Collective neutrino oscillations with tensor network methods

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In collaboration with

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Outline

Collective neutrino oscillations in core-collapse supernovae

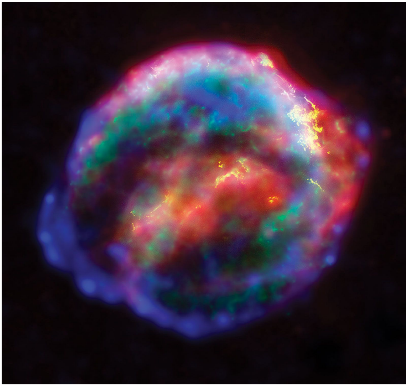
Neutrino many-body problem and governing Hamiltonian under certain approximations

Solution with conventional numerical methods and comparison with the mean-field approximation

Tensor network methods for time-evolution; results

Summary

Neutrinos in core-collapse supernova

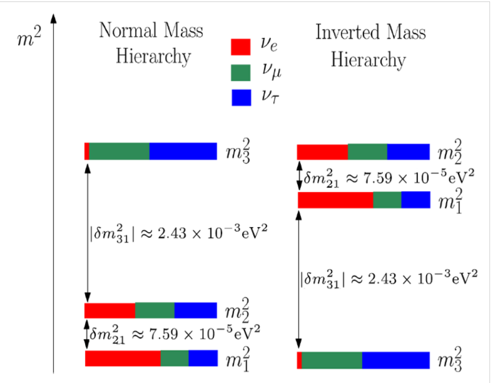


- Energy released in a core-collapse SN: $\Delta E \approx 10^{53}\mathrm{ergs} \approx 10^{59}$ MeV.
- 99% of this energy is carried away by neutrinos and antineutrinos within 10s of seconds.
- This necessitates including the effects of ν-ν interactions.

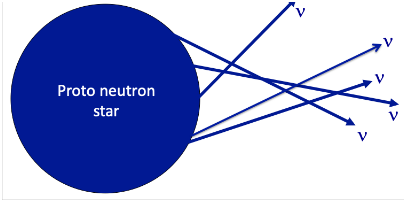
Neutrino flavor oscillations

Neutrinos have mass and vacuum mass eigenstates are different from the flavor states

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle$$



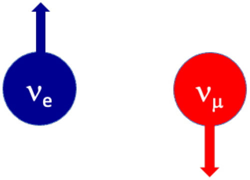
Neutrino Hamiltonian



$$\hat{H} = \sum a^\dagger a + \sum (1 - \cos\phi) a^\dagger a^\dagger a a$$

Neutrino-neutrino interactions lead to novel collective and emergent effects, such as conserved quantities and interesting features in the neutrino energy spectra (spectral “swaps” or “splits”).

Two-flavor settings



$$\hat{J}_+ = a_e^\dagger a_\mu$$
$$\hat{J}_- = a_\mu^\dagger a_e$$
$$\hat{J}_0 = \frac{1}{2}(a_e^\dagger a_e - a_\mu^\dagger a_\mu)$$

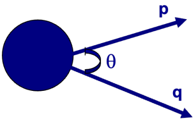
Vacuum oscillations term

$$\begin{aligned}\hat{H}_v &= \frac{m_1^2}{2E} a_1^\dagger a_1 + \frac{m_2^2}{2E} a_2^\dagger a_2 + (\dots)\hat{1} \\ &= \frac{\delta m^2}{4E}\cos 2\theta(-2\hat{J}_0) + \frac{\delta m^2}{4E}\sin 2\theta(\hat{J}_+ + \hat{J}_-) + (\dots)\hat{1}\end{aligned}$$

We neglect matter effects assuming that ν-ν interaction dominates in the region of interest.

Neutrino-neutrino interactions

$$\hat{H}_{\nu\nu} = \frac{\sqrt{2}G_F}{V} \int dp\ dq(1 - \cos\theta_{pq})\vec{J}_p \cdot \vec{J}_q$$



This term makes the physics of a neutrino gas in a core-collapse supernova a genuine many-body problem.

Total Hamiltonian

$$\hat{H} = \int dp \left(\frac{\delta m^2}{2E}\vec{B} \cdot \vec{J}_p\right) + \frac{\sqrt{2}G_F}{V} \int dp\ dq(1 - \cos\theta_{pq})\vec{J}_p \cdot \vec{J}_q$$

$$\vec{B} = (\sin 2\theta, 0, -\cos 2\theta)$$

Single-angle approximation

This problem is "exactly solvable" in single-angle approximation.

$$\hat{H} = \sum_p \omega_p \vec{B} \cdot \vec{J}_p + \mu(r) \vec{J} \cdot \vec{J}$$

$$\mu(r) = \frac{G_F}{\sqrt{2}V} \left[1 - \sqrt{1 - \frac{R_\nu^2}{r^2}} \right]^2$$

$$\omega_p = \frac{\delta m^2}{2p}$$

Bethe-ansatz method gives the exact solution in many-body picture.

Pehlivan *et. al*, PRD (2011)
Cervia *et. al*, PRD (2019)

Entanglement measures

Entropy

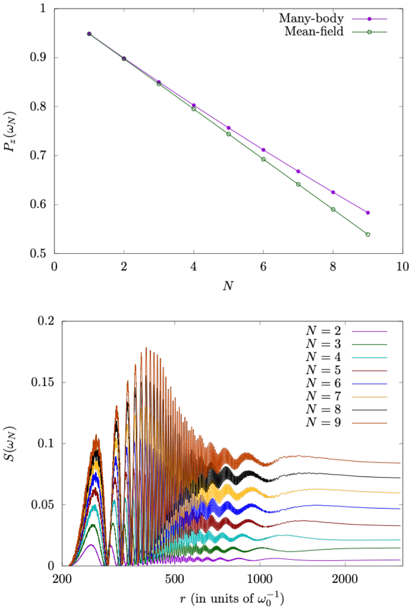
$$\rho = |\psi\rangle \langle\psi|$$
$$\rho_1 = \mathrm{Tr}_2 \rho$$
$$|\psi\rangle = \sum_n \lambda_n |\phi_m^1\rangle |\phi_n^2\rangle$$
$$\rho = \sum_{n,n'} \lambda_n \lambda_{n'} |\phi_n^1\rangle |\phi_n^2\rangle \langle\phi_{n'}^1| \langle\phi_{n'}^2|$$
$$\rho_\alpha = \sum_n |\lambda_n|^2 |\phi_n^\alpha\rangle \langle\phi_n^\alpha| ; \quad \alpha = 1, 2$$
$$S_\alpha = -\mathrm{Tr}(\rho_\alpha \ln \rho_\alpha) = -\sum_n |\lambda_n|^2 \ln |\lambda_n|^2$$

Polarization vector

$$\vec{P}_\omega = 2 \langle\psi| \vec{J}_\omega |\psi\rangle$$
$$\rho = \frac{1}{2}(1 + \vec{\sigma} \cdot \vec{P})$$

$$S = -\frac{1 - |\vec{P}|}{2} \log\left(\frac{1 - |\vec{P}|}{2}\right) - \frac{1 + |\vec{P}|}{2} \log\left(\frac{1 + |\vec{P}|}{2}\right)$$

Results with Bethe-ansatz method



Initial state: $|\nu_e\rangle^{\otimes 9}$

Deviations from the mean-field calculations increase with number of neutrinos N.

Note: Entropy $S = 0$ in mean-field approximation.

Cervia *et. al*, PRD (2019)

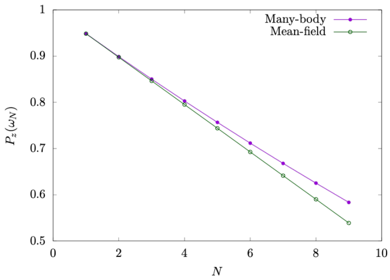
- Bethe ansatz method has numerical instabilities for larger values of N. However, it is very valuable since it leads to the identification of conserved quantities.

- For this reason, the Runge-Kutta technique was explored. This was both to check Bethe ansatz results for N less than 10 and to explore the case with N larger than 10.

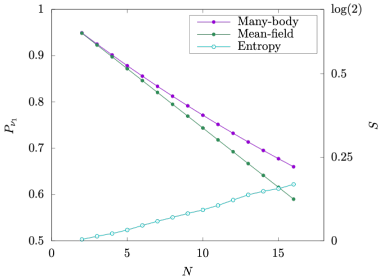
Only a system of 16 neutrinos could be simulated due to memory and time limitations

Results match with the BA ones

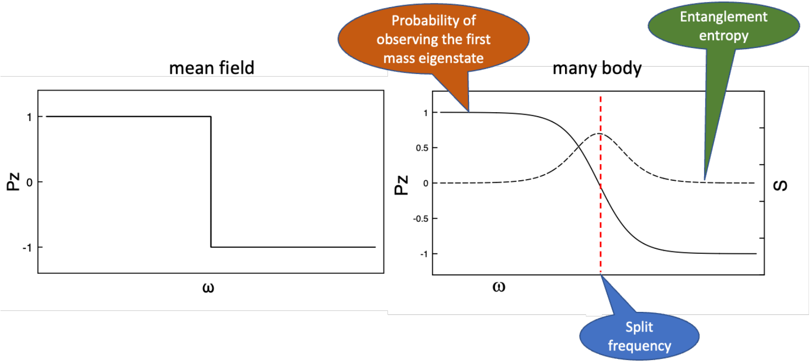
BA Cervia *et al.*, PRD (2019)



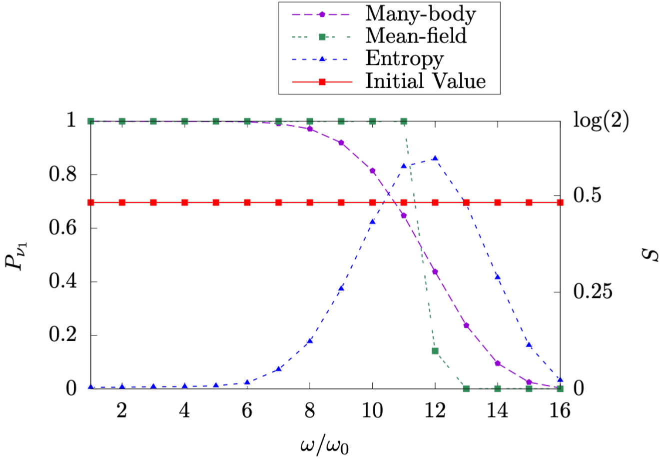
RK4 Patwardhan *et al.*, PRD (2021)



We find that the presence of spectral splits is a good proxy for deviations from the mean-field results.



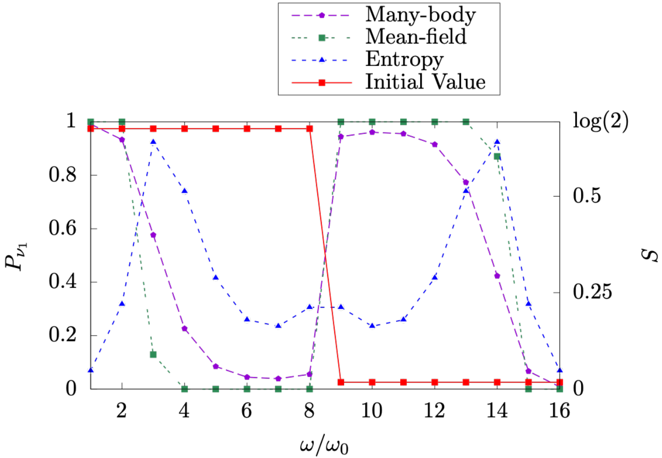
Initial state: $|\nu_e^{\otimes 16}\rangle$



Entropy is maximal in spectral split region.

Patwardhan *et. al*, PRD (2021)

Initial state: $|\nu_e^{\otimes 8}\nu_\mu^{\otimes 8}\rangle$



Patwardhan *et. al*, PRD (2021)

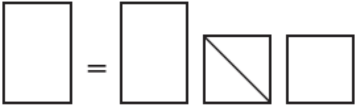
Tensor network methods

Singular value decomposition

SVD of an arbitrary rectangular matrix M of dimension $(N_A \times N_B)$

$$M = USV^\dagger$$

- U is of dimension $(N_A \times \min(N_A, N_B))$ and has left singular vectors $U^\dagger U = I$. Unitary if $N_A \leq N_B$
- S is diagonal of dimension $(\min(N_A, N_B) \times \min(N_A, N_B))$ with non-negative entries called "singular values"
- $V^\dagger$ is of dimension $(\min(N_A, N_B) \times N_B)$ and has right singular vectors $V^\dagger V = I$. Unitary if $N_A \geq N_B$

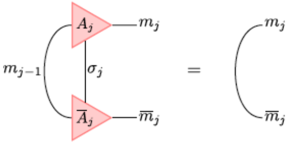


$$|\Psi\rangle = \sum_{\sigma_1 \cdots \sigma_L} c_{\sigma_1 \cdots \sigma_L} |\sigma_1 \cdots \sigma_L\rangle$$

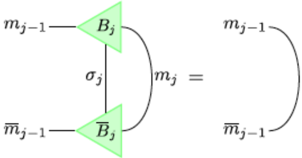
Reshape $c_{\sigma_1 \cdots \sigma_L}$ in rectangular matrix $\psi_{\sigma_1(\sigma_2 \ldots \sigma_L)} = c_{\sigma_1 \cdots \sigma_L}$

$$|\Psi\rangle = \sum_{\sigma_1 \cdots \sigma_L, m_0 \cdots m_L} M^{\sigma_1}_{1;m_0,m_1} \cdots M^{\sigma_L}_{L;m_{L-1},m_L} |\sigma_1 \cdots \sigma_L\rangle$$

Left normalized

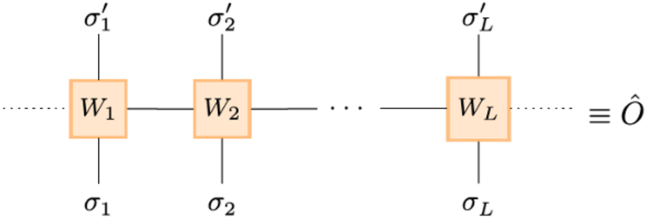


Right normalized



Matrix product operator

$$\hat{O} = \sum_{\sigma_1 \cdots \sigma_L, \sigma'_1 \cdots \sigma'_L} c_{\sigma_1 \cdots \sigma_L, \sigma'_1 \cdots \sigma'_L} \left|\sigma_1 \cdots \sigma_L\right\rangle \left\langle \sigma'_1 \cdots \sigma'_L\right|$$



$$\hat{O} = \sum_{\sigma_1 \cdots \sigma_L, \sigma'_1 \cdots \sigma'_L, w_0 \cdots w_L} W^{\sigma_1 \sigma'_1}_{1; w_0, w_1} \cdots W^{\sigma_L \sigma'_L}_{L; w_{L-1}, w_L} \left|\sigma_1 \cdots \sigma_L\right\rangle \left\langle \sigma'_1 \cdots \sigma'_L\right|$$

U. Schollwöck, Ann. Phys. (2011)

S. Paeckel *et al.* Ann. Phys. (2019)

Time-evolution methods for MPS

TEBD

Approximate $\hat{U} = e^{-i\delta\hat{H}}$ and apply on MPS $|\psi(t)\rangle$ to estimate $|\psi(t+\delta)\rangle$

- Trotter-Suzuki decomposition
- Short-ranged Hamiltonian
- Evolution unitary up to inherent Trotter error but energy is not typically conserved

S. Paeckel *et al.* Ann. Phys. (2019)

MPO $W^{I,II}$

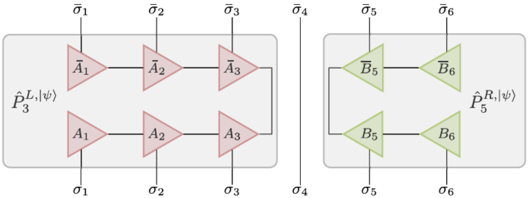
- suited to construct an efficient representation of $\hat{U}(\delta)$ as a matrix-product operator
- Can deal with long-range interactions
- smaller MPOs as compared to TEBD
- evolution not unitary

Time dependent variational principle

$$i\frac{d}{dt}|\psi(t)\rangle = \hat{P}_{T_{|\psi\rangle}} H(t)|\psi(t)\rangle$$

$$\hat{P}_{T_{|\psi\rangle}} = \sum_{j=1}^{L} \hat{P}_{j-1}^{L,|\psi\rangle} \otimes \hat{\mathbf{1}}_j \otimes \hat{P}_{j+1}^{R,|\psi\rangle} - \sum_{j=1}^{L} \hat{P}_{j}^{L,|\psi\rangle} \otimes \hat{P}_{j+1}^{R,|\psi\rangle}$$

$\hat{P}_j^{L,|\psi\rangle}$ projects on left sites



S. Paeckel *et al.* Ann. Phys. (2019)

Time dependent variational principle (continued)

$$i\frac{d}{dt}\psi_C(j) = H_C(j)\psi_C(j)$$

- Suitable for long-range interaction
- Two-site TDVP is necessary for entanglement generation

Errors

- Projection error (zero if the MPS has maximal bond dimension)
- Finite time step error $O(\delta^3)$. Smaller bond dimension, larger time step error
- Truncation error: SVD to split time-evolved two-site tensor in two separate tensors

Changing the time step size affects these errors differently

Global subspace expansion (GSE)

Krylov expansion of time-evolved MPS

$$|\psi(t+\Delta t)\rangle = e^{-i\hat{H}\Delta t}|\psi(t)\rangle \approx \sum_{l=0}^{k-1} \frac{(-i\Delta t)^l}{l!}\hat{H}^l |\psi(t)\rangle$$

$$\mathcal{K}_k(\hat{H}, |\psi\rangle) = \mathrm{span}\{\hat{H}, \hat{H}|\psi\rangle, \ldots, \hat{H}^{k-1}|\psi\rangle\}$$

- Expand the basis at each bond with enlarged bond dimension
- Projection errors are significantly reduced
- Larger time step
- Can even work with one-site TDVP

M. Yang and S.R. White, PRB (2020)

Invariants

- Total J_z
- $h_\omega = -J^z_\omega + 2\mu \sum_{\omega'(\neq\omega)} \frac{\vec{J}_\omega \cdot \vec{J}_{\omega'}}{\omega-\omega'}$

Ehrenfest's theorem

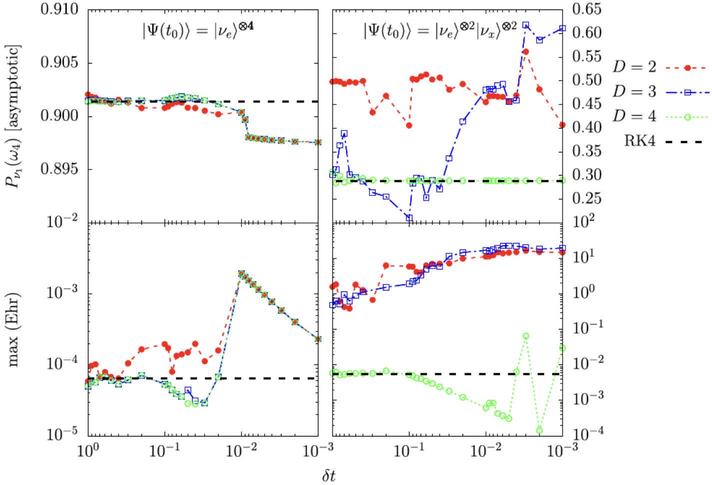
$$\frac{d\langle A\rangle}{dt} = -i\langle [A,H]\rangle + \left\langle \frac{\partial A}{\partial t}\right\rangle$$

$$\mathrm{Ehr}_\omega[\psi(t)] \equiv \frac{1}{2}\frac{dP_z(\omega)}{dt} - 2\mu\frac{d}{dt}\left\langle \sum_{\omega'(\neq\omega)} \frac{\vec{J}_\omega \cdot \vec{J}_{\omega'}}{\omega-\omega'}\right\rangle$$

$$\max\ \mathrm{Ehr}[\psi] \equiv \max_t \max_\omega |\mathrm{Ehr}_\omega[\psi(t)]|$$

Cervia *et al*, PRD (2022)

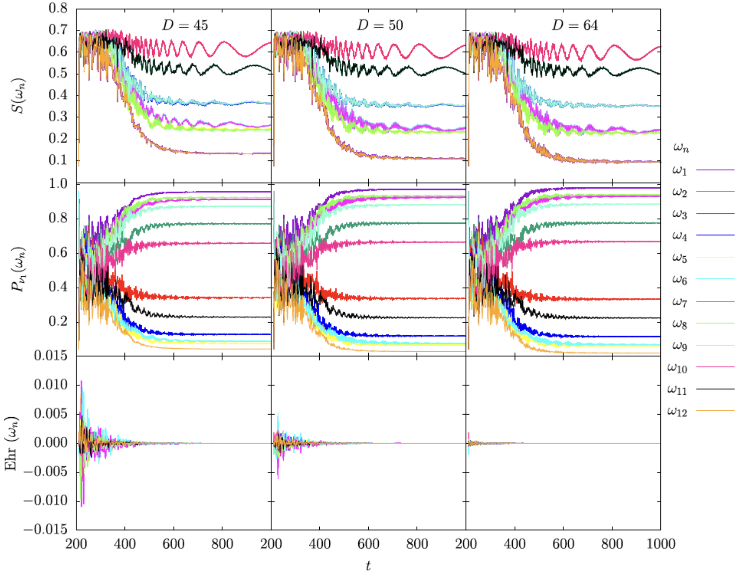
Step size and bond dimension



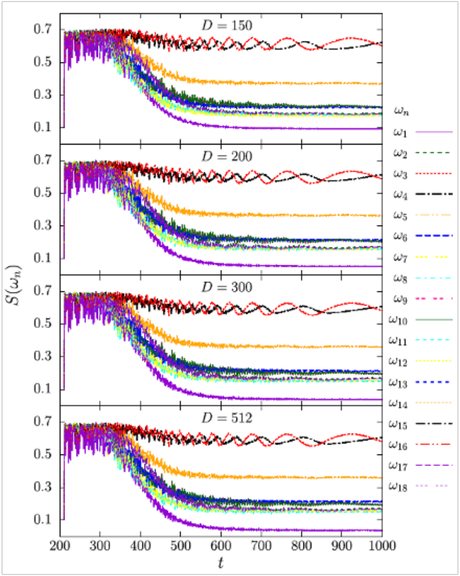
Reducing time step does not always reduce errors.

Cervia *et al*, PRD (2022)

Results for $N = 12$ with mixed initial state



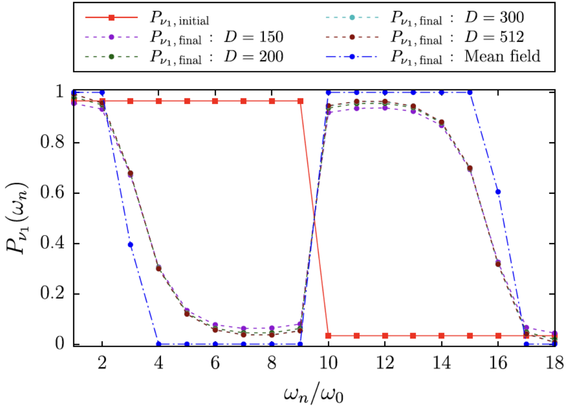
Results for $N = 18$ with mixed initial state



Initial state:
$|\nu_e\rangle^{\otimes 9} |\nu_\mu\rangle^{\otimes 9}$

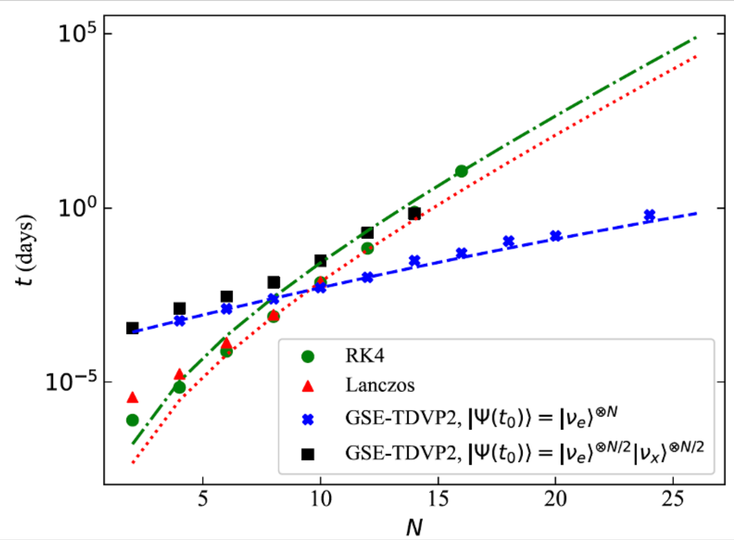
Reducing the bond dimension by $\sim 60\%$ yields qualitatively similar results.

Spectral split wrt bond dimension



- Reducing the bond dimension does not affect the position and width of spectral split
- Range of P_{ν_1} decreases with decrease in bond dimension indicating the overestimation of entanglement

Scaling of different methods



GSE-TDVP2 scales polynomially wrt N for all ν_e initial states

Summary

- Collective neutrino oscillations play a crucial role in supernovae physics and astrophysical nucleosynthesis.
- Mean-field approximations revealed many interesting features, but the results deviate from the many-body treatment.
- The conventional numerical methods like Range-Kutta has limits on the number of neutrinos in a systems
- The tensor network methods are helpful in studying ~ 40 neutrinos. But the computational complexities depend on the initial state.