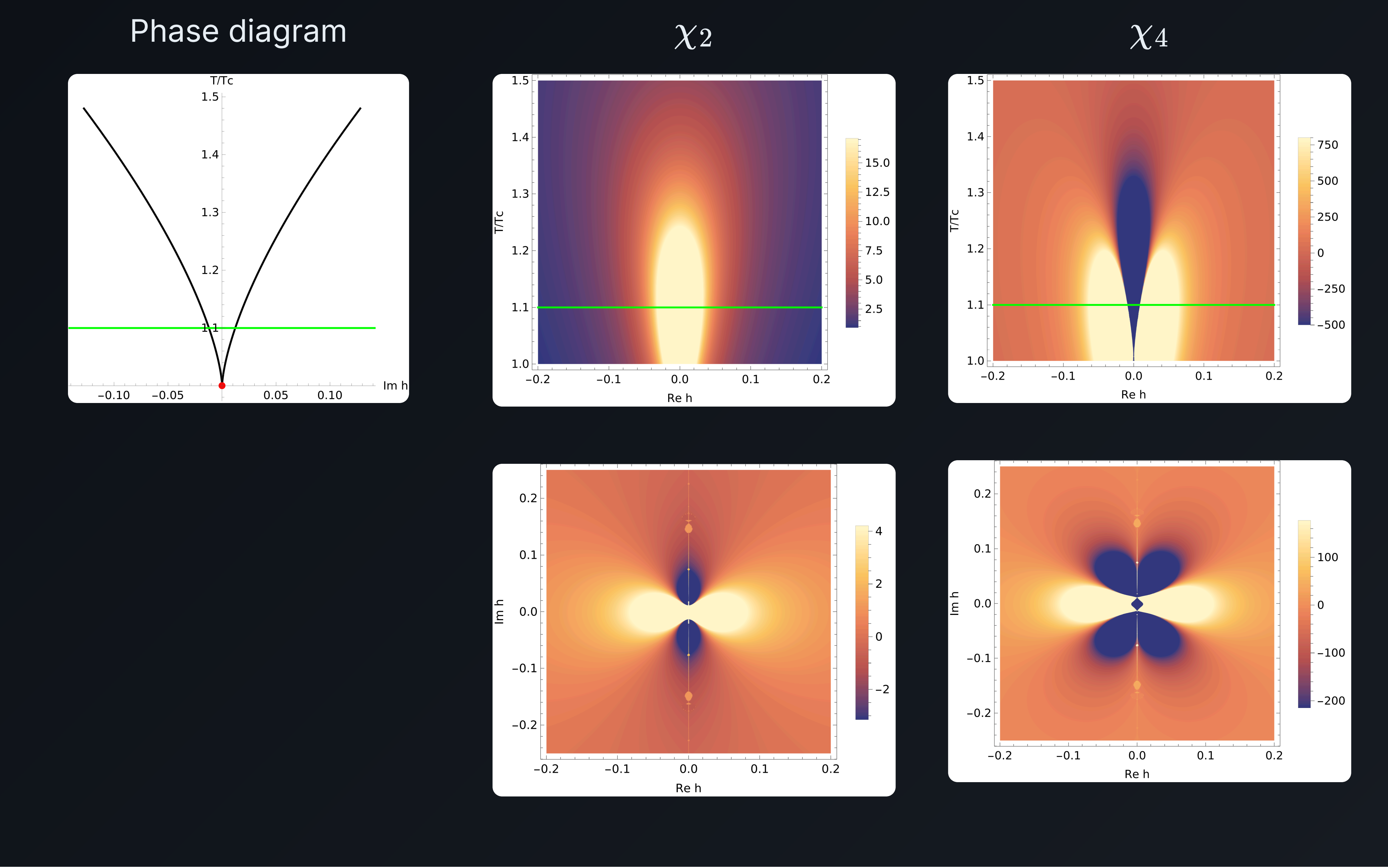
Phase diagram



χ_2





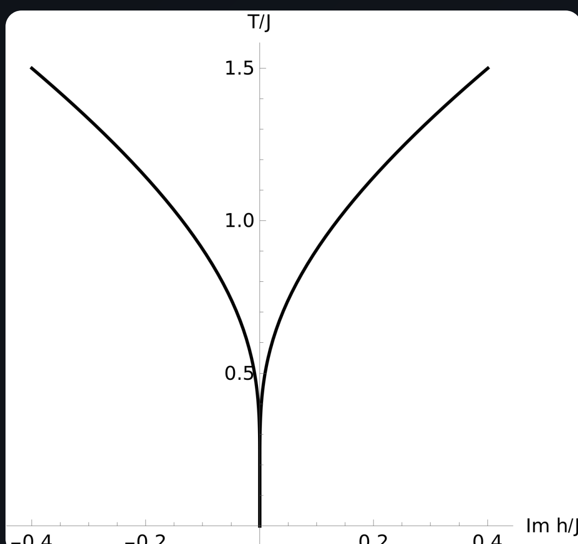


Phase diagram
χ_2
χ_4
T/Tc
Im h
Re h

- Every CP has YLEs. Is the reverse true?
- Does existence of YLEs imply an existence of a finite T critical point?
- Unfortunate no... Canonical example: one-dimensional Ising model

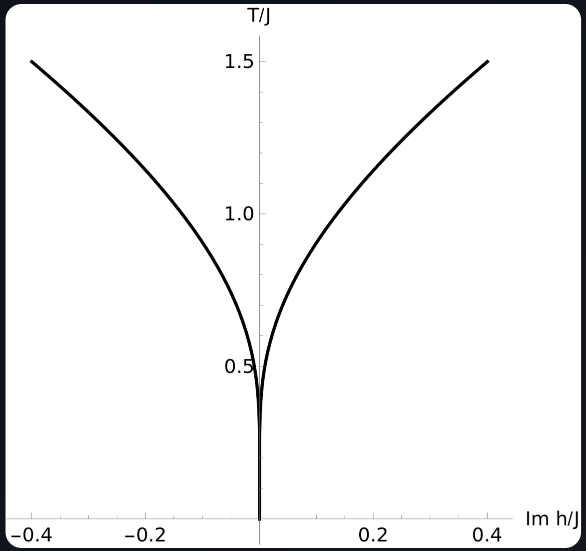
- Every CP has YLEs. Is the reverse true?
- Does existence of YLEs imply an existence of a finite T critical point?
- Unfortunate no... Canonical example: one-dimensional Ising model

Phase diagram

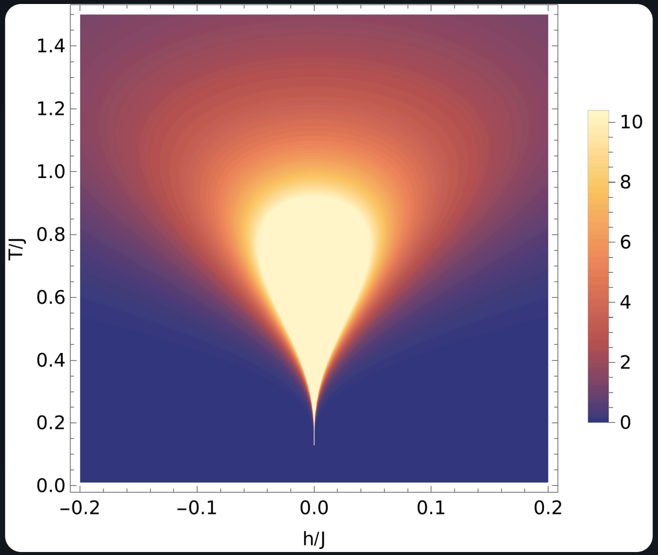


- Every CP has YLEs. Is the reverse true?
- Does existence of YLEs imply an existence of a finite T critical point?
- Unfortunate no... Canonical example: one-dimensional Ising model

Phase diagram

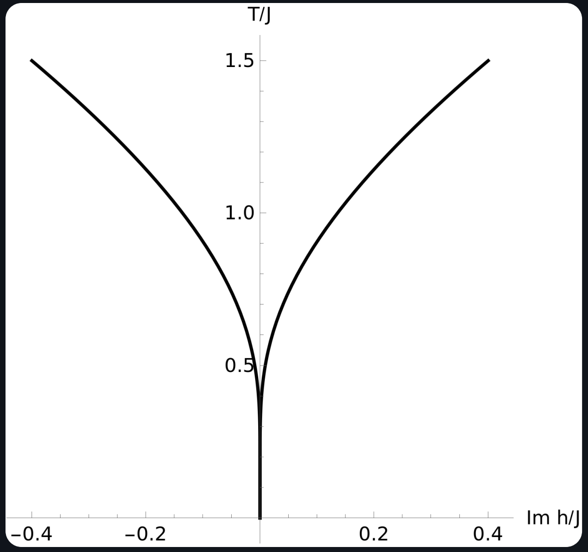


χ_2

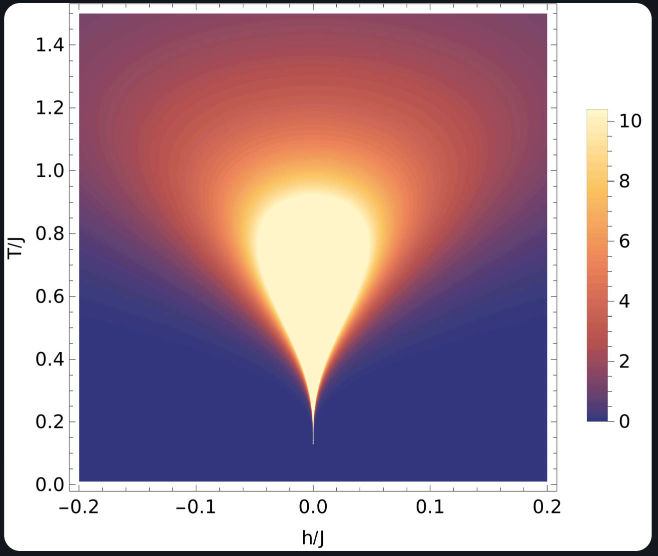


- Every CP has YLEs. Is the reverse true?
- Does existence of YLEs imply an existence of a finite T critical point?
- Unfortunate no... Canonical example: one-dimensional Ising model

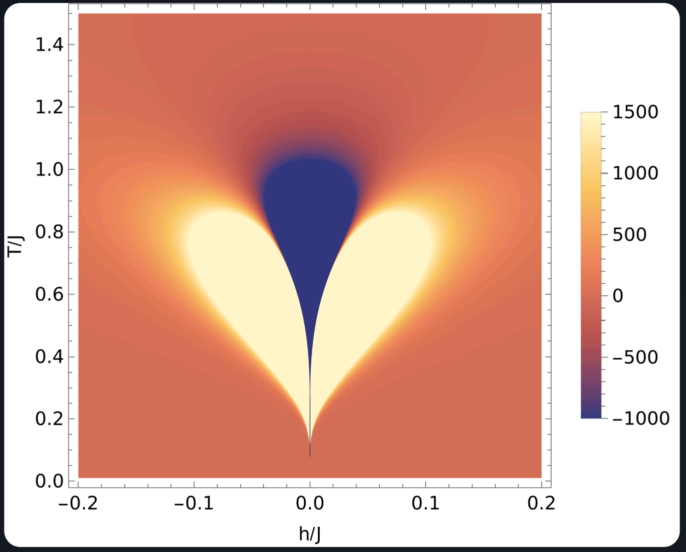
Phase diagram



χ_2



χ_4

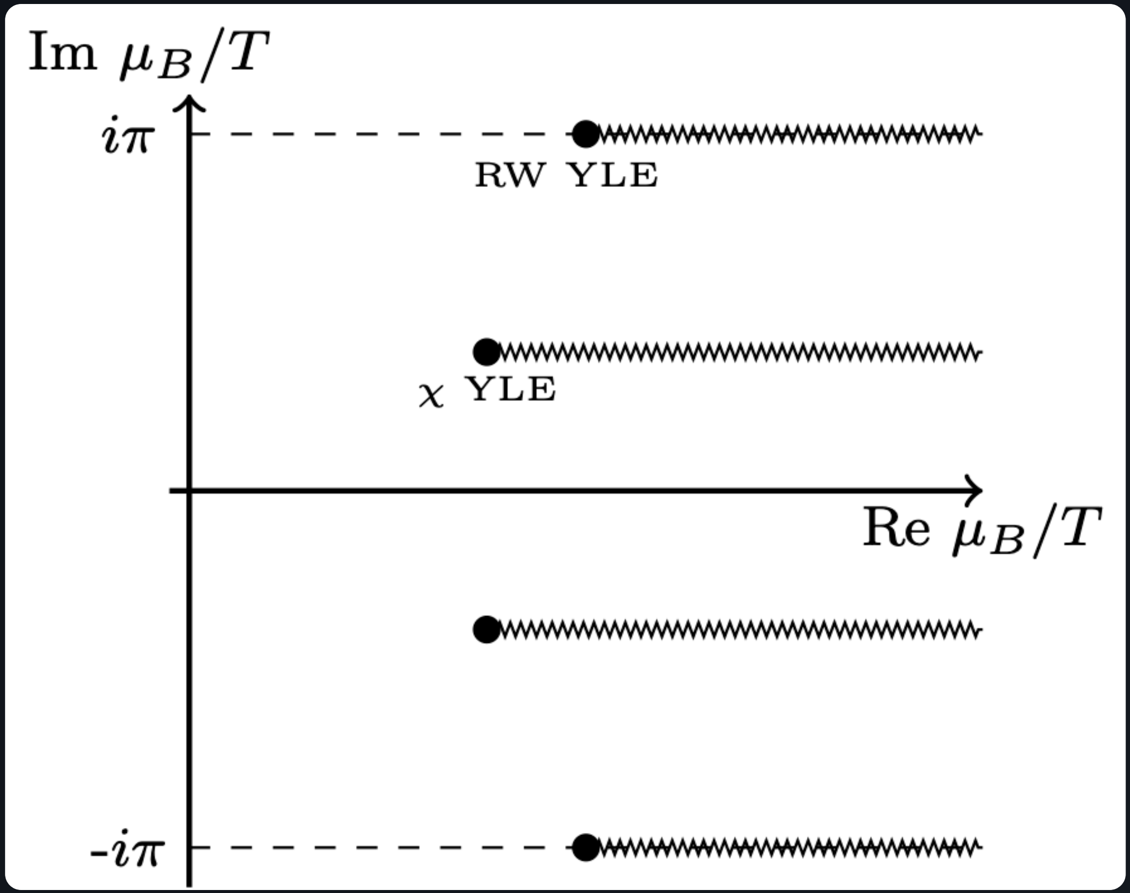


YLE $\to$ CP

Thus:

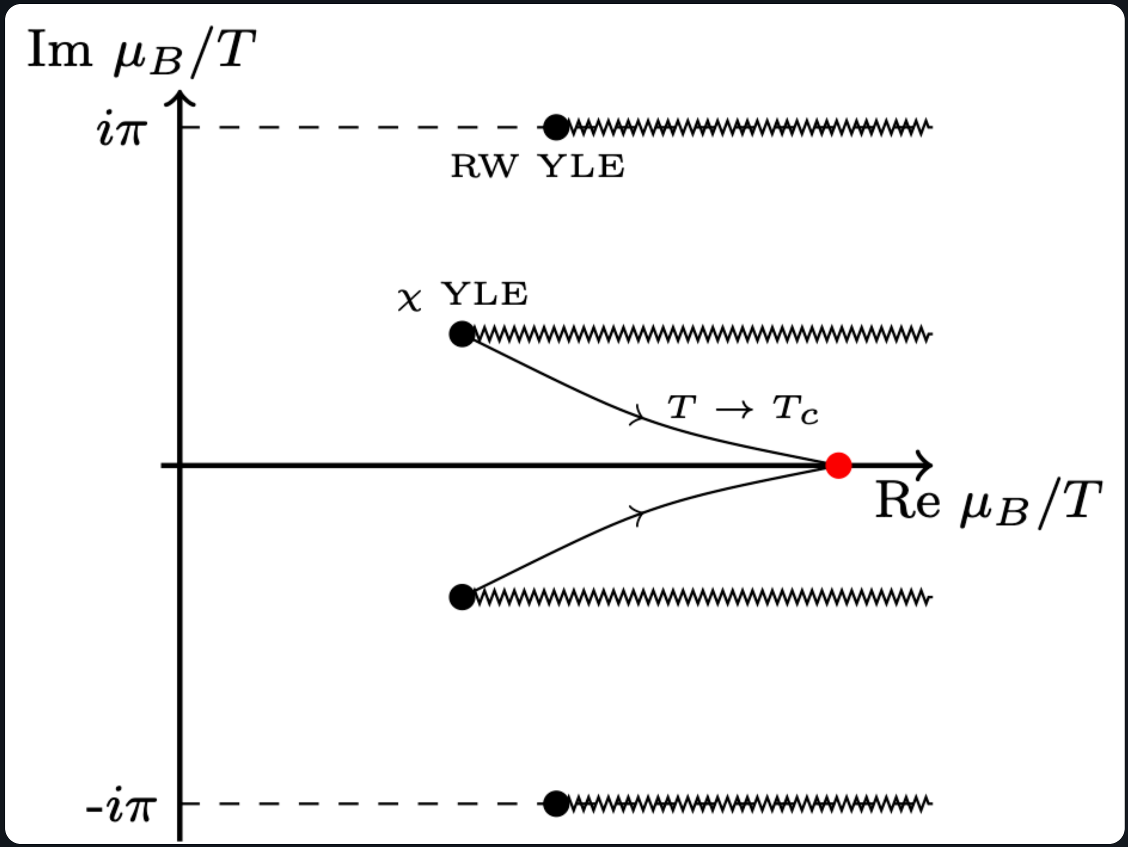
- locate and trace YLE as a function of T
- find when they approach real values of thermodynamic parameters (e.g. μ)

QCD: Expected analytic structure $T_c < T < T_{RW}$

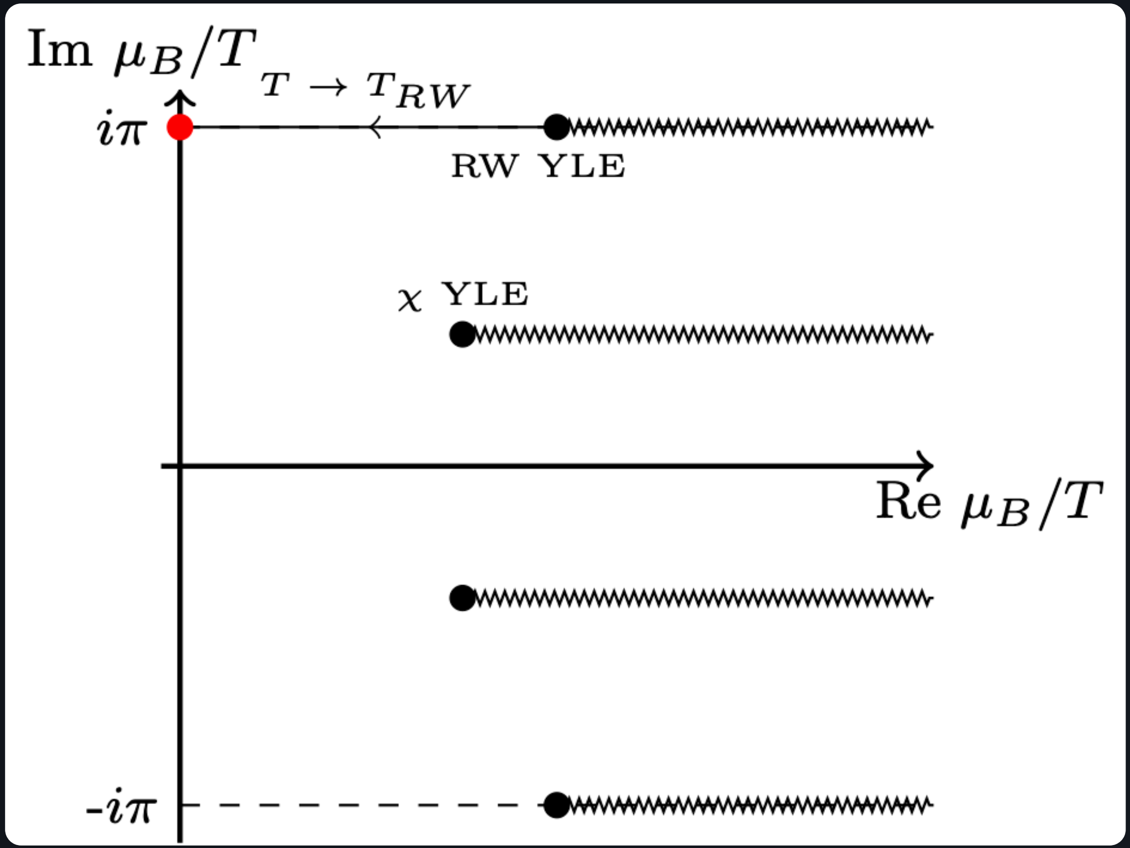


M. Stephanov, hep-lat/0603014

QCD: Expected analytic structure $T \to T_c$



QCD: Expected analytic structure $T \to T_{RW}$

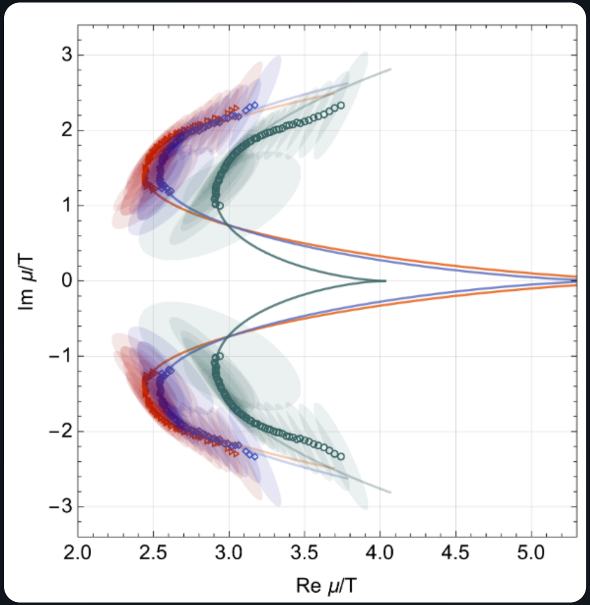


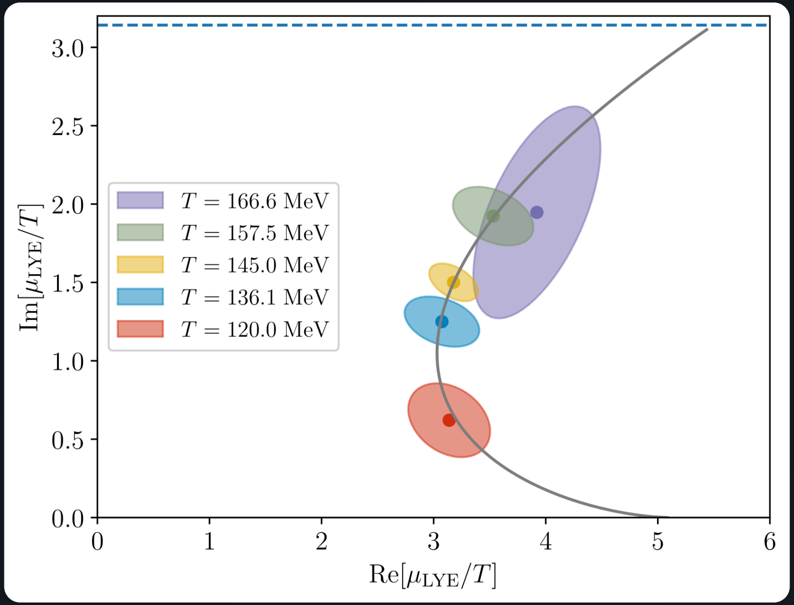
- Roberge-Weiss critical point is at purely imaginary chemical potential
- Direct access in lattice QCD; provides a check on methods

Tracing YLE singularity: chiral critical point

Lattice input:

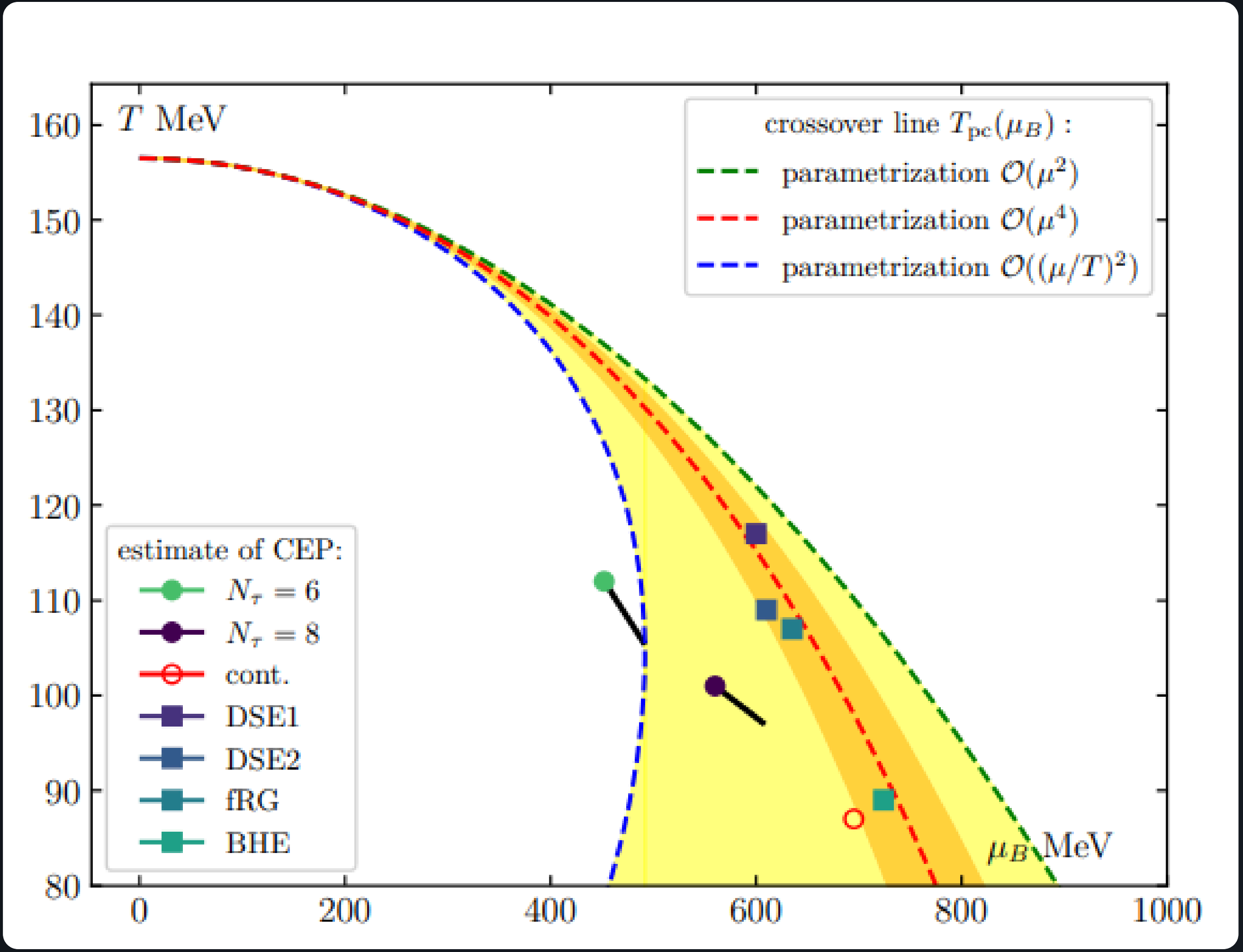
- Taylor series coeff. + conformal Pade
- Im μ + Pade





G. Basar, 2312.06952
D. Clarke et. al., 2405.10196

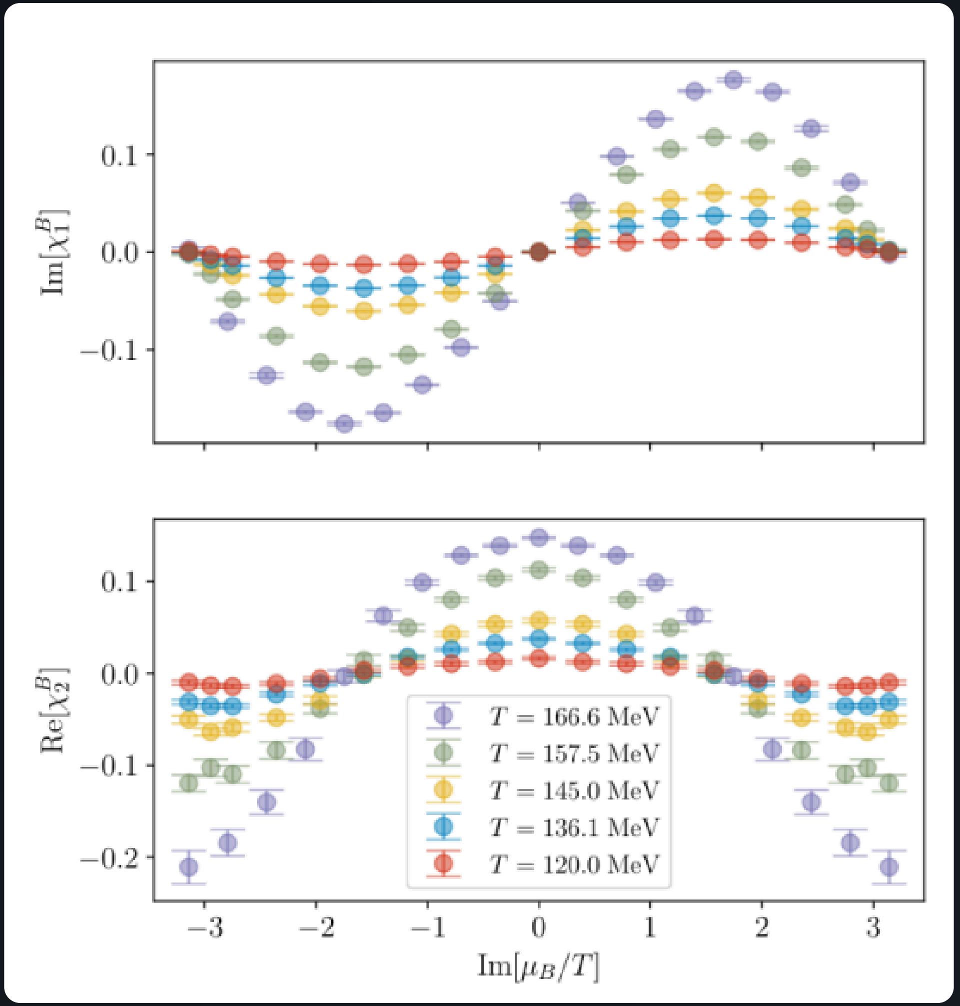
Tracing YLE singularity: estimate for critical point



Clarke et al, 2405.10196

Lattice: numerics

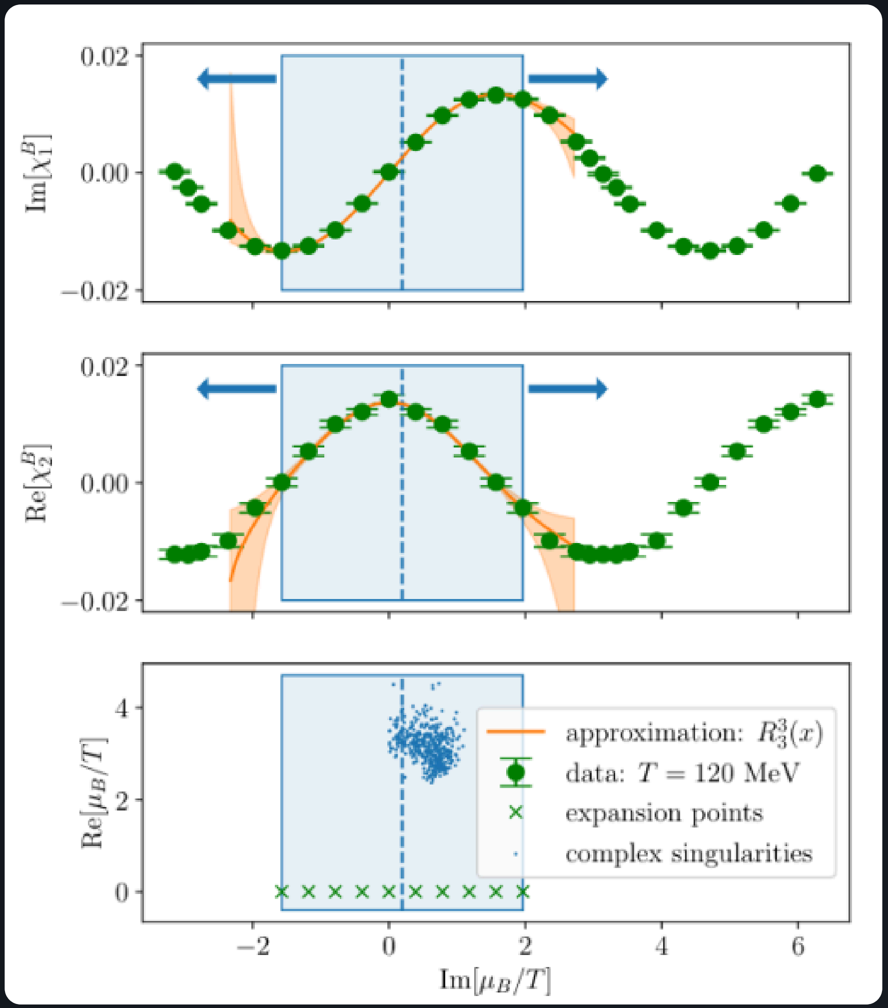
- HISQ action, N_f = 2 + 1 flavours
- $36^3 \times 6$ lattices
- Observable: $\chi^B_{1,2}(T, \mu_B)$
- 5 temperatures: $T = 120.0, 136.1, 157.5, 166.6$ MeV
- 10 imaginary chemical potentials between 0 and $i\pi$



From the presentation by Zambello at ECT* 2025

Details: sliding window

- Approximate $\chi^B_{1,2}(T, \mu_B)$ with a rational function
- Calculate zeros of the numerator and denominator
- Remove near cancelling pairs, keep poles in the first quadrant closest to the center of the interval
- Bootstrap over the data
- Iterate over 55 intervals of varying length (from π to 2π) and center
- Perform scaling fits on randomly chosen intervals and collect the results for further analysis (10^5 fits)



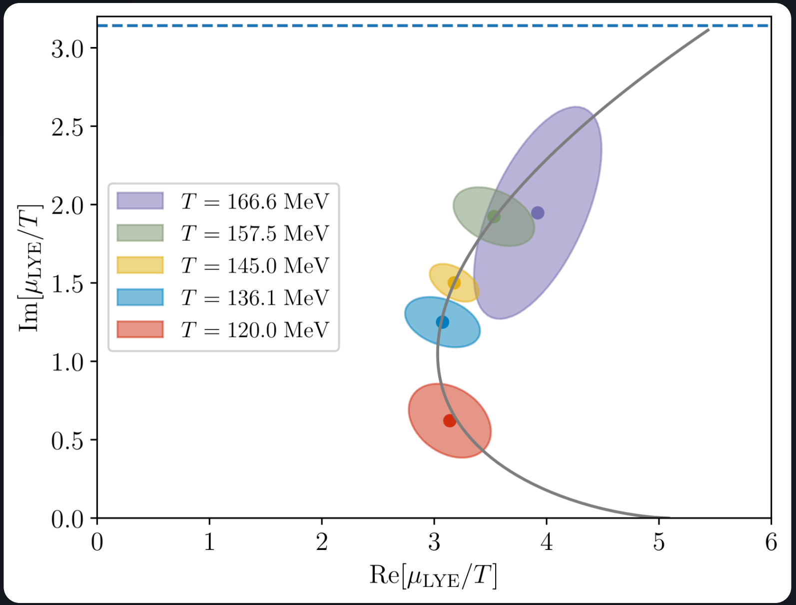
Details: extrapolation to lower T

- At measured T, Im $\mu_{YLE} \neq 0$
- Extrapolated using the universal property of YLE:

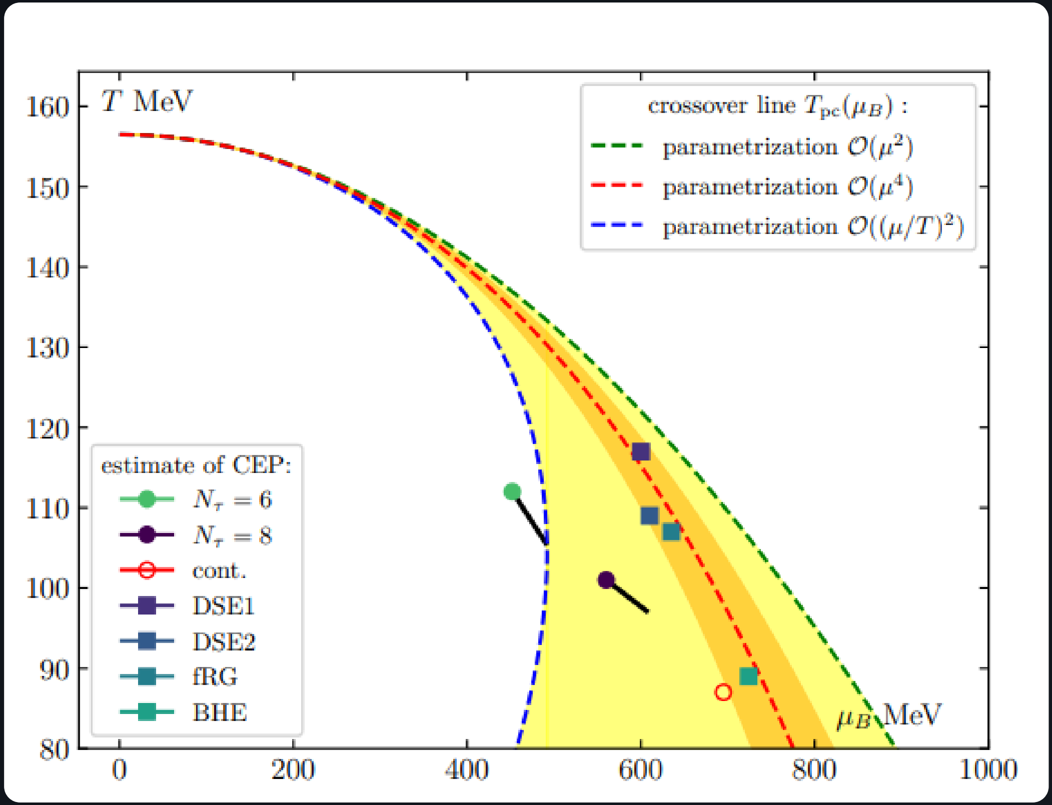
1. $t/h^{1/\beta\delta} = z_c =$ universal number
2. and linear mapping

$$t = \alpha_t(T - T_{CEP}) + \beta_t(\mu_B - \mu_{CEP})$$

$$h = \alpha_h(T - T_{CEP}) + \beta_h(\mu_B - \mu_{CEP})$$



Tracing YLE singularity: estimate for critical point



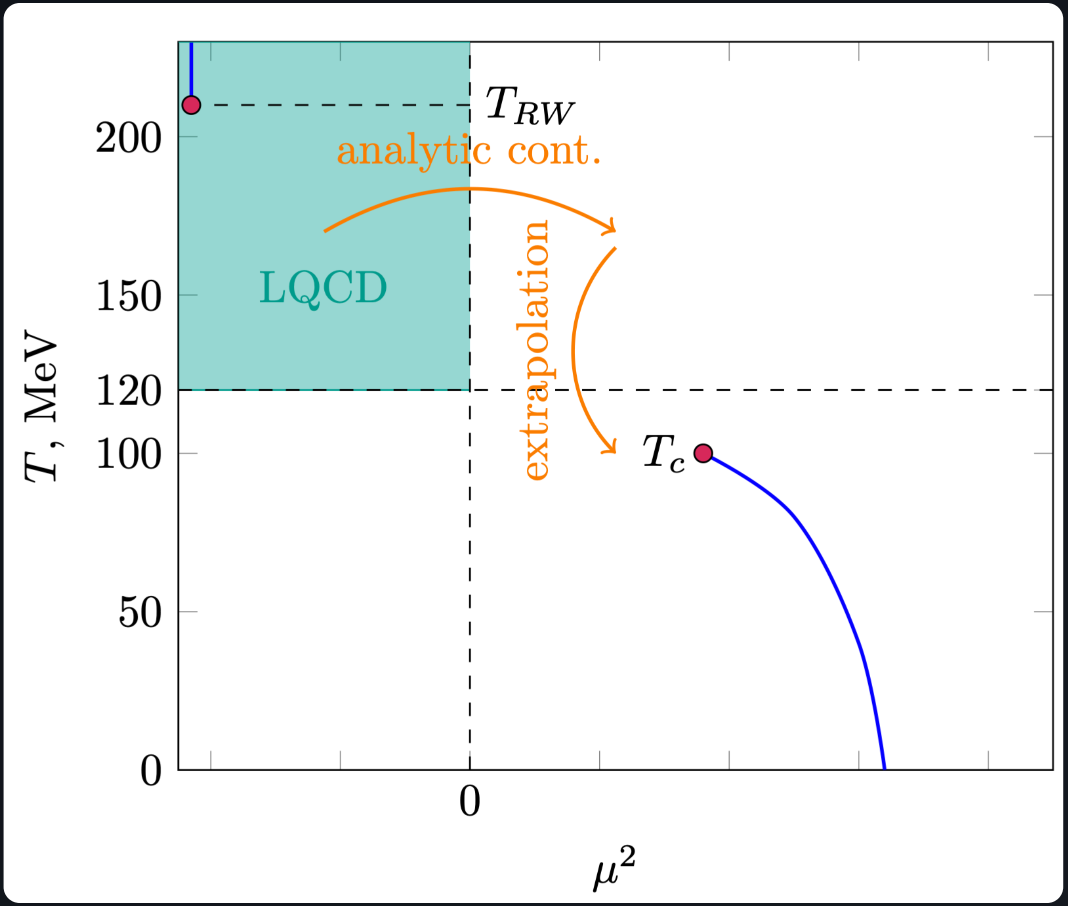
- Somewhat consistent with $16^3 \times 8$ study by Adam et al, arXiv: 2507.13254:

"84% probability the CP either lies below T = 103 MeV or does not exist"

- Consistent with Basar, 2312.06952: $T \approx 100$ MeV, $\mu_c \approx 580$ MeV

Clarke et al, 2405.10196

Tracing YLE singularity: systematics



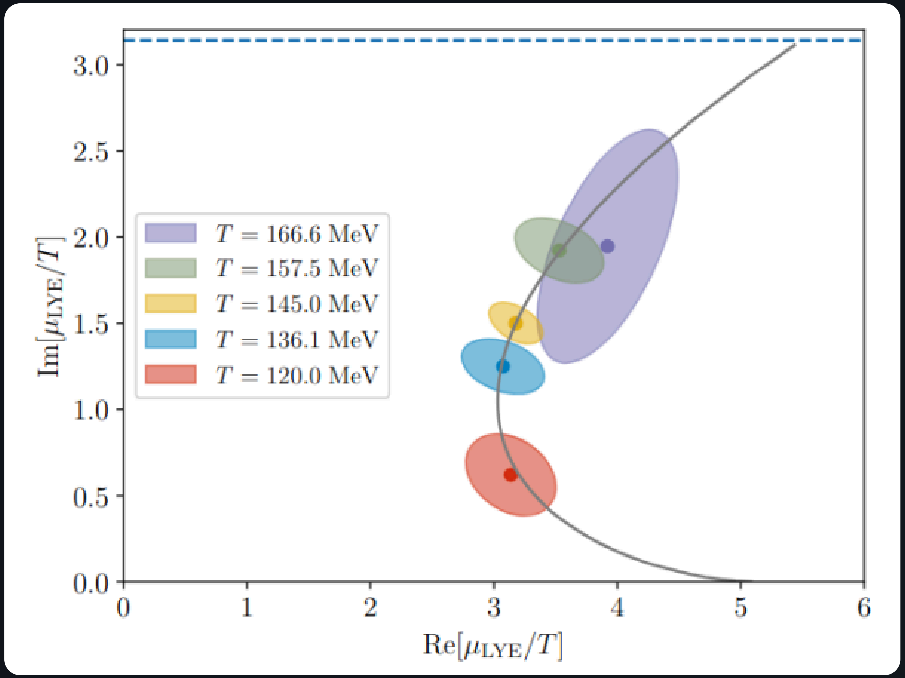
⚠️ Double extrapolation

YLE trajectory: LQCD vs DSE

Lattice: Clarke et al, 2405.10196
DSE: Z.-Y. Wan, Y. Lu, F. Gao, and Y.-X. Liu, 2504.12964

Quantitatively the YLE trajectories are very different

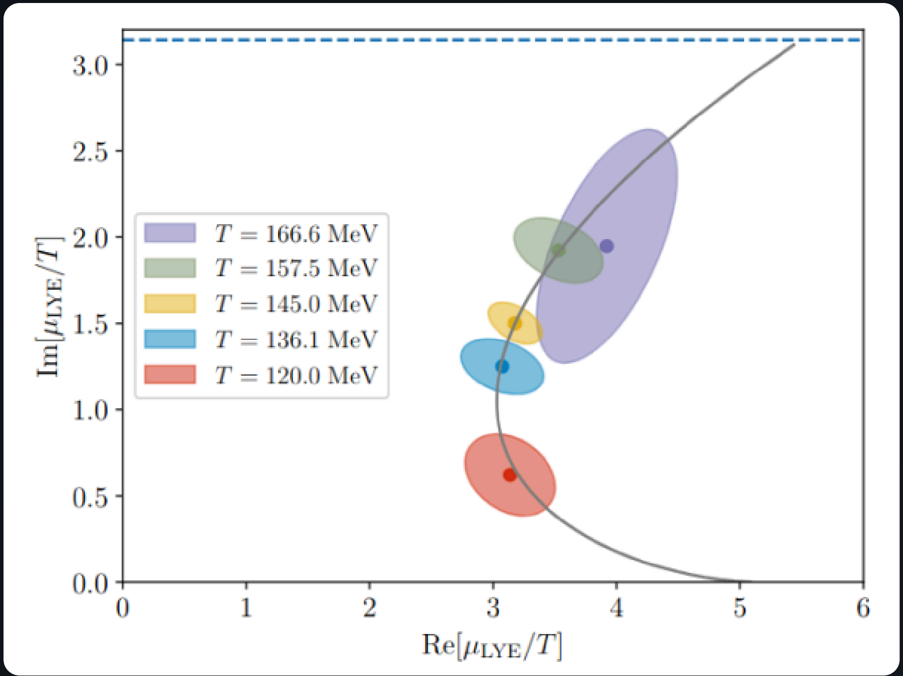
YLE trajectory: LQCD vs DSE

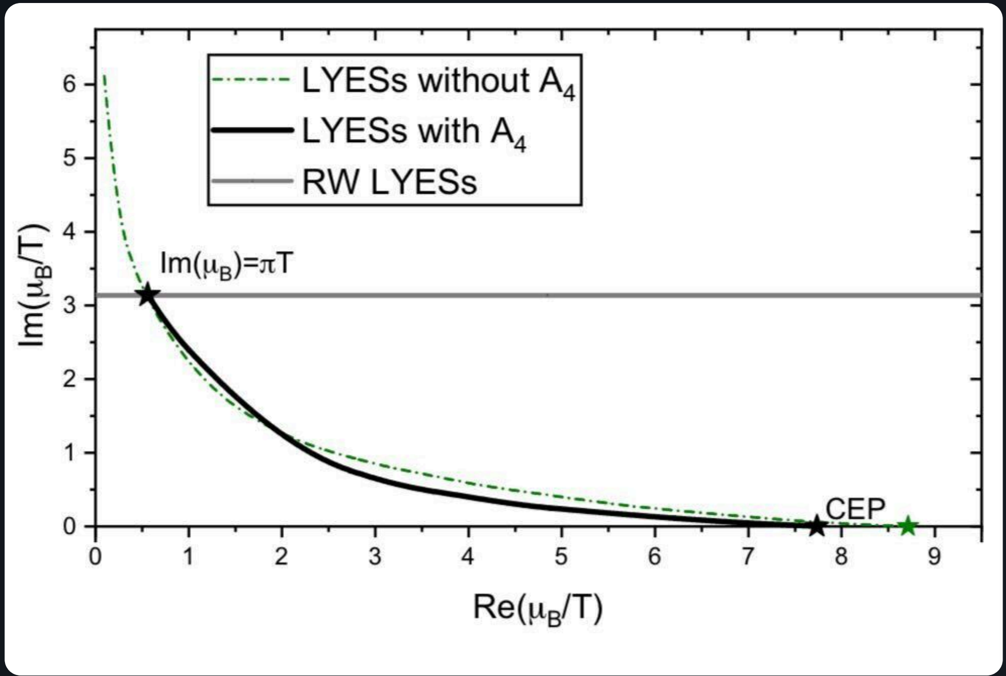


Lattice: Clarke et al, 2405.10196
DSE: Z.-Y. Wan, Y. Lu, F. Gao, and Y.-X. Liu, 2504.12964

Quantitatively the YLE trajectories are very different

YLE trajectory: LQCD vs DSE





Lattice: Clarke et al, 2405.10196
DSE: Z.-Y. Wan, Y. Lu, F. Gao, and Y.-X. Liu, 2504.12964

Quantitatively the YLE trajectories are very different

Conclusions

- Encouraging results

Several sources of systematical uncertainty:

- Double extrapolation
- Linear mapping to $(t, h) \to (T, \mu)$
- Applying $t/h^{1/\beta\delta} = z_c$ in a wide range of temperatures 120 MeV $\leq T \leq$ 166.6 MeV with $T_c \approx 100$ MeV
- The last two points apply to the Roberge-Weiss critical point too (166.6 MeV $\leq T \leq$ 201.4 MeV); the extracted $T_{RW} \approx 211$ MeV coincides with direct lattice simulations
- Continuum limit

YLE trajectory in DSE is qualitatively different from LQCD

Backup

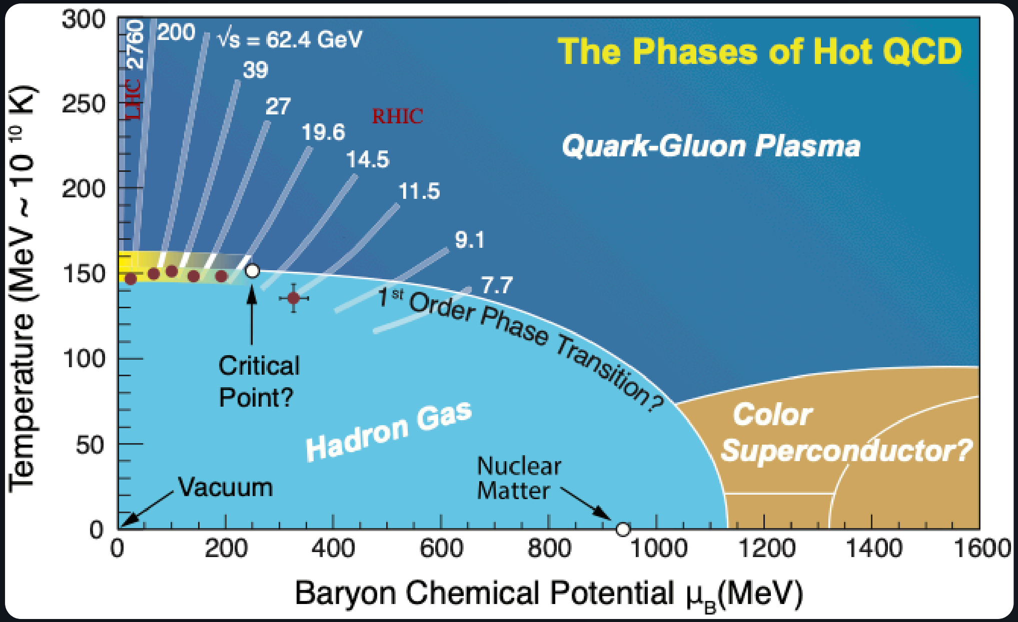
Universal location of YLE

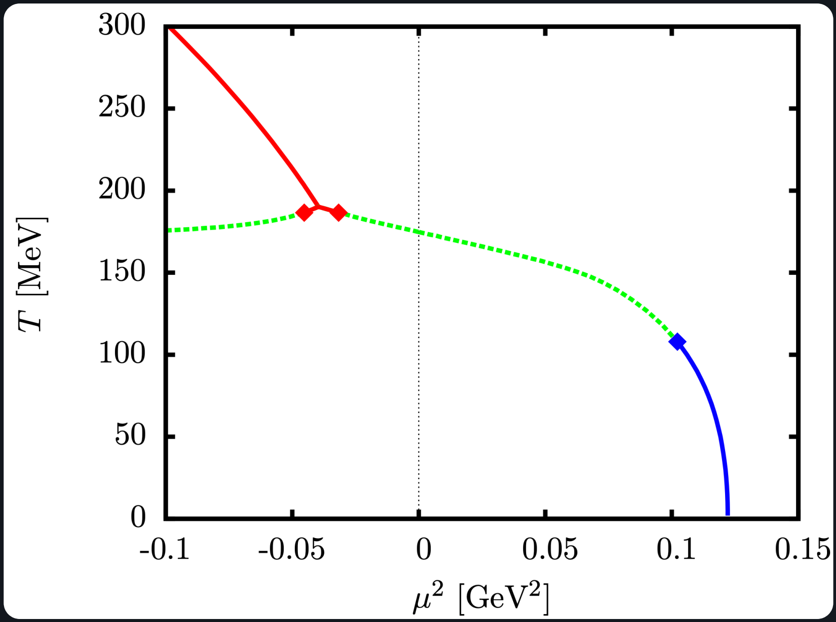
For each universality class:

- universal magnetic equation of state
- **universal location of YLE singularity**
- mapping to QCD requires non-universal parameters

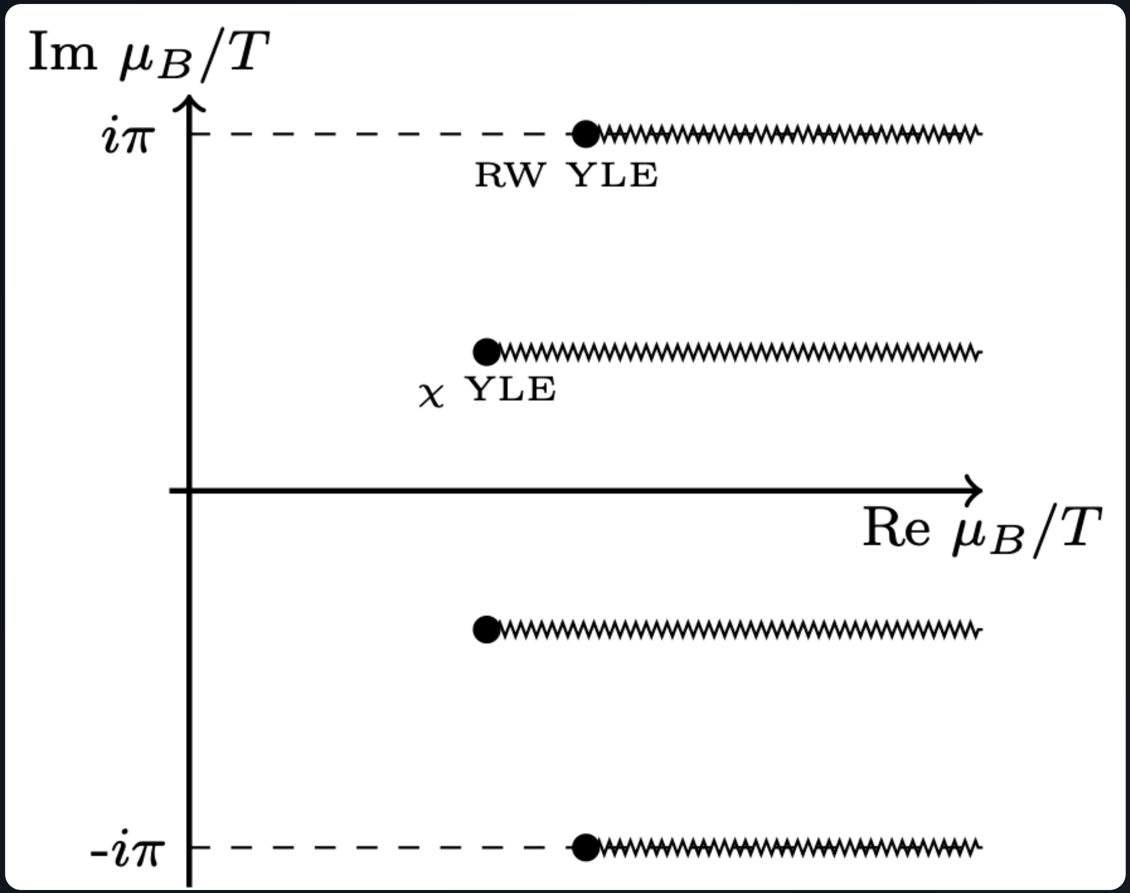
d	1	2	3	4
$\lvert z_c \rvert / R_\chi^{1/\gamma}$	1	1.32504(2)	1.621(4)	$3/2^{2/3}$

Back to QCD phase diagram



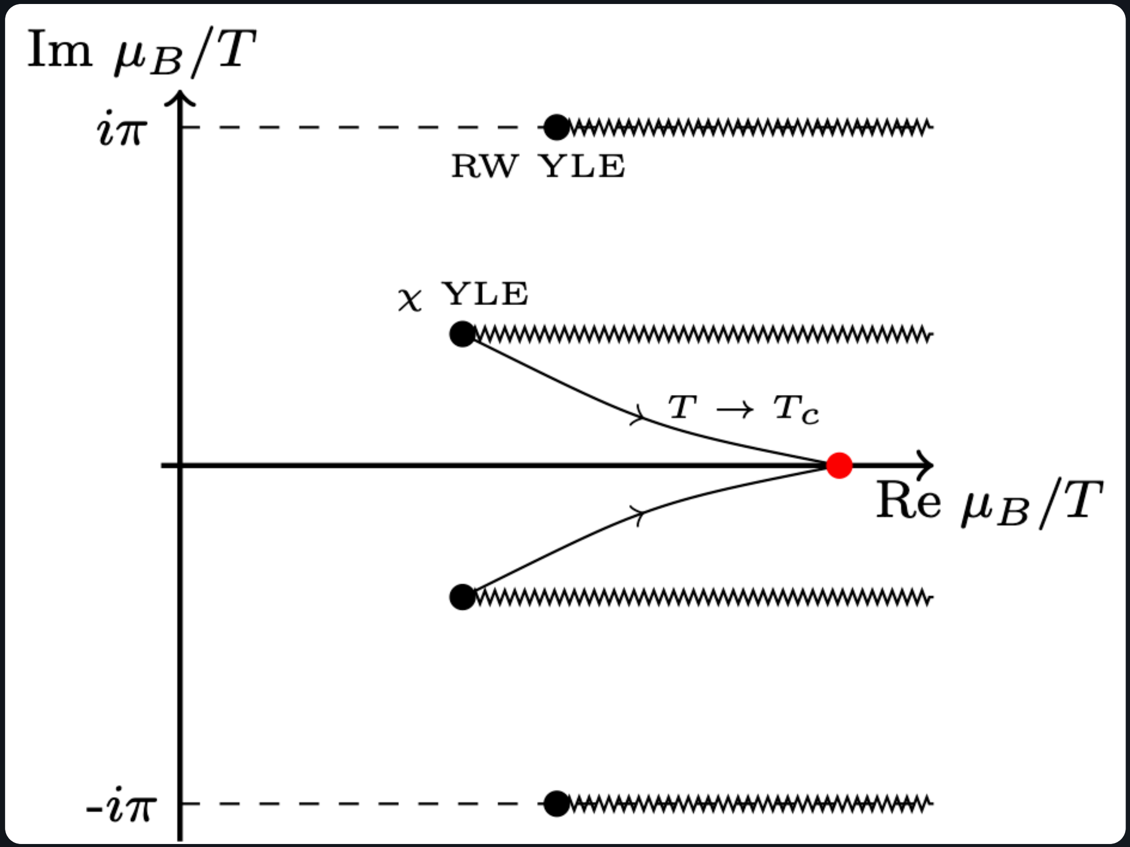


QCD: Expected analytic structure $T_c < T < T_{RW}$

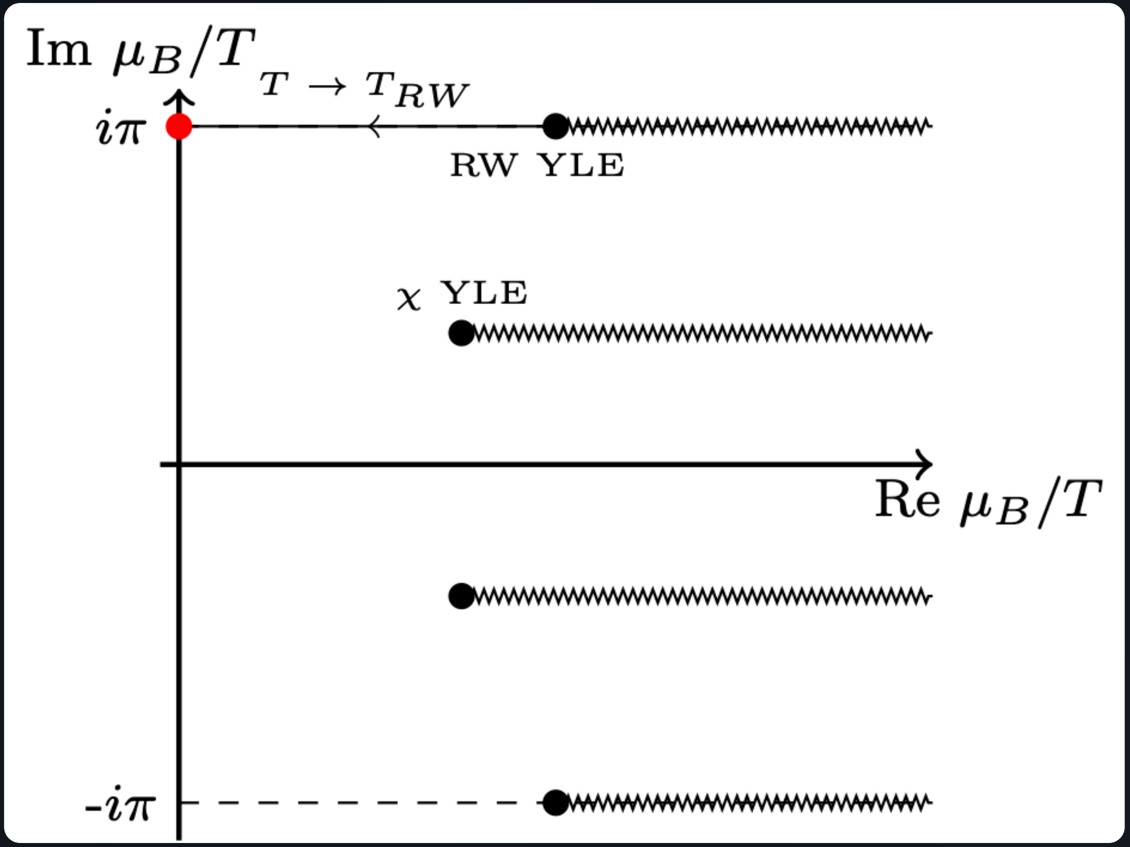


M. Stephanov, hep-lat/0603014

QCD: Expected analytic structure $T \to T_c$



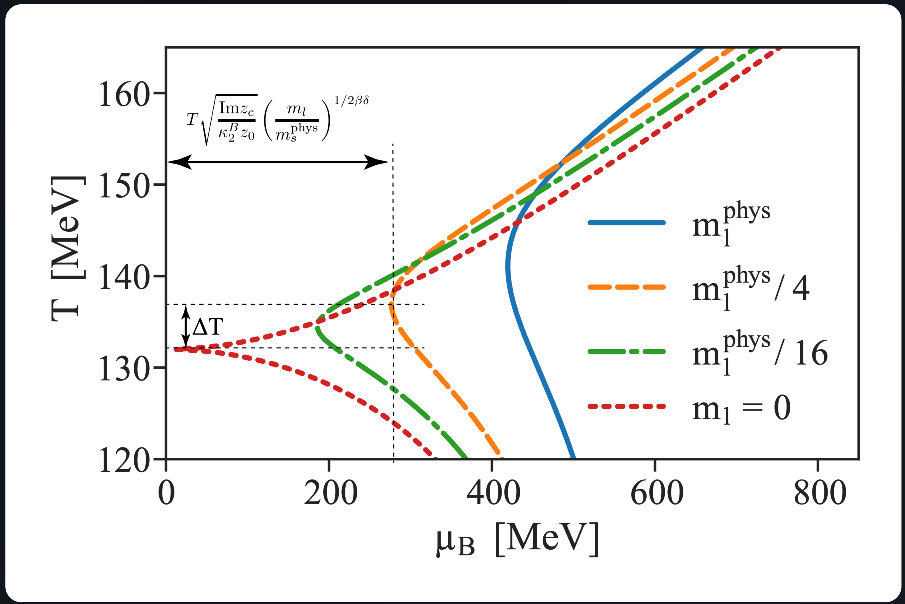
QCD: Expected analytic structure $T \to T_{RW}$



Naive mapping

Near chiral transition

$$z = z_0 \left(\frac{m_l}{m_s}\right)^{-\frac{1}{\beta\delta}} \times \left[\frac{T - T_c}{T_c} + \kappa_2^B \left(\frac{\mu}{T_c}\right)^2 + \kappa_4^B \left(\frac{\mu}{T_c}\right)^4 + \dots\right],.$$

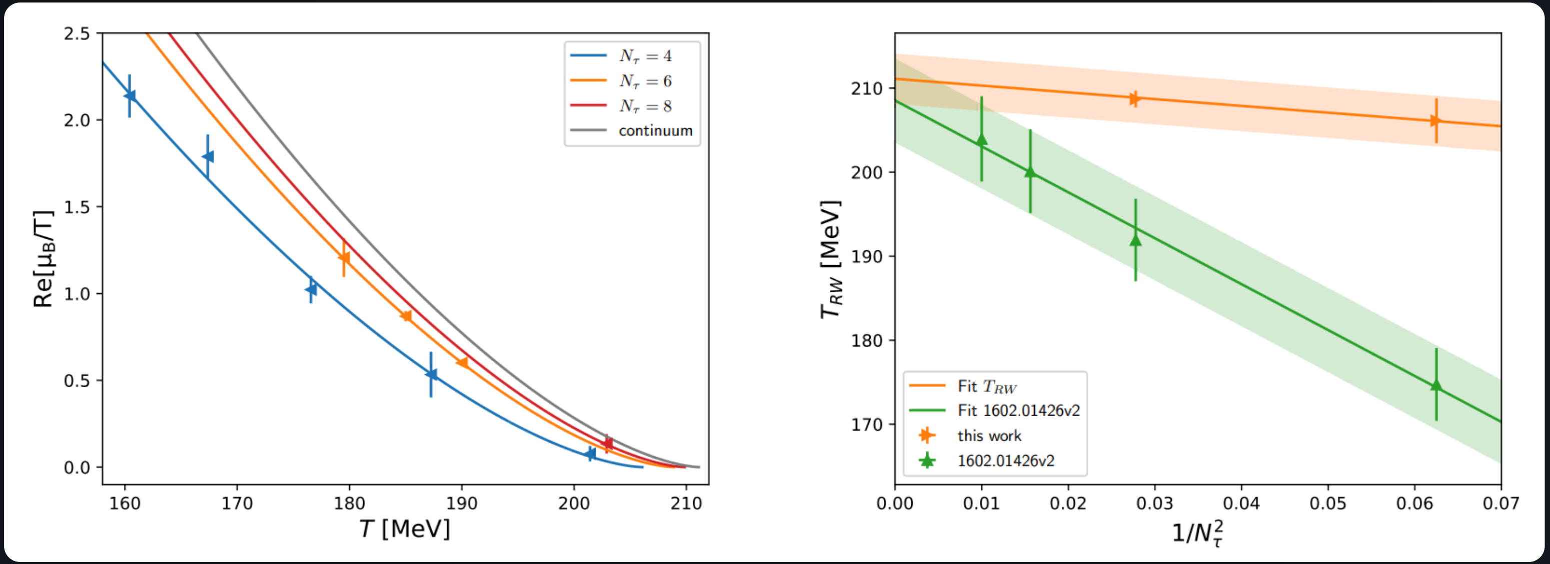


S. Mukherjee, V.S., 1909.04639

Tracing YLE singularity: RW critical point

Lattice QCD and indirect methods to locate YLE:

input from Im μ & analytic continuation

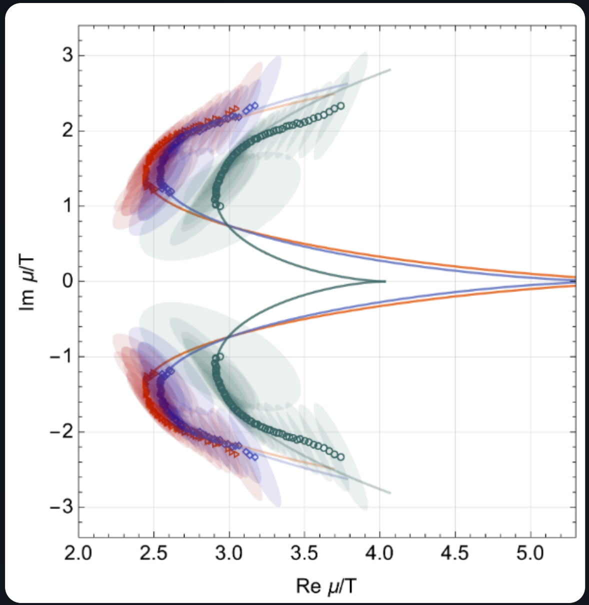


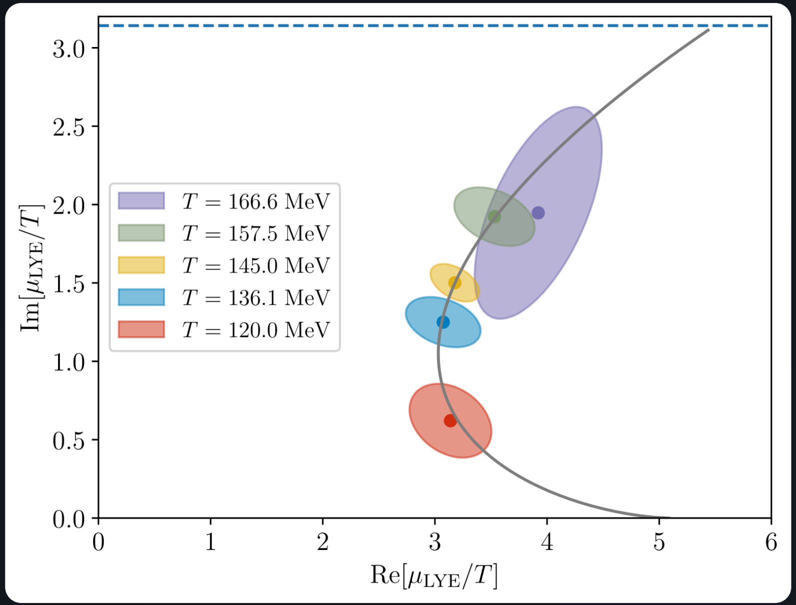
$$z = z_c \to \mathrm{Re}\mu_{YLE} \propto (T_{RW} - T)^{\beta\delta} \quad \rightsquigarrow T_{RW} = 211.1 \pm 3.1\mathrm{MeV}.$$

C. Schmidt et al, 2209.04345

Tracing YLE singularity: chiral critical point

Lattice input from Taylor series coeff. at $\mu = 0$ or Im μ & analytic continuation

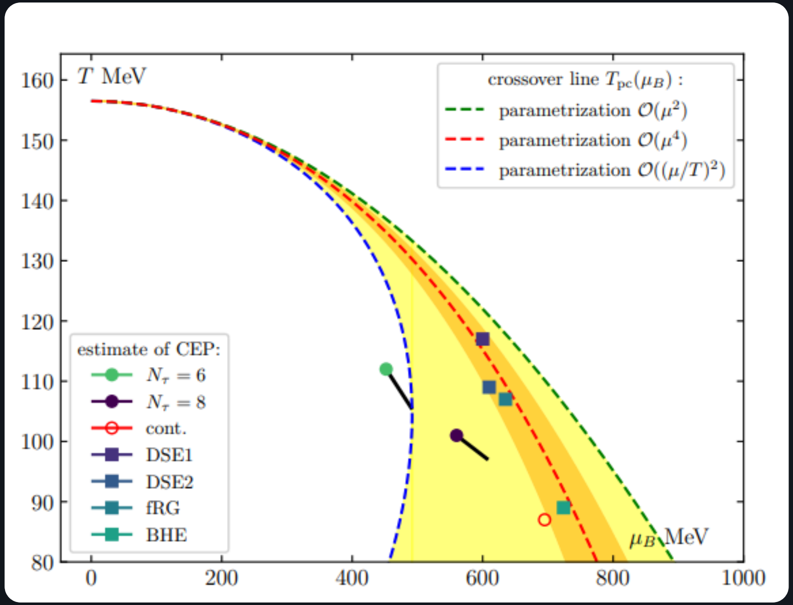




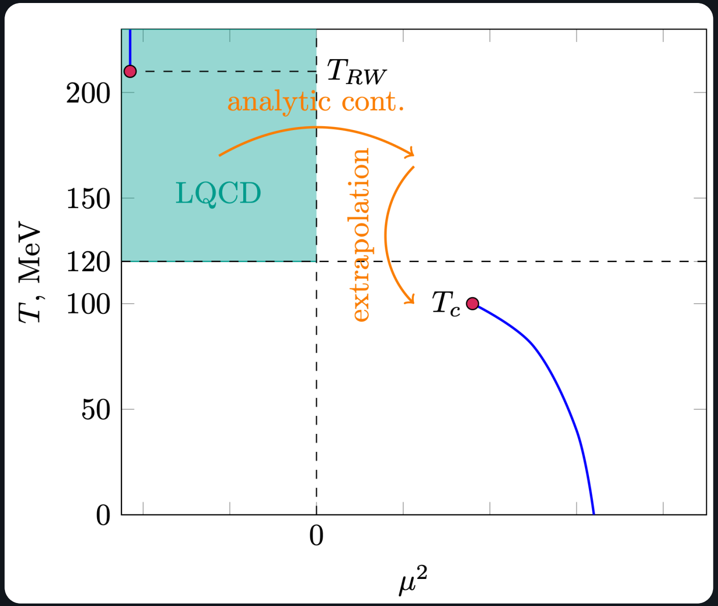
G. Basar, 2312.06952
D. Clarke et. al., 2405.10196

Tracing YLE singularity: estimate for critical point

$$T_c \approx 110 \text{ MeV}, \mu_c \approx 650 \text{ MeV}$$



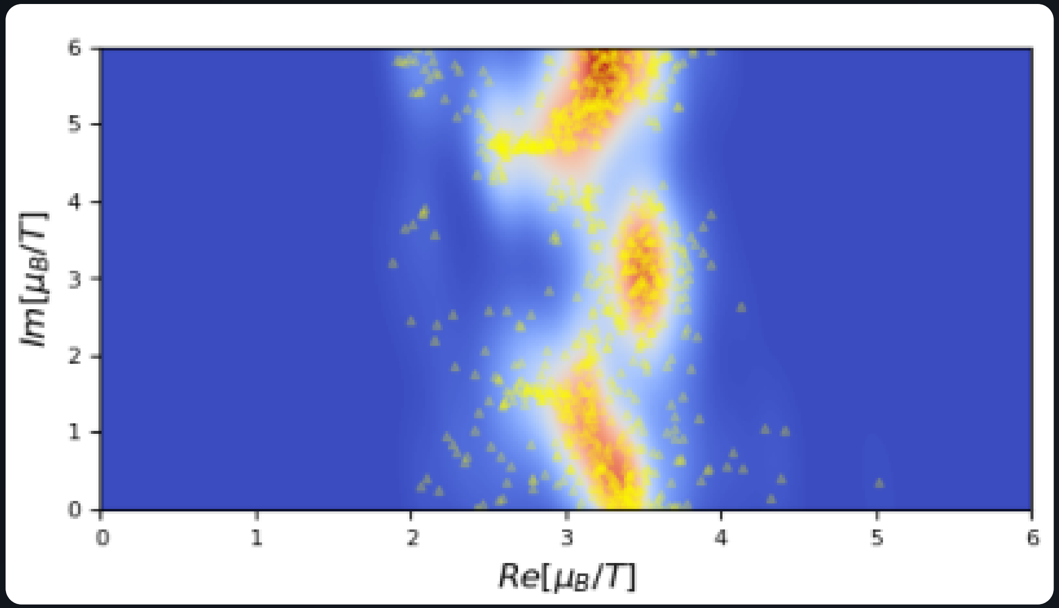
Tracing YLE singularity: systematics



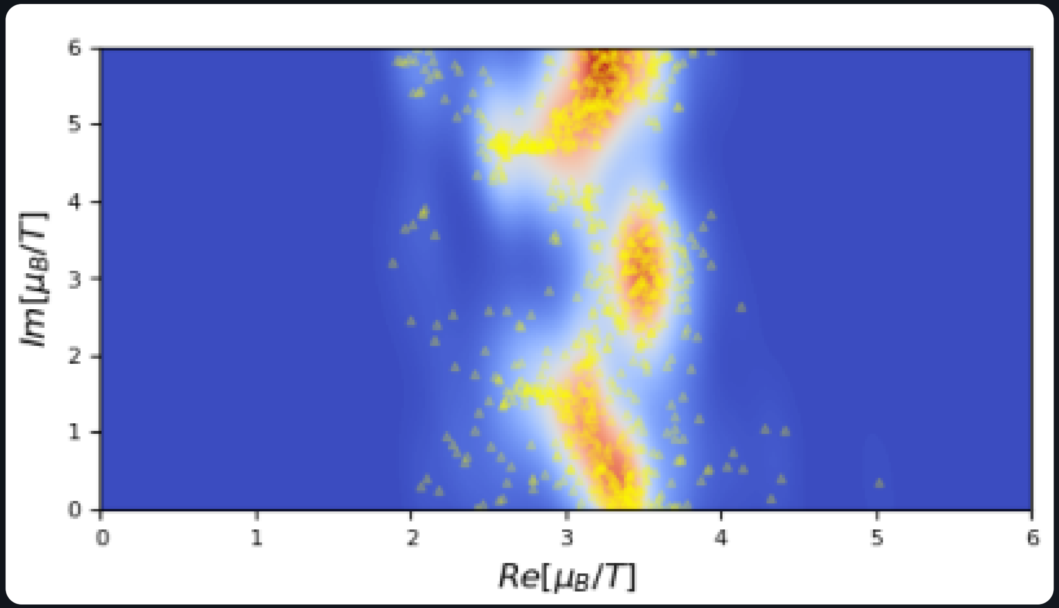
⚠️ Double extrapolation

Interval dependence

Interval dependence

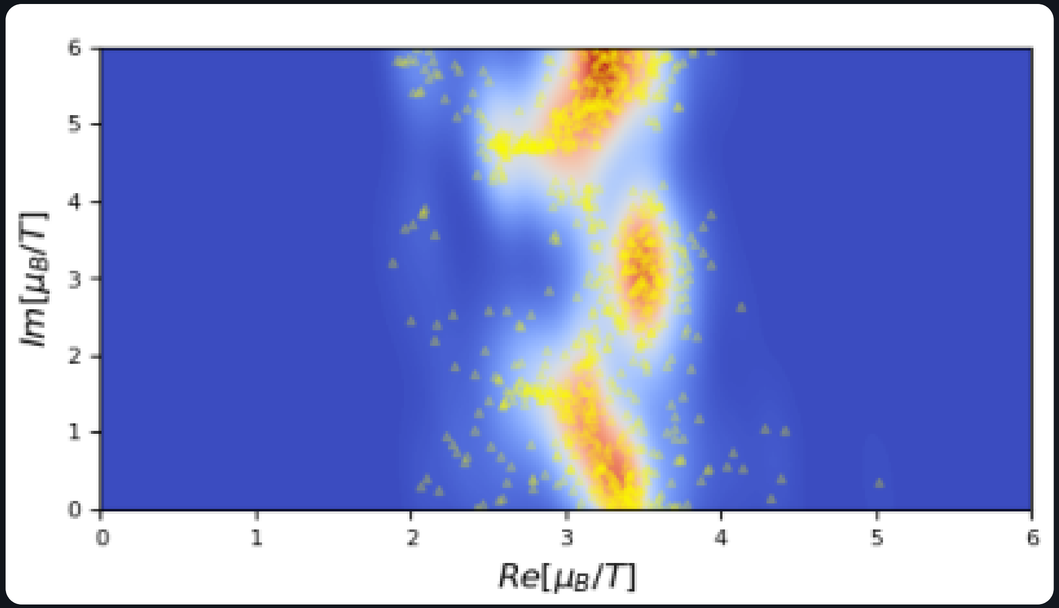


Interval dependence



- Are there exact results?

Interval dependence

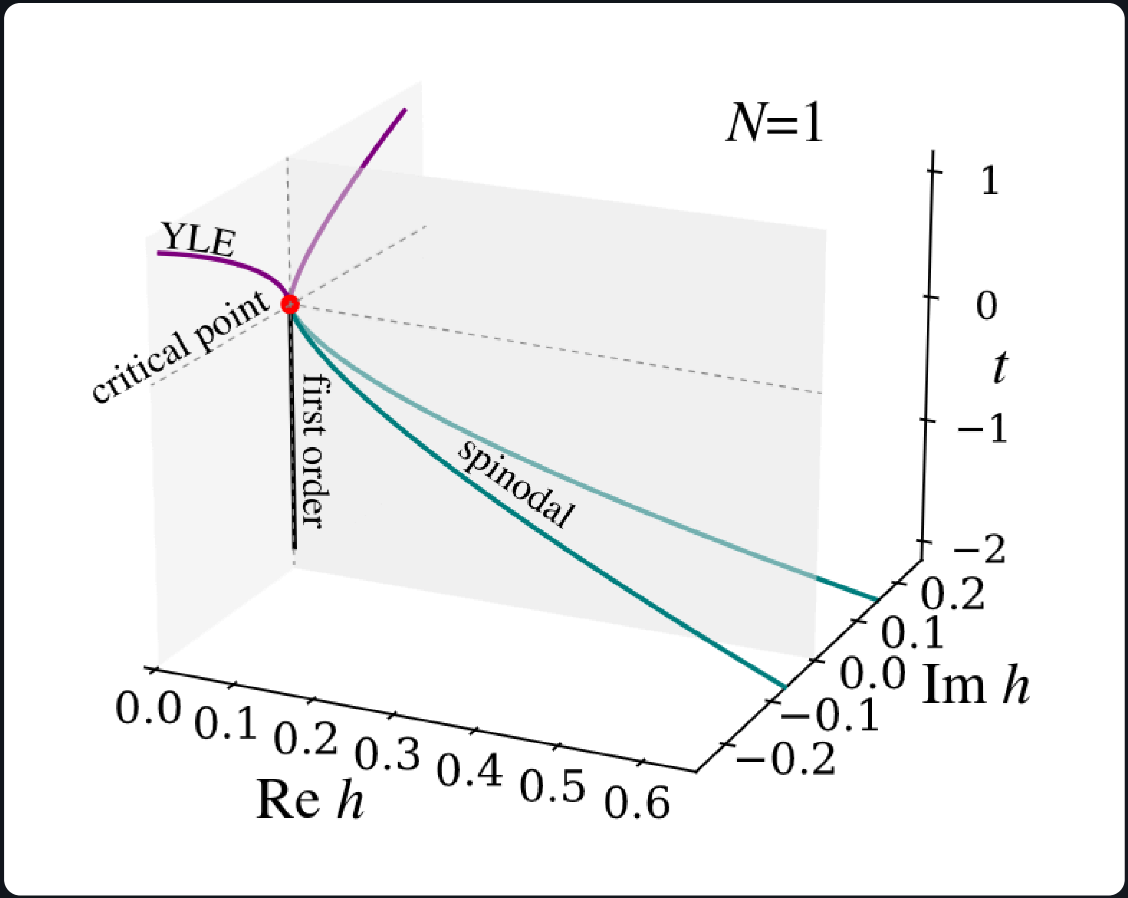


- Are there exact results?
- Interpolating from imaginary line vs Taylor series: "radius of convergence" is due to the singularity closest to the line vs closest to the expansion point

Second LYZ

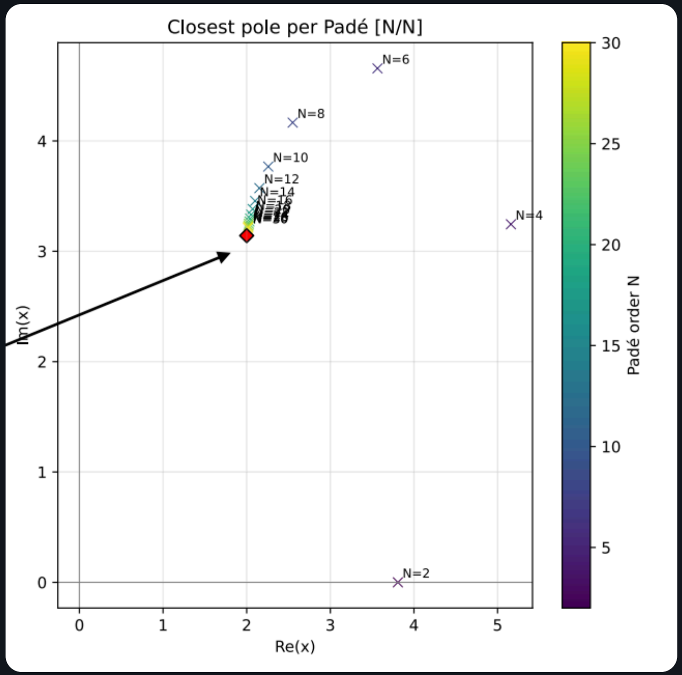
- Any hope to perform finite volume analysis?

Re μ_{YLE} and the relation to the crossover line



Lattice analyses of LYZ gives the parametrization for Re μ_{YLE} and Im μ_{YLE} near the point where the latter quantity vanishes. Does the line $\mu = Re\mu_{YLE}(T)$ follow the "crossover" line?

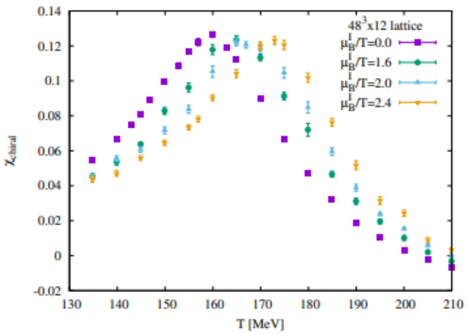
Re μ_{YLE} and the relation to the crossover line

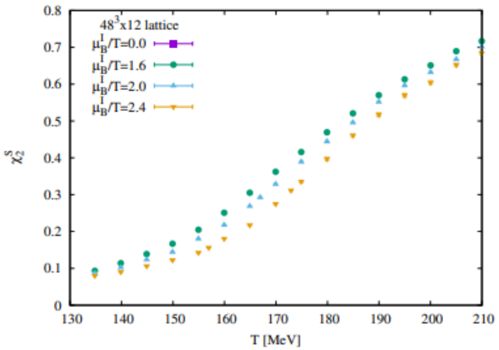


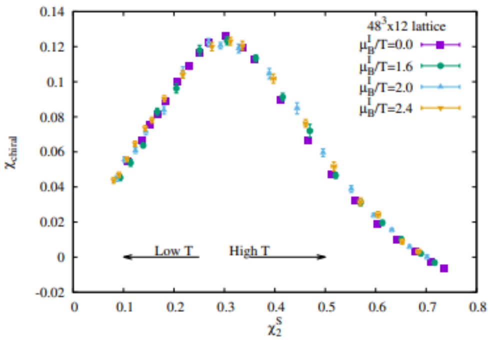
- Current QCD calculations use Pade[3,3], Pade[4,4], Pade[5,5]
- If the shape of the curve depends on the position of the singularity, it may fake the YLE trajectory

Scaling for strange neutral eos?

- Chiral susceptibility peaks at T_c
- The peak shifts with μ_B as expected
- Strange susceptibility $\chi_2^S(T)$ is monotonic in T
- *Collapse plot:* chiral susceptibility as a function of starngeness susceptibility is μ_B independent



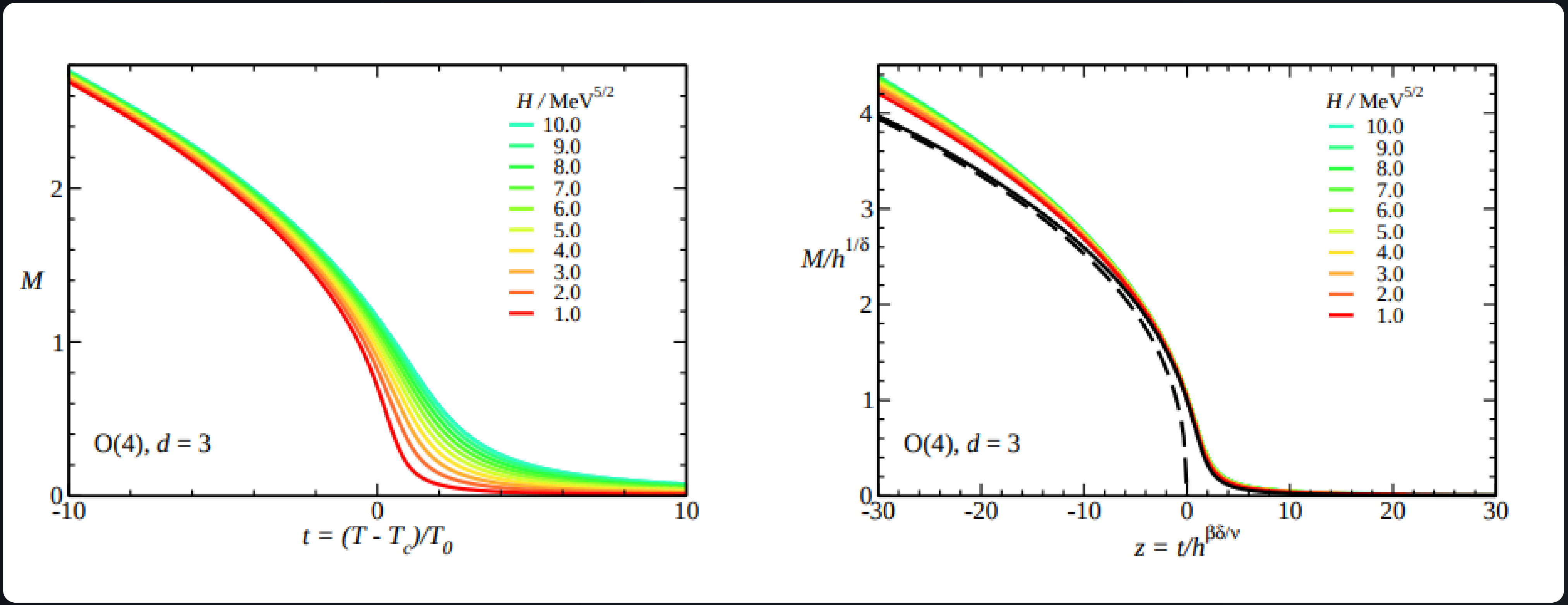




Lession from imaginary μ_B: $\chi_2^S \approx 0.3$ marks T_c for various μ_B

- What is the origin?
- Very large values of z...

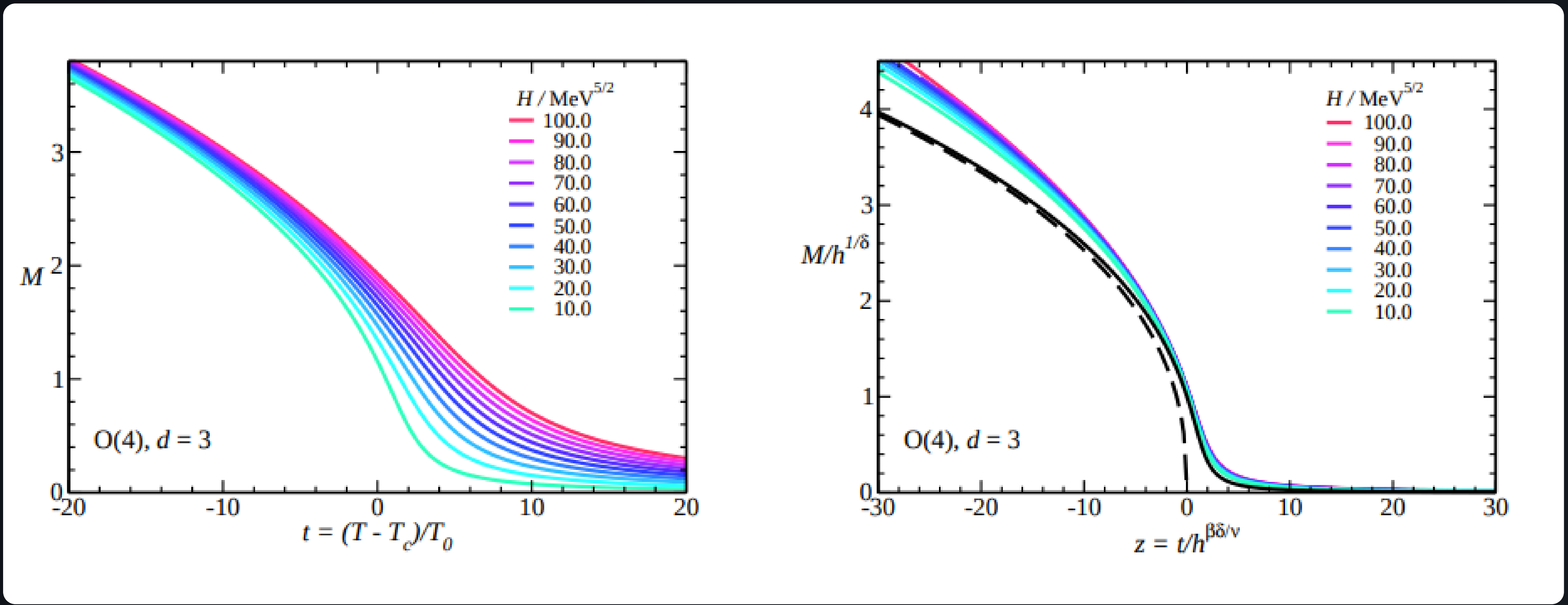
Scaling: Model



- Why not the pion mass...
- Very large values of z...

Braun and Klein

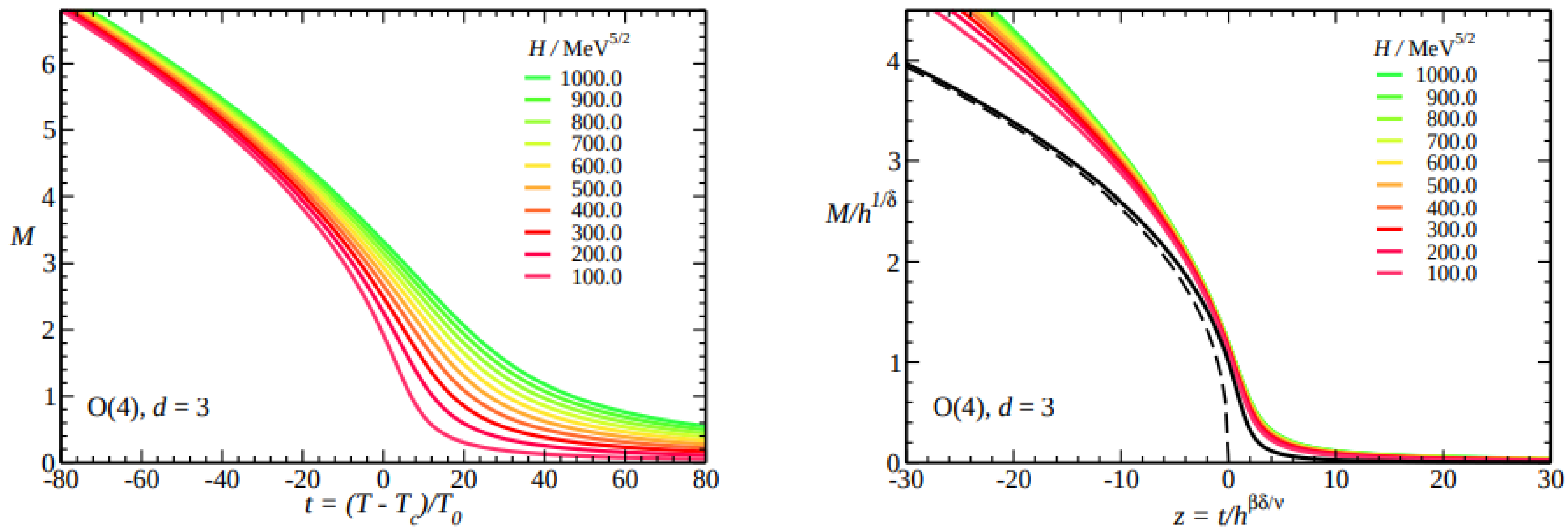
Scaling: Model



- Why not the pion mass...
- Very large values of z...

Braun and Klein

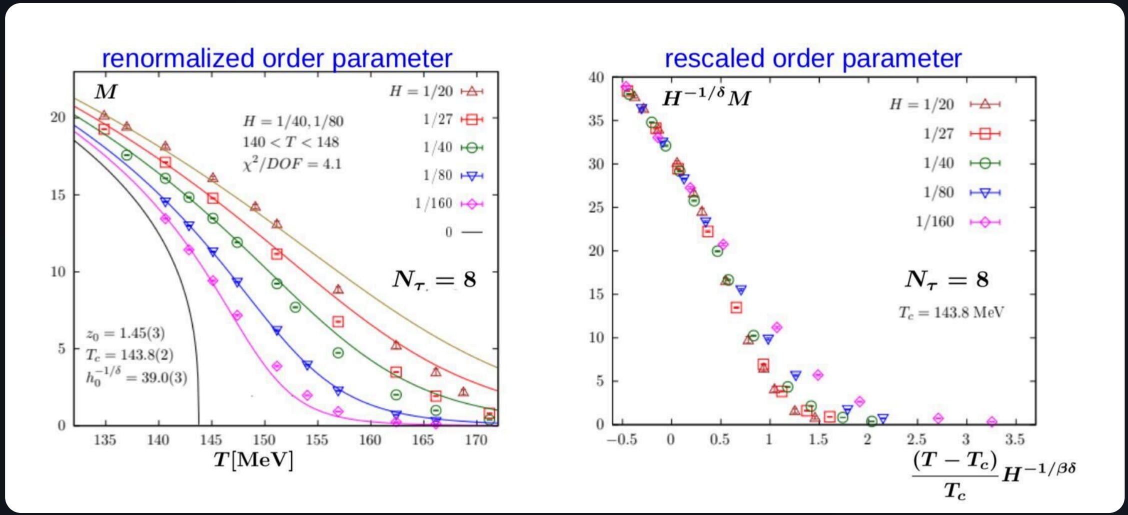
Scaling: Model



- Why not the pion mass...

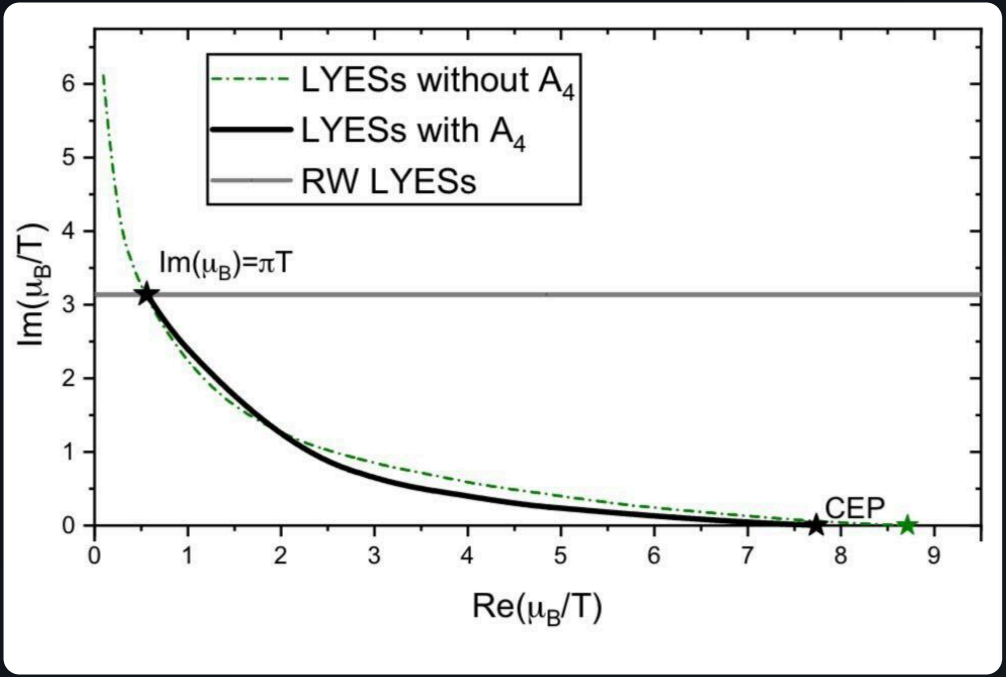
Braun and Klein

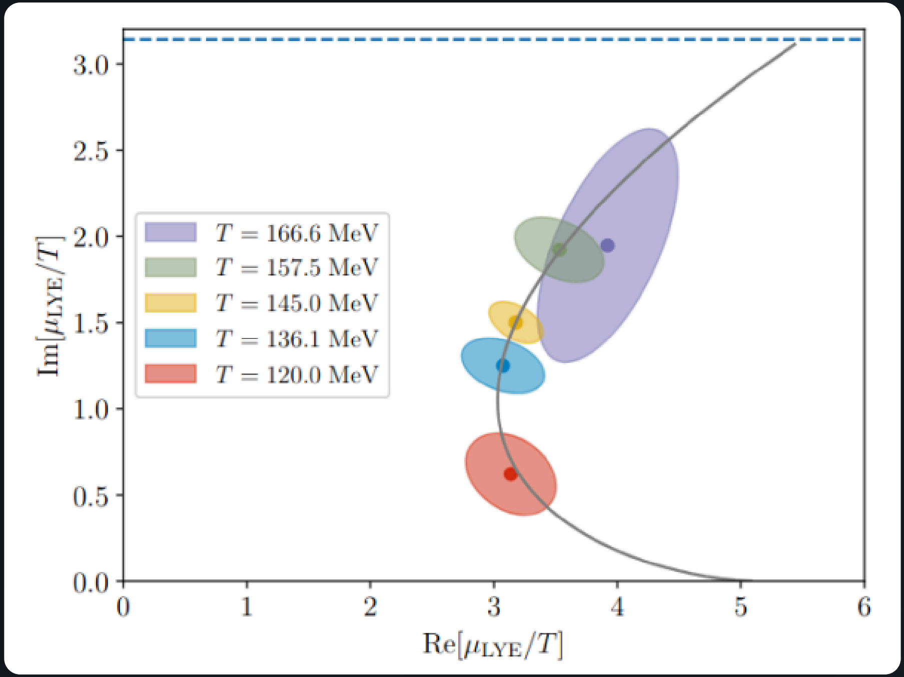
Scaling: LQCD



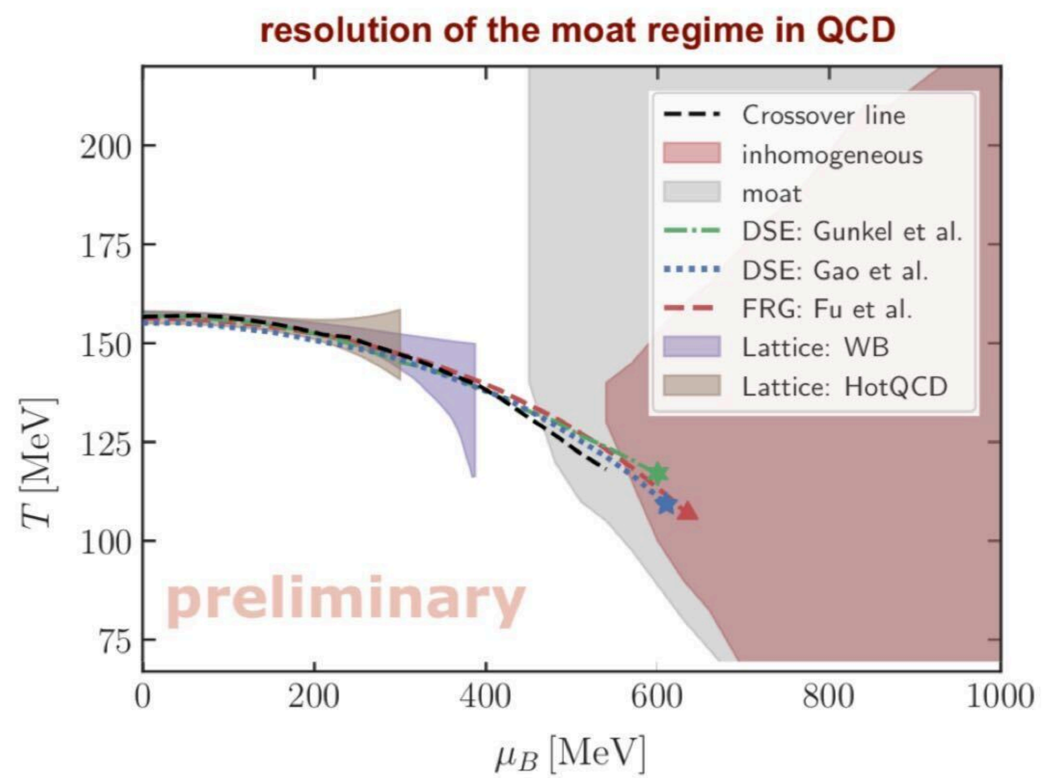
- The scaling is not perfect
- Quite significant deviation rate at small masses
- Why not to use the conventional normalization...

YLE trajectory... DSE, LQCD, what about FRG?





Lifshitz point; corresponding YLE?



JMP, Rennecke, Sattler, in preparation