Relativistic invariance in a finite volume: three-body sector

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Plan

- Introduction
- Quantization in the moving frame: essentials
- Higher partial waves: Thomas-Wigner rotation
- 3-particle Lellouch-Lüscher formalism in the moving frame
- Conclusions, outlook

How does one interpret the relativistic invariance of the QC?

- Quantization condition operates with the on-shell amplitudes $\rightarrow$ three-dimensional scattering equations should be used
- A finite box breaks Lorentz/rotational invariance $\rightarrow$ only infinite volume
- Scalar particles, S-wave:

$$\text{Particle-dimer amplitude} \quad : \quad \mathscr{M}(P,p;Q,q)=\mathscr{M}(P',p';Q',q')$$

$$\text{Three-particle amplitude} \quad : \quad T(p_1,p_2,p_3;p_1,q_2,q_3)=T(p_1',p_2',p_3';p_1',q_2',q_3')$$

- Enables to describe the data taken in different moving frames by using the relativistic-invariant three-body force $\rightarrow$ less independent fitting parameters

- The way out: write down the scattering equations in the manifestly Lorentz-invariant form

RFT: modifying the propagator (same in FVU or NREFT)

- Three-particle scattering amplitude, infinite volume (ignoring cutoff):

$$
\begin{aligned}
iT &= \left(1 - i\mathcal{K}_2(iF + iG)\right)^{-1} i\mathcal{K}_2 \\
iG_{p\ell' m';k\ell m} &= \frac{1}{2w_k} \mathscr{Y}_{\ell' m'}(\mathbf{k}^*_{2,p}) iS_3^0(\mathbf{p},\mathbf{k}) \mathscr{Y}^*_{\ell m}(\mathbf{p}^*_{2,k}) \\
iF_{p\ell' m';k\ell m} &= (2\pi)^3 \delta^3(\mathbf{p}-\mathbf{k}) \frac{1}{2} \int \frac{d^3\mathbf{a}}{(2\pi)^3 2w_a} \mathscr{Y}_{\ell' m'}(\mathbf{a}^*_{2,k}) iS_3^{i\varepsilon}(\mathbf{p},\mathbf{k}) \mathscr{Y}^*_{\ell m}(\mathbf{a}^*_{2,k})
\end{aligned}
$$

- Modification of the three-dimensional propagator ($\tilde{p}, \tilde{q}$ are on-shell momenta):

$$
\begin{aligned}
iS_3^{i\varepsilon}(\mathbf{p},\mathbf{k}) &= \frac{1}{2w(\mathbf{K}-\mathbf{p}-\mathbf{q})(w(\mathbf{p}) + w(\mathbf{q}) + w(\mathbf{K}-\mathbf{p}-\mathbf{q}) - K^0)} \\
&- \underbrace{\frac{1}{2w(\mathbf{K}-\mathbf{p}-\mathbf{q})(w(\mathbf{p}) + w(\mathbf{q}) - w(\mathbf{K}-\mathbf{p}-\mathbf{q}) - K^0)}}_{\text{low-energy polynomial}} = \frac{1}{m^2 - (\tilde{p}+\tilde{q}-K)^2}
\end{aligned}
$$

What can go wrong?

- Conjecture: low-energy polynomial can be removed by renormalization
- The kernel G is singular at high momenta, breaks unitarity already at threshold

$$w(\mathbf{p}) + w(\mathbf{q}) - w(\mathbf{K} - \mathbf{p} - \mathbf{q}) - K^0 = 0 \quad \text{for} \quad K^0 - 3m \ll m$$

  ... has the solution with $\mathbf{p}, \mathbf{q} \sim m$

- Possible solution: low cutoff excludes the singularities
  $\hookrightarrow$ Cutoff should be chosen of order of the particle mass m, cutoff dependence of the solution cannot be investigated for the whole range of cutoffs

An alternative formalism (lowest-order only)

(F. Müller, J.-Y. Pang, AR and J.-J. Wu, JHEP 02 (2022) 158, JHEP 02 (2023) 214)

- Choose "quantization axis" in direction of an arbitrary unit vector v^μ, $v^2 = 1$
- The Lagrangian:

$$\mathscr{L} = \phi^\dagger(i(v\partial) - w_v)(2w_v)\phi + \sigma T^\dagger T + \frac{f_0}{2}(T^\dagger\phi\phi + \text{h.c.}) + h_0 T^\dagger T\phi^\dagger\phi$$

- Here, $w_v = \sqrt{m^2 + \partial^2 - (v\partial)^2}$
- The propagator:

$$\langle 0|T\phi(x)\phi^\dagger(x)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ik(x-y)}}{2w_v(k)(w_v(k) - (vk) - i\varepsilon)}$$

- Matching: $f_0 \to$ scattering length, $h_0 \to$ three-body amplitude

Two-particle sector

- Dimer: an alternative description of an infinite bubble sum; dummy field in the path integral
- Mathematically equivalent to the standard treatment – not an approximation

dimer :

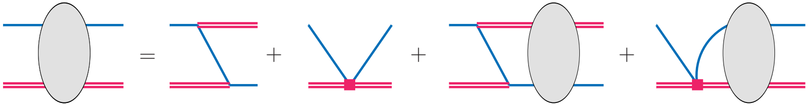
- Threshold expansion:

$$\frac{1}{2w_v(k)(w_v(k)-vk)} = \frac{1}{m^2-k^2} - \underbrace{\frac{1}{2w_v(k)(w_v(k)+vk)}}_{\text{low-energy polynomial}}$$

- The loop is v-independent!

$$\text{loop: } I(s) = \text{const} + \frac{\sigma}{16\pi^2}\ln\frac{\sigma-1}{\sigma+1}\,, \qquad \sigma = \left(1-\frac{4m^2}{s+i\varepsilon}\right)^{1/2}$$

Relativistic invariant scattering equation



Bethe-Salpeter equation

$$\mathcal{M}(p,q) = Z(p,q) + 8\pi \int \frac{d^3 k_\perp}{(2\pi)^3 2 w_v(k)} \theta(\Lambda^2 + m^2 - (vk)^2) Z(p,k) \tau(K-k) \mathcal{M}(k,q)$$

$$\tau(P) = \frac{2\sqrt{P^2}}{k^* \cot\delta(k^*) - ik^*} \qquad k^* = \sqrt{\frac{P^2}{4} - m^2}$$

$$Z(p,q) = \frac{1}{2w_v(K-p-q)(w_v(p) + w_v(q) + w_v(K-p-q) - (vK) - i\varepsilon)} + \tilde{H}_0 + \cdots$$

Relativistic invariance

- The two-body propagator $\tau(K-k)$ does not depend on v^μ
- The solution of the scattering amplitude is relativistic invariant:

$$\mathscr{M}(p,q;K;v) = \mathscr{M}(p',q';K';v')$$

- Relativistic invariance is achieved by expressing v^μ in terms of the external momenta

$$\text{a natural choice:} \quad v^\mu = \frac{K^\mu}{\sqrt{K^2}}$$

- Three-body amplitude expressed through particle-dimer amplitude $\to$ relativistic-invariant

Crucial issues

- NREFT + threshold expansion guarantee a manifest Lorentz-invariance of the two-body amplitude and elastic unitarity; high-energy input is hidden in the EFT couplings

  Beware of the spurious subthreshold poles at the hard scale:

  Ebert, Hammer and AR, EPJA 57 (2021) 12, 332

  Pang, Ebert, Hammer, Müller, AR and Wu, JHEP 07 (2022) 019
- The v-independence of the two-body input is crucial in the proof of relativistic invariance in the 3-body sector
- The three-body force is parameterized by a low-energy polynomial, no singularities in the complex plane
- Explicitly fulfills two- and three-body unitarity; no contradiction with the decoupling theorem
- Relativistic invariance and unitarity are maintained for all values of the cutoff

Higher partial waves, derivative couplings

- The dimer field with an arbitrary (integer) spin

$$T_{\ell m}=\sum_{\mu_i,\nu_i}(c^{-1})^{\ell m}_{\mu_1\cdots\mu_\ell}\underline{\Lambda}^{\mu_1}_{\nu_1}\cdots\underline{\Lambda}^{\mu_\ell}_{\nu_\ell}T^{\nu_1\cdots\nu_\ell}\,,\qquad \underline{\Lambda}^{\mu}_{\nu}v^{\nu}=v^{\mu}_0=(1,\mathbf{0})$$

- (Symmetric) dimer field obeys the constraints

$$v_{\mu_i}T^{\mu_1\cdots\mu_\ell}=0,\qquad T^{\mu_1\cdots\mu_i\cdots\mu_\ell}_{\mu_i}=0$$

- Interaction of a dimer with two particles

$$\mathscr{L}_2=\sum_{\ell m}\sigma_\ell T^\dagger_{\ell m}T_{\ell m}+\sum_{\ell m}(T^\dagger_{\ell m}O_{\ell m}+\text{h.c.})$$

Two-particle vertices

- Generalization of the on-shell three-momentum to moving frames:

$$\bar{w}_\perp^\mu = \bar{w}^\mu - v^\mu(v\bar{w})\,, \qquad \bar{w}^\mu = \Lambda^\mu_\nu w^\nu\,, \qquad w^\mu = v^\mu w_v + i(\partial^\mu - v^\mu(v\partial))$$

- The boost Λ renders the total momentum of the pair P^μ parallel to v^μ

$$\Lambda^\mu_\nu(v,u)u^\mu = v^\mu\,, \qquad u^\mu = \frac{P^\mu}{\sqrt{P^2}}\,, \quad P^\mu = \tilde{p}_1^\mu + \tilde{p}_2^\mu \text{ (on-shell)}$$

- The vertices:

$$\begin{aligned} O &= \frac{f_0^{(0)}}{2}\phi^2 + \frac{f_0^{(2)}}{4}(\phi\bar{w}_\perp^\mu\bar{w}_{\perp\mu}\phi - \bar{w}_\perp^\mu\phi\bar{w}_{\perp\mu}\phi) + \cdots \\ O^{\mu\nu} &= \frac{f_2^0}{2}\Big(3(\phi\bar{w}_\perp^\mu\bar{w}_\perp^\mu\phi - \bar{w}_\perp^\mu\phi\bar{w}_\perp^\nu\phi) - (g^{\mu\nu} - v^\mu v^\nu)(\phi\bar{w}_\perp^\lambda\bar{w}_{\perp\lambda}\phi - \bar{w}_\perp^\lambda\phi\bar{w}_{\perp\lambda}\phi)\Big) + \cdots \end{aligned}$$

...and so on

Three-particle force

$$\mathscr{L}_3 = \sum_{\ell m, \ell' m'} \sum_{L L' J M} T^{\dagger}_{\ell' m'} \left(\mathscr{Y}^{JM}_{L'\ell'}(\underline{w}, m')\phi^{\dagger}\right) T^{\ell'\ell}_{JL'L}(\Delta, \overrightarrow{\Delta}_T, \overleftarrow{\Delta}_T) \left(\left(\mathscr{Y}^{JM}_{L\ell}(\underline{w}, m)\right)^{*}\phi\right) T_{\ell m}$$

$$\mathscr{Y}^{JM}_{L\ell}(\mathbf{k}, m) = \langle L(M-m), \ell m | JM \rangle \mathscr{Y}_{L(M-m)}(\mathbf{k}) \,, \quad \underline{w}^{\mu} = \underline{\Lambda}^{\mu}_{\nu} w^{\nu}$$

- The three-body force is parameterized by effective couplings

$$T^{\ell'\ell}_{JL'L}(\Delta, \overrightarrow{\Delta}_T, \overleftarrow{\Delta}_T) = h_0 + h_1 \Delta + h_2(\overrightarrow{\Delta}_T + \overleftarrow{\Delta}_T) + \cdots$$

$$\underbrace{\Delta = K^2 - (3m)^2}_{\text{3-body system}}, \qquad \underbrace{\Delta_T = P^2 - (2m)^2}_{\text{2-body subsystem}}$$

- Number of independent couplings depends on the detailed dynamics of the system!

The Bethe-Salpeter equation

$$\mathscr{M}_{\ell' m',\ell m} = Z_{\ell' m',\ell m} + \sum_{\ell''} Z_{\ell' m',\ell'' m''} S_{\ell''} \mathscr{M}_{\ell'' m'',\ell m}$$

- The two-body propagator

$$S_\ell(s) = -\frac{1}{\sigma_\ell - f_\ell^2(s)\frac{1}{2}\, p^{2\ell}(s) I(s)}$$

- The driving term

$$\begin{aligned} Z_{\ell' m',\ell'' m''}(p,q) &= \frac{4\pi\ (\mathscr{Y}_{\ell' m'}(\tilde{\mathbf{p}}))^*\ f_{\ell'}(s_p)\ f_\ell(s_q) \mathscr{Y}_{\ell m}(\tilde{\mathbf{q}})}{2w_v(K-p-q)(w_v(p)+w_v(q)+w_v(K-p-q)-vK-i\varepsilon)} \\ &+ \quad 4\pi \sum_{LL'} \sum_{JM} \mathscr{Y}_{JM}^{L'\ell'}(\underline{\mathbf{p}}, m')\, T_{JL'L}^{\ell'\ell}(\Delta, \Delta_p, \Delta_q) \left(\mathscr{Y}_{JM}^{L\ell}(\underline{\mathbf{q}}, m)\right)^* \end{aligned}$$

Relativistic invariance of the framework

- Two types of momenta: $\tilde{p} = \underline{\Lambda}(v)\Lambda(v,u)p$ and $\underline{p} = \underline{\Lambda}(v)p$
- Wigner-Thomas rotation

$$\underline{\Lambda}(v_\Omega) = R\underline{\Lambda}(v)\Omega^{-1}\,, \qquad R = R(\Omega, v)$$

- Lorentz-transformation of the momenta

$$\underline{p} \to \underline{\Lambda}(v_\Omega)p_\Omega = R\underline{\Lambda}(v)\Omega^{-1}\Omega p = R\underline{p}\,, \qquad \tilde{p} \to R\tilde{p}$$

- Lorentz-transformation of the kernel

$$Z_{\ell' m', \ell m}(\Omega p, \Omega q, \Omega K) = \sum_{m''' m''} \mathscr{D}^{(\ell')}_{m' m'''}(R) Z_{\ell' m''', \ell m''}(p, q, K) \left(\mathscr{D}^{(\ell)}_{m'' m}(R)\right)^*$$

$$\to \quad \mathscr{M}_{\ell' m', \ell m}(\Omega p, \Omega q, \Omega K) = \sum_{m''' m''} \mathscr{D}^{(\ell')}_{m' m'''}(R) \mathscr{M}_{\ell' m''', \ell m''}(p, q, K) \left(\mathscr{D}^{(\ell)}_{m'' m}(R)\right)^* \quad \checkmark$$

Relativistic invariant three-body QC

$$\det\left(\mathscr{A}\right) = 0$$

$$\mathscr{A}_{\ell' m', \ell m}(p,q) = 2w(\mathbf{p})\delta_{\mathbf{pq}}\big(S^L_{\ell' m', \ell m}(K-p)\big)^{-1} - \frac{1}{L^3} Z_{\ell' m', \ell m}(p,q)$$

- Even in a finite volume, dimer propagator S^L does not depend on v^μ
- Projection on the irreps of the cubic group and its subgroups can be done in a standard manner
- Meaning of the relativistic invariance in a finite volume: Parameterizing the three-body force in a Lorentz-invariant manner and fitting it to data in different frames, the finite-volume corrections to the extracted effective couplings will be exponentially suppressed.

Three-particle decays

(F. Müller and AR, JHEP 03 (2021) 152, F. Müller, J.-Y. Pang, AR and J.-J.Wu, JHEP 02 (2023) 214)

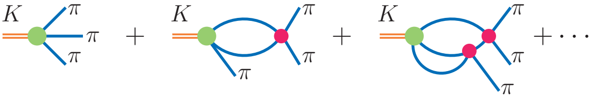
a) Decays through the weak or electromagnetic interactions; isospin-breaking decays: pole on the real axis

Example: $K \to 3\pi$

b) Decays through strong interactions, the pole moves into the complex plane

Example: $N(1440) \to \pi\pi N$

- Final-state interactions lead to the irregular volume-dependence in the matrix element



An analog of the LL formula in the three-particle sector?

The Lagrangian describing decays, and the LL formula

- The Lagrangian

$$\begin{aligned}\mathscr{L}_K &= K^\dagger(i(v\partial) - w_v^K)(2w_v^K)K \\ &+ \sqrt{4\pi}\sum_{\ell m}\frac{(-1)^\ell}{\sqrt{2\ell+1}}\left(K^\dagger G_\ell(\Delta_T)((\mathscr{Y}_{\ell,-m}(\underline{w}))^*\phi)\,T_{\ell m} + \text{h.c.}\right)\end{aligned}$$

- The effective couplings

$$G_\ell(\Delta_T) = G_\ell^{(0)} + G_\ell^{(1)}\Delta_T + \cdots$$

- The LL formula relates matrix elements in a finite and infinite volume

$$\begin{aligned}L_\alpha^{3/2}\langle n|J_K^\dagger(0)|0\rangle &= \sum_{\ell,i} a_\ell^{(i)}(K_\alpha, L_\alpha)\,G_\ell^{(i)} \\ \langle\pi(k_1)\pi(k_2)\pi(k_3)|J_K^\dagger(0)|0\rangle &= \sum_{\ell,i} A_\ell^{(i)}(K)\,G_\ell^{(i)}\end{aligned}$$

Conclusions, outlook

- Quantization condition contains on-shell quantities and thus defines an inherently three-dimensional formalism
- A global fit to the lattice data in different moving frames requires manifestly Lorentz-invariant framework. How to reconcile these two properties?
- Relativistic invariant quantization condition is defined by a consistent relativistic-invariant decoupling of high and low-energy scales in the three-dimensional formalism, which does not break unitarity, etc
- This can be achieved by quantizing the system in an arbitrary moving frame and fixing its velocity in terms of the external momenta at the end
- The Lorentz-invariant parameterization of the three-body force should be used. The number of independent couplings at a given order is determined by dynamics
- Can be implemented in RFT and FVU as well