Three spinning particles and some takeaways

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Overview

- Part 1: three neutrons in a finite volume

Three relativistic neutrons in a finite volume

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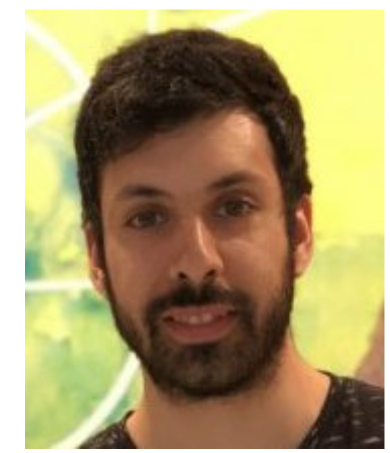


arXiv:2303.10219

- Part 2: takeaways from the workshop—a personal summary

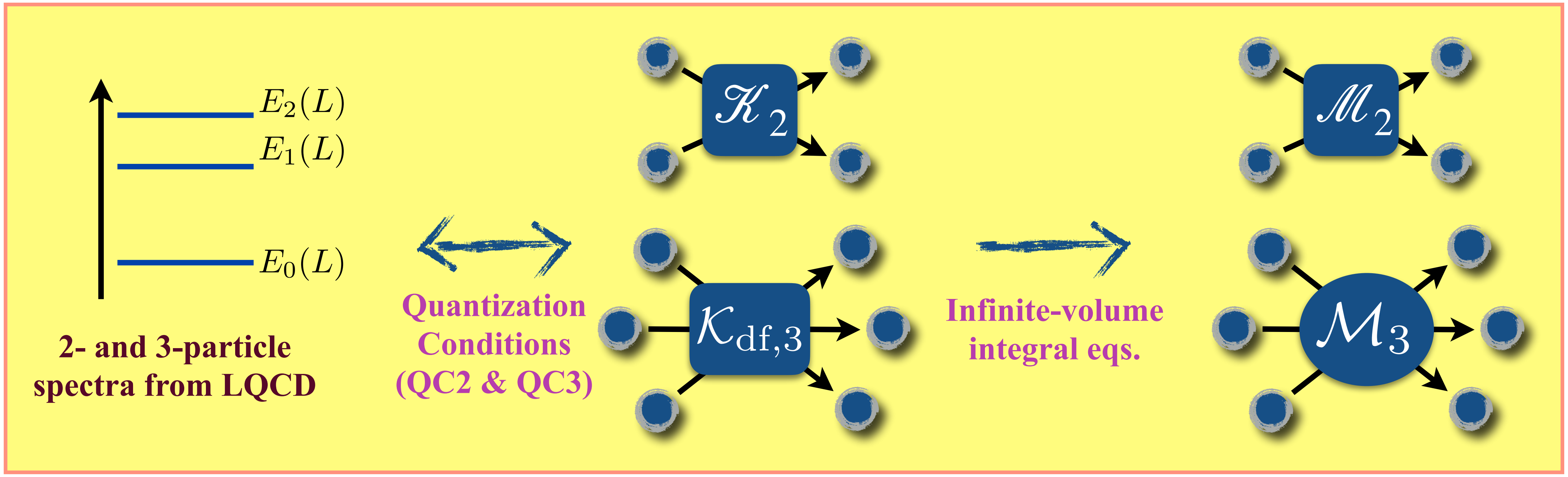
Outline: Part 1

- Motivation: why study 3 (relativistic) neutrons in a box
- Background: 3-particle formalisms
- Overview of Relativistic Field Theory (RFT) approach
- New issues with spin ½ fermions
- Final result
- Threshold expansion for $\mathcal{K}_{\mathrm{df},3}$
- Ongoing/future work

Motivation

- Determine 3 neutron interaction from first principles using LQCD
  - Important for neutron star EoS, heavy nuclei, …
- Incorporating spin into 3-particle formalism in a simple setting
  - Extensions to 3 nucleon interactions in isosymmetric QCD should be straightforward
  - Important step on the way to studying Roper: $N(1440) \to \pi N, \pi\pi N$
- Want relativistic approach since, for heavier than physical pions, the first inelastic threshold (where the formalism breaks down) can occur for relativistic nucleons
  - And for future applications such as the Roper, relativistic effects needed

3-particle formalism

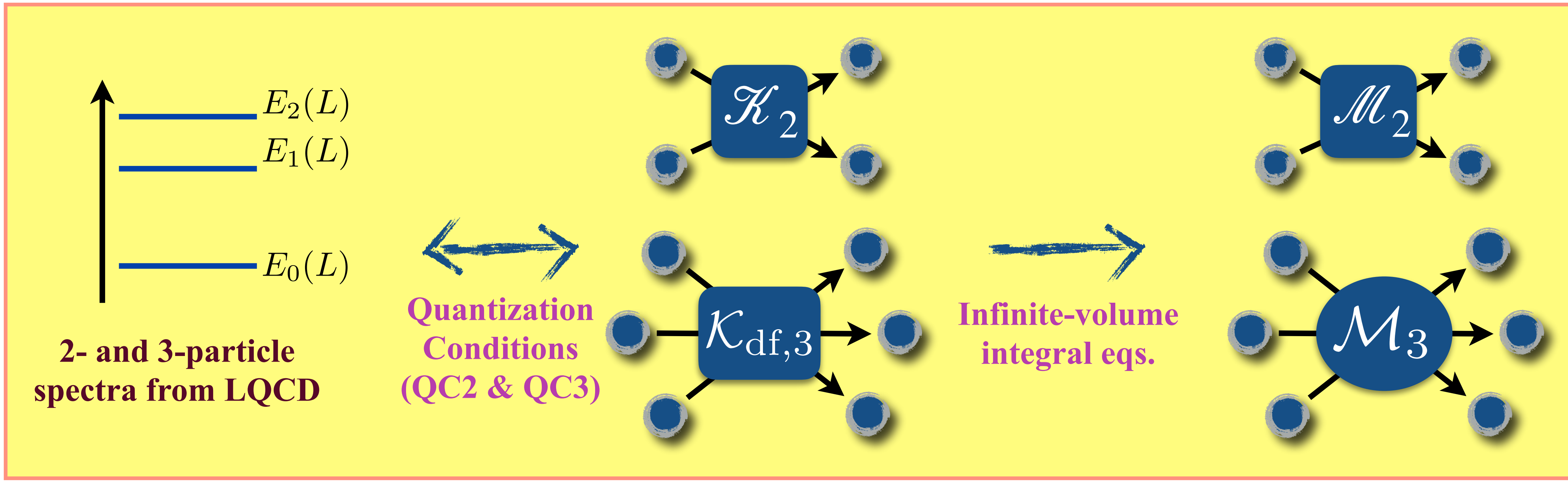


- $\mathcal{K}_{\mathrm{df},3}$ is a real, infinite-volume (but scheme-dependent) K matrix that is smooth aside from possible 3-particle resonance poles; integral equations ensure unitarity of $\mathcal{M}_3$

- Parametrize $\mathcal{K}_2$ and $\mathcal{K}_{\mathrm{df},3}$ in an "effective-range-like expansion" about threshold and determine parameters by fitting spectrum

- With multiple frames and waves, there is not a 1-to-1 relation between energies and phase shifts, so a global fit is required

3-particle formalism



- Formalism exists for arbitrary choices of spinless particles [References in backup slides]
  - QC3 implemented for 3 identical scalars ($3\pi^+$, $3K^+$), $3\pi(I=1)$, $\pi^+\pi^+K^+$, $K^+K^+\pi^+$, and ϕ^4 theory [Talks by Döring, Romero-López & Rusetsky]
  - Integral equations solved for identical scalars including two-particle and three-particle bound states and resonances [Talks by Dawid, Döring & Islam]
- Three approaches used in derivation: generic Relativistic Field Theory (RFT) [Romero-López], (relativized) NREFT [Rusetsky], and Finite-volume Unitarity (FVU) [Döring]
  - Formally equivalent up to technical details
  - We use the RFT approach

Overview of RFT approach

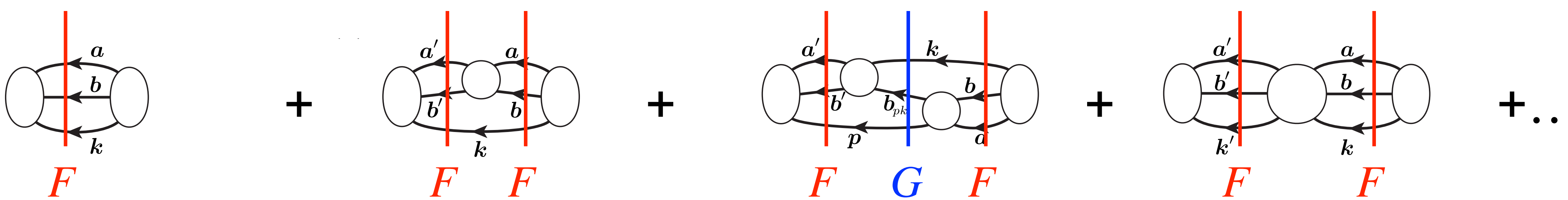
[Hansen & SRS]

- All (symmetric) RFT QC3s have the same form; all that varies are the matrix indices

$$\det\left[F_3^{-1}(E,\mathbf{P},L)+\mathcal{K}_{\mathrm{df},3}(E^*)\right]=0$$

$$F_3=\frac{F}{3}-F\frac{1}{\overline{\mathcal{K}}_{2,L}^{-1}+F+G}F$$

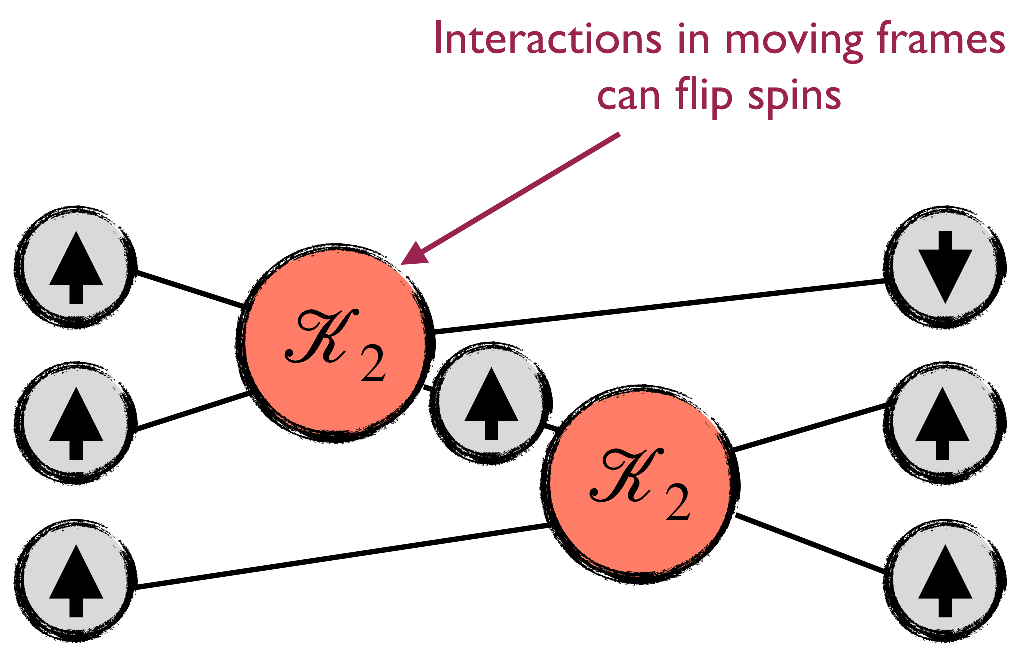
- Derived by determining power-law volume dependence of finite-volume 3-particle correlation functions to all orders in a skeleton expansion in a generic relativistic EFT



  - Volume dependence arises from 3-particle cuts

New features for spin ½

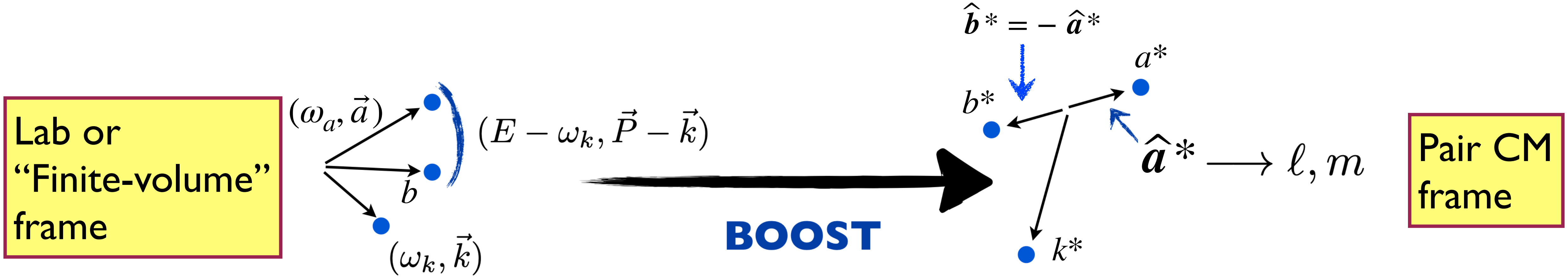
- Extra spin degree of freedom—gives extra matrix indices
- Total spin is conserved in NR limit; no longer true in relativistic system, due to Wigner rotations induced by boosts



- Antisymmetry of states due to Fermi statistics
- Inclusion of spin is much more complicated than for 2-particle QC [Briceño]

3-particle coordinates

- 3 scalars with total momentum $(E, \overrightarrow{P})$



[$\vec{k}$ of the spectator] x [ℓm of the "pair" or "dimer"]

- In finite volume, $\vec{k} = (2\pi/L)\mathbb{Z}^3 \Rightarrow$ matrix indices $\{k, \ell, m\}$
- What changes when include spin?

Describing spin ½ states

- Standard moving spin states: boost from CMF; corresponds to spinor $u(\boldsymbol{p}, s)$

$$|\boldsymbol{p}, sm_s\rangle = U(L(\boldsymbol{\beta}_p))\,|\boldsymbol{0}, sm_s\rangle \qquad\qquad L(\boldsymbol{\beta}_p) = R(\theta_p, \hat{\boldsymbol{n}}_p)\cdot L(\beta_p\hat{\boldsymbol{z}})\cdot R(\theta_p, \hat{\boldsymbol{n}}_p)^{-1}$$

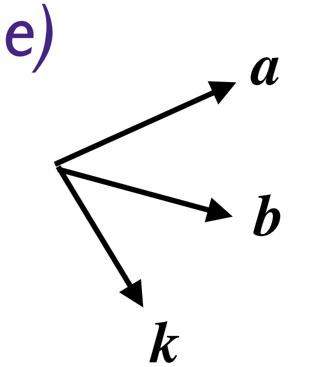
$$\equiv |\boldsymbol{p}, m_s(\boldsymbol{p})\rangle$$ for spin ½

  - Key property: rotates as nonrelativistic 2-component spinor

$$U(R)\,|\boldsymbol{p}, s\,m_s\rangle = |R\boldsymbol{p}, s\,m_s'\rangle\,\mathcal{D}^{(s)}_{m_s', m_s}(R)$$

- Lab-frame description of 3 spin-½ particles (*lab-axis frame)*

$$|\boldsymbol{k}, m_s(\boldsymbol{k})\rangle \otimes |\boldsymbol{a}, m_s(\boldsymbol{a})\rangle \otimes |\boldsymbol{b}, m_s(\boldsymbol{b})\rangle$$



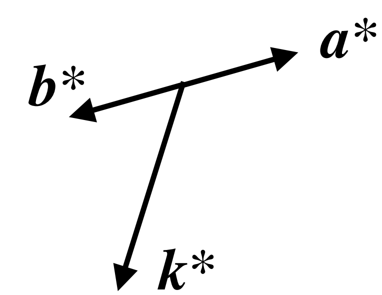
  - Natural choice for $\mathscr{K}_{\mathrm{df},3}$
  - Collect spin indices into vector: $\boldsymbol{m}_s = (m_s(\boldsymbol{k}), m_s(\boldsymbol{a}), m_s(\boldsymbol{b}))$

Describing spin ½ states

- To combine spins of pair with orbital angular momentum ℓ, need a & b in pair CMF

$$|\boldsymbol{a}^*, m_s(\boldsymbol{a}^*)\rangle \equiv U(L(\boldsymbol{\beta}_{a^*}))\,|\boldsymbol{0}, m_s\rangle \quad \text{and} \quad |\boldsymbol{b}^*, m_s(\boldsymbol{b}^*)\rangle \equiv U(L(\boldsymbol{\beta}_{b^*}))\,|\boldsymbol{0}, m_s\rangle$$

  - Thus introduce *dimer-axis frame* spin indices



$$|\boldsymbol{k}, m_s(\boldsymbol{k})\rangle \otimes |\boldsymbol{a}^*, m_s(\boldsymbol{a}^*)\rangle \otimes |\boldsymbol{b}^*, m_s(\boldsymbol{b}^*)\rangle$$

  - Natural choice for $\mathscr{K}_2$, and for QC3
  - Collect spin indices into vector: $\boldsymbol{m}_s^* = (m_s(\boldsymbol{k}), m_s(\boldsymbol{a}^*), m_s(\boldsymbol{b}^*))$

- Relation between spin components involves Wigner rotations, e.g.

$$\begin{aligned}
|\boldsymbol{a}^*, m_s(\boldsymbol{a})\rangle &\equiv U(L(-\boldsymbol{\beta}_{P-k}))\,|\boldsymbol{a}, m_s(\boldsymbol{a})\rangle \\
&= U(L(-\boldsymbol{\beta}_{P-k}))U(L(\boldsymbol{\beta}_a))\,|\boldsymbol{0}, m_s\rangle \\
&= U(L(\boldsymbol{\beta}_{a^*}))U(R_a))\,|\boldsymbol{0}, m_s\rangle \\
&= |\boldsymbol{a}^*, m_s'(\boldsymbol{a}^*)\rangle \mathscr{D}(R_a)_{m_s' m_s}
\end{aligned}$$

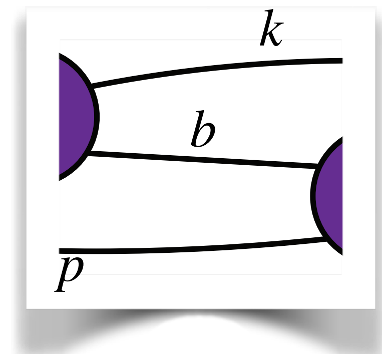
Wigner rotation

Spin ½ Wigner D-matrix representing Wigner rotation

Impact on G

$$\det\left[F_3^{-1}(E,\mathbf{P},L)+\mathscr{K}_{\mathrm{df},3}(E^*)\right]=0$$

$$F_3=\frac{F}{3}-F\frac{1}{\overline{\mathscr{K}}_{2,L}^{-1}+F+\boxed{G}}F$$



- Arises when spectator is switched
- Spin components conserved in lab frame

$$\Delta_{L,\alpha\beta}(b)=i\frac{(\not{b}+m)_{\alpha\beta}}{b^2-m^2+i\epsilon}+R_{L,\alpha\beta}(b)$$

$$(\not{b}+m)_{\alpha\beta}\Big|_{b^0=\omega_b}=\sum_{r=1}^{2}u_\alpha^r(\boldsymbol{b})\overline{u}_\beta^r(\boldsymbol{b})$$

Fully dressed propagator

Nonsingular residue

- Leads to Wigner D-matrices when express in *dimer-axis frame*

Spin indices match in lab frame

$$[\mathbf{G}^{\mathsf{lab}}]_{p\ell'm'\boldsymbol{m}_s';k\ell m\boldsymbol{m}_s}(E,\boldsymbol{P},L)\equiv-\delta_{m_s'(\boldsymbol{p}),m_s(\boldsymbol{p})}\delta_{m_s'(\boldsymbol{k}),m_s(\boldsymbol{k})}\delta_{m_s'(\boldsymbol{b}),m_s(\boldsymbol{b})}\times\frac{i}{4\omega_p\omega_k L^6}\frac{H(\boldsymbol{p})H(\boldsymbol{k})}{b^2-m^2}\frac{4\pi\mathcal{Y}_{\ell'm'}(\boldsymbol{k}_p^*)\mathcal{Y}_{\ell m}^*(\boldsymbol{p}_k^*)}{q_{2,p}^{*\ell'}\,q_{2,k}^{*\ell}}$$

Sign from Fermi Statistics

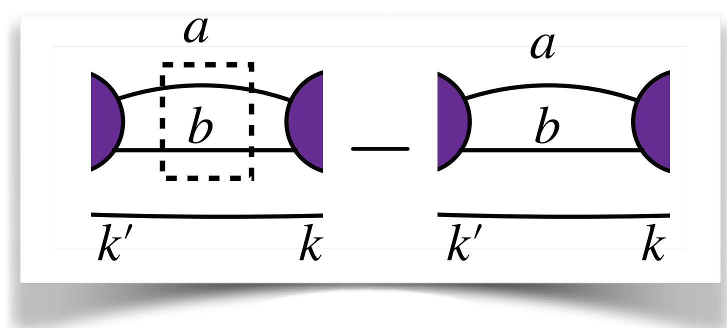
$$\mathbf{G}_{p\ell'm'\boldsymbol{m}_s'^*;k\ell m\boldsymbol{m}_s^*}=\mathcal{D}^{(p,k)\dagger}_{\boldsymbol{m}_s'^*\boldsymbol{m}_s''}\,\mathbf{G}^{\mathsf{lab}}_{p\ell'm'\boldsymbol{m}_s'';k\ell m\boldsymbol{m}_s'''}\,\mathcal{D}^{(k,p)}_{\boldsymbol{m}_s'''\boldsymbol{m}_s^*}$$

Product of two Wigner D-matrices (one for each Member of pair)

Impact on F

$$\det\left[F_3^{-1}(E,\mathbf{P},L) + \mathscr{K}_{\mathrm{df},3}(E^*)\right] = 0$$

$$F_3 = \frac{F}{3} - F\frac{1}{\overline{\mathscr{K}}_{2,L}^{-1} + F + G}F$$



Spin indices match
in lab frame

$$[\mathbf{F}^{\mathsf{lab}}]_{k'\ell' m' \boldsymbol{m}'_s; k\ell m \boldsymbol{m}_s}(E,\boldsymbol{P},L) \equiv \delta_{\boldsymbol{m}'_s \boldsymbol{m}_s}\delta_{k'k}\frac{iH(\boldsymbol{k})}{2\omega_k L^3}\frac{1}{2}\left[\frac{1}{L^3}\sum_{\boldsymbol{a}} - \mathrm{p.v.}\int_{\boldsymbol{a}}\right]$$
$$\times\frac{4\pi\mathcal{Y}_{\ell' m'}(\boldsymbol{a}_k^*)\mathcal{Y}^*_{\ell m}(\boldsymbol{a}_k^*)}{2\omega_a(b^2-m^2)}\frac{1}{(q^*_{2,k})^{\ell+\ell'}}$$

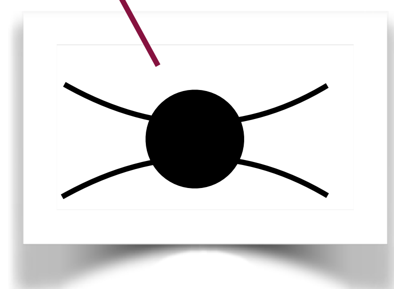
$$\mathbf{F} = \mathbf{F}^{\mathsf{lab}}$$

Wigner D-matrices cancel

Impact on $\mathscr{K}_2$

$$\det\left[F_3^{-1}(E,\mathbf{P},L)+\mathscr{K}_{\mathrm{df},3}(E^*)\right]=0$$

$$F_3=\frac{F}{3}-F\frac{1}{\boxed{\overline{\mathscr{K}}_{2,L}^{-1}}+F+G}F$$



- Naturally expressed in *dimer-axis frame*

$$[\mathbf{K}_2]_{k'\ell'm'\boldsymbol{m}_s'^*;k\ell m\boldsymbol{m}_s^*}(E,\boldsymbol{P})=i\delta_{k'k}2\omega_k L^3\mathcal{K}_2^{(\ell'm'\boldsymbol{m}_s'^*,\ell m\boldsymbol{m}_s^*)}(E^*_{2,k})$$

$$\mathcal{K}_2^{(\ell'm'\boldsymbol{m}_s'^*,\ell m\boldsymbol{m}_s^*)}(E^*_{2,k})=\delta_{m_s'(\boldsymbol{k})m_s(\boldsymbol{k})}\ \mathcal{K}_2^{[\ell'm'm_s'(\boldsymbol{a}'^*)m_s'(\boldsymbol{b}'^*)],\,[\ell m m_s(\boldsymbol{a}^*)m_s(\boldsymbol{b}^*)]}(E^*_{2,k})$$

- Can convert $\mathscr{K}_2$ indices to total dimer spin: $\{\ell m s\mu_s\}$
  - Antisymmetry $\Rightarrow$ $s=0$ and $s=1$ have opposite parities and do not mix
- And then to total dimer angular momentum: $\{j\mu_j\}$
  - $s=0 \;\Rightarrow\;$ even $\ell=j \;\Rightarrow$ single channel described by phase shift
  - $s=1 \;\Rightarrow\;$ odd $\ell \;\Rightarrow j=\ell-1,\ell,\ell+1 \;\Rightarrow$ for even $j>0$ have two-channel mixing

Final results

- Quantization condition (**boldface** quantities absorb factors of i, 2ω, L^3)

$$\det_{\boldsymbol{k},\ell,m,\boldsymbol{m}_s^*}[\boldsymbol{F}_3^{-1} - \boldsymbol{K}_{\mathrm{df},3}] = 0 \qquad \boldsymbol{F}_3 = \frac{\boldsymbol{F}}{3} + \boldsymbol{F}\frac{1}{\boldsymbol{K}_2^{-1} - \boldsymbol{F} - \boldsymbol{G}}\boldsymbol{F}$$

  - In practice, must truncate in ℓ so that matrices have finite dimension

- Integral equations relating $\boldsymbol{K}_{\mathrm{df},3}$ and $\mathscr{M}_3$ take similar form to those for scalar particles, aside from extra spin indices and Wigner D-matrices

- Range of validity for (isosymmetric) QCD

$$\sqrt{4m_N^2 - m_\pi^2} + m_N < E_3^* < 3m_N + m_\pi$$

2-particle subchannel LH cut Inelastic threshold

Threshold expansion for $\mathcal{K}_{\rm df,3}$

- Need parametrization of $\mathcal{K}_{\rm df,3}$ in order to apply QC3 in practice
- Expand about threshold; analogous to effective-range expansion for $\mathcal{K}_2$
  - Similar to NR expansion in pionless EFT, except using relativistic fields
- Method: use neutron field operators $\mathcal{N}$
  - Write down all operators of the form $(\overline{\mathcal{N}}\mathcal{N})^3$ with arbitrary gamma-matrix structure and 0, 2, 4,… derivatives, requiring Lorentz and parity invariance
  - Take matrix elements of these operators between lab-frame states, leading to completely antisymmetric expressions in terms of Dirac spinors (in lab frame)
  - Determine which are independent
  - Insert NR expression for Dirac spinors, and expand in 3-momenta

Operators without derivatives

- 12 operators (have not used Fierz identities)

$$
\begin{aligned}
\mathcal{O}_{\rm SSS}(x) &= [\overline{\mathcal{N}}(x)\mathcal{N}(x)]^3\,,\\
\mathcal{O}_{\rm SPP} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm SVV} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\gamma_\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\mathcal{N}]\,,\\
\mathcal{O}_{\rm SAA} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\gamma_\mu\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm STT} &= [\overline{\mathcal{N}}\mathcal{N}][\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\mu\nu}\mathcal{N}]\,,\\
\mathcal{O}_{\rm PVA} &= [\overline{\mathcal{N}}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma_\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm PTT'} &= [\overline{\mathcal{N}}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\mu\nu}\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm TVV} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\nu\mathcal{N}]\,,\\
\mathcal{O}_{\rm TAA} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm T'VA} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\gamma^\mu\mathcal{N}][\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N}]\,,\\
\mathcal{O}_{\rm TTT} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N}][\overline{\mathcal{N}}\sigma_\rho{}^\mu\mathcal{N}]\,,\\
\mathcal{O}_{\rm TT'T'} &= [\overline{\mathcal{N}}\sigma_{\mu\nu}\mathcal{N}][\overline{\mathcal{N}}\sigma^{\nu\rho}\gamma_5\mathcal{N}][\overline{\mathcal{N}}\sigma_\rho{}^\mu\gamma_5\mathcal{N}]\,,
\end{aligned}
$$

- All lead to **identical** $3\to 3$ amplitudes (*cf.* four independent forms for $2\to 2$)
- Insert NR on-shell form, and find leading contribution to $\mathscr{K}_{\rm df,3}$ involves 2 derivatives:

$$
u_k = \sqrt{2\omega_k}\begin{pmatrix}\chi_k\\ \frac{\boldsymbol{\sigma}\cdot\boldsymbol{k}}{\omega_k+m}\chi_k\end{pmatrix} \quad\Rightarrow\quad \mathcal{K}_A = \overline{\mathcal{A}}\left[(\chi^\dagger_{k'}\,\boldsymbol{\sigma}\cdot\boldsymbol{k'}\,\boldsymbol{\sigma}\cdot\boldsymbol{k}\,\chi_k)(\chi^\dagger_{a'}\chi_a)(\chi^\dagger_{b'}\chi_b)\right]
$$

antisymmetrization

Operators with 2 derivatives

- For consistency, need to consider operators with 2 derivatives
  - Using Fierz identities and EoM, find 22 independent operators

$$
\begin{aligned}
SSS &= (\partial^\mu \overline{\mathcal{N}} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\mathcal{N})\,,\\
SPP &= (\partial^\mu \overline{\mathcal{N}} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})\,,\\
PSP &= (\partial^\mu \overline{\mathcal{N}}\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})\,,\\
SVV &= (\partial^\mu \overline{\mathcal{N}} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\mathcal{N})\,,\\
VSV &= (\partial^\mu \overline{\mathcal{N}}\gamma_\nu \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\mathcal{N})\,,\\
ASA &= (\partial^\mu \overline{\mathcal{N}}\gamma_\nu\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N})\,,\\
TST &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\nu\rho} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})\,,\\
PVA &= (\partial^\mu \overline{\mathcal{N}}\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N})\,,\\
VAP &= (\partial^\mu \overline{\mathcal{N}}\gamma^\nu \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})\,,\\
APV &= (\partial^\mu \overline{\mathcal{N}}\gamma^\nu\gamma_5 \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_\nu\mathcal{N})\,,
\end{aligned}
$$

$$
\begin{aligned}
SVV' &= (\partial^\mu \overline{\mathcal{N}} \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\mathcal{N})\,,\\
TST' &= (\partial_\nu \overline{\mathcal{N}}\sigma_{\mu\rho} \partial^\mu \mathcal{N})(\overline{\mathcal{N}}\mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})\,,\\
PVA' &= (\partial^\mu \overline{\mathcal{N}}\gamma_5 \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\mathcal{N})(\overline{\mathcal{N}}\gamma^\nu\gamma_5\mathcal{N})\,,\\
VTV' &= (\partial^\mu \overline{\mathcal{N}}\gamma_\rho \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\mathcal{N})\,,\\
ATA' &= (\partial^\mu \overline{\mathcal{N}}\gamma_\rho\gamma_5 \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})(\overline{\mathcal{N}}\gamma_\mu\gamma_5\mathcal{N})\,,\\
TTT' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\eta} \partial^\eta \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\nu\rho}\mathcal{N})\,,\\
TTT'' &= (\partial_\mu \overline{\mathcal{N}}\sigma^{\mu\nu} \partial_\eta \mathcal{N})(\overline{\mathcal{N}}\sigma^{\eta\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\rho\nu}\mathcal{N})\,,\\
TTT'' &= (\partial_\eta \overline{\mathcal{N}}\sigma^{\mu\nu} \partial_\mu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\eta\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\rho\nu}\mathcal{N})\,,\\
TT_5P' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\rho} \partial_\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\nu\rho}\gamma_5\mathcal{N})(\overline{\mathcal{N}}\gamma_5\mathcal{N})\,,\\
T_5VA' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\nu}\gamma_5 \partial^\nu \mathcal{N})(\overline{\mathcal{N}}\gamma^\rho\mathcal{N})(\overline{\mathcal{N}}\gamma_\rho\gamma_5\mathcal{N})\,,\\
T_5VA'' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\rho\nu}\gamma_5 \partial^\nu \mathcal{N})(\overline{\mathcal{N}}\gamma^\mu\mathcal{N})(\overline{\mathcal{N}}\gamma^\rho\gamma_5\mathcal{N})\,,\\
T_5TT_5' &= (\partial^\mu \overline{\mathcal{N}}\sigma_{\mu\nu}\gamma_5 \partial^\nu \mathcal{N})(\overline{\mathcal{N}}\sigma^{\eta\rho}\mathcal{N})(\overline{\mathcal{N}}\sigma_{\eta\rho}\gamma_5\mathcal{N})
\end{aligned}
$$

- Inserting NR on-shell form, find two independent 2-derivatives forms:

$$\mathcal{K}_A = \overline{\mathcal{A}}\left[(\chi_{k'}^\dagger\, \boldsymbol{\sigma}\cdot\boldsymbol{k'}\, \boldsymbol{\sigma}\cdot\boldsymbol{k}\, \chi_k)(\chi_{a'}^\dagger\chi_a)(\chi_{b'}^\dagger\chi_b)\right] \qquad \mathcal{K}_B = \overline{\mathcal{A}}\left[\boldsymbol{k'}\cdot\boldsymbol{k}(\chi_{k'}^\dagger\chi_k)(\chi_{a'}^\dagger\chi_a)(\chi_{b'}^\dagger\chi_b)\right]$$

Same as from 0-derivative operators

Summary for $\mathcal{K}_{\text{df},3}$

- 0-derivative operators contribute

Dimensionless combination

Unknown, dimensionless constant

$$m_N^2 \mathcal{K}_{\text{df},3}^{\text{lab}} \supset \frac{c_0}{m_N^2} \mathcal{K}_A + \mathcal{O}(\boldsymbol{k}^4/m_N^4)$$

- 2-derivative operators imply

$$m_N^2 \mathcal{K}_{\text{df},3}^{\text{lab}} \supset \frac{c_1}{\Lambda_{\text{EFT}}^2} \mathcal{K}_A + \frac{c_2}{\Lambda_{\text{EFT}}^2} \mathcal{K}_B + \mathcal{O}(\boldsymbol{k}^4/m_N^2 \Lambda_{\text{EFT}}^2)$$

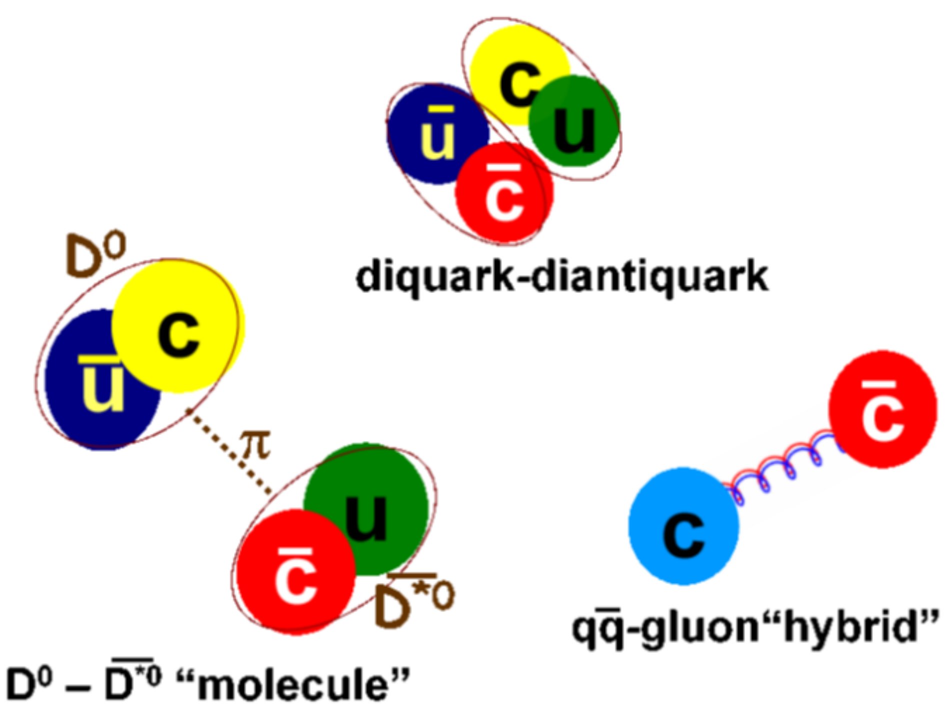
- Since expect $\Lambda_{\text{EFT}} \sim m_\pi$, the 2-derivative operators dominate
- The form of the allowed operators could more easily have been determined directly using a NR expansion, but this would lose the implications of relativity at higher order

Summary & Outlook for 3N

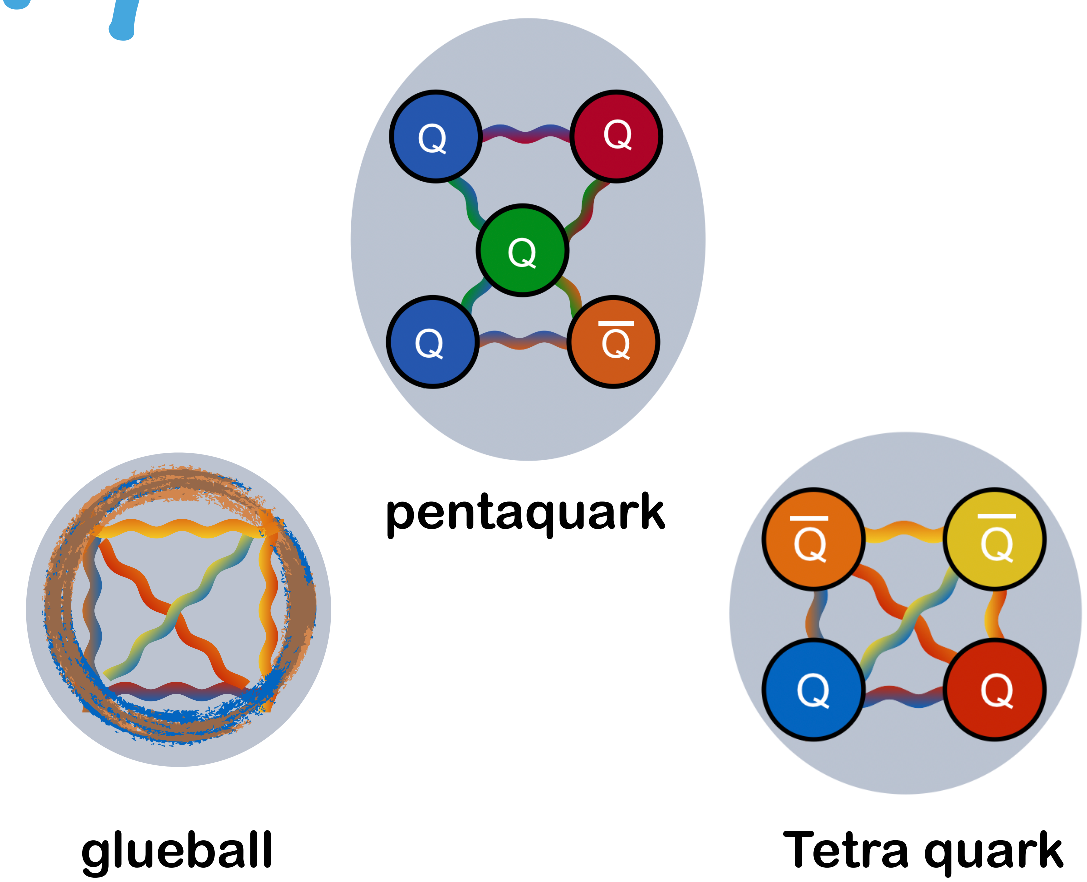
- Including spin in the formalism involves additional subtleties not present for 2 particles
  - Wigner rotations and fermion signs
- Implementing the QC3 is underway for toy interactions
- Various generalizations should be straightforward
  - 3 nucleons of arbitrary isospin
  - $N\pi\pi$ at maximal isospin (no 3-particle resonance, but includes $\Delta\pi$)
  - $N\pi\pi + N\pi$ (for the Roper)
  - Higher spins (e.g. ρ if stable)—though hard to think of applications
- Need to extend methods for solving integral equations
  - Need to relate parameters in $\mathscr{K}_{\mathrm{df},3}$ to those in chiral EFTs used to study light nuclei

End of part 1

Part 2: Takeaways from the workshop—a personal summary



[R. Lebed]



[V. Mathieu]

Personal perspective

- Long, long ago I worked on glueballs and hybrids using the MIT bag model

LBL-14865
Preprint

MEIKTONS: MIXED STATES OF QUARKS AND GLUONS

Michael Chanowitz and Stephen Sharpe

August 1982

Report number: RL-82-088

Q anti-Q G Hermaphrodite Mesons in the MIT Bag Model

Ted Barnes (Rutherford), F.E. Close (Rutherford), F. de Viron (Louvain U.)
Oct, 1982

Nuclear Physics B222 (1983) 211–244

HYBRIDS: MIXED STATES OF QUARKS AND GLUONS*

Michael CHANOWITZ and Stephen SHARPE

Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA

Received 20 September 1982
(Revised 26 January 1983)

Nuclear Physics B224 (1983) 241–264

$Q\bar{Q}G$ HYBRID MESONS IN THE MIT BAG MODEL

T BARNES and F E CLOSE

Rutherford Appleton Laboratory, Chilton, Didcot, OXON OX11 OQX, England

F de VIRON

Dept de Physique Theorique, University Catholique de Louvain, B-1348 Louvain-La-Neuve, Belgium

Received 19 October 1982
(Revised 3 February 1983)

Personal perspective

- Long, long ago I worked on glueballs and hybrids using the MIT bag model

Nuclear Physics B222 (1983) 211–244

HYBRIDS: MIXED STATES OF QUARKS AND GLUONS*

Michael CHANOWITZ and Stephen SHARPE

Lawrence Berkeley Laboratory, University of California, Berkeley, California 94720, USA

Received 20 September 1982
(Revised 26 January 1983)

PHYSICS LETTERS Volume 132B, number 4,5,6 1 December 1983

GLUEBALLS AND MEIKTONS WHICH DECAY TO MULTI-KAON FINAL STATES

Michael S. CHANOWITZ and Stephen R. SHARPE [1]
Lawrence Berkeley Laboratory, University of California, Berkeley, CA 94720, USA

Received 19 August 1983

PHYSICAL REVIEW D VOLUME 30, NUMBER 5 1 SEPTEMBER 1984

Hunting a hidden hadron: Is there a scalar glueball below 1 GeV?

Stephen R. Sharpe
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts 02138

R. L. Jaffe
Center for Theoretical Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

M. R. Pennington
Physics Department, University of Durham, Durham DH1 3LE, United Kingdom
(Received 23 April 1984)

The answer to the question in the title is "probably not." But if there is, it must either lie above

GLUEBALLS: A STATUS REPORT ON THE THEORY

Stephen R. Sharpe (Harvard U.)
May, 1984

11 pages
Part of Proceedings, 6th International Conference on High-Energy Physics at Vanderbilt: High-Energy Interactions : Nashville, Tennessee, April 6-8, 1984

GLUEBALLS AND OTHER EXOTICA IN $p\bar{p}$ ANNIHILATION

Stephen R. Sharpe [1]

Physics Department, FM-15, University of Washington, Seattle, WA 98195

Published in: *eConf* C860410 (1986) 165
Contribution to: 1st Workshop on Antimatter Physics at Low-energy

Annu. Rev. Nucl. Part. Sci. 1990. 40: 327–55

NON-QUARK-MODEL MESONS

T. H. Burnett[1]

CERN, European Organization for Particle Physics, 1211 Geneva 23, Switzerland

Stephen R. Sharpe

Physics Department, University of Washington, Seattle, Washington 98195, USA

KEY WORDS: glueball, meikton, hybrid, exotic.

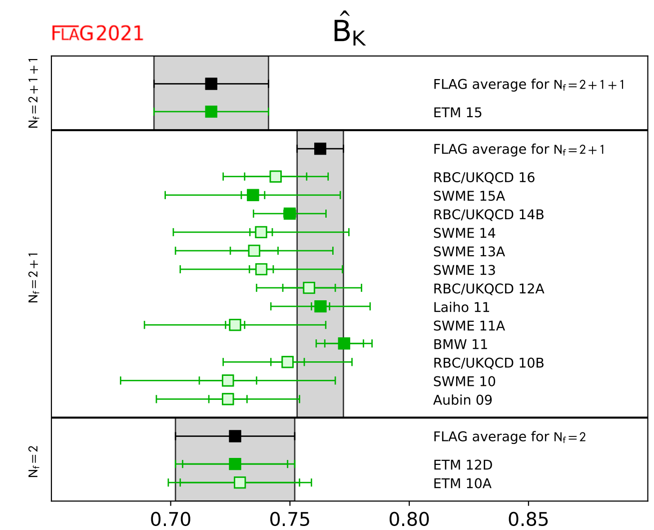
Personal perspective

- Long, long ago I worked on glueballs and hybrids using the MIT bag model
- I moved to LQCD to get more reliable results for properties of exotics (!!)
- LQCD has now become a precision tool for a range of quantities

FLAG Review 2021

Flavour Lattice Averaging Group (FLAG)

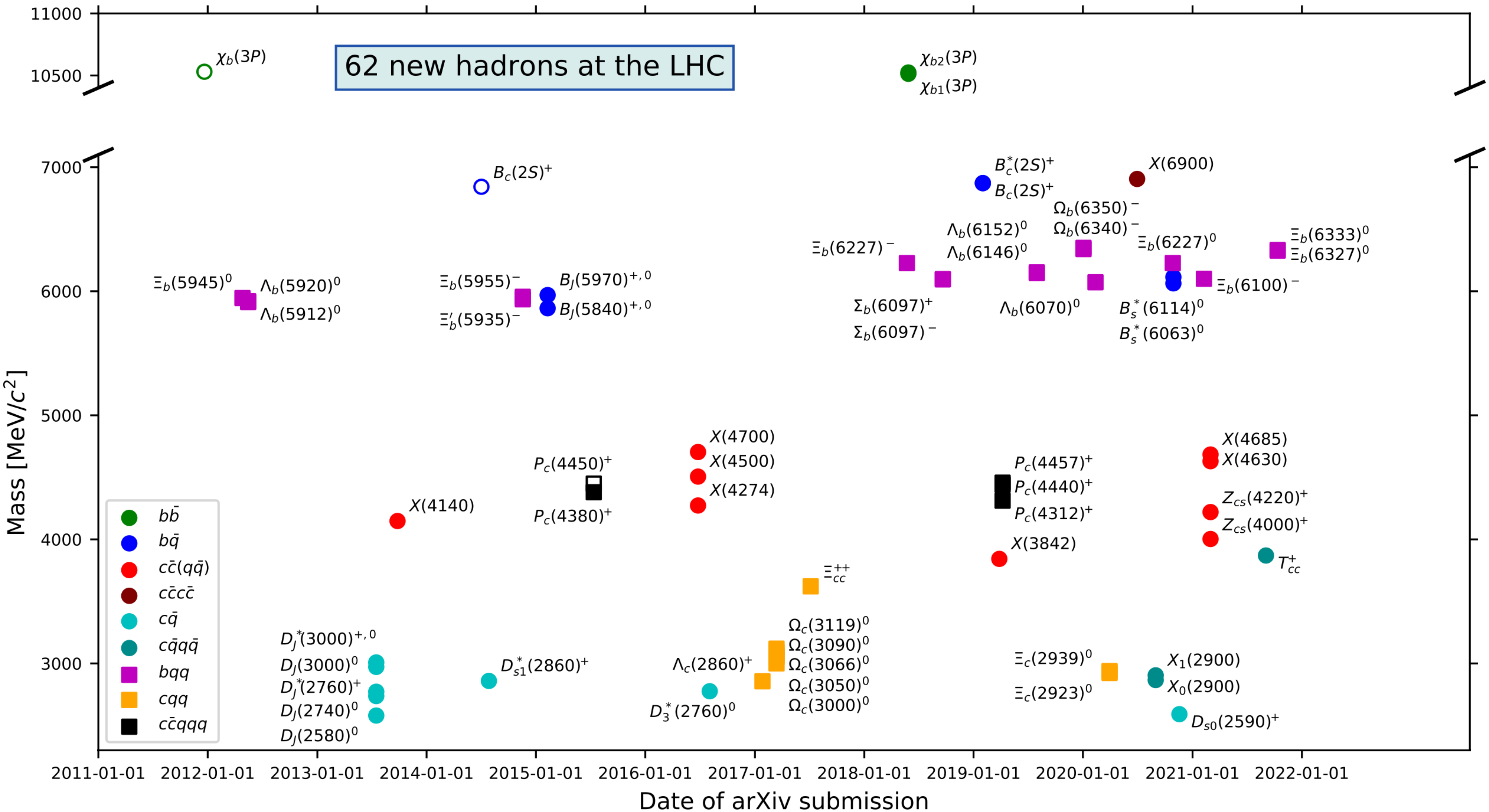
Y. Aoki[1], T. Blum[2,3], G. Colangelo[4], S. Collins[5], M. Della Morte[6], P. Dimopoulos[7,8], S. Dürr[9,10], X. Feng[11,12,13,14], H. Fukaya[15], M. Golterman[16], Steven Gottlieb[17], R. Gupta[18], S. Hashimoto[19,20], U. M. Heller[21], G. Herdoiza[22], P. Hernandez[23], R. Horsley[24], A. Jüttner[25,26,27,a], T. Kaneko[19,20], E. Lunghi[17], S. Meinel[28], C. Monahan[29,30], A. Nicholson[31], T. Onogi[15], C. Pena[22], P. Petreczky[32], A. Portelli[24], A. Ramos[23], S. R. Sharpe[33], J. N. Simone[34], S. Simula[35], S. Sint[36], R. Sommer[37,38], N. Tantalo[39], R. Van de Water[34], U. Wenger[4,27], H. Wittig[40]



- Can LQCD + amplitude analysis + EFTs + models lead to a new level of understanding of resonances in general & exotics in particular?
- Can LQCD calculations for few-nucleon systems contribute to our understanding of large nuclei, neutron stars, etc?

Cornucopia of exotics

[I. Danilkin talk]

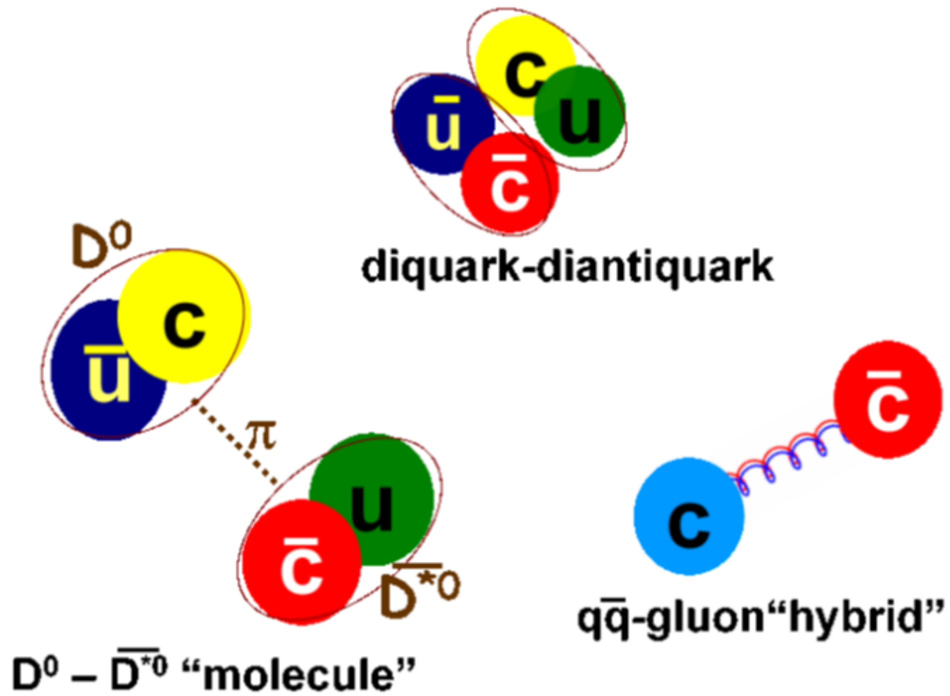


+ upcoming results from GLUEX, …[Dobbs, Shepherd]

+ data from Babar, Belle, COMPASS, …

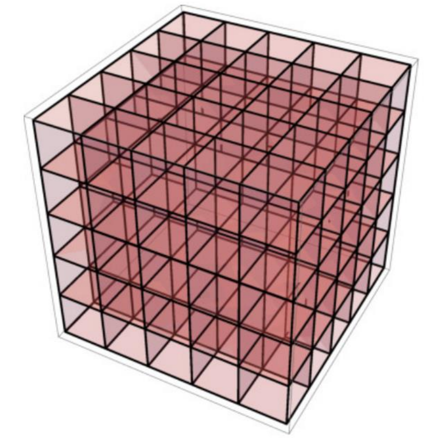
Abundance of ideas/models

- NR quark model [Segovia]
- Constituent glue [Swanson]
- Heavy-quark hybrids (flux excitations) [Mohapatra]
- Diquark quasiparticle [Lebed]
- Molecules [Ikeno, Hanhart]
- Nonresonant kinematical enhancements



- Which are most "useful"? Can some be falsified? Systematic errors?
- Can specific observables be predicted that can be calculated with LQCD?
- Can we tell a "story" of exotics?
- Can they be connected to a deeper understanding of QCD/confinement?

Abundance of theoretical/ computational tools



- Functional methods [Huber, Fischer]
- Born-Oppenheimer [Bruschini]
- Data-driven analytic continuation [Tripolt]
- Dispersion relations [Palaez, Hanhart]
- EFTs [Bruschini, Hanhart]
- Bootstrapping QCD [Guerrieri]
- Light front [Shuryak]
- Holography [Brodsky]
- Amplitude analysis [Mikhasenko]

- LQCD related/driven
  - Spectrum & form-factors [Athenodorou, Nicholson, Hanlon, Walker-Loud, Dudek, Prelovsek, Mohler, Pefkou]
  - Finite-volume formalism [Romero-López, Mai, Rusetsky, Ortega-Gama, SRS]
  - Integral equations [Dawid, Islam, Döring]

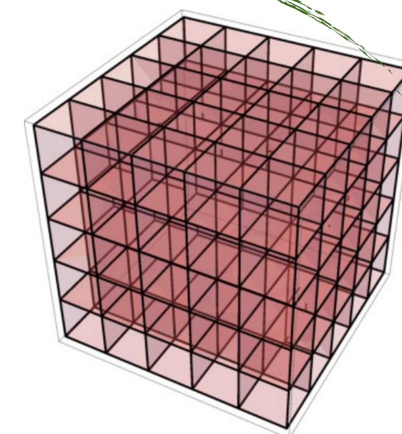
- LQCD is a powerful tool with a limited domain: How can we effectively combine other methods with LQCD?

Vision & Synergies

Dispersion relations
EFTs
Models

LQCD constrains theory.
Theory extends LQCD

LQCD



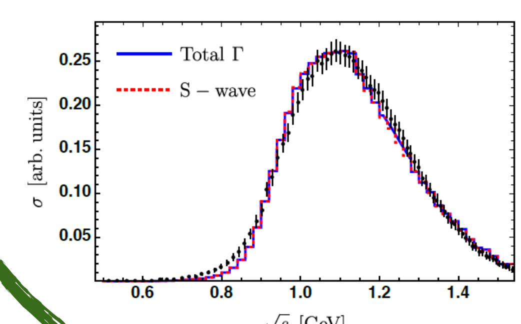
Playground with many knobs to turn

A story for all resonances & exotics

Expt constrains theory.
Theory predicts new phenomena

FV formalism.
Rough predictions aid experimental searches

Experiments + PWA

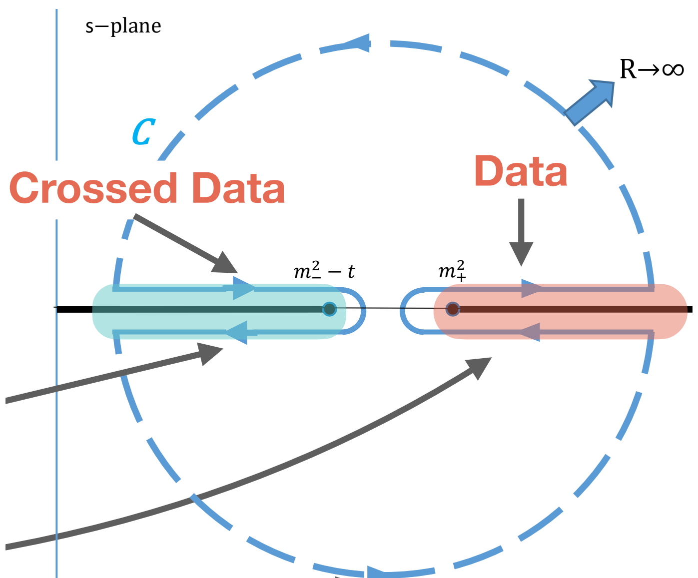


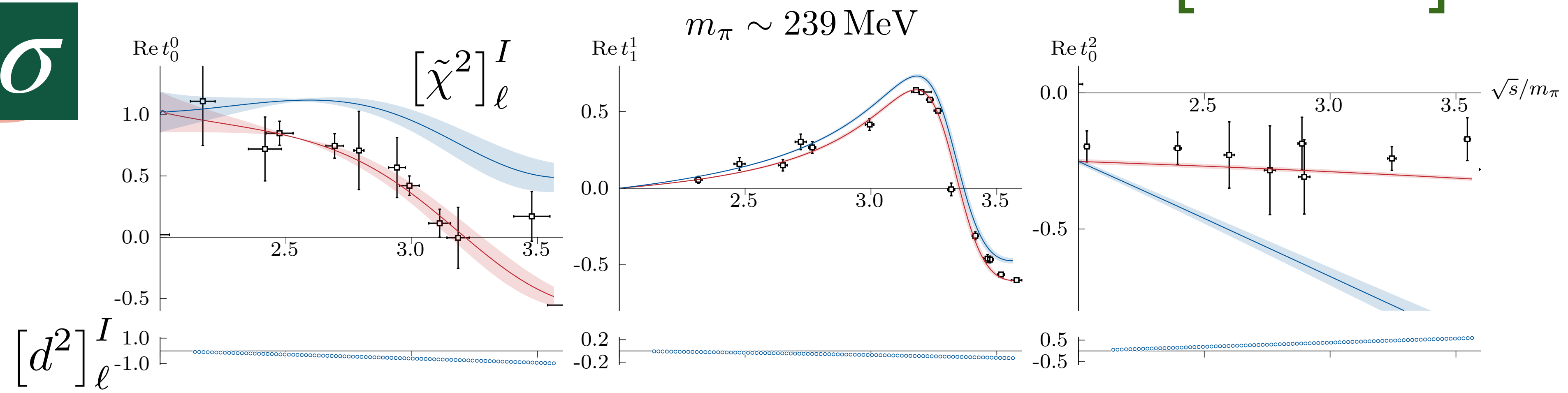
A few personal highlights (& caveats)

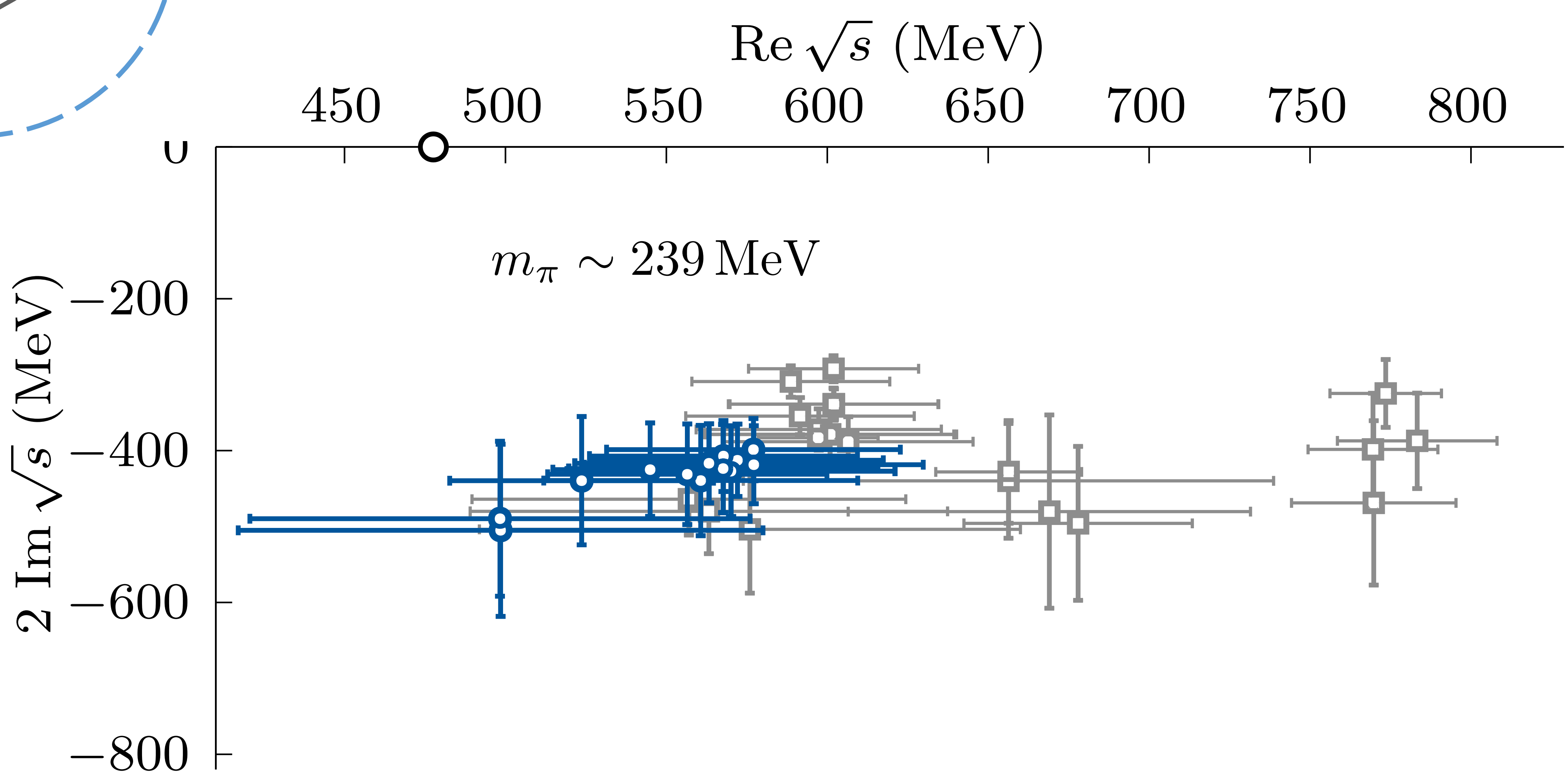
Combining LQCD with crossing

[A. Rodas]

Dispersive σ



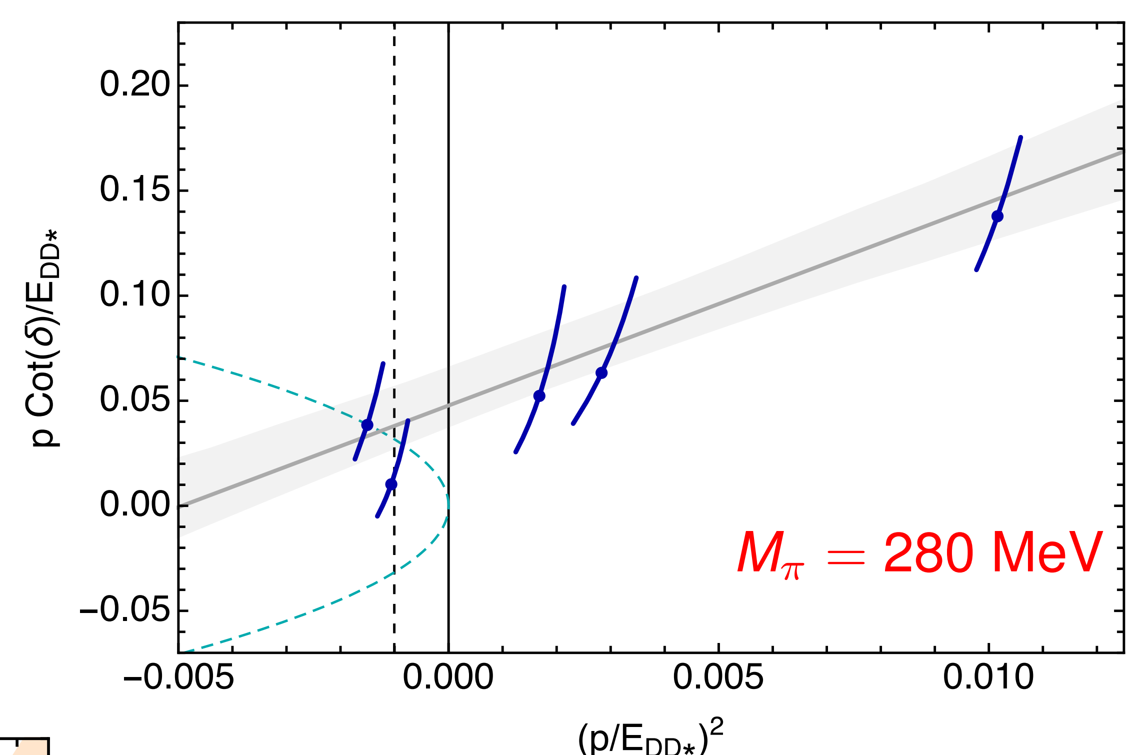




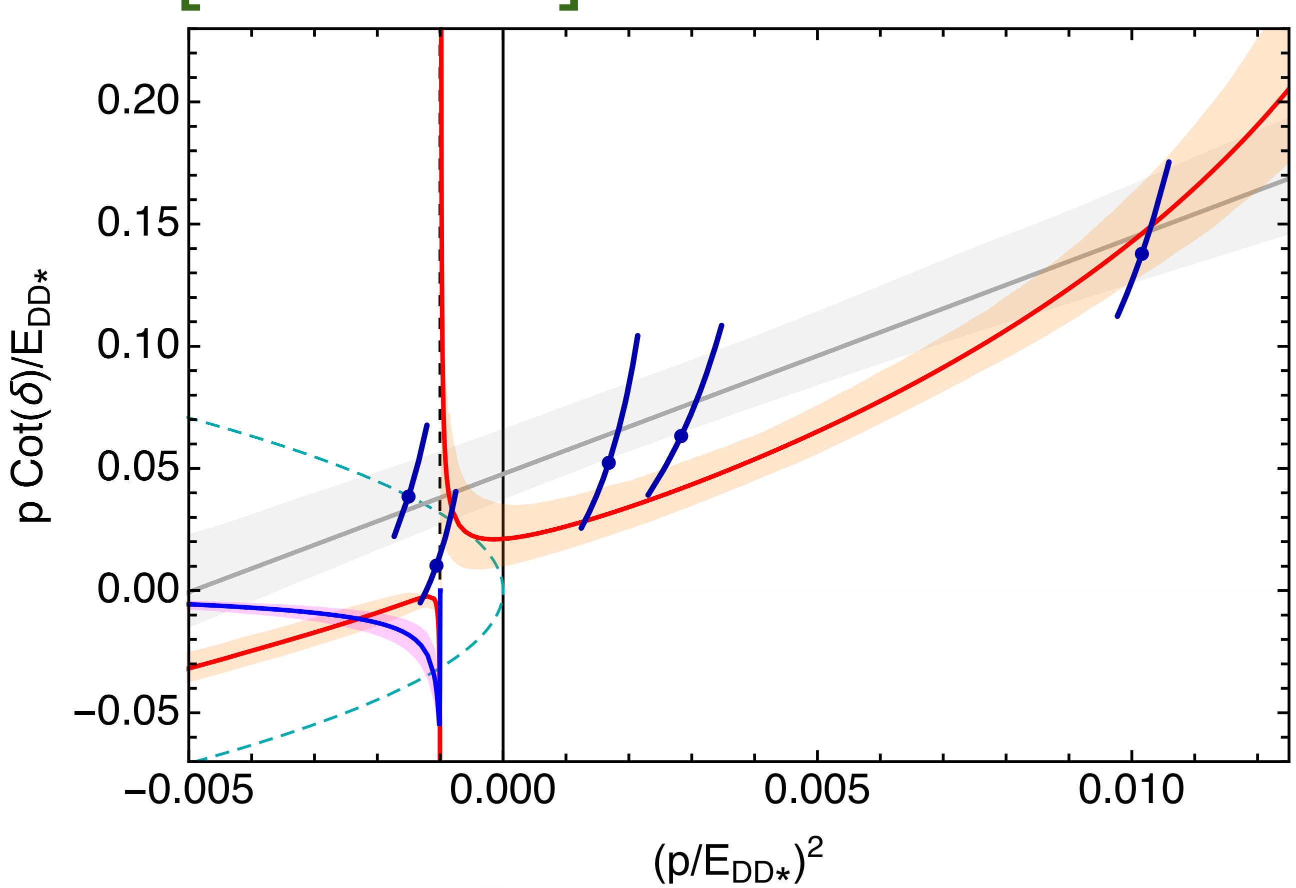
Beware of LH cuts!

- LQCD results for DD^* (with stable D^*)
- Channel of $T_c(3872)$
- Found virtual bound state assuming ERE

Padmanath & Prelovsek, PRL129(2022)032002



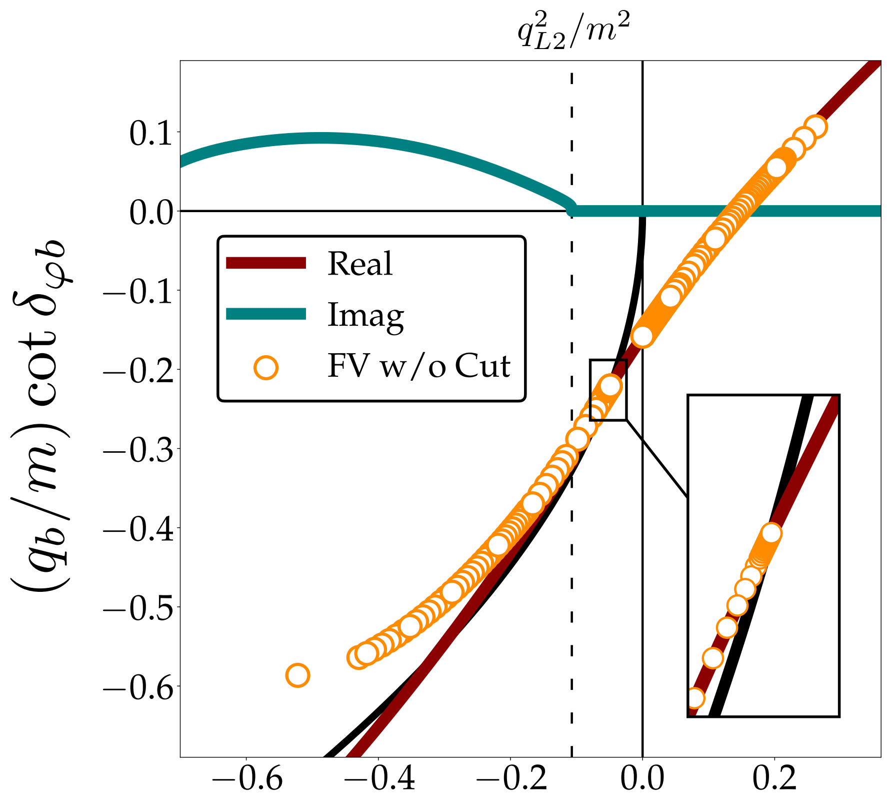
[C. Hanhart]



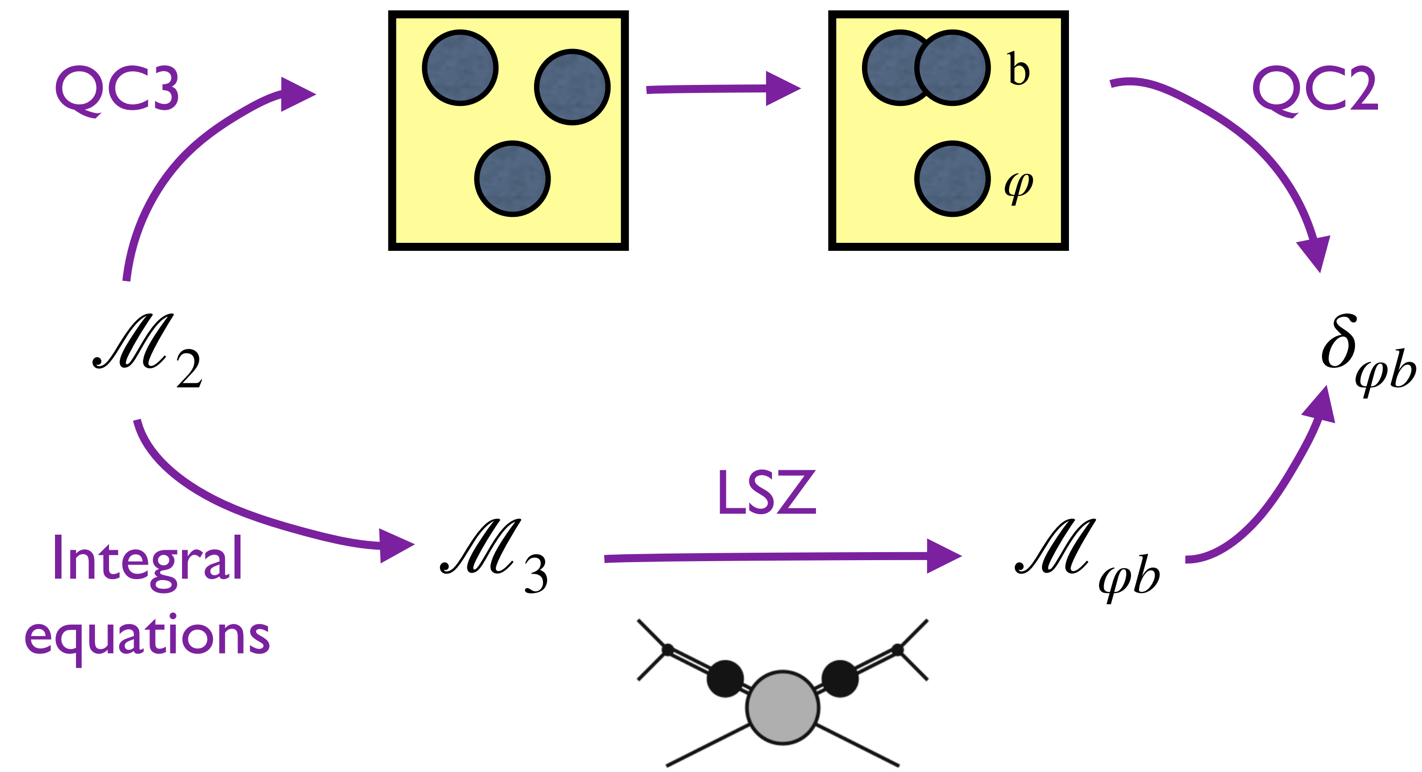
- ERE (and QC2) fail below left-hand cut
- Fit including OPE shows very different behavior
- Two virtual bound states

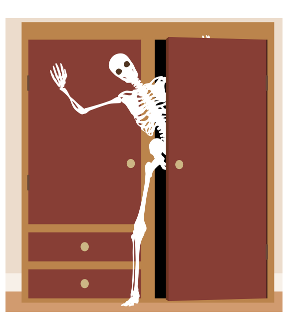
Progress in 3-particle formalisms

[Islam, Dawid]



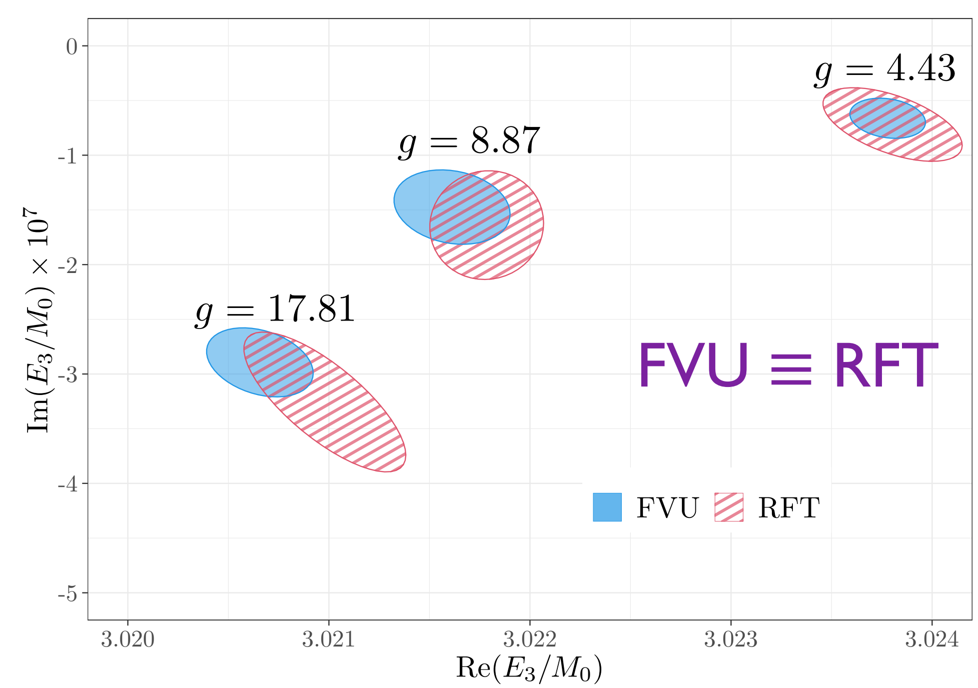
$ma = 2$





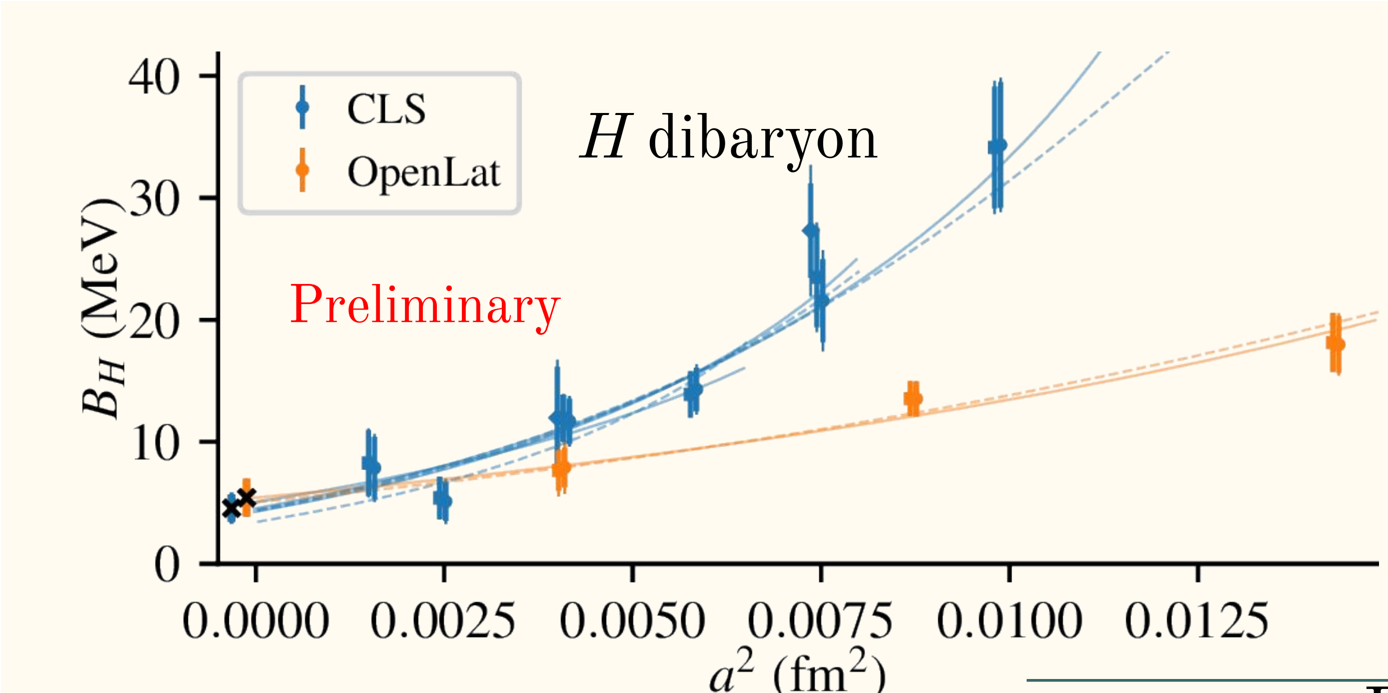
- QCs fail beyond relevant LH cut
- Smooth cutoff (RFT) leads to problems with analytic continuation
- Symmetric vs asymmetric $\mathcal{K}_{\mathrm{df},3}$

[Mai]

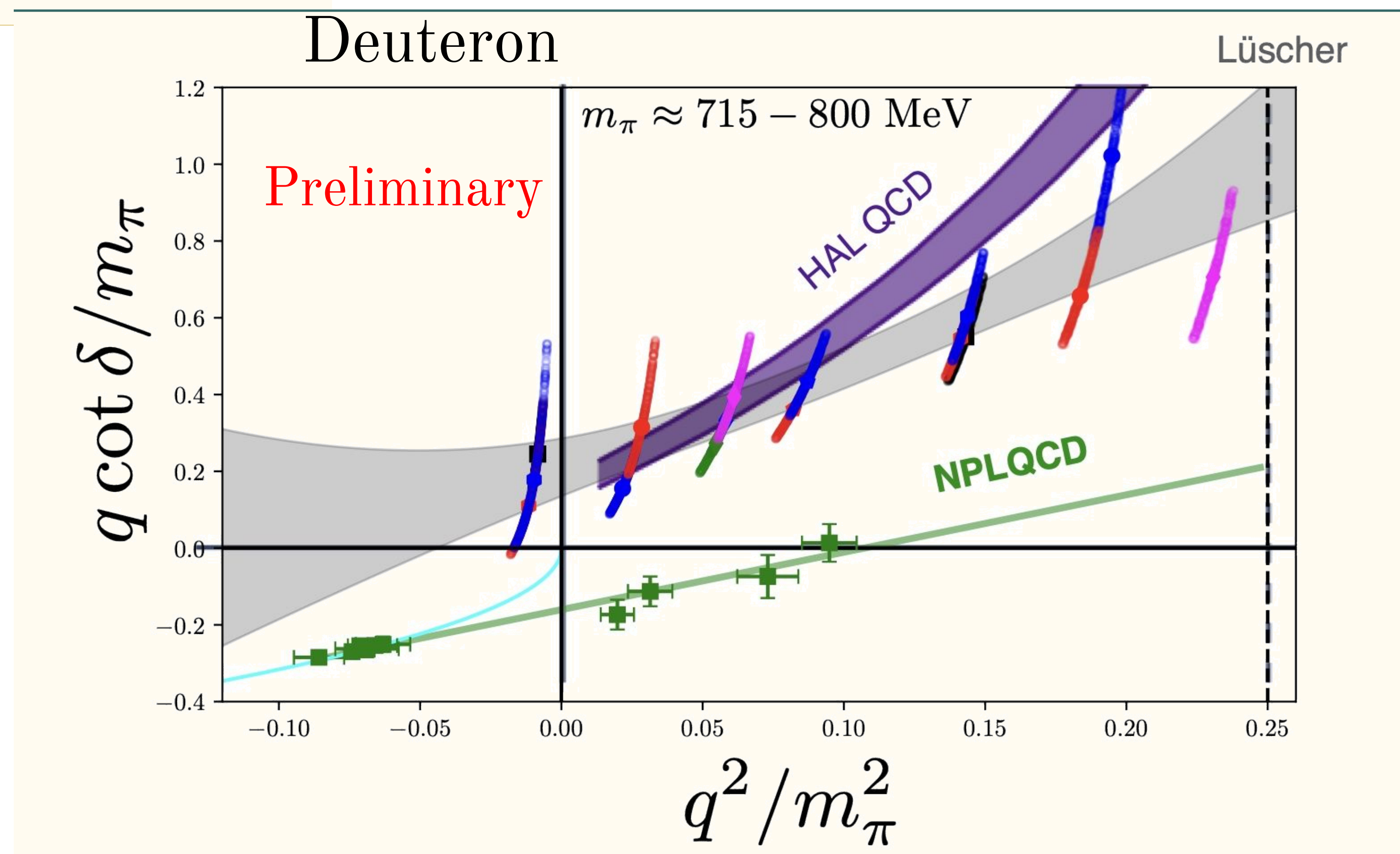


LQCD for NN matures

[Nicholson, Hanlon, Walker-Loud]



Discretization errors not well controlled in most LQCD calculations related to resonances



Closing thoughts

The questions we want to address [A. Pilloni]

- What «understanding» mean?
  What would be the acceptable end of the hadron quest?
- Once we have the determined the spectrum and interactions of hybrids/XYZ/glueballs etc., what do we really want to learn?
- What level of complementarity can we expect between Lattice QCD and experimental data in the next decade?
- Is the present model of collaboration between theory and experiment efficient?
- Could AI technology provide groundbreakingly different tools?

- What «understanding» mean?
  What would be the acceptable end of the hadron quest?

- Once we have the determined the spectrum and interactions of hybrids/XYZ/glueballs etc., what do we really want to learn?

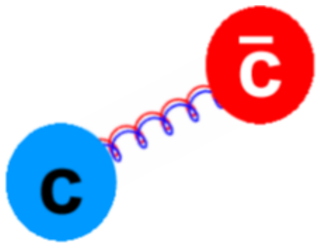
- We are a long way from reaching the "end"! Nevertheless, these are good questions.
- Some answers:
  - Understanding the different types of collective phenomena/important degrees of freedom that occur in a strongly-interacting theory (having a "story")
  - Having tools that will transfer to other (BSM?) strongly-interacting theories
  - Having tools that will allow first-principles calculations of electroweak processes (e.g. CP-violation in D decays)
  - We should not be afraid of needing multiple pictures to encompass the exotic zoo

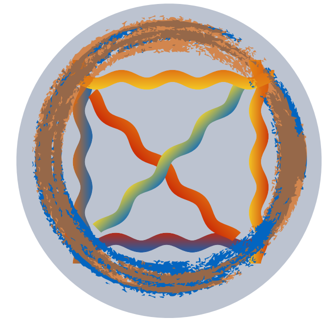
- What level of complementarity can we expect between Lattice QCD and experimental data in the next decade?

- Area with tremendous potential growth!
  - LQCD calculations are hard, and move relatively slowly
    - Focus on key quantities: resonances with 3-particle decays; 2N & 3N
  - Use LQCD as a playground
    - Varying quark masses allows access to different regimes
    - 3-particle scattering is not possible experimentally

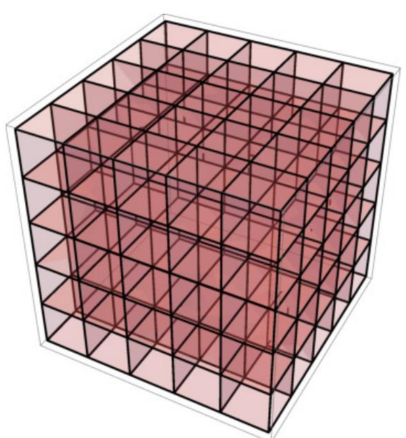
- Is the present model of collaboration between theory and LQCD and experiment efficient?

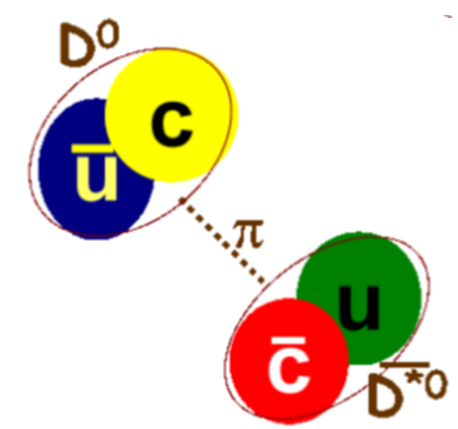
- Making this work is essential!
  - JPAC and (hopefully) ExoHad are excellent models for this
  - Workshops like this one are crucial
    - Include more experimentalists?
    - Summer schools/lecture series: train experimentalists and theorists side by side?
  - Help build the case for future experimental upgrades

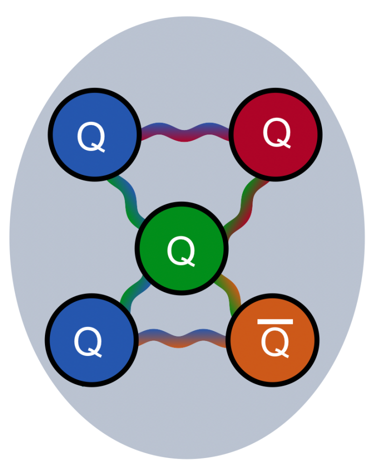




Many thanks to Alessandro, Gernot, Raúl, and to the INT, and to all of you!







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RFT 3-particle papers

Max Hansen & SRS:

"Relativistic, model-independent, three-particle quantization condition,"

arXiv:1408.5933 (PRD) [HS14]

"Expressing the 3-particle finite-volume spectrum in terms of the 3-to-3 scattering amplitude,"

arXiv:1504.04028 (PRD) [HS15]

"Perturbative results for 2- & 3-particle threshold energies in finite volume,"

arXiv:1509.07929 (PRD) [HSPT15]

"Threshold expansion of the 3-particle quantization condition,"

arXiv:1602.00324 (PRD) [HSTH15]

"Applying the relativistic quantization condition to a 3-particle bound state in a periodic box,"

arXiv: 1609.04317 (PRD) [HSBS16]

"Lattice QCD and three-particle decays of Resonances,"

arXiv: 1901.00483 (Ann. Rev. Nucl. Part. Science) [HSREV19]

Raúl Briceño, Max Hansen & SRS:

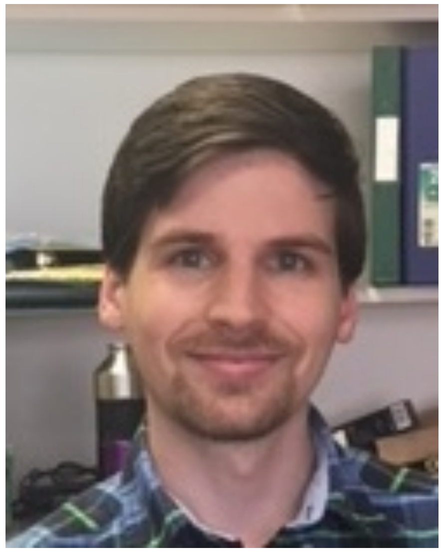
"Relating the finite-volume spectrum and the 2-and-3-particle S-matrix for relativistic systems of identical scalar particles," arXiv:1701.07465 (PRD) [BHS17]

"Numerical study of the relativistic three-body quantization condition in the isotropic approximation," arXiv:1803.04169 (PRD) [BHS18]

"Three-particle systems with resonant sub-processes in a finite volume," arXiv:1810.01429 (PRD 19) [BHS19]

SRS

"Testing the threshold expansion for three-particle energies at fourth order in φ^4 theory," arXiv:1707.04279 (PRD) [SPT17]



Tyler Blanton, Fernando Romero-López & SRS:

"Implementing the three-particle quantization condition including higher partial waves," arXiv:1901.07095 (JHEP) [BRS19]

"I=3 three-pion scattering amplitude from lattice QCD," arXiv:1909.02973 (PRL) [BRS-PRL19]

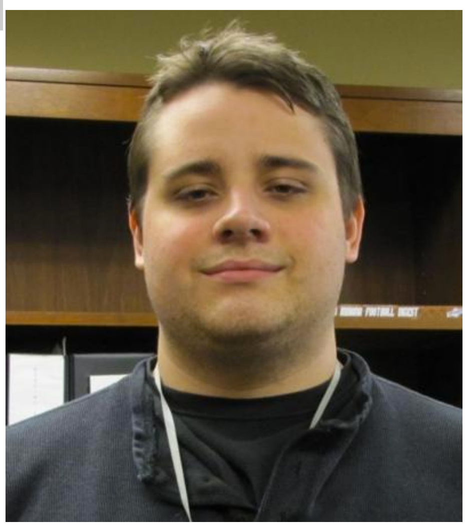
"Implementing the three-particle quantization condition for $\pi^+\pi^+K^+$ and related systems" 2111.12734 (JHEP)

Tyler Blanton, Raúl Briceño, Max Hansen, Fernando Romero-López, SRS:

"Numerical exploration of three relativistic particles in a finite volume including two-particle resonances and bound states", arXiv:1908.02411 (JHEP) [BBHRS19]

Raúl Briceño, Max Hansen, SRS & Adam Szczepaniak:

"Unitarity of the infinite-volume three-particle scattering amplitude arising from a finite-volume formalism," arXiv:1905.11188 (PRD)



Andrew Jackura, S. Dawid, C. Fernández-Ramírez, V. Mathieu, M. Mikhasenko, A. Pilloni, SRS & A. Szczepaniak:

"On the Equivalence of Three-Particle Scattering Formalisms," arXiv:1905.12007 (PRD)

Max Hansen, Fernando Romero-López, SRS:

"Generalizing the relativistic quantization condition to include all three-pion isospin channels", arXiv:2003.10974 (JHEP) [HRS20]

"Decay amplitudes to three particles from finite-volume matrix elements," arXiv: 2101.10246 (JHEP)

Tyler Blanton & SRS:

“Alternative derivation of the relativistic three-particle quantization condition,”

arXiv:2007.16188 (PRD) [BS20a]

“Equivalence of relativistic three-particle quantization conditions,”

arXiv:2007.16190 (PRD) [BS20b]

“Relativistic three-particle quantization condition for nondegenerate scalars,”

arXiv:2011.05520 (PRD)

“Three-particle finite-volume formalism for $\pi^+\pi^+K^+$ & related systems,” arXiv:2105.12904 (PRD)

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“$3\pi^+$ & $3K^+$ interactions beyond leading order from lattice QCD,” arXiv:2106.05590 (JHEP)

Zack Draper, Drew Hanlon, Ben Hörz, Colin Morningstar, Fernando Romero-López & SRS

“Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD,” arXiv:2302.13587





<u>Zach Draper, Max Hansen, Fernando Romero-López & SRS:</u>

"Three relativistic neutrons in a finite volume,"

arXiv:2303.10219

<u>Jorge Baeza-Ballesteros, Johan Bijnens, Tomas Husek, Fernando Romero-López, SRS & Mattias Sjö:</u> "The isospin-3 three-particle K-matrix at NLO in ChPT," arXiv:2303.13206

Other work

★ Implementing RFT integral equations

- M.T. Hansen et al. (HADSPEC), 2009.04931, PRL [Calculating $3\pi^+$ spectrum and using to determine three-particle scattering amplitude]
- A. Jackura et al., 2010.09820, PRD [Solving s-wave RFT integral equations in presence of bound states]
- S. Dawid, Md. Islam and R. Briceño, 2303.04394 [Analytic continuation of 3-particle amplitues]

★ Reviews

- A. Rusetsky, 1911.01253 [LATTICE 2019 plenary]
- M. Mai, M. Döring and A. Rusetsky, 2103.00577 [Review of formalisms and chiral extrapolations]
- F. Romero-López, 2112.05170, [Three-particle scattering amplitudes from lattice QCD]

★ Other numerical simulations

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- M. Fischer et al., 2008.03035, Eur.Phys.J.C [$2\pi^+$ & $3\pi^+$ at physical masses]
- M. Garofolo et al., 2211.05605, JHEP [3-body resonances in φ^4 theory]

Other work

★ **NREFT approach**

- H.-W. Hammer, J.-Y. Pang & A. Rusetsky, 1706.07700, JHEP & 1707.02176, JHEP [Formalism & examples]
- M. Döring et al., 1802.03362, PRD [Numerical implementation]
- J.-Y. Pang et al., 1902.01111, PRD [large volume expansion for excited levels]
- F. Müller, T. Yu & A. Rusetsky, 2011.14178, PRD [large volume expansion for I=1 three pion ground state]
- F. Romero-López, A. Rusetsky, N. Schlage & C. Urbach, 2010.11715, JHEP [generalized large-volume exps]
- F. Müller & A. Rusetsky, 2012.13957, JHEP [Three-particle analog of Lellouch-Lüscher formula]
- J-Y. Pang, M. Ebert, H-W. Hammer, F. Müller, A. Rusetsky, 2204.04807, JHEP, [Spurious poles in a finite volume]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2110.09351, JHEP [Relativistic-invariant formulation of the NREFT three-particle quantization condition]
- J. Lozano, U. Meißner, F. Romero-López, A. Rusetsky & G. Schierholz, 2205.11316, JHEP [Resonance form factors from finit volume correlation functions with the external field method]
- F. Müller, J-Y. Pang, A. Rusetsky, J-J. Wu, 2211.10126, JHEP [3-particle Lellouch-Lüscher formalism in moving frames]

Alternate 3-particle approaches

★ **Finite-volume unitarity (FVU) approach**

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- M. Mai et al., 1706.06118, EPJA [unitary parametrization of M_3 involving R matrix; used in FVU approach]
- A. Jackura et al., 1809.10523, EPJC [further analysis of R matrix parametrization]
- M. Mai & M. Döring, 1807.04746, PRL [3 pion spectrum at finite-volume from FVU]
- M. Mai et al., 1909.05749, PRD [applying FVU approach to $3\pi^+$ spectrum from Hanlon & Hörz]
- C. Culver et al., 1911.09047, PRD [calculating $3\pi^+$ spectrum and comparing with FVU predictions]
- A. Alexandru et al., 2009.12358, PRD [calculating $3K^-$ spectrum and comparing with FVU predictions]
- R. Brett et al., 2101.06144, PRD [determining $3\pi^+$ interaction from LQCD spectrum]
- M. Mai et al., 2107.03973, PRL [three-body dynamics of the $a_1(1260)$ from LQCD]
- D. Dasadivan et al., 2112.03355, PRD [pole position of $a_1(1260)$ in a unitary framework]

★ **HALQCD approach**

- T. Doi et al. (HALQCD collab.), 1106.2276, Prog.Theor.Phys. [3 nucleon potentials in NR regime]