LQCD- and HRG-based approaches in the search for the critical point

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Taylor Expansion

- $\frac{P(T,\mu_B)}{T^4} = \frac{P(T,0)}{T^4} + \Sigma_i \frac{1}{i!}\, \chi_i^B \left(\frac{\mu_B}{T}\right)^i$

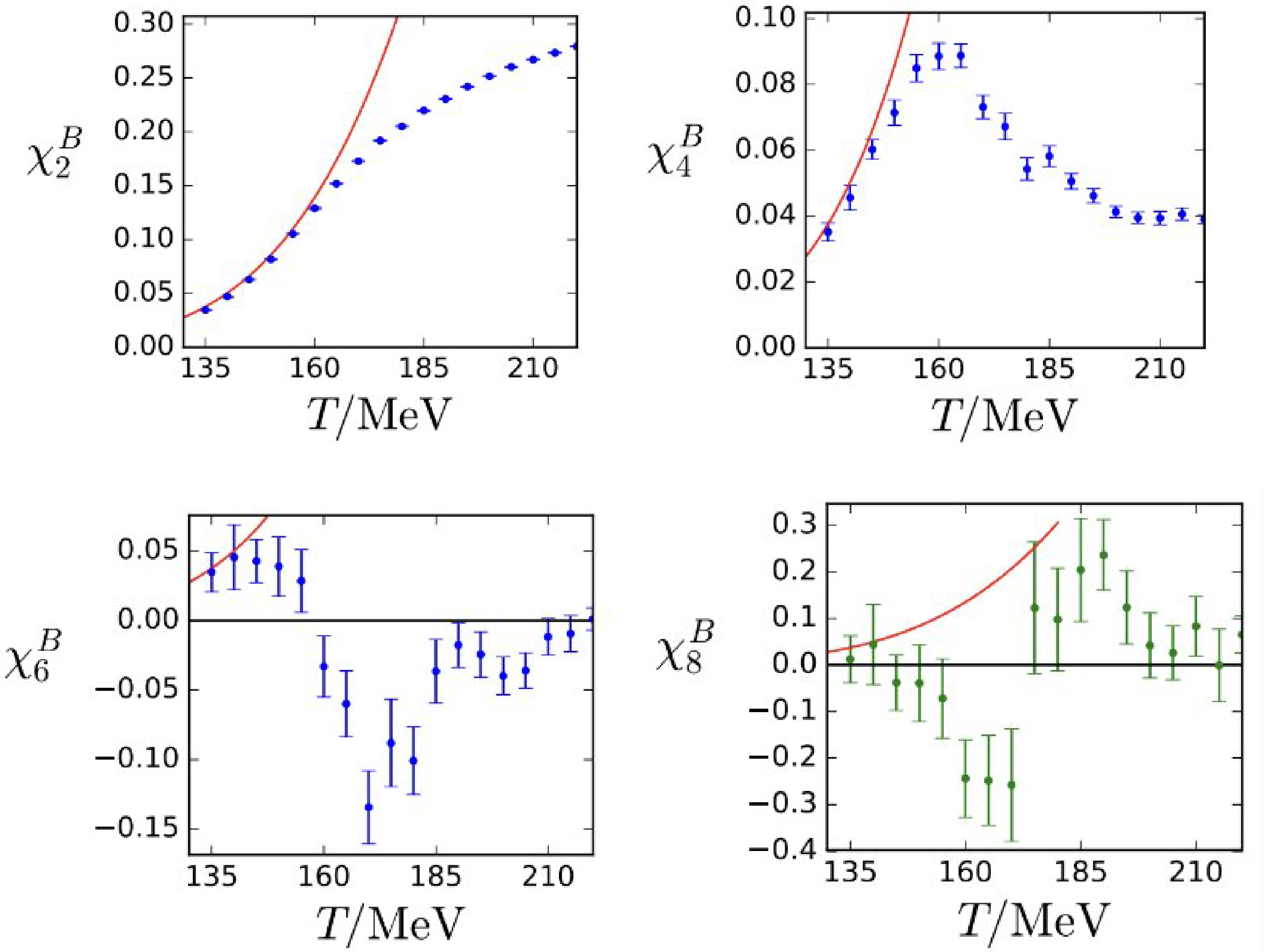
Here, χ_i^B (susceptibilities) are the coefficients

of the expansion given by $\chi_i^B = \frac{\partial^i P(T,\mu_B)/T^4}{\partial\left(\frac{\mu_B}{T}\right)^i}\Big|_{\frac{\mu_B}{T}=0,\,T}$

Borsanyi, S. et al High Energy Physics.9(8), 1-16.(2012)

- ***Limitations***:

1. Expansion at constant T, missing out the curvature of transition line.
2. Computationally difficult to obtain higher order susceptibilities.
3. Large errors due to higher order terms; expansion typically applied at $\frac{\mu_B}{T} \leq 2.5$. No CP in this region.



Borsanyi, S. et al JHEP 10 205 (2018); Bazavov, A et al PhysRevD.95, 054504 (2017)

T' - expansion

New method to re-sum the Taylor expansion based on the following ansatz:

$$\frac{T\chi_1^B(T,\mu_B)}{\mu_B} = \chi_2^B(T',0)$$

where the shifted temperature

$$T' = T\left(1 + \kappa_2^B\left(\frac{\mu_B}{T}\right)^2 + \kappa_4^B\left(\frac{\mu_B}{T}\right)^4\right)$$

$$\kappa_2^B(T) = \frac{\chi_4^B(T)}{6T\chi_2'^B(T)}$$

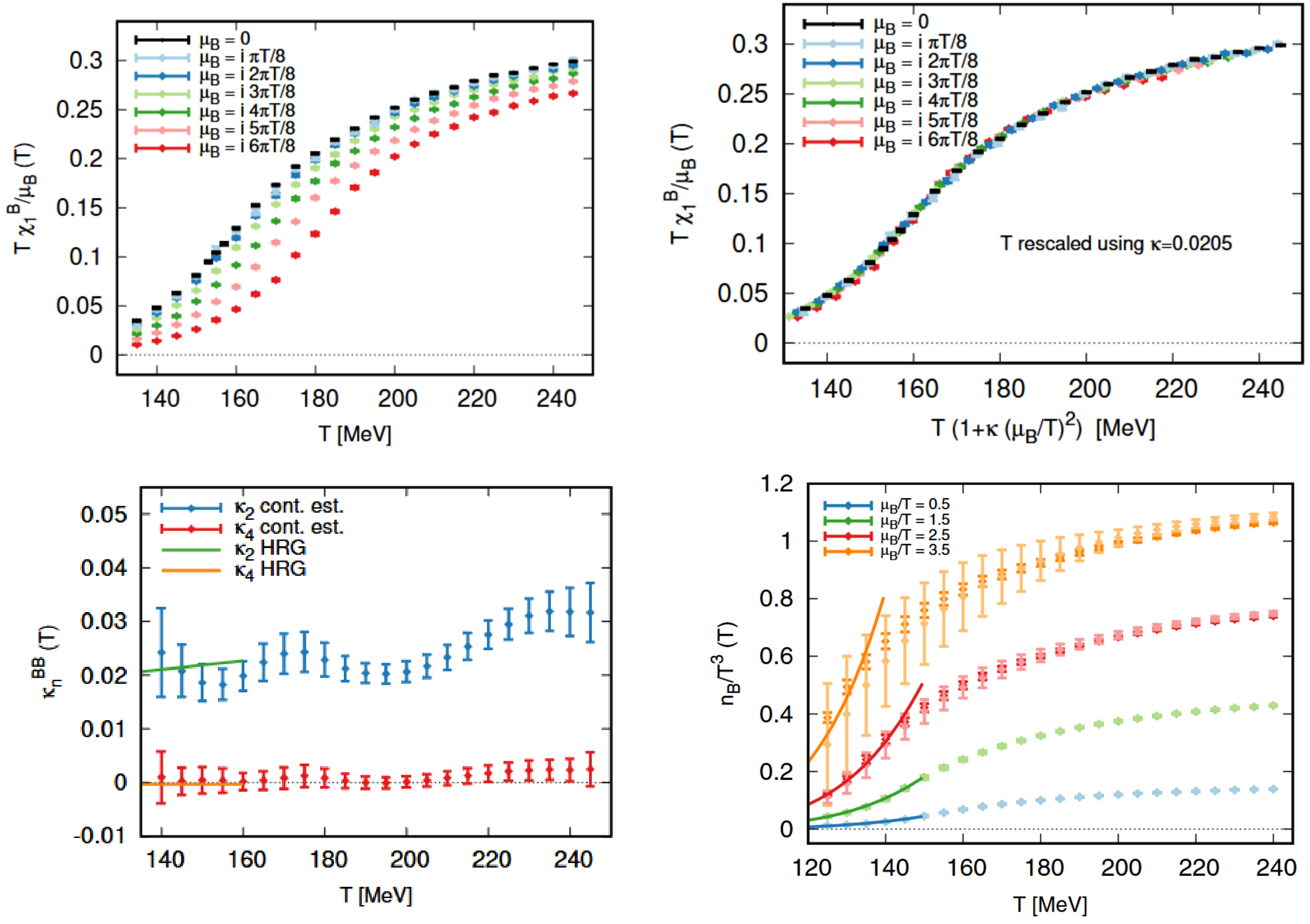
Limitations:

1. This expansion is applied up to $\frac{\mu_B}{T} \leq 3.5$

2. Does not give description of the CP, suggests crossover up to $\frac{\mu_B}{T} = 3.5$

Recent update: This expansion was performed in the 4D plane of (T, μ_B, μ_Q, μ_S) to obtain an EoS with higher coverage than Taylor expansion.

A. Abuali, J. Jahan, HS et al. - Phys.Rev.D 112 (2025) 5, 054502



Borsanyi et al., PRL 126, 232001 (2021)

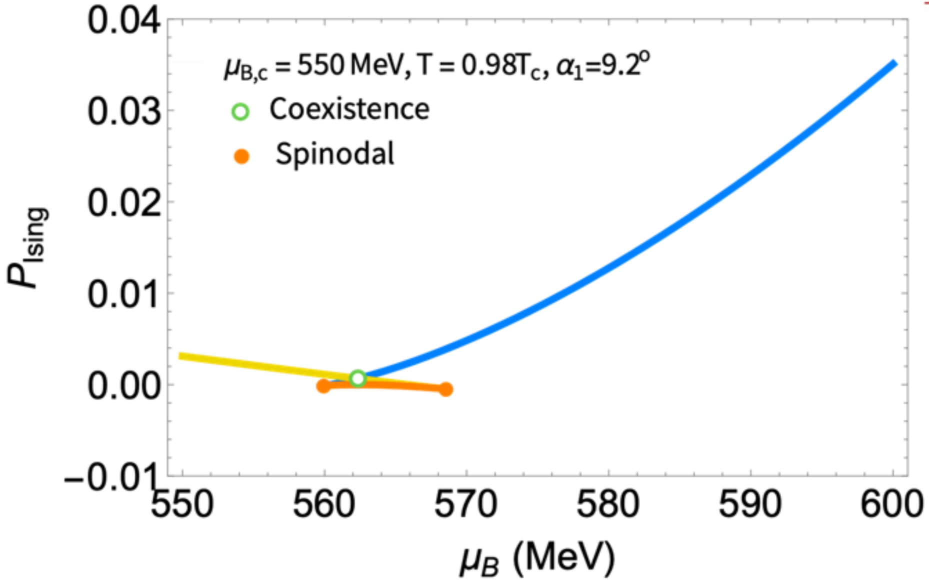
EoS for First Order Regime

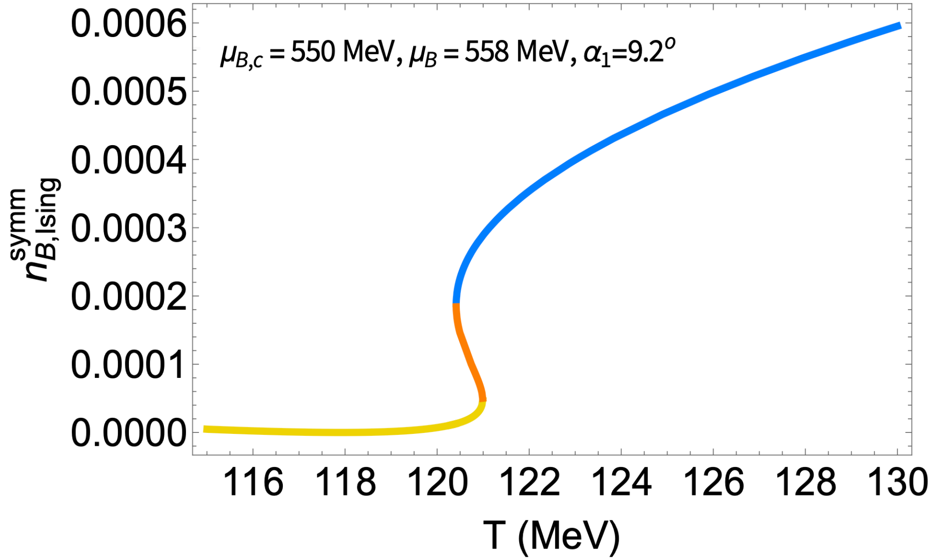
- Considering a mean field Ising model mapped to QCD, we can implement first order features in the phase diagram.
- With Landau theory, we find expected spinodal features in n_B isotherms unlike with 3D Ising where spinodals are Lee-Yang edge singularities in complex plane.
- Mapping parameters including α_1, w, ρ control the shape of critical region and first order features.

$$P(M,r) = h(M,r)M - F(M,r) = \frac{1}{12}(M^4 + 2rM^2)$$

$$(\mathbf{r},\mathbf{h}) \longleftrightarrow (\mathbf{T},\mu_\mathbf{B}) : \quad \frac{T - \mathbf{T_C}}{\mathbf{T_C}} = \mathbf{w}\,(r\rho\,\sin\alpha_1 + h\,\sin\alpha_2)$$

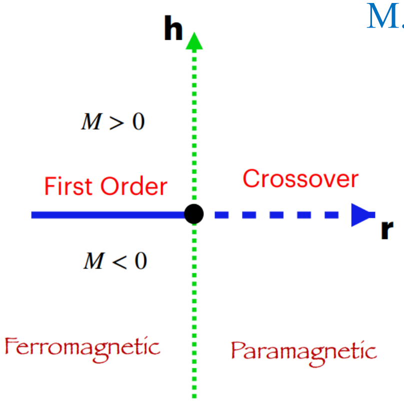
$$\frac{\mu_B - \mu_{\mathbf{BC}}}{\mathbf{T_C}} = \mathbf{w}\,(-r\rho\,\cos\alpha_1 - h\,\cos\alpha_2)$$





J. Karthein, V. Koch, C. Ratti – PRD (2025)

T'-expansion scheme with 3D-Ising critical point



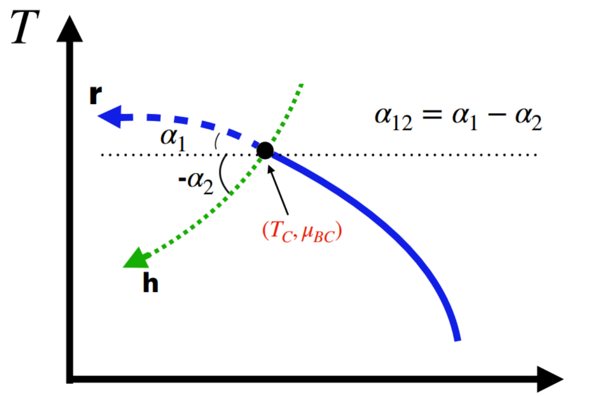
3D Ising coordinates

M. Kahangirwe et al. PRD (2024)

$$\frac{T'-T_0}{T_C T'_T} = -w' h \sin\alpha'_{12}$$

$$\frac{\mu_B^2-\mu_{BC}^2}{2\mu_{BC}T_C} = w'(-r\rho' - h\cos\alpha'_{12})$$

$$T' = T\left[1+\left(\frac{\mu_B}{T}\right)^2 \kappa_2^{BB}(T) + \mathcal{O}\left(\frac{\mu_B}{T}\right)^4\right]$$



QCD coordinates

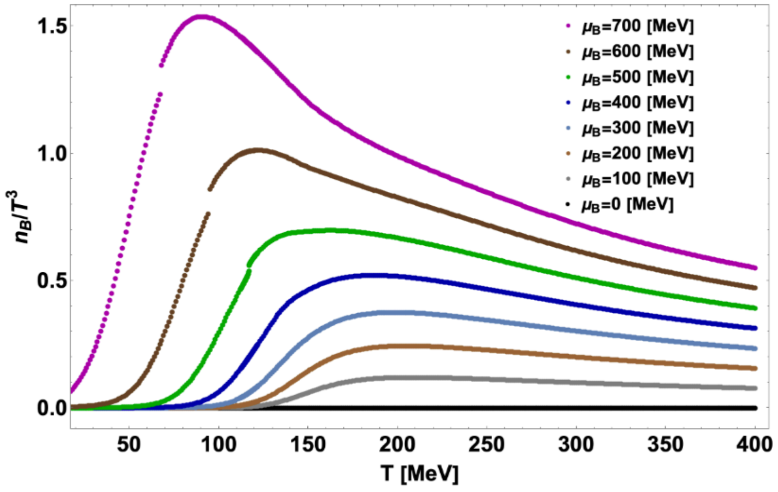
- A critical point in the 3D-Ising model universality class can be incorporated into the T' expansion scheme.

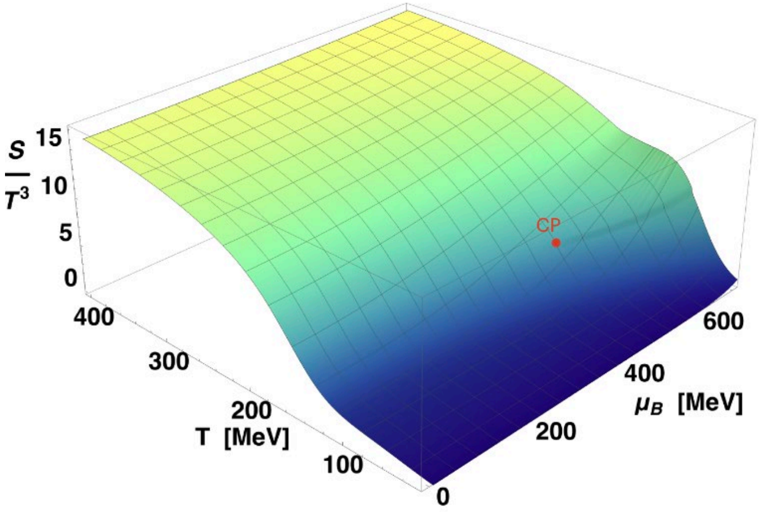
Full Baryon Density

$$\frac{n_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \left(\frac{\mu_B}{T}\right)\chi_{2,lat}^B(T',0)$$

Introducing Critical point

$$T' = T'_{Crit}(T,\mu_B) + T'_{Non-Crit}(T,\mu_B)$$

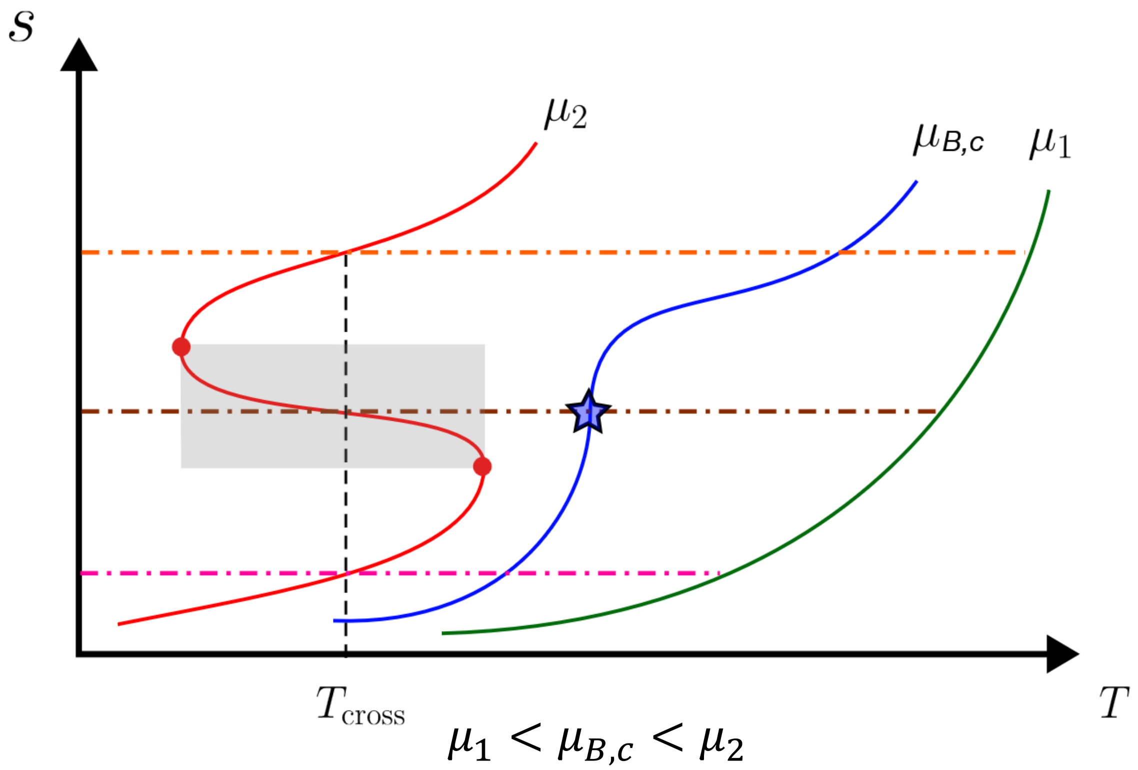


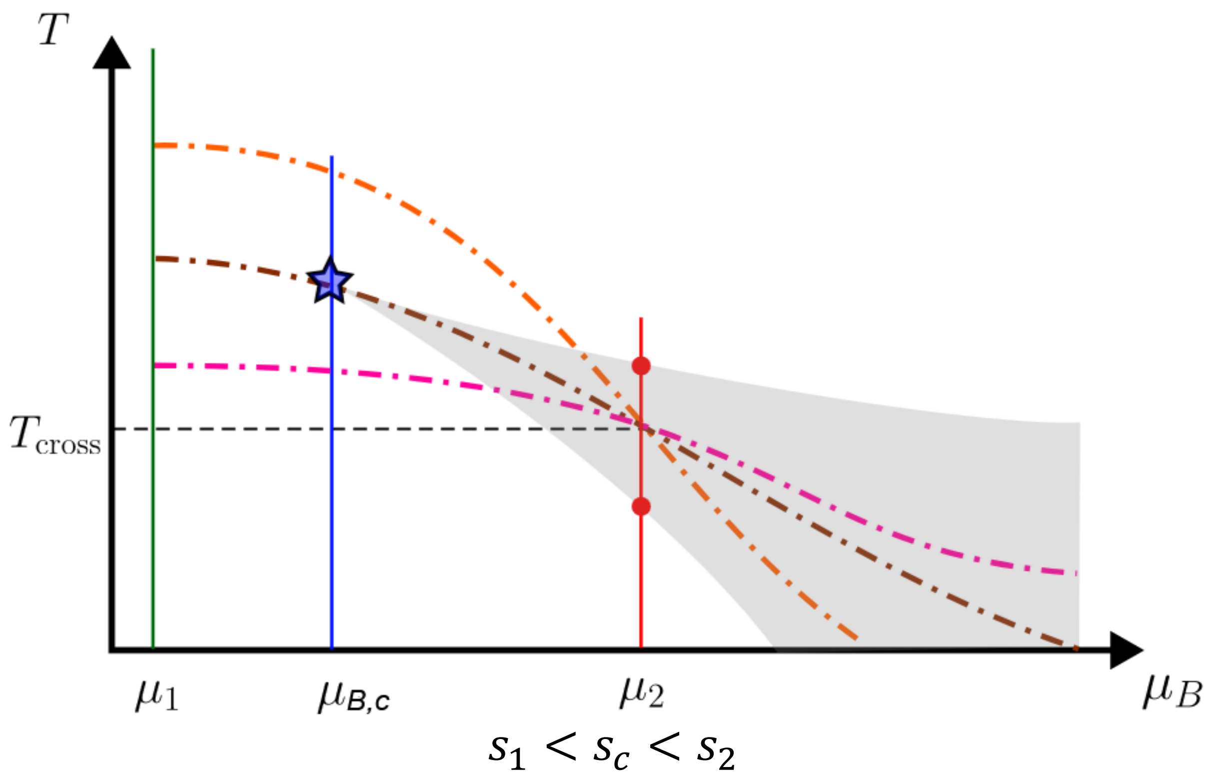


- Equation of state from constant entropy contour method can be compared to this one to constrain free parameters.

Constant entropy density contours

Intuitive picture based on a mean field behavior of the equation of state near a phase transition





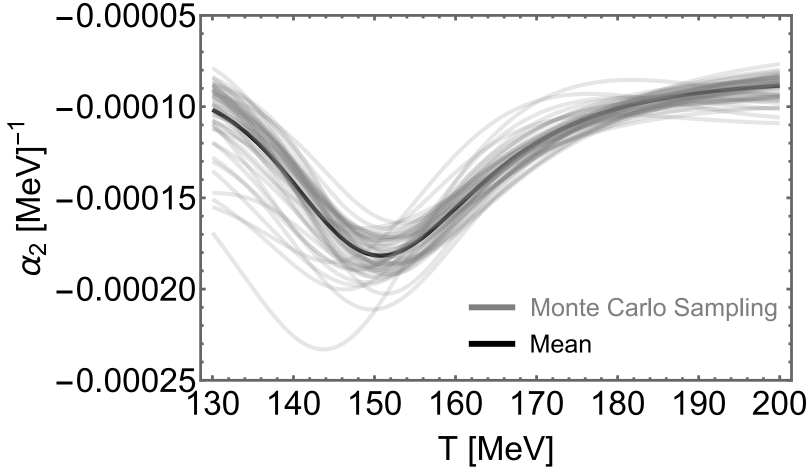
- If the contours of constant entropy density do not cross, it suggests an analytic crossover.
- If the contours of constant entropy density do cross, it suggests a first order phase transition.

HS, M. Hippert, J, Noronha, C. Ratti, V. Vovchenko - arXiv:2410.16026

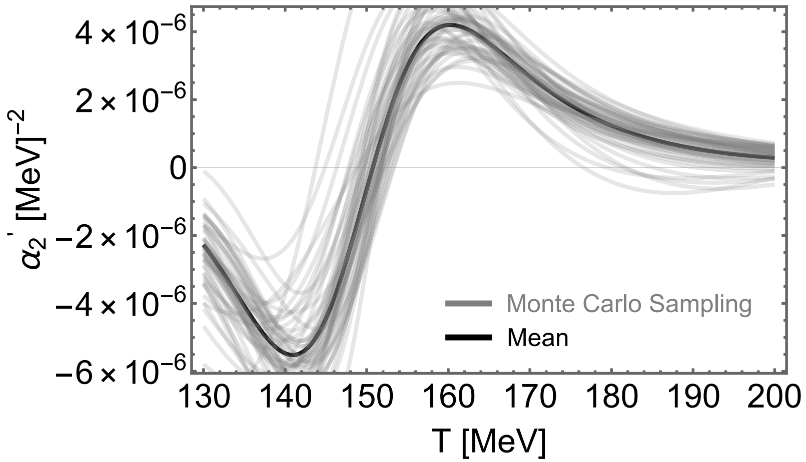
Critical point using α_2

$$T_s(T_0, \mu_B) = T_0 + \alpha_2(T_0) \cdot \frac{\mu_B^2}{2!}$$

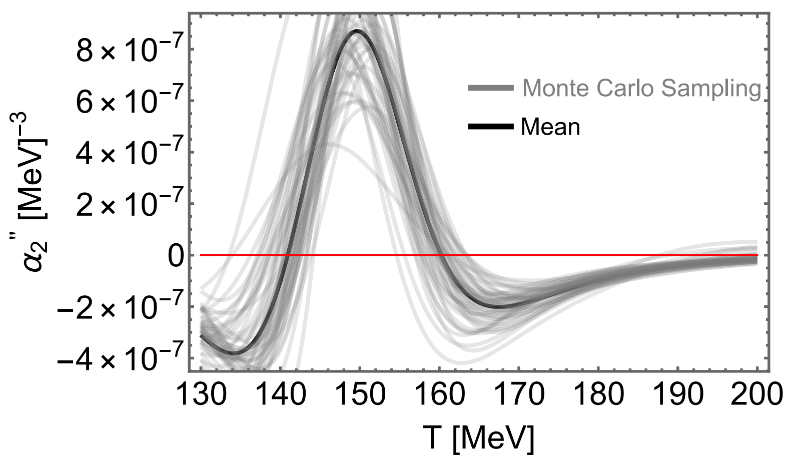
Lattice Parameterization



Lattice Parameterization



Lattice Parameterization



$$\alpha_2(T_0) = -\frac{2T_0\chi_2^B(T_0) + T_0^2\chi_2^{B\prime}(T_0)}{s'(T_0)}$$

$\left(\frac{\partial s}{\partial T}\right)_{\mu_B}$ diverges in first order phase transition.

Hence, $\left(\frac{\partial T_s}{\partial s}\right)_{\mu_B} = 0$

CP is an inflection point

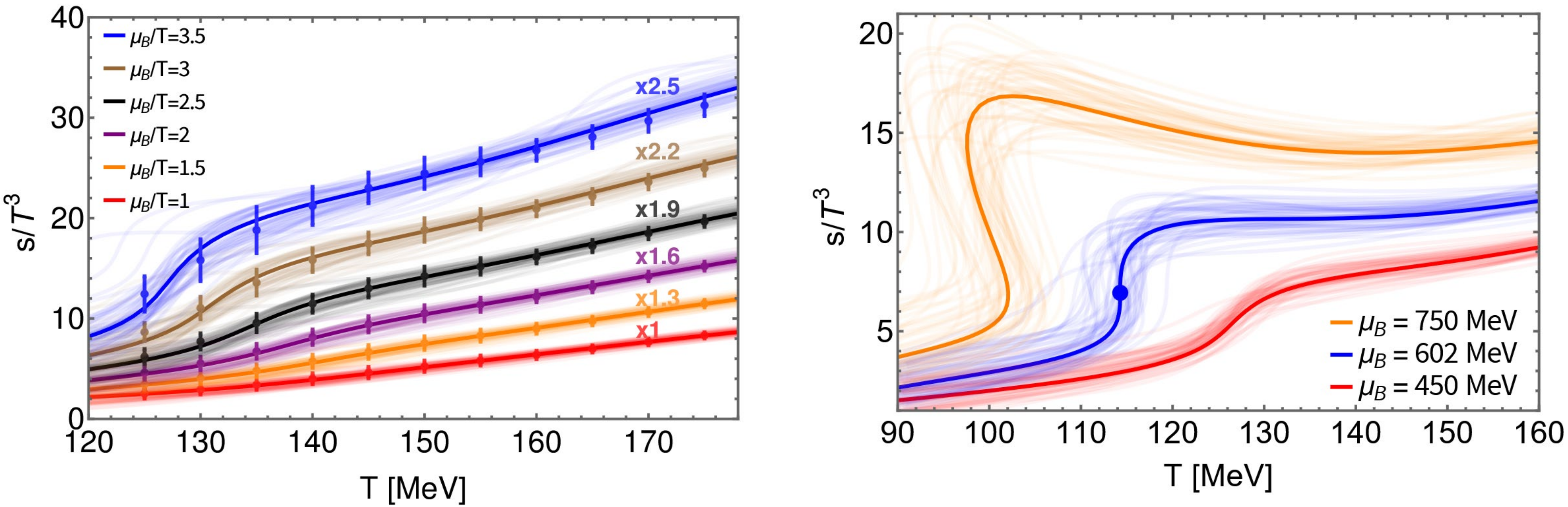
Hence, $\left(\frac{\partial^2 T_s}{\partial s^2}\right)_{\mu_B} = 0$

$$T_c = T_{0c} + \alpha_2(T_{0c}) \cdot \frac{\mu_{Bc}^2}{2} \longleftarrow \Rightarrow \mu_{B,c} = \sqrt{-\frac{2}{\alpha_2'(T_{0c})}} \longleftarrow \Rightarrow T_{0c} \to \alpha_2''(T_{0c}) = 0$$

HS, M. Hippert, J, Noronha, C. Ratti, V. Vovchenko - arXiv:2410.16026

Expansion using constant entropy density contours

Normalized entropy density at finite T and μ_B

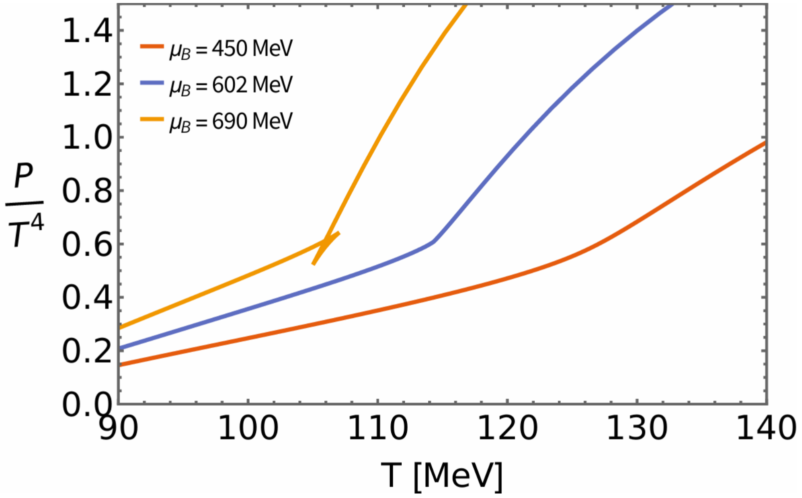


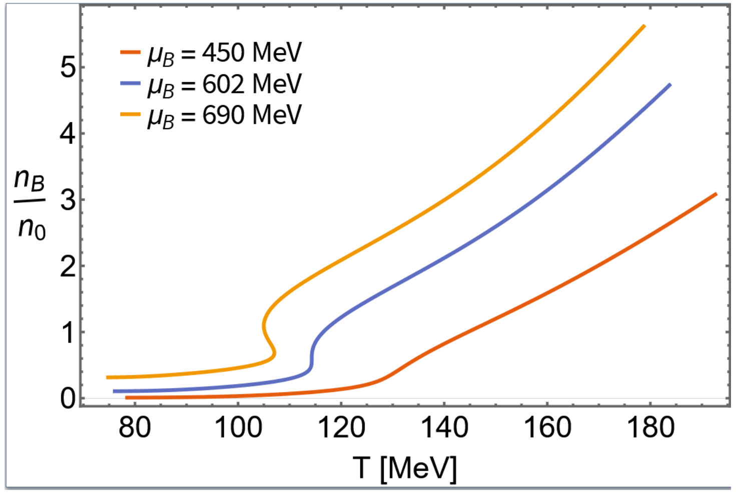
- Shows excellent agreement with available lattice QCD data up to $\frac{\mu_B}{T} \leq 3.5$ from T' expansion scheme.

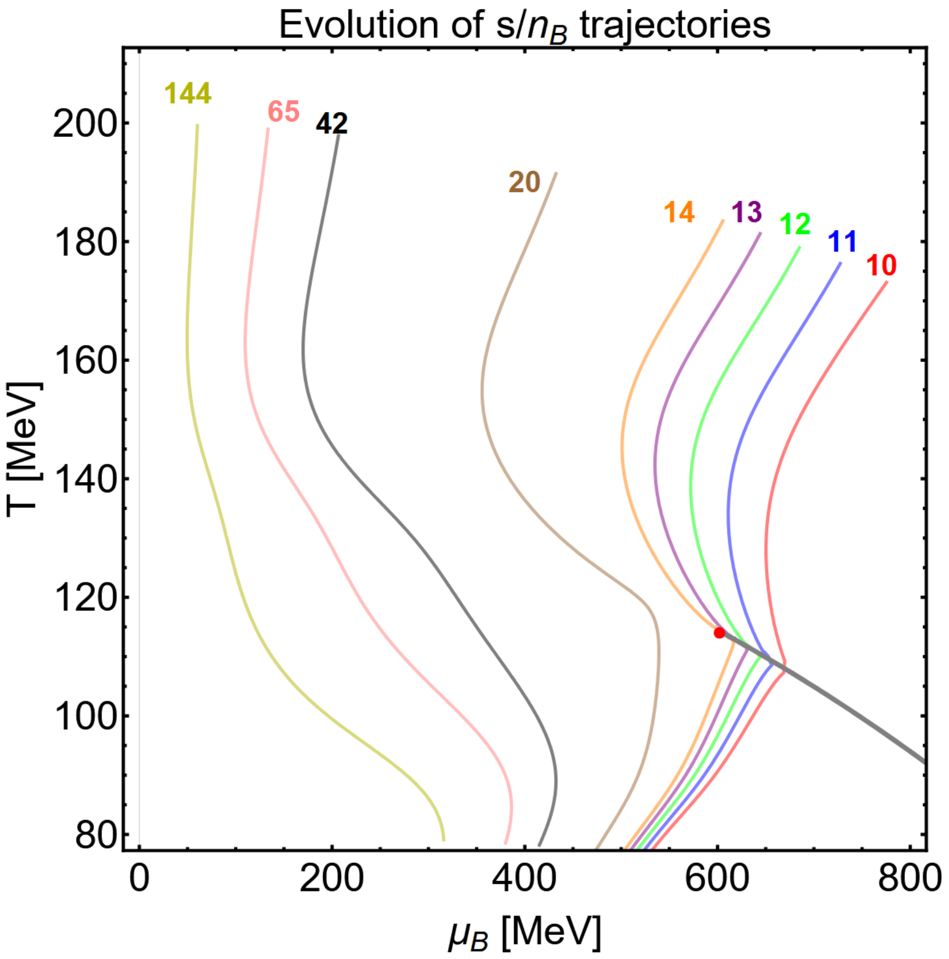
Borsanyi et al., PRL 126, 232001 (2021)

- Provides the first order phase transition region at higher μ_B.
- S-shape suggests a mean field behavior due to the analytic expression of the truncated expansion.

Equation of State with s-contours expansion



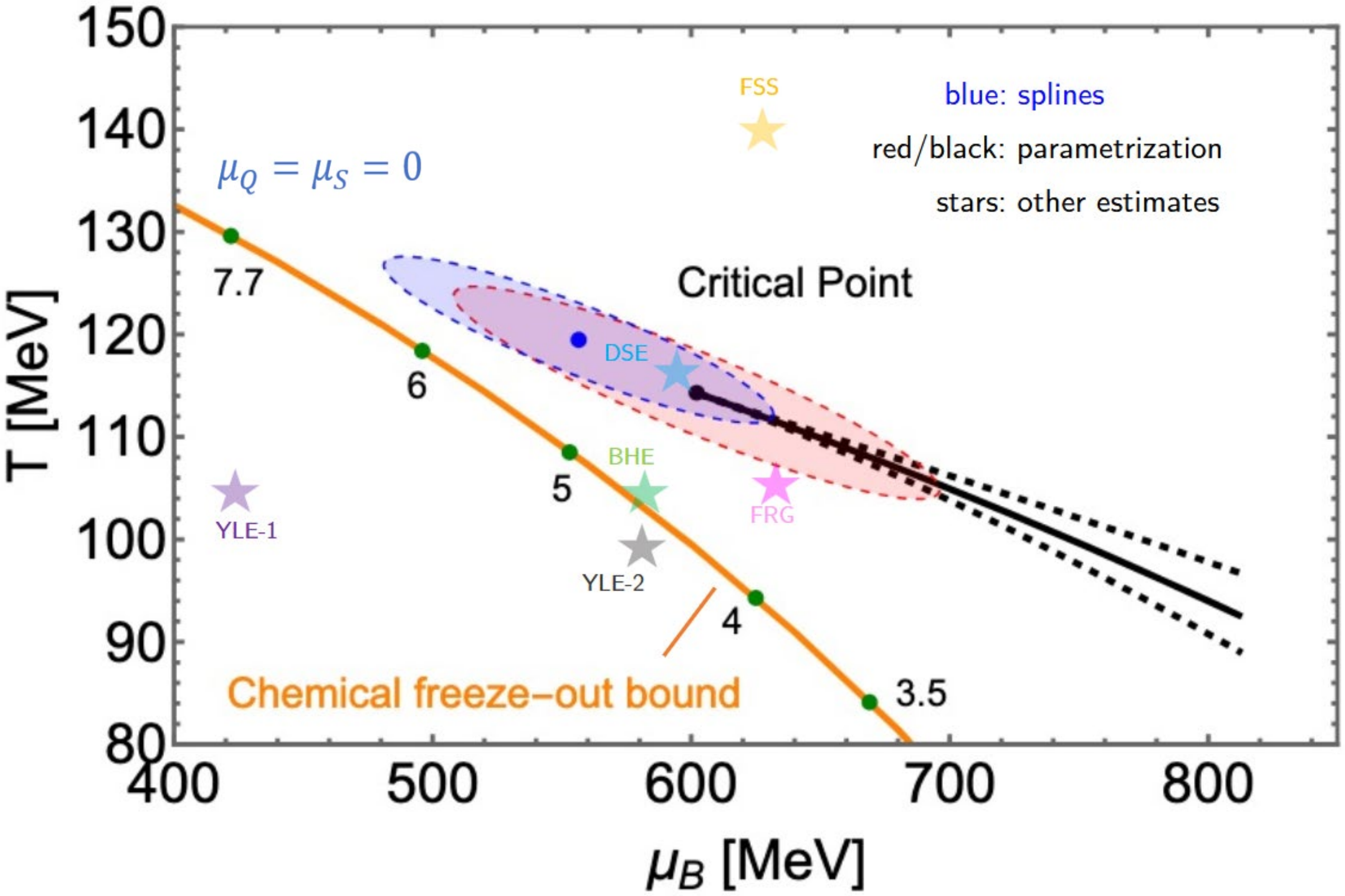




- Pressure, baryon density as a function of T for constant μ_B.
- An HRG baseline at $T = 80$ MeV was used to reproduce the thermodynamics.
- $\frac{s}{n_B}$ isentropic trajectories across the phase diagram attracting towards the CP using s-contours expansion.

HS, M. Hippert, J, Noronha, C. Ratti, V. Vovchenko – Work in preparation

Expansion using constant entropy density contours



CP obtained using the relations:

$$\mu_{B,c} = \sqrt{-\frac{2}{\alpha_2'(T_{0c})}} \qquad \alpha_2''(T_{0c}) = 0$$

CP location @ $O(\mu_B^2)$:

$$\mu_{B,c} = 602 \pm 62 \text{ MeV}$$
$$T_c = 114 \pm 7 \text{ MeV}$$

- YLE-1: D.A. Clarke et al, arXiv:2405.10196
- YLE-2: G. Basar, PRC 110, 015203 (2024)
- BHE: M. Hippert et al., PRD 110, 094006 (2024)
- FRG: W-J. Fu et al., PRD 101, 054032 (2020)
- DSE: P.J. Gunkel et al., PRD 104, 052202 (2021)
- DSE/fRG: Gao, Pawlowski., PLB 820, 136584 (2021)
- FSS: A. Sorensen et al., arXiv:2405.10278

- The green points on the freezeout curve show the center of mass energy $\sqrt{s_{NN}}$ in GeV from RHIC.
- Chemical freeze-out estimate for the case $\mu_Q = \mu_S = 0$.

A. Lysenko et al. arXiv:2408.06473

s – contours in QCD based models

Holographic Blackhole model

True CP :- $(T, \mu_B) \rightarrow (103{,}599)$ MeV

Hippert et. al.- Phys.Rev.D 110 (2024) 9, 094006

Extrapolated CP :- $(T, \mu_B) \rightarrow (104{,}637)$ MeV

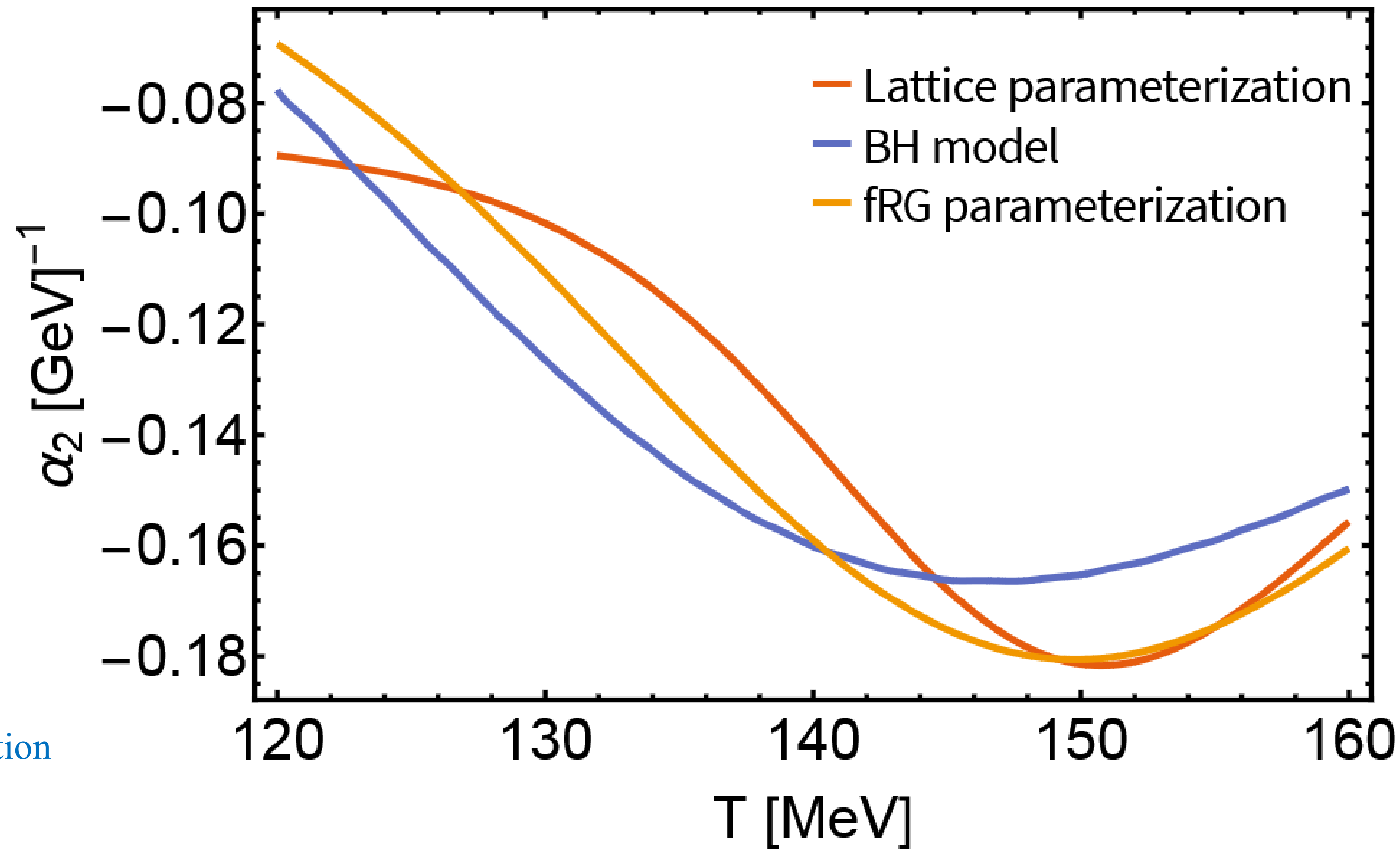
fRG – DSE approach

True CP :- $(T, \mu_B) \rightarrow (103{,}660)$MeV

Y. Lu, J. Pawlowski et. al.- 2504.05099

Extrapolated CP :- $(T, \mu_B) \rightarrow (108{,}629)$MeV

HS, M. Hippert, J, Noronha, C. Ratti, V. Vovchenko – Work in preparation

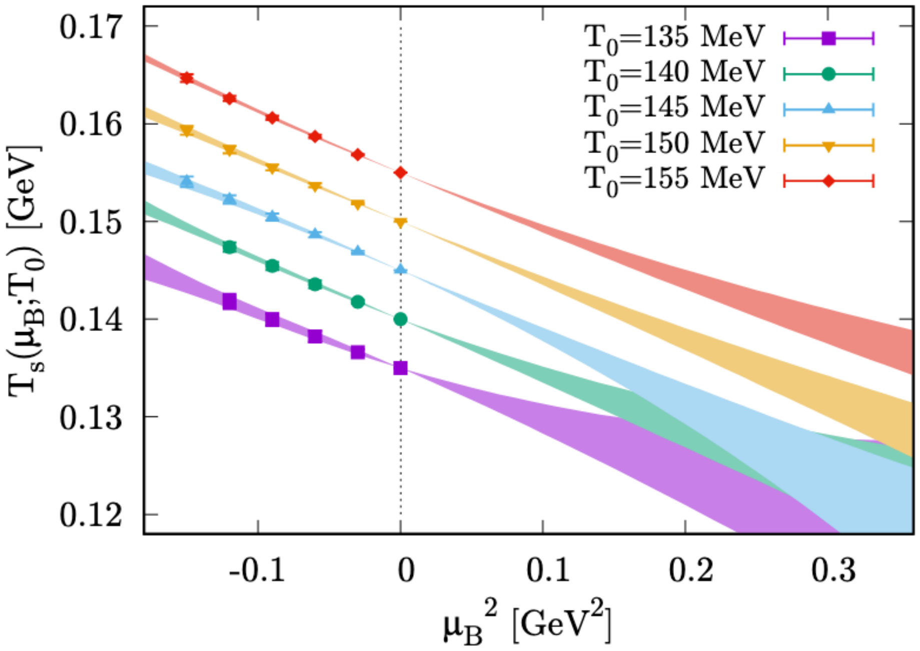


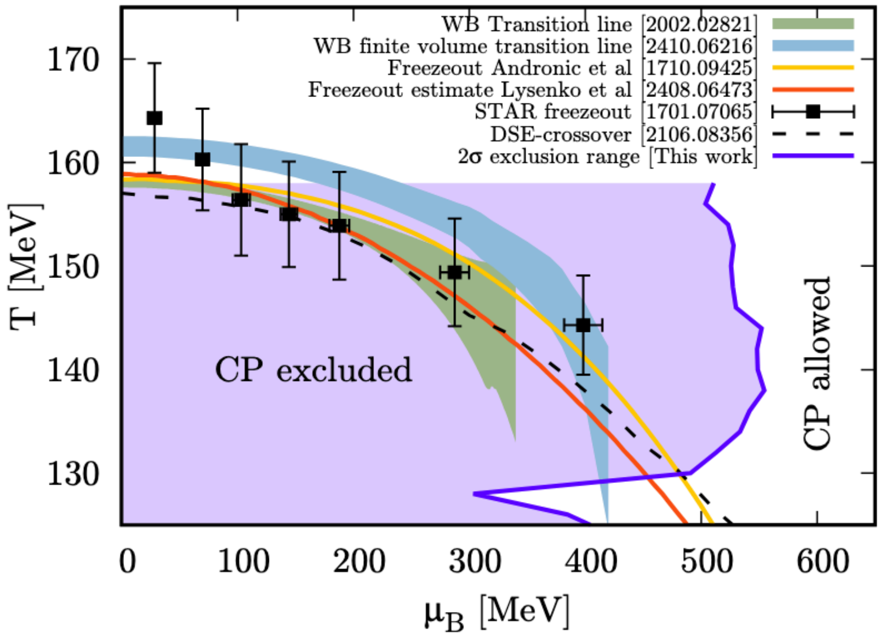
- HRG models do show a fake CP behavior at low T_0 value and the fake CP is obtained at a very high $\mu_B \sim 830$ MeV

M. Marczenko et. al Phys.Rev.D 112 (2025) 3, 034019

Direct extrapolation of LQCD from Imaginary μ_B

- This method was employed to obtain more stringent constraints on the CP location from lattice QCD.
- The new exclusion region for the CP, at the 2σ level, is $\mu_B < 450$ MeV.
- These results are obtained by combining an improved-precision lattice QCD EoS at $\mu_B = 0$ and direct extrapolation of constant entropy contours from imaginary μ_B.

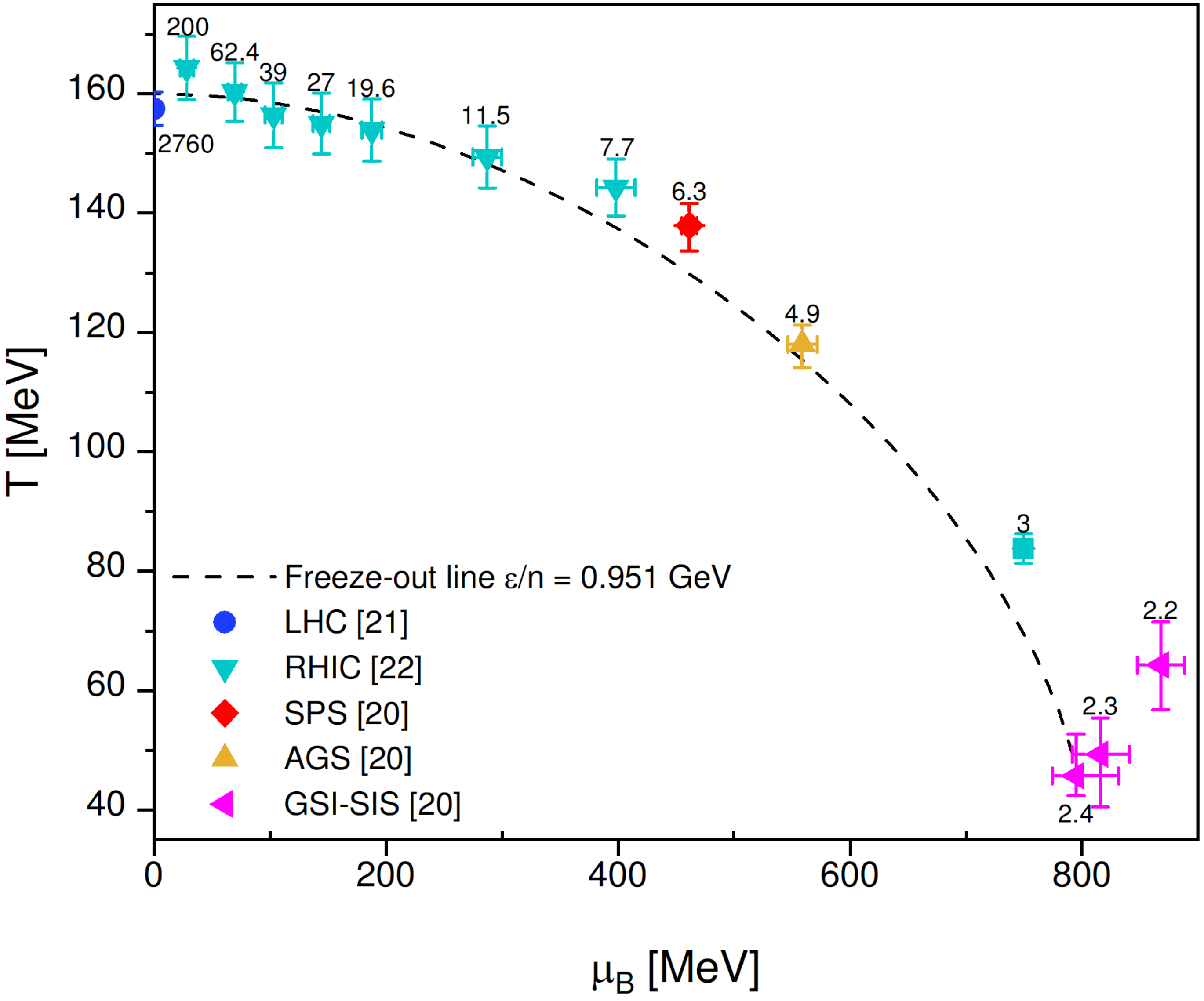




S. Borsanyi et al. arXiv:2502.10267

These results are obtained in the strangeness-neutral scenario

Chemical freeze-out



[Lysenko, R.P., Gorenstein, Vovchenko, PRC, 2025]

$$E/N \equiv \varepsilon/n \sim 0.9 - 1.0 \text{ GeV}$$

[Cleymans, Redlich, PRL 1998]

Ideal-HRG

$$T_{\rm ch}(\mu_B = 0) = 160 \text{ MeV}$$

$$\Rightarrow \quad \frac{\varepsilon}{n} = 0.951 \text{ GeV}$$

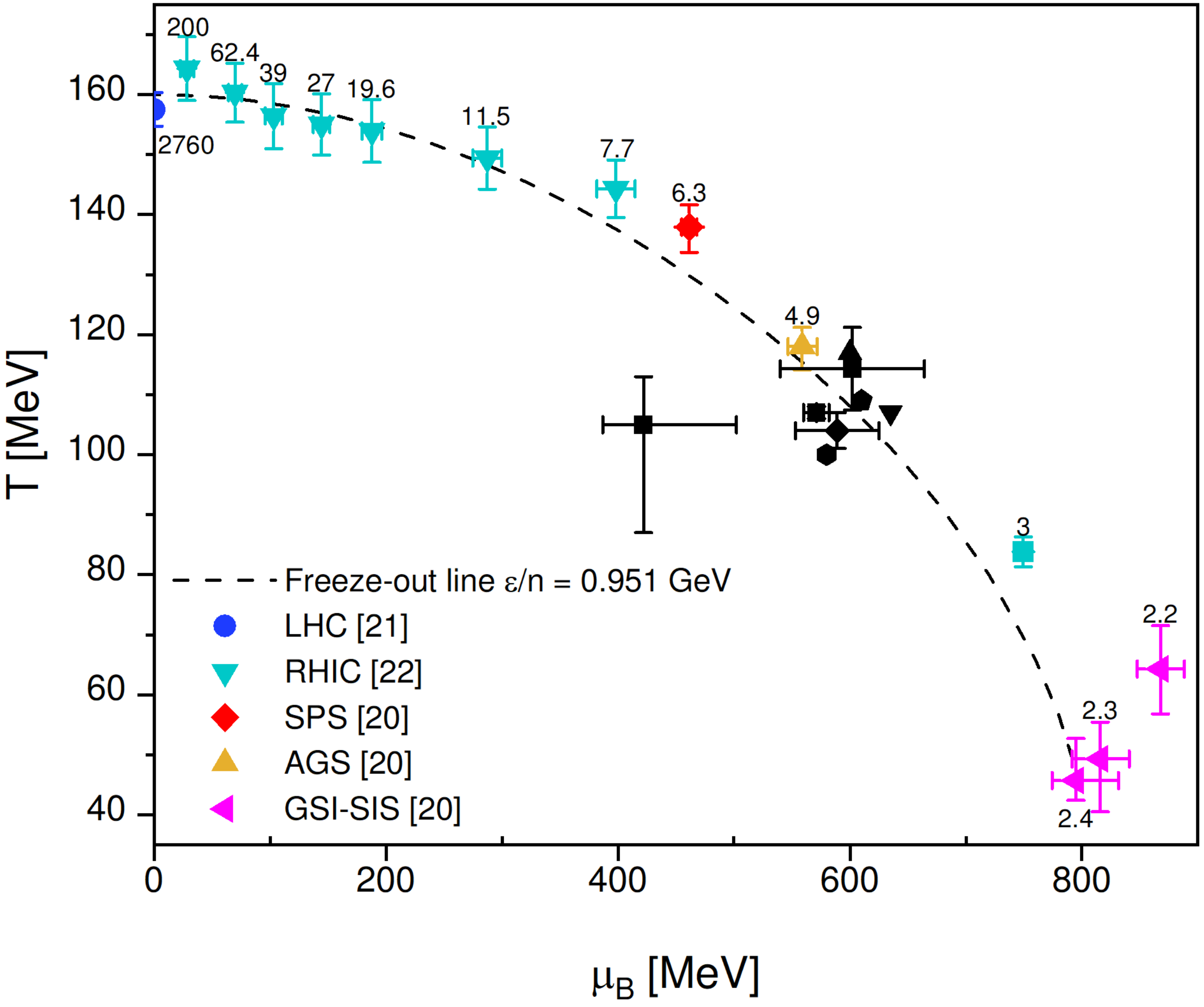
$$\frac{\varepsilon(T,\mu_B,\mu_Q,\mu_S)}{n(T,\mu_B,\mu_Q,\mu_S)} = \text{const},$$

$$\frac{n_Q(T,\mu_B,\mu_Q,\mu_S)}{n_B(T,\mu_B,\mu_Q,\mu_S)} = 0.4,$$

$$n_S(T,\mu_B,\mu_Q,\mu_S) = 0.$$

Thermal-FIST

Chemical freeze-out



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$$\frac{\varepsilon(T,\mu_B,\mu_Q,\mu_S)}{n(T,\mu_B,\mu_Q,\mu_S)} = \text{const},$$

$$\frac{n_Q(T,\mu_B,\mu_Q,\mu_S)}{n_B(T,\mu_B,\mu_Q,\mu_S)} = 0.4,$$

$$n_S(T,\mu_B,\mu_Q,\mu_S) = 0.$$

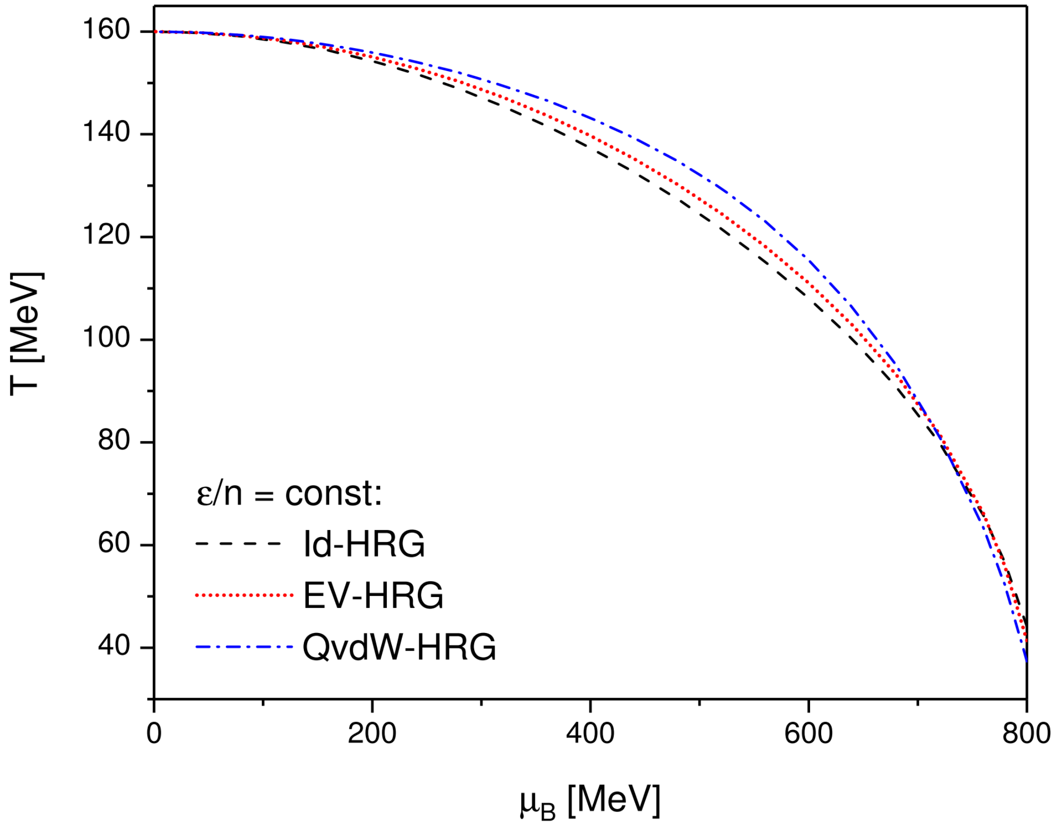
Thermal-FIST

Influence of strangeness neutrality and interactions

Calculations of the FO curve: $S = 0 \text{ and } \langle Q/B \rangle = 0.4 \;\Rightarrow\; \mu_S \sim \mu_B/3 \text{ and } \mu_Q \sim -\mu_B/30$

Theoretical estimates of the CP: $\mu_S = \mu_Q = 0$

We assume that at the freeze-out $(\varepsilon/n)_{\mathrm{FO}} = 0.951 \text{ GeV} \neq f(\mu_S, \mu_Q)$



[Lysenko, R.P., Gorenstein, Vovchenko, PRC, 2025]

Baryon-baryon interactions:

QvdW-HRG: $a = 329 \text{ MeV} \cdot \text{fm}^3,\; b = 3.42 \text{ fm}^3$

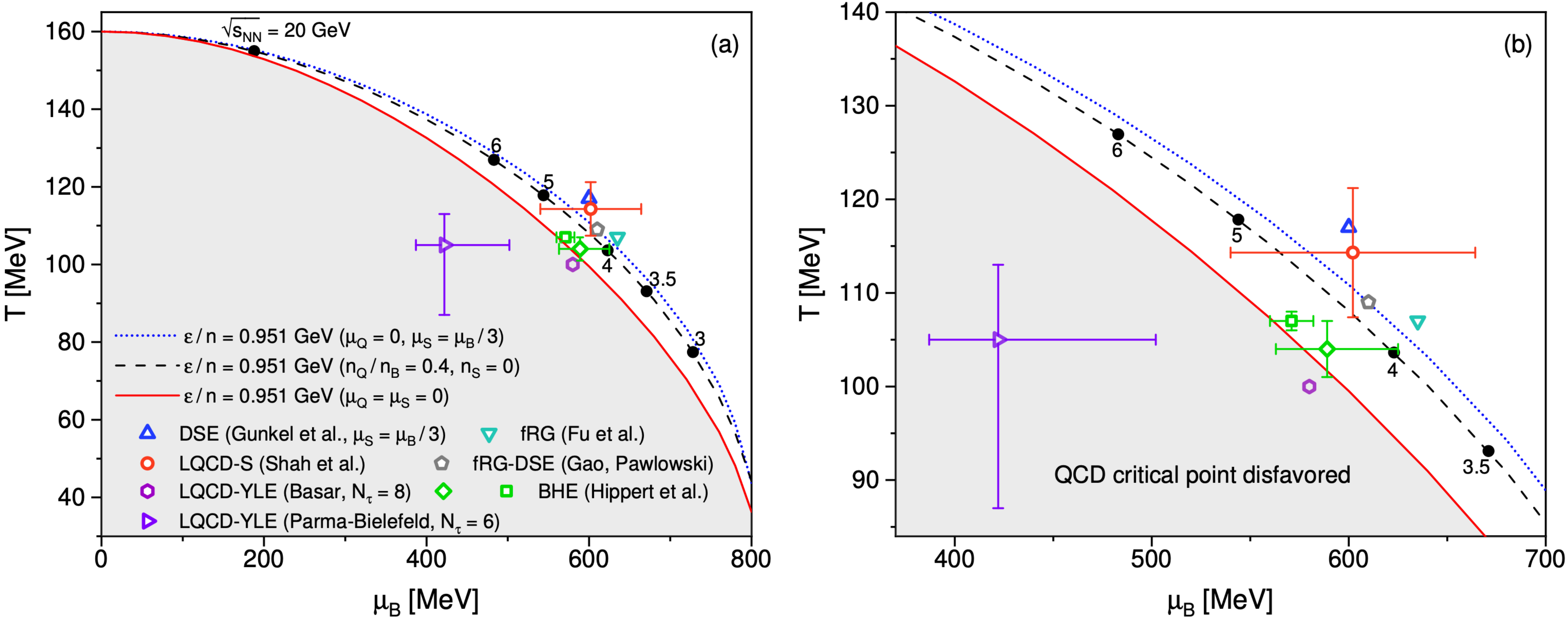
EV-HRG: $b = 1 \text{ fm}^3$

$T_{\mathrm{ch}}(\mu_B = 0) = 160 \text{ MeV} \;\Longrightarrow$

- Id-HRG: 0.951 GeV
- EV-HRG: 0.946 GeV
- QvdW-HRG: 0.942 GeV

Thus ε/n =0.951 GeV line from Ideal HRG at $\mu_Q, \mu_S = 0$ is a data-driven lower bound on the location of QCD CP.

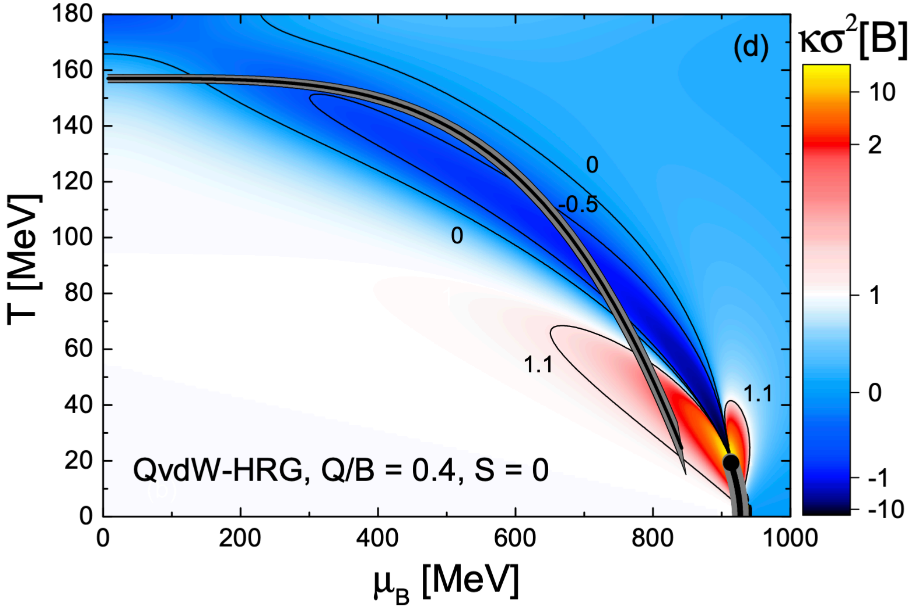
Lower bound on the location of QCD CP



$\mu_B \cong \frac{c}{1 + d\sqrt{s_{\rm NN}}} \Rightarrow \sqrt{s_{\rm NN}} = 3.5 \div 5$ GeV most promising in search for the CP (RHIC FXT, CBM at FAIR)

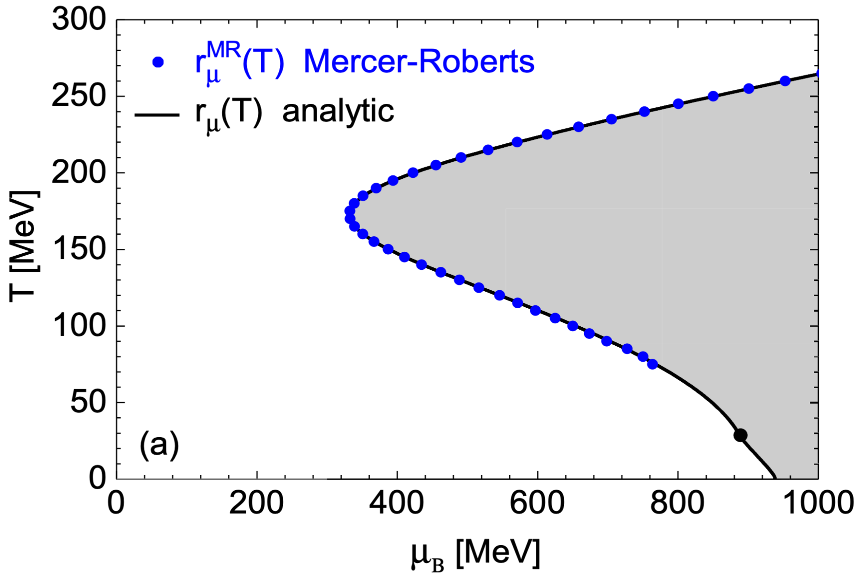
Interacting HRG

Net-baryon fluctuations



[R.P., Vovchenko, Motornenko, Gorenstein, Stoecker, PRC, 2019]

Radius of convergence



[Savchuk, Vovchenko, R.P., Gorenstein, Stoecker, PRC, 2020]

Remnants of the nuclear CP shine across the phase diagram

Outlook: Merging with high temperature/density equations of state, such as Holography

muses

Summary

Lattice Based Summary:

- EoS from lattice QCD expansion available in market for hydrodynamic simulations, with and without an insertion of CP. Lattice QCD based EoS available in 4D with an extended coverage than Taylor expansion.

- The latest 4D equation of state derived from lattice QCD extrapolation features a smooth crossover and extends the range of applicability to higher μ_B beyond the limits of standard Taylor expansions.

- Constant s contours expansion by construction might provide a CP signature. The method when applied to lattice data excludes a CP for $\mu_B < 450$ MeV for heavy ion conditions and predicts a CP at $(T\ ;\ \mu_B) \rightarrow (114 \pm 7\ ; 602 \pm 62)$ MeV at order of μ_B^2 for $\mu_Q = \mu_S = 0$.

- However, with the constant s contours expansion, more model-based analysis is required, with a possibility to improve the method (analytic continuation, higher order coefficients?)

HRG Based Summary:

- Line $\epsilon/n \approx 0.95$ GeV calculated within Ideal HRG for $\mu_S = \mu_Q = 0$ sets the lower bound in temperature on the location of QCD CP.

- Interactions within HRG are required to describe nuclear ground state, fluctuations and for merging with other equations of state.

Backup slides

Testing the convergence properties of the T'-expansion scheme

Taylor Expansion

$$\frac{n_B(T,\mu_B)}{T^3} = \sum_{n=0}^{\infty} \frac{1}{(2n-1)!} \chi_{2n}(T,\mu_B=0) \left(\frac{\mu_B}{T}\right)^{2n-1}$$

$$\kappa_2^B = \frac{1}{T\chi_2'^B}\left(\frac{\chi_4^B}{3!}\right)$$

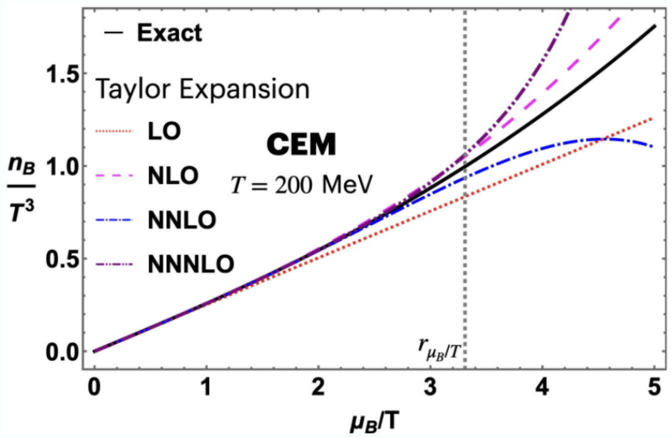
T'-Expansion Scheme

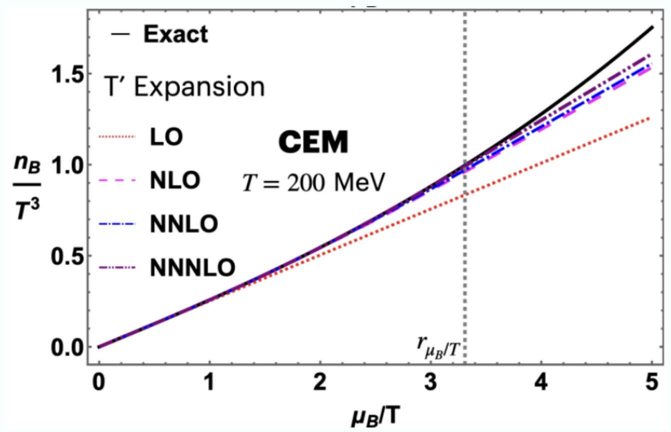
$$\frac{n_B(T,\mu_B)}{T^3} = \chi_1^B(T,\mu_B) = \frac{\mu_B}{T}\chi_2^B(T',0)$$

$$T'(T,\mu_B) = T\left[1 + \kappa_2^B(T)\,\hat{\mu}_B^2 + \kappa_4^B(T)\,\hat{\mu}_B^4 + \mathcal{O}\left(\hat{\mu}_B\right)^6\right]$$

Steps

- We select an effective model that possesses an analytical structure eg . **Cluster Expansion Model (CEM)**
- Compute all necessary Taylor coefficients $\chi_2, \chi_4, \ldots$ and $\kappa_2, \kappa_4, \ldots$ for T'-expansion schemes .
- Compare n_B/T^3 from the two expansions order by order.





The T'-expansion scheme outperforms the Taylor expansion in models for which this scaling holds:

$$\chi_2' \sim \chi_4 \quad \longrightarrow \quad \frac{\partial}{\partial T} \sim \frac{\partial^2}{\partial \mu_B^2}$$

However, T'-expansion provides limited benefits in the hadronic phase at low T and large muB

M. Kahangirwe et al., PRD (2025)

Caveats in entropy contours expansion

- This is a proof of principle: It is an expansion that can describe the critical point by following constant s contours, but accuracy depends on how good the truncated expansion is.

- In our work, we truncate the expansion at order μ_B^2. The μ_B^4 order is given by:

$$T_s(T_0;\mu_B) = T_0 + \frac{\alpha_2(T_0)\mu_B^2}{2!} + \frac{\alpha_4(T_0)\mu_B^4}{4!}$$

$$\alpha_4(T_0) = \left(\frac{\partial^4 T}{\partial \mu_B^4}\right)_s \Bigg|_{T=T_0,\mu_B=0} = -\frac{\chi_4^{B'}(T_0)}{s'(T_0)} - \frac{6\alpha_2(T_0)\left[2\chi_2^B(T_0) + 4T\chi_2^{B'}(T_0) + T^2\chi_2^{B''}(T_0)\right]}{s'(T_0)} - \frac{3s''(T_0)\,\alpha_2^2(T_0)}{s'(T_0)}$$

- To calculate the CP based on $\alpha_4(T_0)$, one also requires to calculate $\alpha_4'(T_0)$ and $\alpha_4''(T_0)$

- Difficult to estimate the CP location as higher T derivatives lead to larger error bars.

- We suggest to look for constant s contours at imaginary chemical potential .

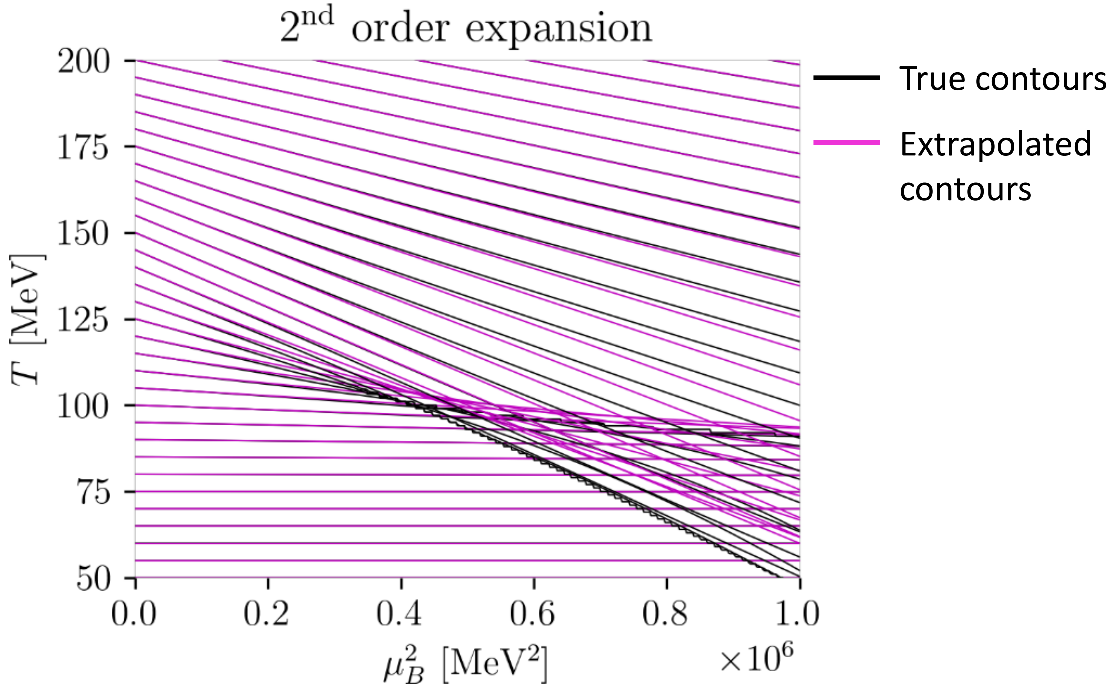
- This was recently applied in the strangeness neutral direction (we have $\mu_Q = \mu_S = 0$) by the WB lattice QCD Collaboration

  Excludes CP < 450 MeV

  S. Borsanyi et al. arXiv:2502.10267

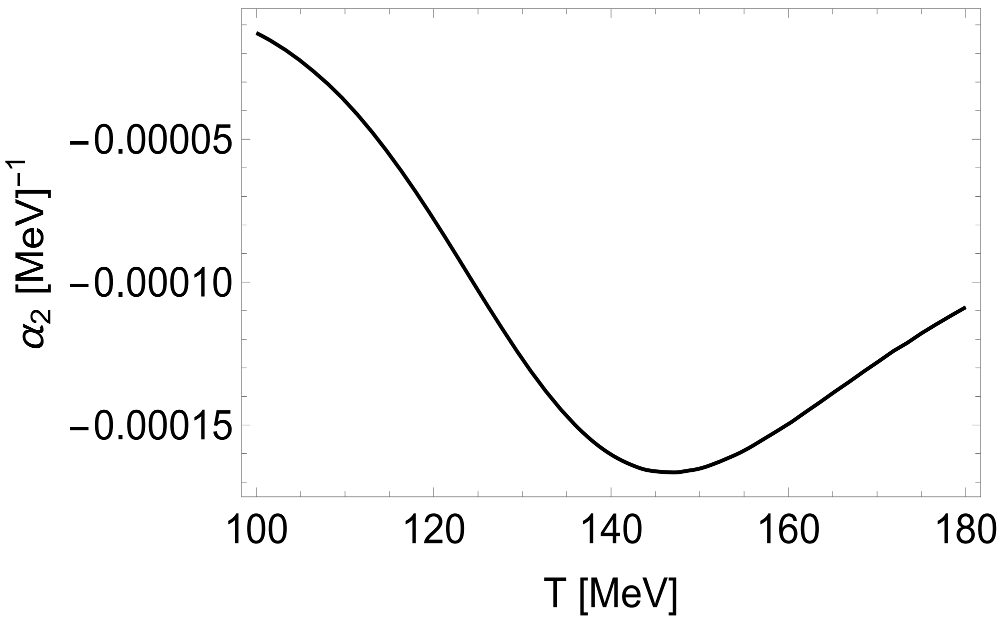
Holographic Black-hole Model

- Good description of lattice + QGP phenomenology
- Contours of s are nearly quadratic



- True CP → $(T_c, \mu_{Bc})\sim(103, 599)$ MeV
- Extrapolated CP → $(T_c, \mu_{Bc})\sim(104, 637)$MeV

α_2 from BH model at $\mu_B = 0$



- Analysis of method properties and performance within a variety of models is in progress.

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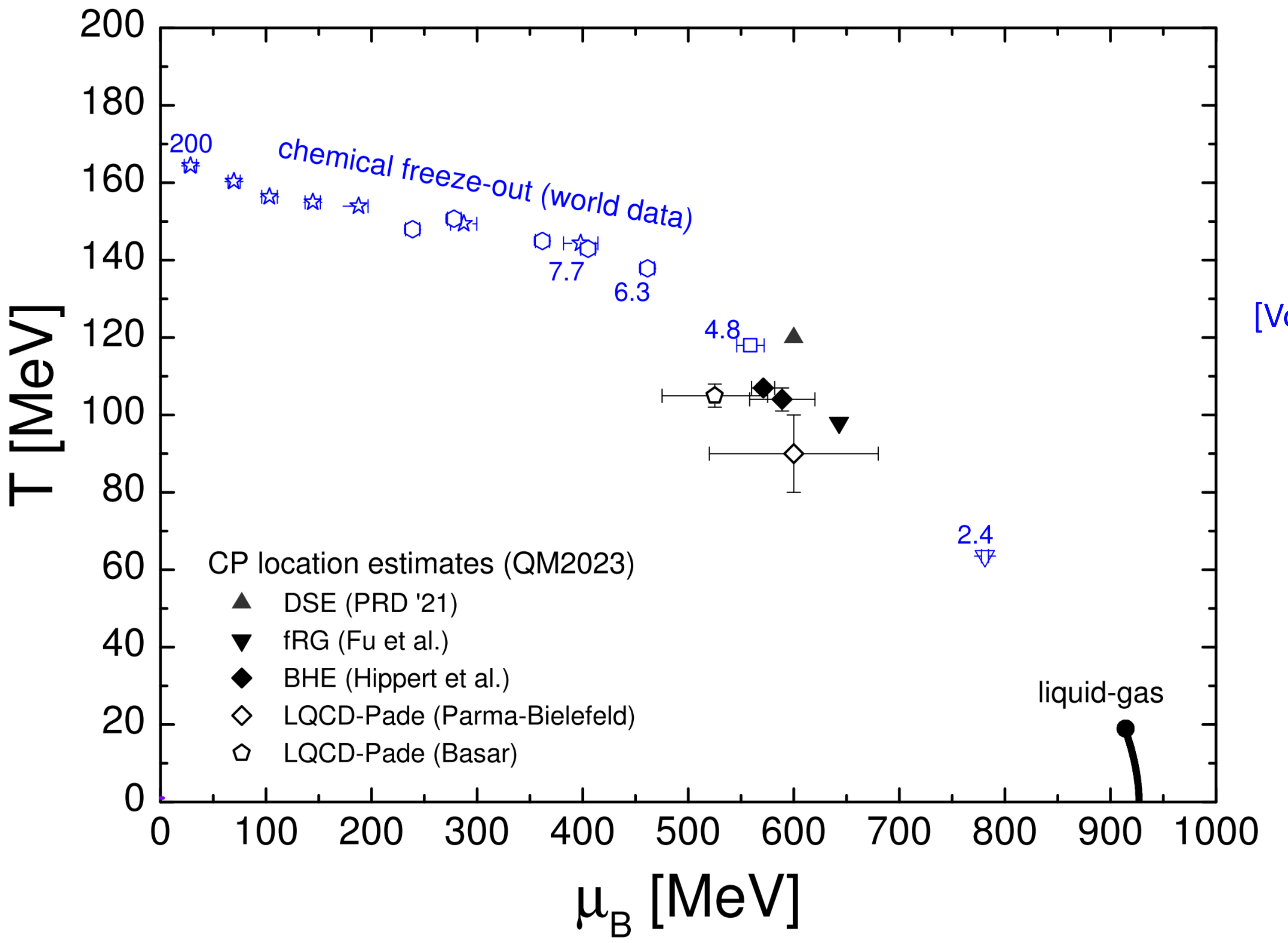
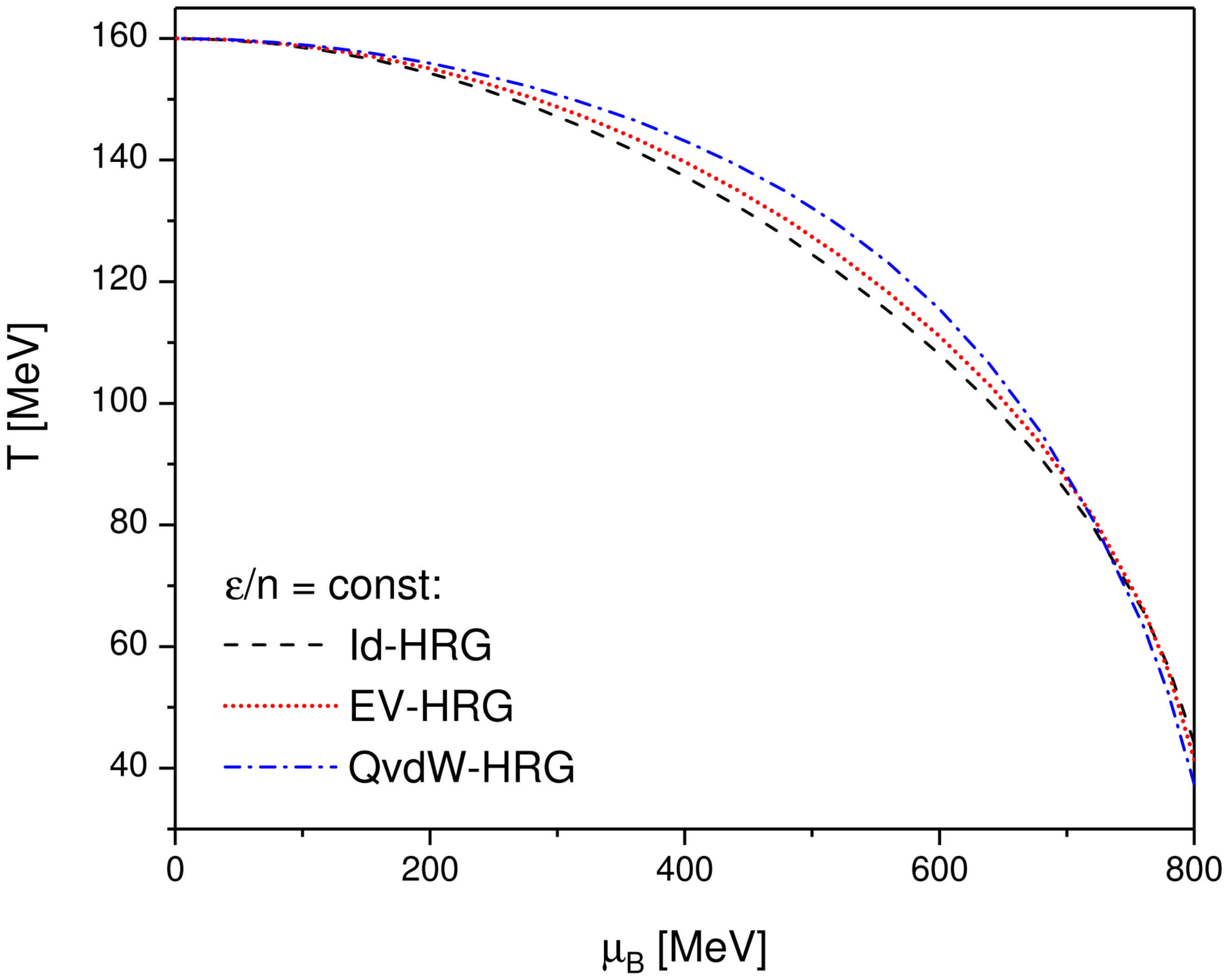
First-principle QCD CP location estimates (from QM2023)



[Vovchenko, QM2023]

$$T_{\mathrm{CP}} \sim 90 - 120 \text{ MeV and } \mu_{B,\mathrm{CP}} \sim 400 - 600 \text{ MeV}$$

Influence of $\mu_Q, \mu_S = 0$ and interactions on the curve of ε/n = const



[Lysenko, R.P., Gorenstein, Vovchenko, PRC, 2025]

QvdW-HRG: $a = 329\ \mathrm{MeV} \cdot \mathrm{fm}^3,\ b = 3.42\ \mathrm{fm}^3$

EV-HRG: $b = 1\ \mathrm{fm}^3$

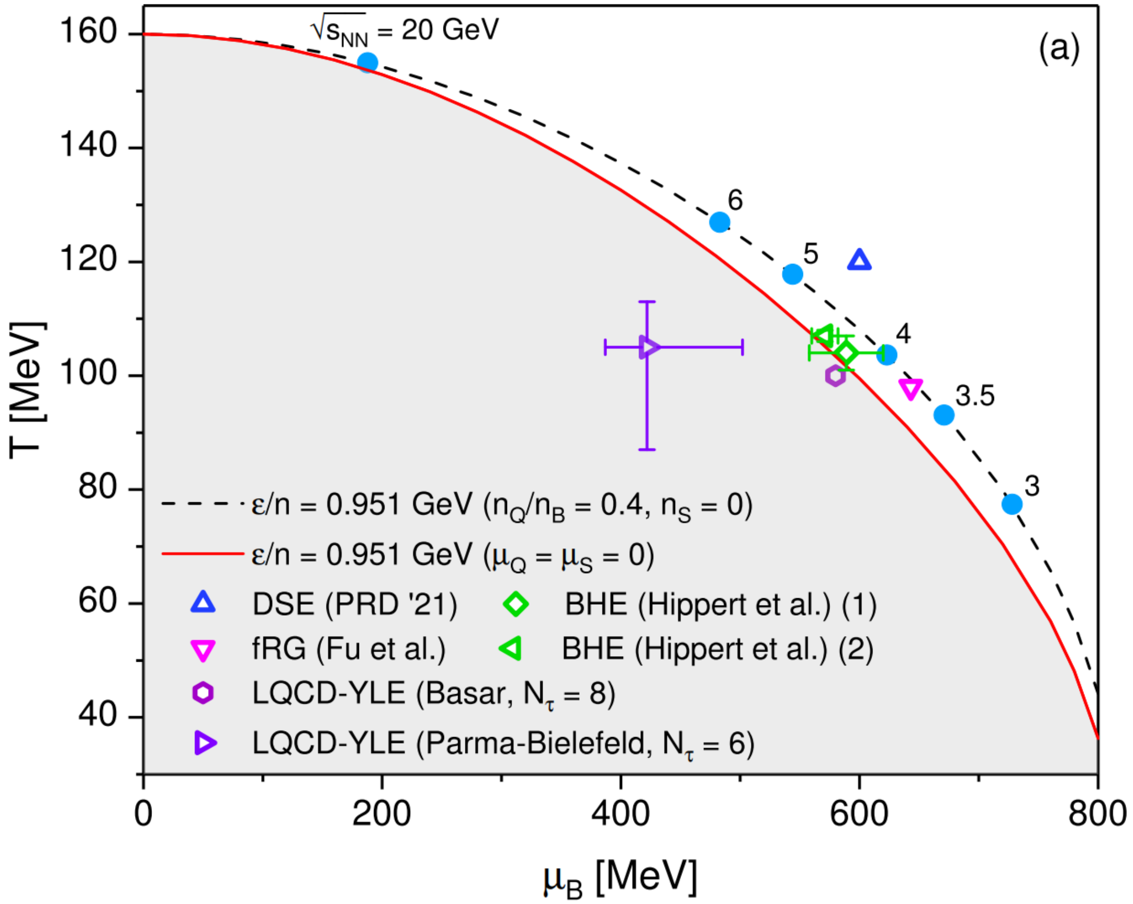
$$T_{\mathrm{ch}}(\mu_B = 0) = 160\ \mathrm{MeV}$$

$\Rightarrow$

- Id-HRG: 0.951 GeV
- EV-HRG: 0.946 GeV
- QvdW-HRG: 0.942 GeV

Apparent contradiction

- Id-HRG calculations and the line of constant ε/n = 0.95 GeV in Ideal-HRG provide the lower bound on the freeze-out line in HIC
- Freeze-out line provides a lower bound for the possible location of QCD CP



$$\mu_B \cong \frac{c}{1 + d\sqrt{s_{\mathrm{NN}}}}$$

Different conditions in theory vs experiment

Theoretical estimates of the CP location: $\mu_S = \mu_Q = 0$

Vs

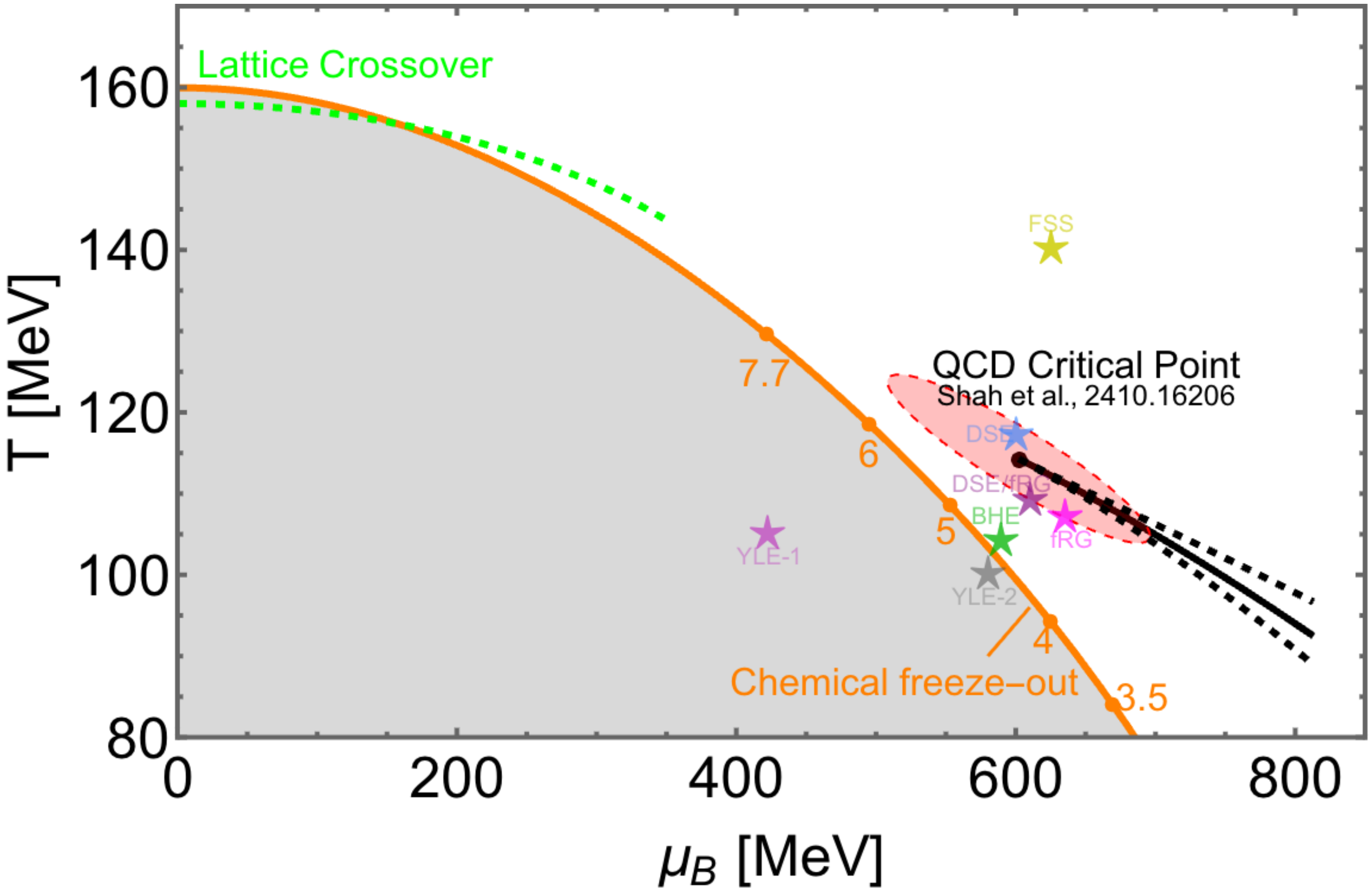
Calculations of the freeze-out curve: $S = 0 \text{ and } \langle Q/B \rangle = 0.4 \;\Rightarrow$

$\mu_S \sim \mu_B/3 \text{ and } \mu_Q \sim -\mu_B/30$

We assume that at the freeze-out $(\varepsilon/n)_{\mathrm{FO}} = 0.951 \text{ GeV} \neq f(\mu_S, \mu_Q)$

Id-HRG $\rightarrow$ Hypothetical freeze-out line for $\mu_S = \mu_Q = 0$

First-principle QCD CP location estimates (Alternative version)



[Adapted from Shah et al., arXiv:2410.XXXXX]

$$T_{\mathrm{CP}} \sim 90 - 120 \text{ MeV and } \mu_{B,\mathrm{CP}} \sim 400 - 600 \text{ MeV}$$

Interacting HRG

QvdW-HRG

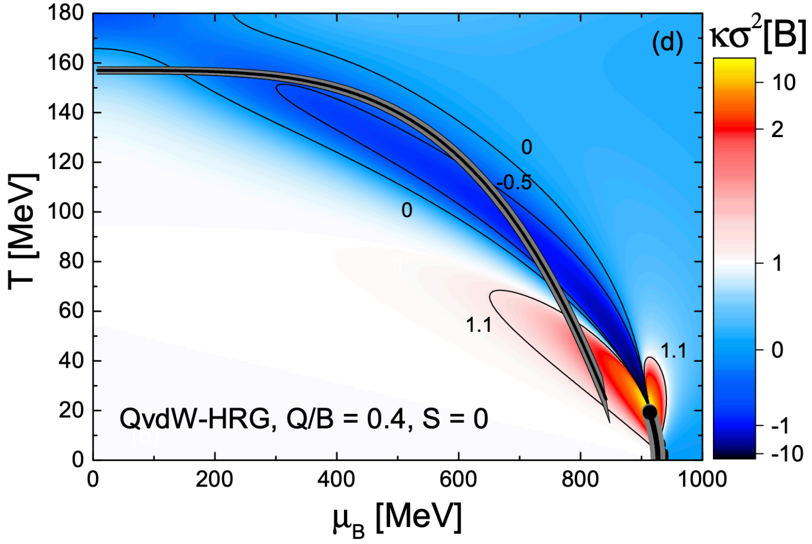
$$p(T,\mu) = p_B + p_{\bar{B}} + p_M$$

$$p_B\,(T,\mu) = \sum_{j \in B} p_j^{id}\left(T, \mu_j^{B*}\right) - a n_B^2$$

$$\mu_j^{B*} = \mu_j - b\, p_B - a\, b\, n_B^2 + 2\, a\, n_B$$

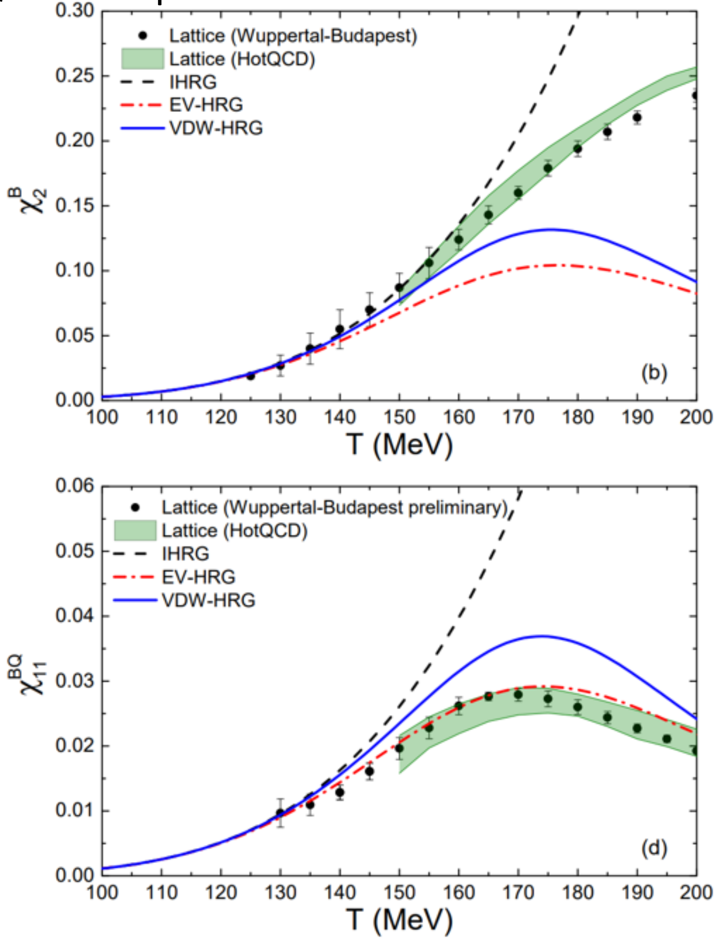
a - attraction, b - repulsion

Nuclear CP fluctuations:



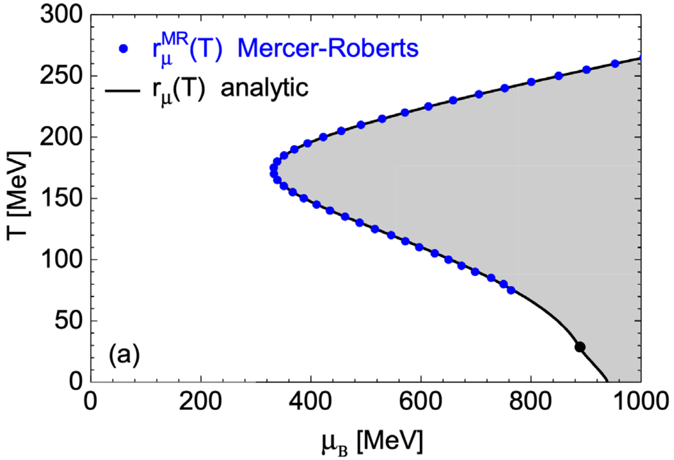
[R.P., Vovchenko, Motornenko, Gorenstein, Stoecker, PRC, 2019]

LQCD susceptibilities:



[Vovchenko, Gorenstein, Stoecker, PRL, 2017]

Radius of convergence:



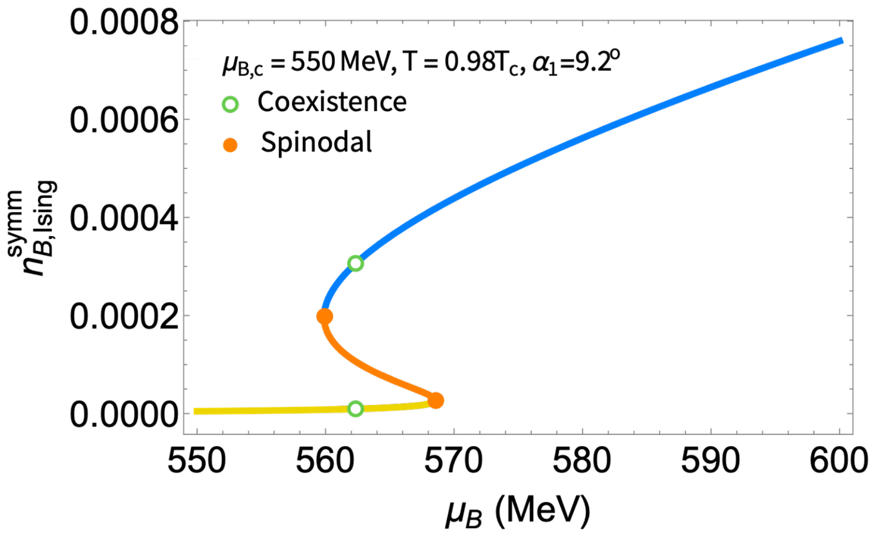
[Savchuk, Vovchenko, R.P., Gorenstein, Stoecker, PRC, 2020]

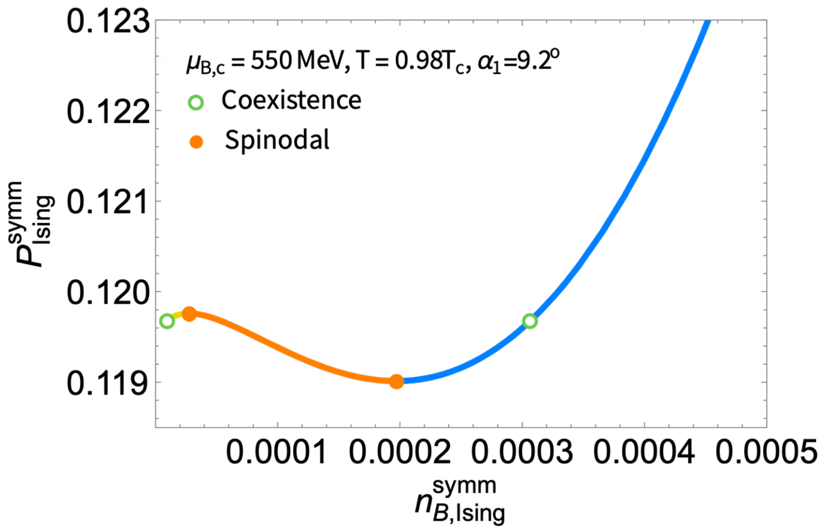
Extensions:

- Isospin dependent interactions
- Excluded volume for mesons, flavor-dependent interactions
- Modifications to the excluded volume (e.g. Carnahan-Starling) and attraction based on EoS's of real gases

➤ With Landau theory, we find expected spinodal features in n_B isotherms unlike with 3D Ising where spinodals are Lee-Yang edge singularities in complex plane

➤ Mapping parameters including α_1, w, ρ control the shape of critical region and first order features





J.M. Karthein, *V. Koch, C. Ratti, PRD (2025)*

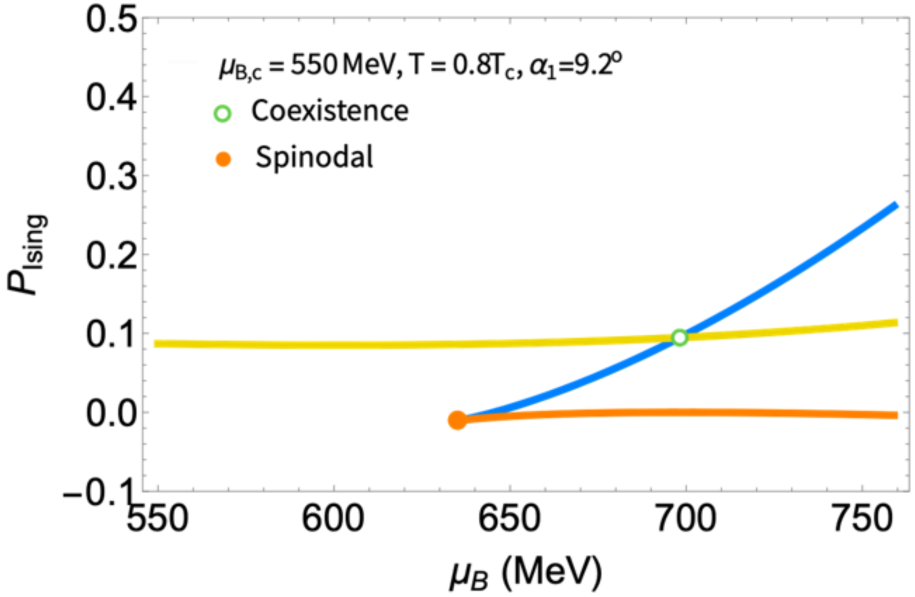
EoS for First Order Regime

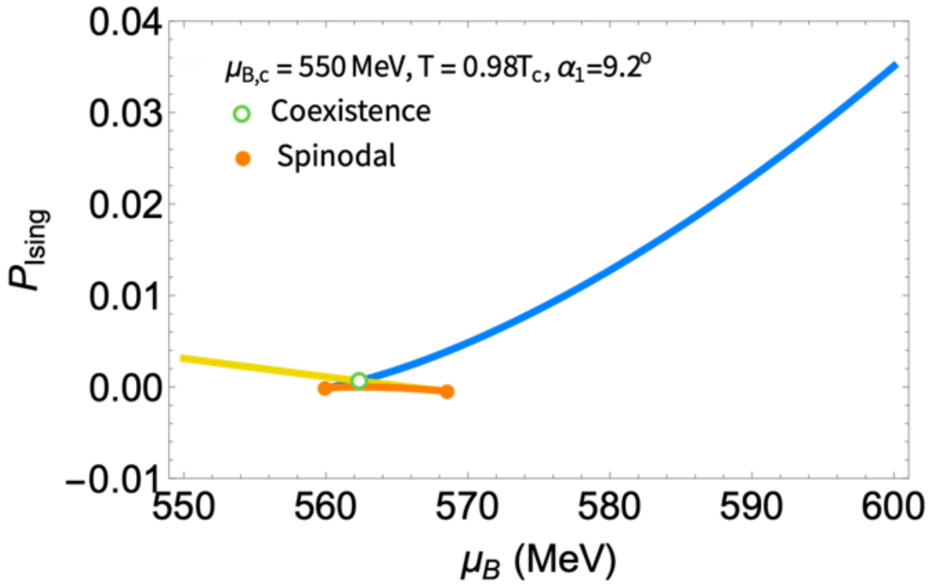
- Considering a mean field Ising model mapped to QCD, we can implement first order features in the phase diagram.

$$P(M,r) = h(M,r)M - F(M,r) = \frac{1}{12}(M^4 + 2rM^2)$$

$$(\mathbf{r},\mathbf{h}) \longleftrightarrow (\mathbf{T},\mu_\mathbf{B}): \quad \frac{T-\mathbf{T_C}}{\mathbf{T_C}} = \mathbf{w}\,(r\rho\,\sin\alpha_1 + h\,\sin\alpha_2)$$

$$\frac{\mu_B-\mu_{\mathbf{BC}}}{\mathbf{T_C}} = \mathbf{w}\,(-r\rho\,\cos\alpha_1 - h\,\cos\alpha_2)$$

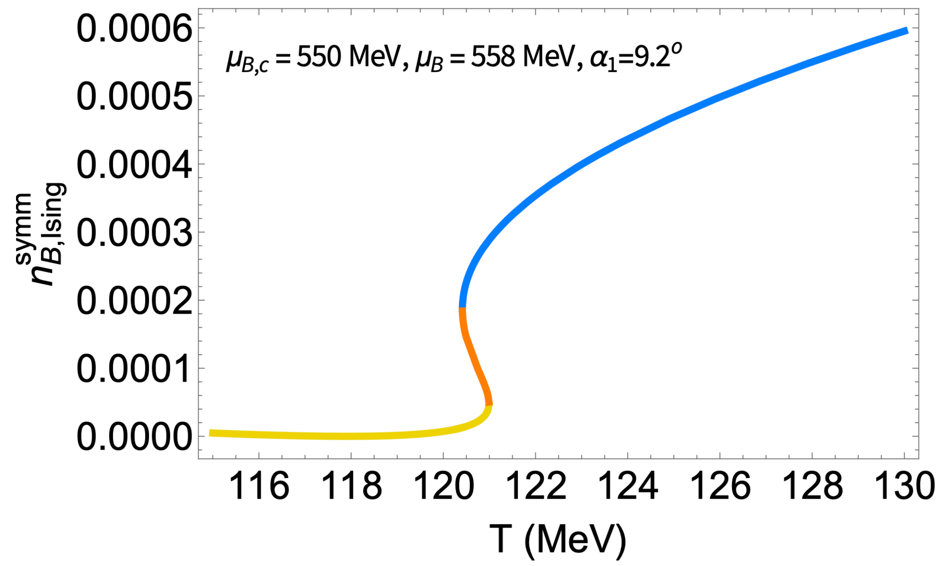




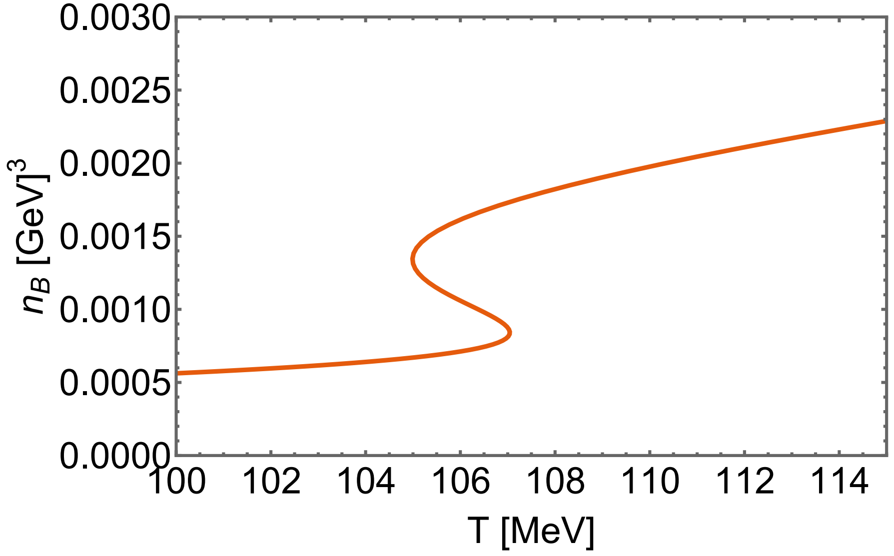
EoS for First Order Regime

- With Landau theory, we find expected spinodal features in n_B isotherms unlike with 3D Ising where spinodals are Lee-Yang edge singularities in complex plane.

- Mapping parameters including α_1, w, ρ control the shape of critical region and first order features.



J. Karthein, V. Koch, C. Ratti – PRD (2025)



HS, M. Hippert, J, Noronha, C. Ratti, V. Vovchenko – Work in preparation

- Comparison of EoS behavior between Taylor expansion mapped with mean field Ising and s- contours Expansion.