Simulating stochastic fluids

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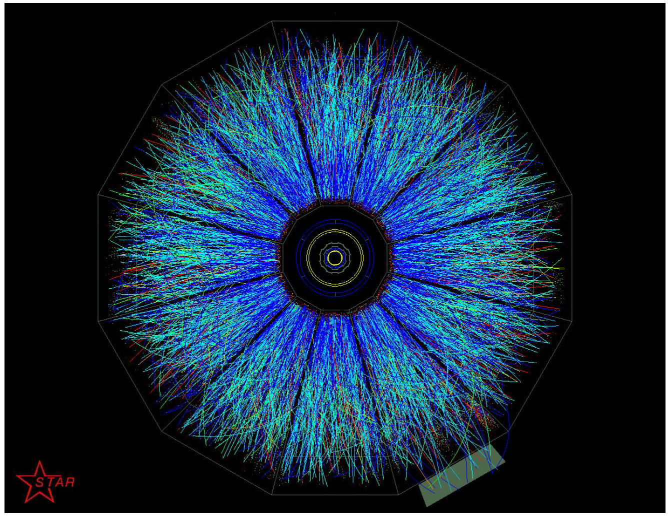
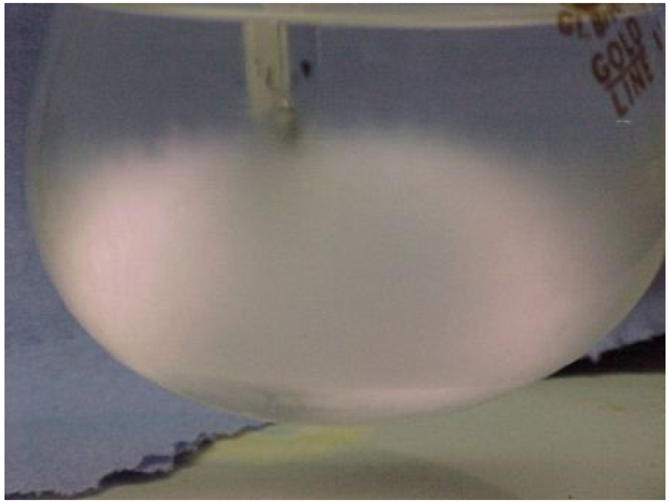




With J. Chao, C. Chattopadhyay, J. Ott, V. Skokov

References: 2204.02433, 2302.00720, 2304.07279.

Dynamical Theory

What is the dynamical theory near the critical point?

The basic logic of fluid dynamics still applies. Important modifications:

- Critical equation of state.
- Stochastic fluxes, fluctuation-dissipation relations.
- Possible Goldstone modes (chiral field in QCD?)

Outline:

1. Stochastic field theories: Diffusion of a conserved charge.
2. What if fluctuations are large? Functional methods, the nPI action.
3. Numerical approaches to stochastic diffusion.

1. Stochastic diffusion

Consider diffusion of a conserved charge

$$\partial_0\psi + \vec{\nabla}\cdot\vec{j} = 0 \qquad \vec{j} = -D\nabla\psi + \ldots$$

Introduce noise and non-linear interactions

$$\partial_0\psi = \kappa\nabla^2\frac{\delta\mathcal{F}}{\delta\psi} + \xi$$

$$\mathcal{F} = \int d^dx\,\left[\frac{\gamma}{2}(\vec{\nabla}\psi)^2 + \frac{m^2}{2}\psi^2 + \frac{\lambda}{3}\psi^3 + \frac{u}{4}\psi^4\right]$$

$$\langle\xi(x,t)\xi(x',t')\rangle = \kappa T\nabla^2\delta(x-x')\delta(t-t') \qquad D = \kappa m^2$$

Equilibrium distribution

$$P[\psi] \sim \exp\left(-\frac{\mathcal{F}[\psi]}{k_BT}\right)$$

Stochastic Field Theory

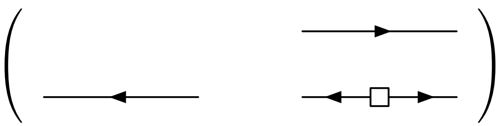
Stochastic effective lagrangian

$$\mathcal{L} = \tilde{\psi}\left(\partial_0 - D\nabla^2\right)\psi + \tilde{\psi}DT\nabla^2\tilde{\psi} + \tilde{\psi}D\lambda\nabla^2\psi^2 + \ldots$$

Diffusion Noise Interactions

Matrix propagator

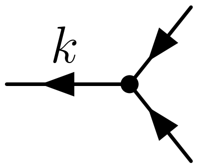
$$\begin{pmatrix} \langle\tilde{\psi}\tilde{\psi}\rangle & \langle\tilde{\psi}\psi\rangle \\ \langle\psi\tilde{\psi}\rangle & \langle\psi\psi\rangle \end{pmatrix} = \begin{pmatrix} 0 & G_R \\ G_A & G_S \end{pmatrix} =$$



Analytic structure of the Schwinger-Keldysh propagator

Interaction vertex

$D\lambda k^2$



What are the rules for constructing more general vertices?

Time reversal invariance

Stochastic theory must describe detailed balance

$$\frac{P(\psi_1 \to \psi_2)}{P(\psi_2 \to \psi_1)} = \exp\left(-\frac{\Delta\mathcal{F}}{k_B T}\right)$$

Related to T-reversal symmetry

$$\begin{aligned} \psi(t) &\to \psi(-t) \\ \tilde{\psi}(t) &\to -\left[\tilde{\psi}(-t) + \frac{\delta\mathcal{F}}{\delta\psi}\right] \end{aligned}$$

$$\mathcal{L} \to \mathcal{L} + \frac{d\mathcal{F}}{dt}$$

Ward identities: Fluctuation-Dissipation relations

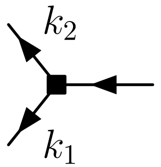
$$2\kappa\,\mathrm{Im}\left\{k^2\langle\psi(\omega,k)\tilde{\psi}(-\omega,-k)\rangle\right\} = \omega\langle\psi(\omega,k)\psi(-\omega,-k)\rangle$$

New and non-classical interactions

At this order (Ψ^3, ∇^2) there is one more interaction

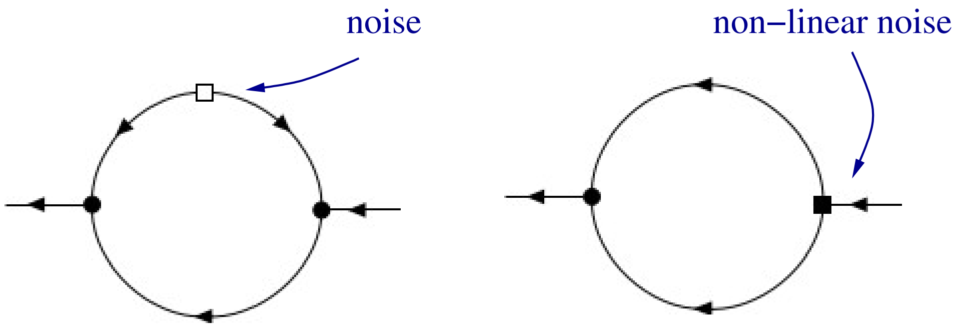
Multiplicative noise : $\mathcal{L} \sim D\lambda_D \psi(\vec{\nabla}\tilde{\psi})^2$

$D\lambda_D k_1 \cdot k_2$



Non-linear noise vertex

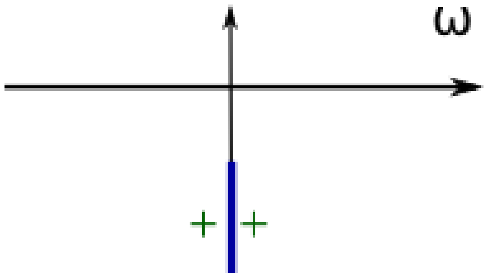
Retarded self energy



Contribute to (non-critical) order parameter relaxation

$$\Sigma(\omega, k) = \frac{\lambda'}{32\pi} \left(i\lambda' \omega k^2 + \lambda_D \left[i\omega - Dk^2 \right] k^2 \right) \sqrt{k^2 - \frac{2i\omega}{D}}$$

Analytical structure



Diffusive cut dominates over (split) diffusive pole.

Even higher order: Non-linear noise with no contribution to constitutive equations.

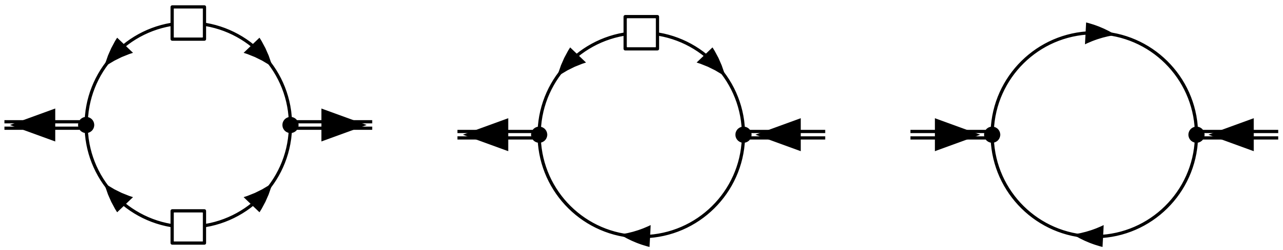
Chao, T.S. [2008.01269], see also Chen-Lin et al. [1811.12540] and Jain & Kovtun [2009.01356]

2. 1PI effective action

Consider 1PI effective action

$$\Gamma[\Psi, \tilde{\Psi}] = W[J, \tilde{J}] - \int dt\, d^3x \left(J\Psi + \tilde{J}\tilde{\Psi} \right) \qquad \frac{\delta W}{\delta J} = \langle \psi \rangle = \Psi\,,$$

Loop expansion



“Classical” equation of motion

$$(\partial_t - D\nabla^2)\Psi - \frac{D\lambda}{2}\nabla^2\Psi^2 + \int d^3x\, dt\, \Psi(x', t')\Sigma(x, t; x', t') = 0$$

$$\Sigma(t, x) \sim \frac{\lambda^2 \theta(t)}{t^{3/2}} \exp\left(-\frac{Dk^2 t}{2} \right)$$

2PI effective action

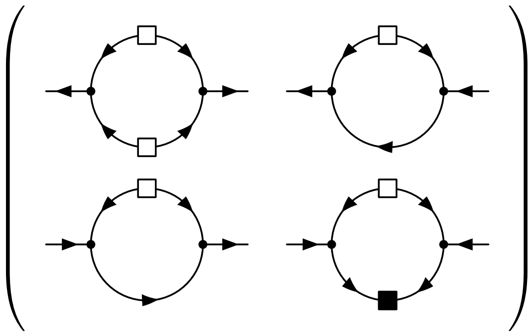
Consider 2PI effective action

$$\Gamma[\Psi_a, G_{ab}] = W[J_a, K_{ab}] - J_A \Psi_A - \frac{1}{2} K_{AB} \left[\Psi_A \Psi_B + G_{AB}\right]$$

Matrix propagator G_{ab}, Bilocal source K_{ab}

Equation of motion for Ψ_a unchanged, but Σ_{ab} satisfies Dyson-Schwinger equation

$$\begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} =$$



Gap equation (mixed representation)

Consider mixed representation $\Sigma(t,k^2)$. Free propagator $G_R^0 = \Theta(t)e^{-tDk^2}$

$$\Sigma(t,k^2) \sim \lambda^2 \int d^3k'\, G(t,k-k')G(t,k')$$

Have to determine G from Dyson equation (matrix structure suppressed)

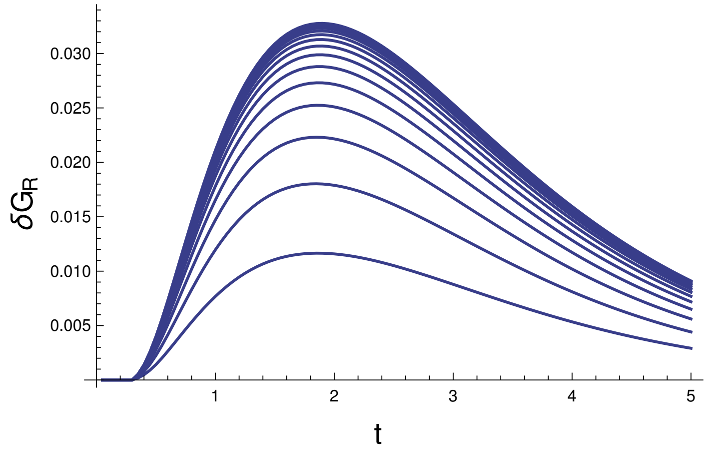
$$G(t,k^2) = G_0(t,k^2) - \int dt_1\, dt_2\, G_0(t_1,k^2)\Sigma(t_2-t_1,k^2)G_(t-t_2,k^2)$$

Short time singularities regulated by Pauli-Vilars "Diffuson"

$G_R(t,k^2)$ at fixed k is non-perturbative for

$$t > \frac{\tau}{(k\xi)^2}\log\left(\frac{2}{\alpha g_3^2(k\xi)^2}\right)$$

$D = \xi^4/\tau,\ g_3 = \lambda_3\xi^{3/2}T^{1/2}$



3. Stochastic relaxation/diffusion

Stochastic relaxation equation ("model A")

$$\partial_t \psi = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = \Gamma T \delta(x-x')\delta(t-t')$$

Naive discretization

$$\psi(t+\Delta t) = \psi(t) + (\Delta t)\left[-\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \sqrt{\frac{\Gamma T}{(\Delta t) a^3}}\theta\right] \qquad \langle \theta^2 \rangle = 1$$

Noise dominates as $\Delta t \to 0$, leads to discretization ambiguities in the equilibrium distribution.

Idea: Use Metropolis update

$$\psi(t+\Delta t) = \psi(t) + \sqrt{2\Gamma(\Delta t)}\theta \qquad p = min(1, e^{-\beta \Delta \mathcal{F}})$$

Stochastic relaxation

Central observation

$$\begin{aligned}\langle \psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})\rangle &= -(\Delta t)\,\Gamma\,\frac{\delta \mathcal{F}}{\delta \psi} + O\left((\Delta t)^2\right)\\ \langle [\psi(t+\Delta t,\vec{x}) - \psi(t,\vec{x})]^2\rangle &= 2(\Delta t)\,\Gamma\,T + O\left((\Delta t)^2\right)\ .\end{aligned}$$

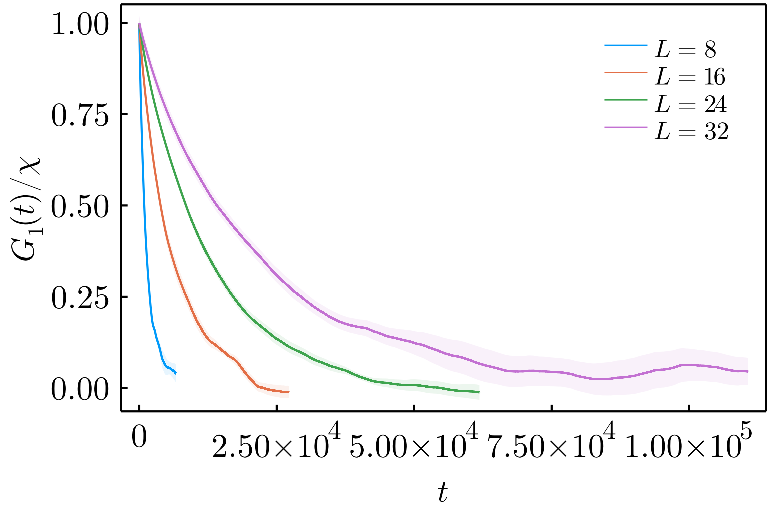
Metropolis realizes both diffusive and stochastic step. Also

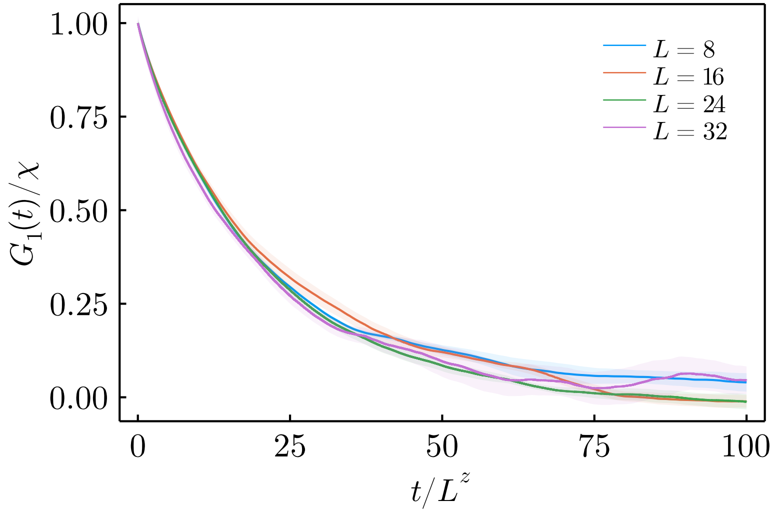
$$P[\psi] \sim \exp(-\beta \mathcal{F}[\psi])$$

Note: Still have short distance noise; need to adjust bare parameters such as Γ, m^2, λ to reproduce physical quantities.

Dynamic scaling (model A)

Correlation functions at T_c, $V = L^3$, $L = 8, 16, 24, 32$



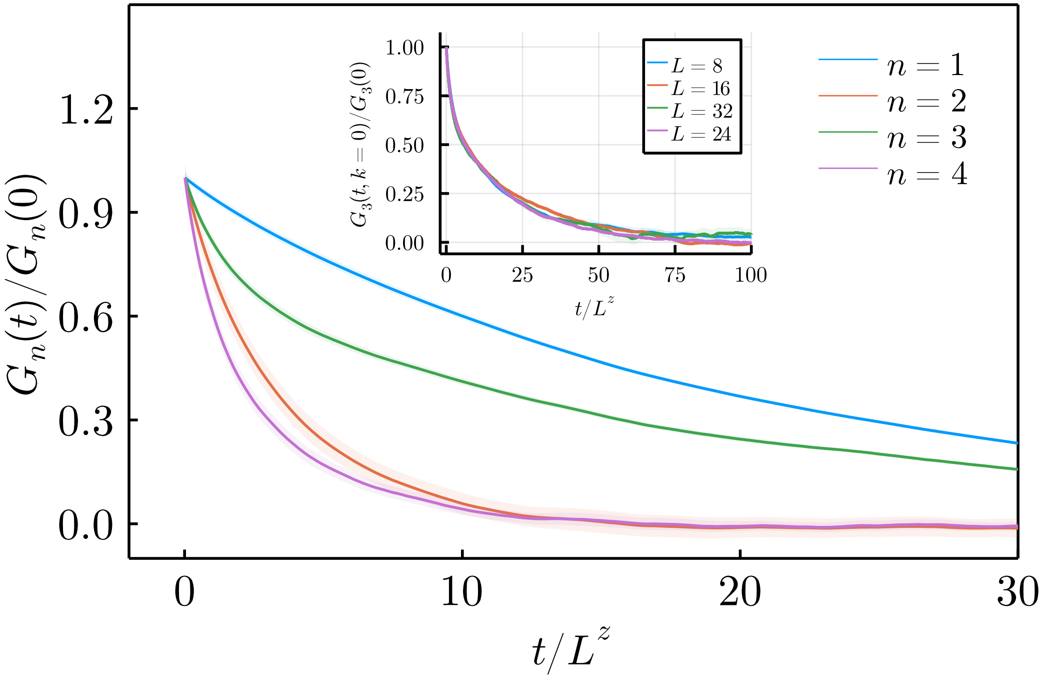


$$G_1(t) = \langle M(0)M(t)\rangle \qquad M(t) = \int d^3x\, \psi(x)$$

Dynamic critical exponent $z = 2.026(56)$.

Correlation functions of higher moments

Correlation functions at T_c



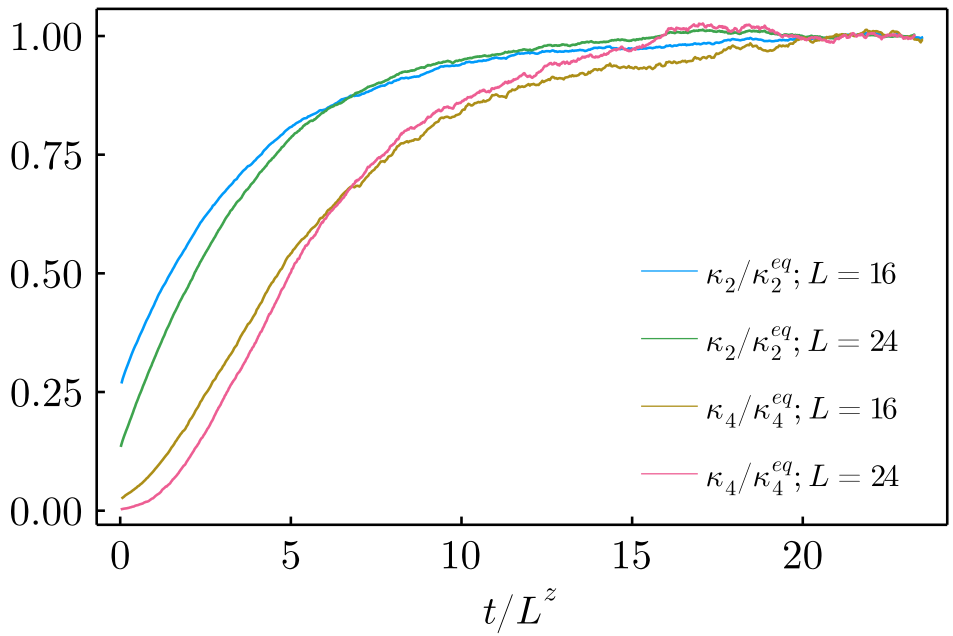
Inset: Dynamic scaling of $G_3(t)$ with $z = 2.026(56)$.

$$G_n(t) = \langle M^n(0) M^n(t) \rangle \qquad M(t) = \int d^3x \, \psi(x)$$

Dynamic scaling holds for all n, but decay constant depends on n.

Relaxation after a quench

Thermalize at $T > T_c$. Study evolution at T_c



$$C_n(t) = \langle\langle M^n(t) \rangle\rangle_{M(0)} (n = 2, 4) \qquad M(t) = \int d^3x\, \psi(x)$$

Observe separate early ("slip") and late ("dynamical") exponents.

Stochastic diffusion (model B)

Stochastic diffusion equation ("model B")

$$\partial_t \psi = \Gamma \nabla^2 \frac{\delta \mathcal{F}}{\delta \psi} + \zeta \qquad \langle \zeta(x,t)\zeta(x',t')\rangle = -\Gamma T \nabla^2 \delta(x-x')\delta(t-t')$$

Write this as a conservation law

$$\partial_t \psi + \vec{\nabla} \cdot \vec{\jmath} = 0, \qquad \vec{\jmath} = -\Gamma \frac{\delta \mathcal{F}}{\delta \psi} + \vec{\xi}$$

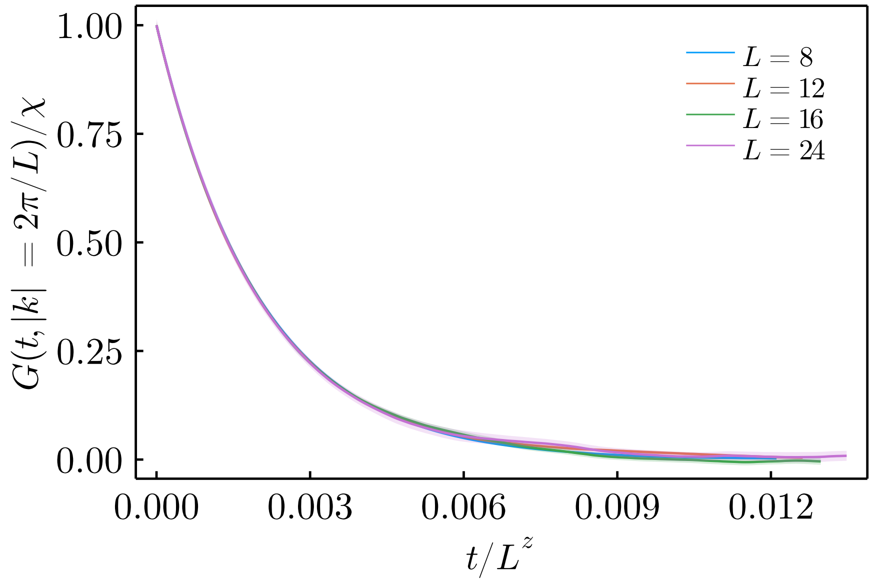
and update fluxes $q_\mu = \hat{n}_\mu \cdot \vec{\jmath}$

$$\begin{aligned} \psi(t+\Delta t, \vec{x}) &= \psi(t, \vec{x}) + q_\mu \,, \\ \psi(t+\Delta t, \vec{x}+\hat{\mu}) &= \psi(t, \vec{x}+\hat{\mu}) - q_\mu \,, \end{aligned}$$

with $q_\mu = \sqrt{2\Gamma\Delta t}\,\theta$.

Dynamic scaling (model B)

Correlation functions at T_c, $V = L^3$, $L = 8, 12, 16, 24$

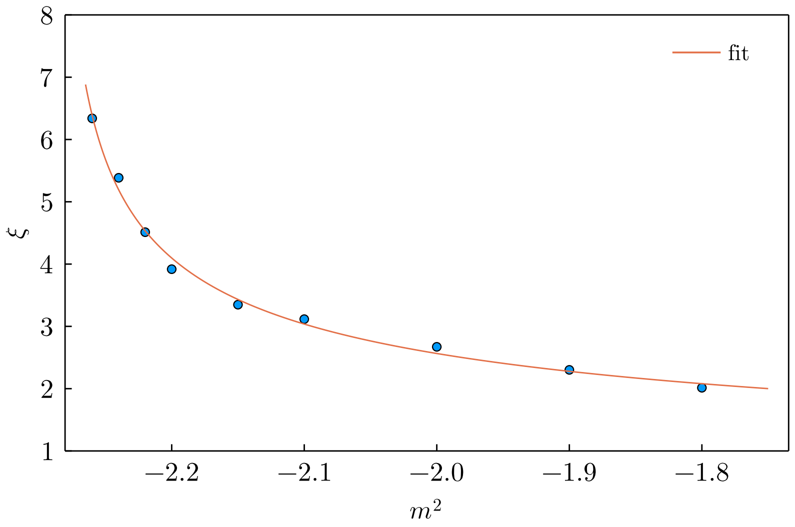


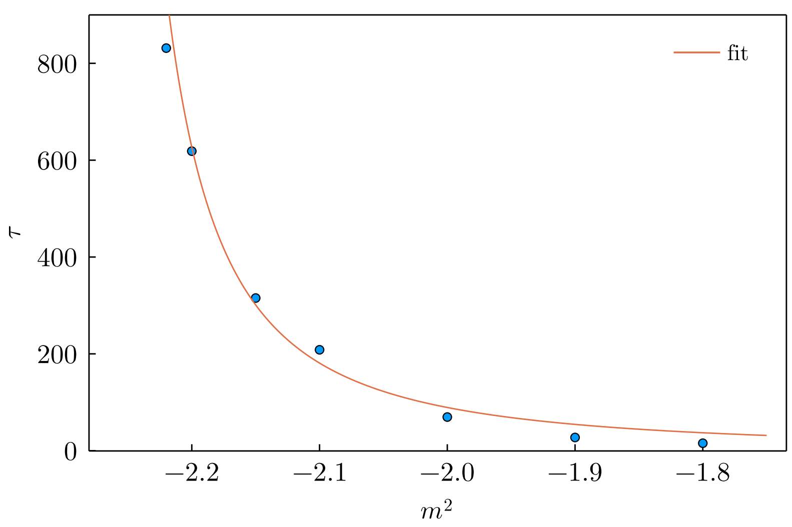
$$G_1(t, k) = \langle M(0, k) M(t, k) \rangle \qquad M(t, k) = \int d^3x \, \psi(x) e^{ix \cdot k}$$

Dynamic critical exponent $z = 3.972(2)$.

Stochastic diffusion

Correlation length and relaxation time as a function of m^2

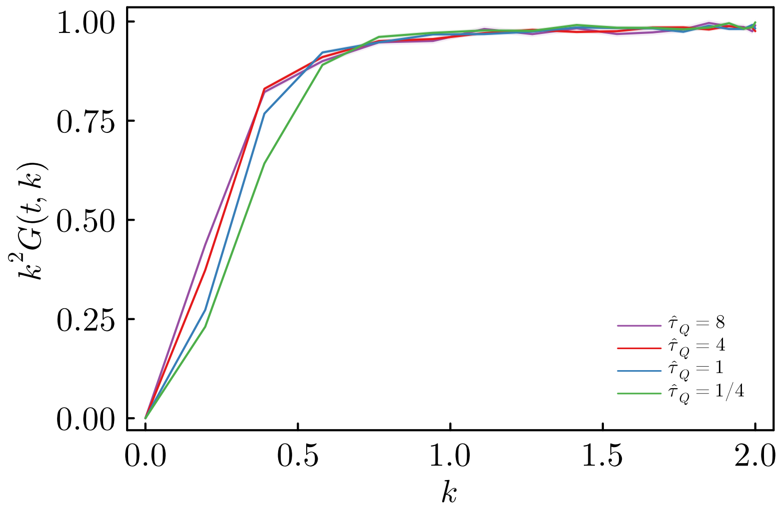


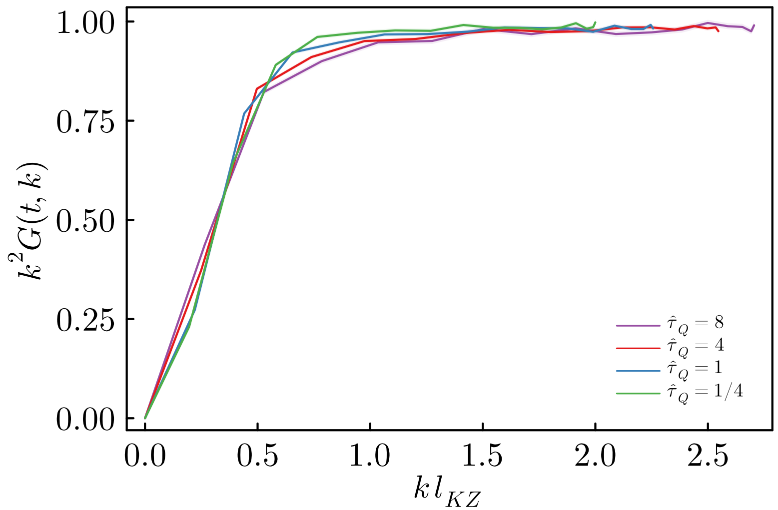


$$\xi \sim \frac{1}{(m^2 - m_c^2)^\nu} \quad (\nu \simeq 0.54) \qquad\qquad \tau \sim \xi^z \quad (z \simeq 3.73)$$

Kibble-Zurek scaling

Sweep $m^2 = m_c^2[1 + \Gamma_Q(t - t_c)]$ $(\tau_Q = \Gamma_Q^{-1})$





Expect scaling with $\tau_{KZ} \sim \tau_Q^{2/3}$ and $l_{KZ} \sim \tau_Q^{1/6}$

Summary

Dynamical evolution of fluctuations is important.

Old and new ideas about effective actions on the Keldysh contour. In principle allows systematic derivation of hydro equations for n-point functions.

Alternative approach: Direct simulation of stochastic fluid dynamics. New idea: Ignore back-reaction, and use Metropolis (or heat bath?) algorithm.