Two and three mesons at maximal isospin from lattice QCD

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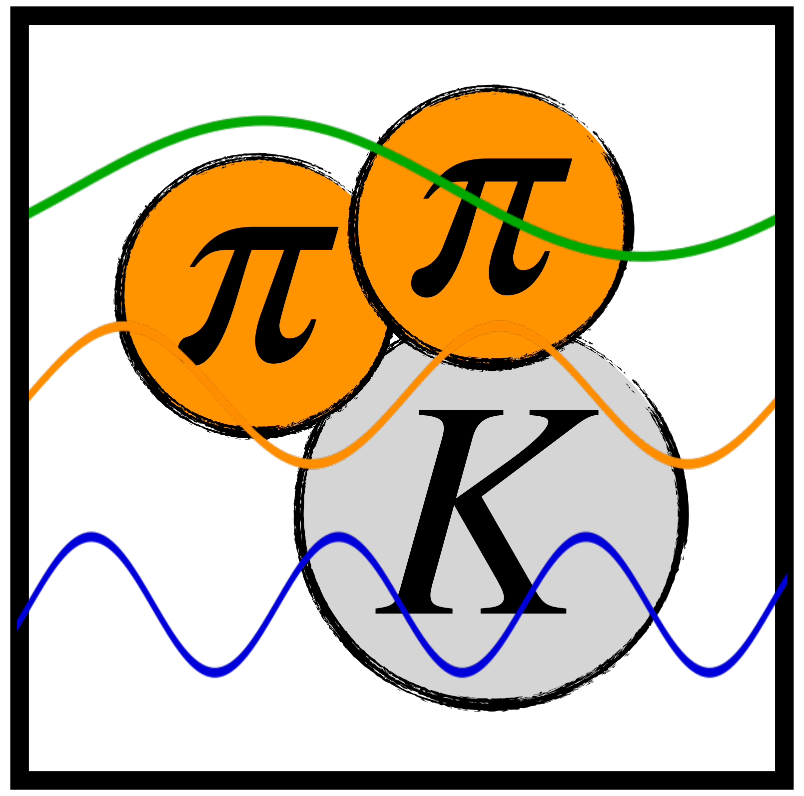
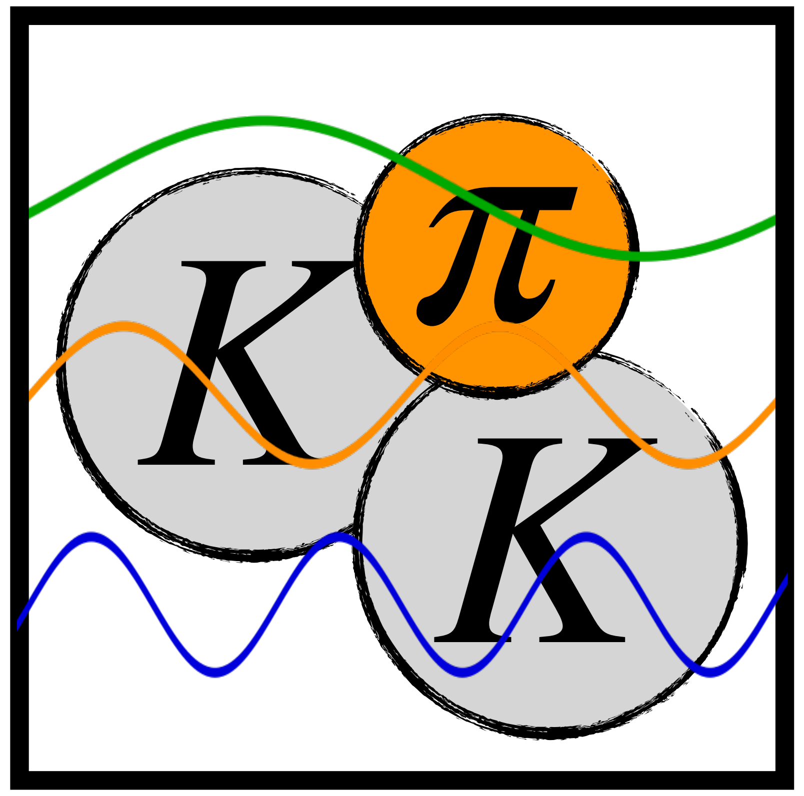
Seattle, 22nd March 2023





References

Interactions of two and three mesons including higher partial waves from lattice QCD

[arXiv:2106.05590]

Tyler D. Blanton[1] , Andrew D. Hanlon[2,3,4] , Ben Hörz[5] , Colin Morningstar[6] , Fernando Romero-López[7] , and Stephen R. Sharpe[1]

Interactions of πK, $\pi\pi K$ and $KK\pi$ systems at maximal isospin from lattice QCD

[arXiv:2302.13587]

Zachary T. Draper[1] , Andrew D. Hanlon[2] , Ben Hörz[3] , Colin Morningstar[4] , Fernando Romero-López[5] , and Stephen R. Sharpe[1]

The isospin-3 three-particle K-matrix at NLO in ChPT

[to appear]

Jorge Baeza-Ballesteros,[a] Johan Bijnens,[b] Tomáš Husek,[b,c] Fernando Romero-López,[d] Stephen R. Sharpe,[e] and Mattias Sjö[b]

Multi-hadron interactions

- Lattice QCD offers a systematically improvable approach to investigate the hadron spectrum
- Finite volume and Euclidean spacetime complicate the study of multi-hadron systems

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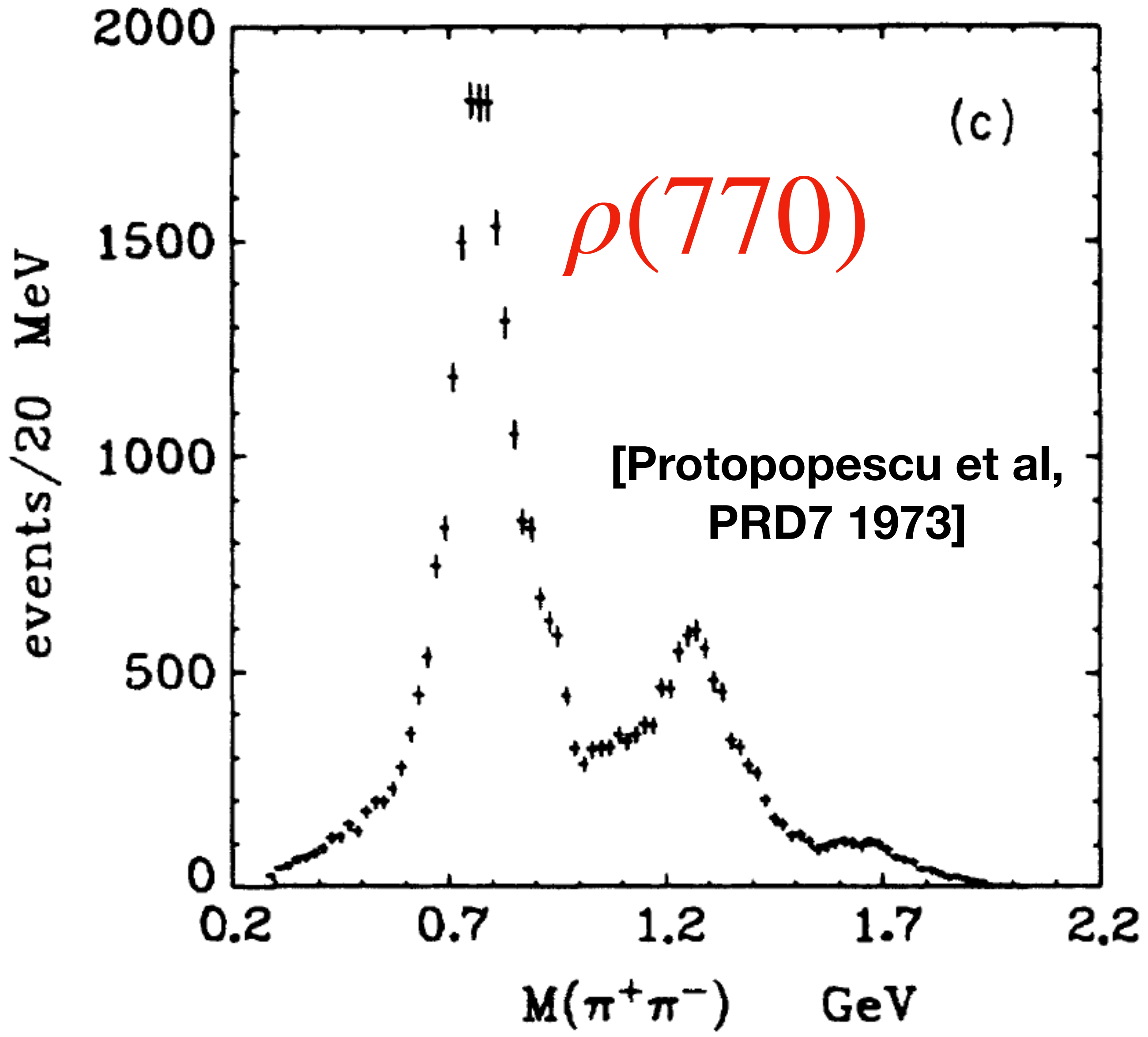
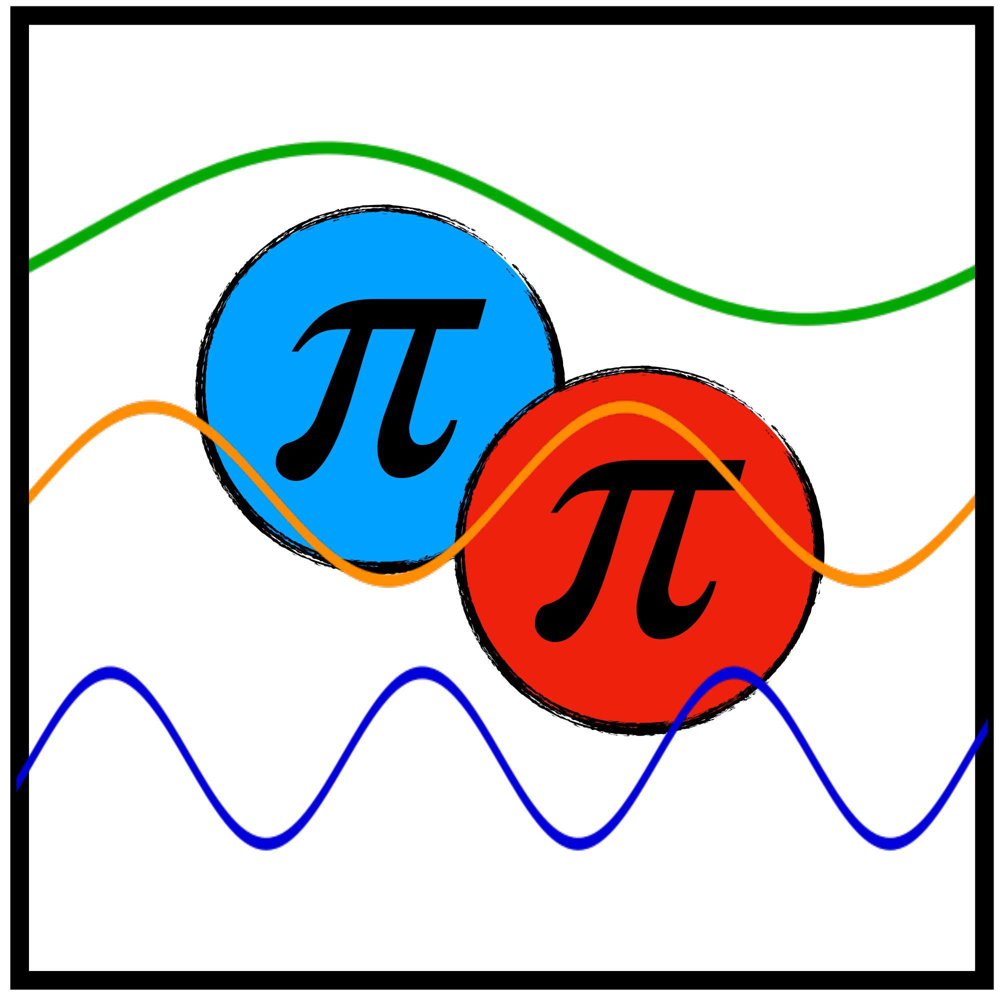
- Finite volume and Euclidean spacetime complicate the study of multi-hadron systems
- Enormous progress in the two-particle formalism & application.

[Talks by J. Dudek, A. Hanlon, D. Mohler, A. Nicholson]
[Talks by F. Ortega-Gama, S. Prelovsek, A. Rodas, A. Walker-Loud]



Two-particle finite-volume formalism

[Lüscher, 89']



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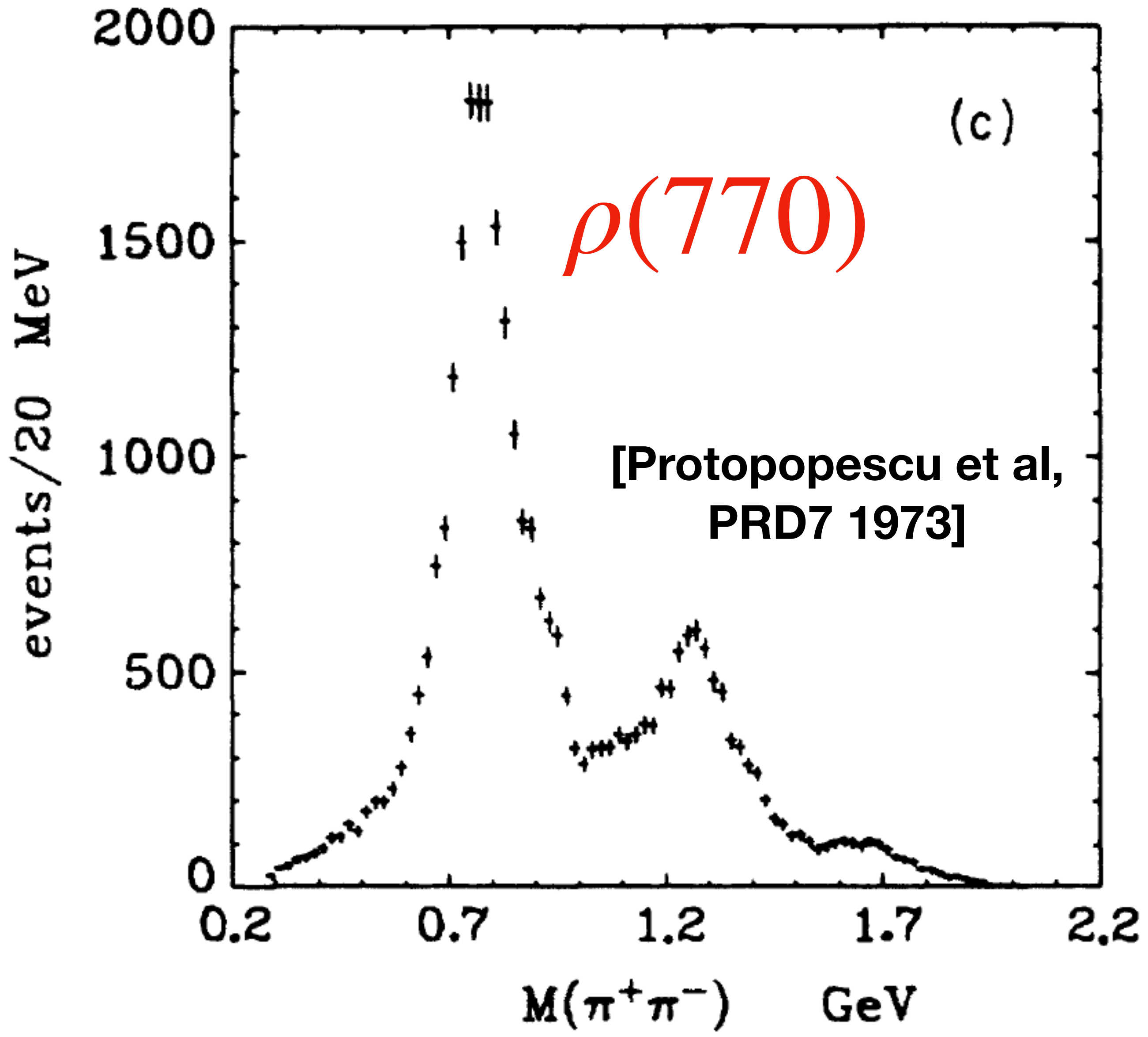

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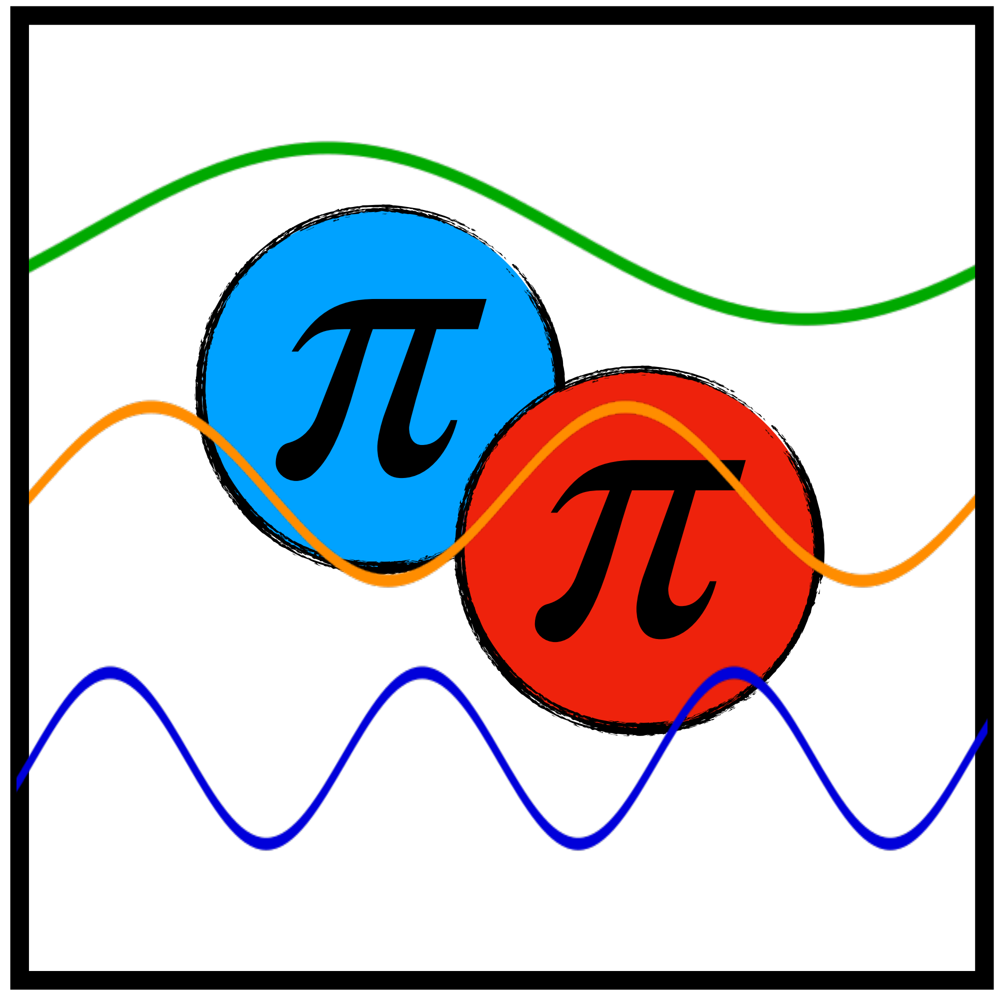
[Lüscher, 89']

and beyond two?



Three-particle processes

Many reasons to study the three-body problem in finite volume

- The two-body formalism is restricted to few interesting resonances
- The puzzle of the Roper resonance

$$N(1440) \to \Delta\pi \to N\pi\pi$$

- Exotic hadrons

$$T_{cc} \to DD^*, DD\pi$$

- Many-body nuclear physics: 3N force, tritium nucleus
- CP violation: $K \to 3\pi, \quad K^0 \leftrightarrow 3\pi \leftrightarrow \overline{K}^0$

[Talk by D. Pefkou]

Resonance	$I_{\pi\pi\pi}$	J^P
$\omega(782)$	0	1^-
$h_1(1170)$	0	1^+
$\omega_3(1670)$	0	3^-
$\pi(1300)$	1	0^-
$a_1(1260)$	1	1^+
$\pi_1(1400)$	1	1^-
$\pi_2(1670)$	1	2^-
$a_2(1320)$	1	2^+
$a_4(1970)$	1	4^+

(with $\geq 3\pi$ decay modes)

A three-particle era

- Formalism has been developed for several cases of interest

e.g. $\pi^+\pi^0\pi^-, K^+K^+\pi^+, NNN$

- Generic Relativistic Field Theory (RFT) [Hansen, Sharpe, PRD 2014 & 2015] [Talk by S. Sharpe]
- Non-Relativistic EFT (NREFT) [Hammer, Pang, Rusetsky, JHEP 2017] x 2 [Talk by A. Rusetsky]
- Finite-Volume Unitarity (FVU) [Mai, Döring, EPJA 2017] [Talks by M. Döring & M. Mai]

- Applications to simple system have been achieved

[Alexandrou et al, PRD 2020], [Blanton et al., PRL 2020 & JHEP 2021], [Brett et al, PRD 2021], [Culver et al, PRD 2021], [Fischer et al, EPJC 2021], [Garofalo et al, JHEP 2023], [Hansen et al, PRL 2021], [Mai et a PRL 2019 & 2021]

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- Important benchmark systems: three pseudoscalar mesons at maximal isospin
  - Implement formalism and explore its features
  - Test fitting strategies to extract three-body K matrix
  - Interpret results in combination with EFTs

Outline

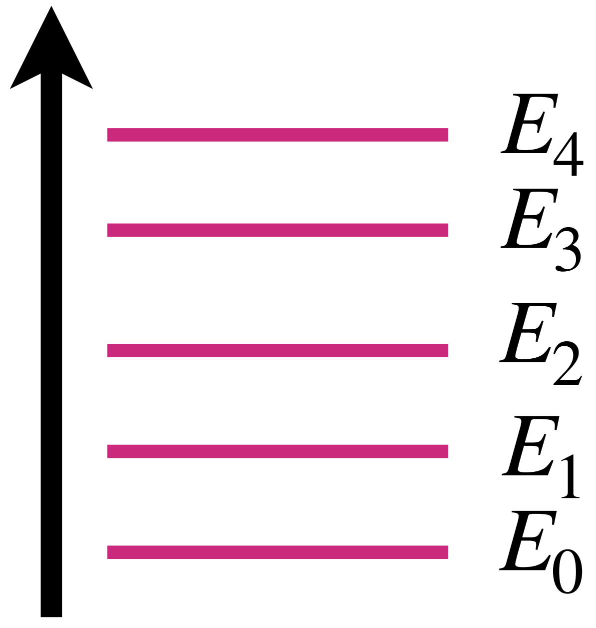
1. Results for 3K and 3π systems
2. 3π systems at NLO in ChPT
3. Results for mixed systems: πK, $\pi\pi K$, $K\pi\pi$

3K and 3π systems

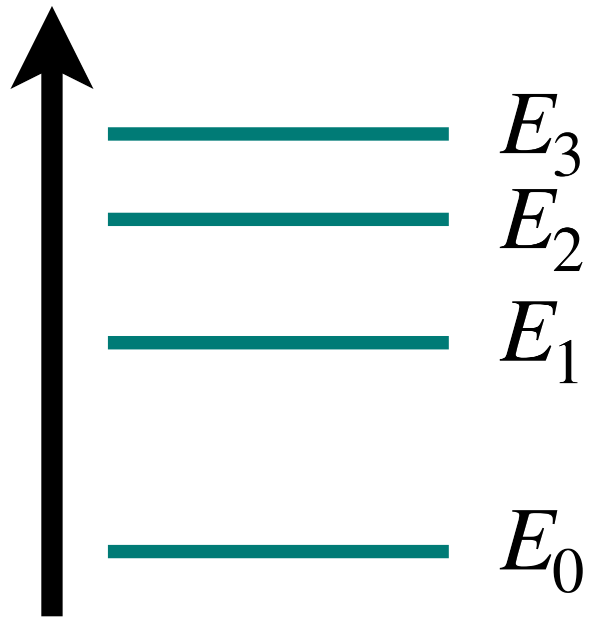
Relativistic three-particle formalism

[Hansen, Sharpe, PRD 2014 & 2015]

2π Spectrum



3π Spectrum



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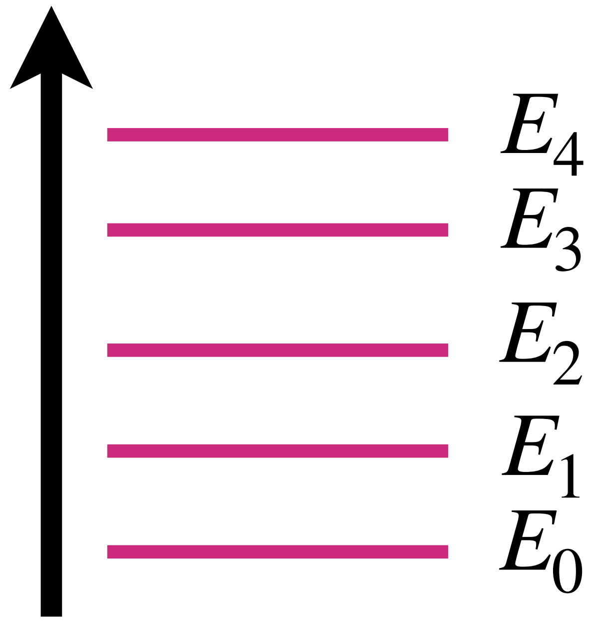

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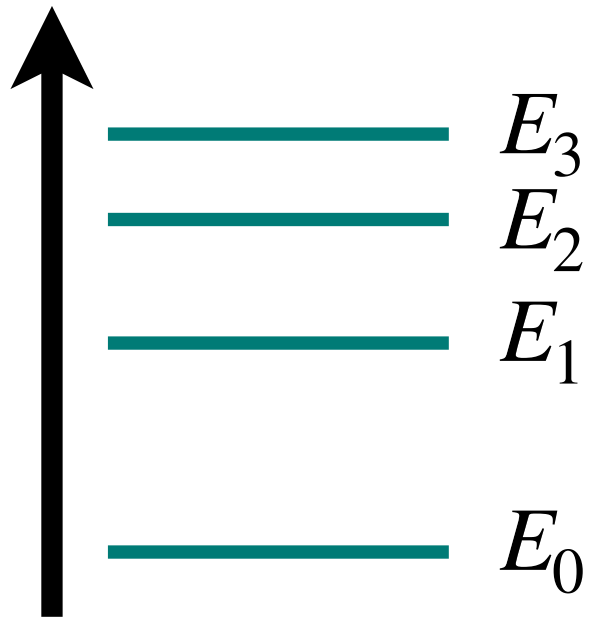

Quantization conditions

$$\det_{\ell m}\left[\mathcal{K}_2 + F_2^{-1}\right] = 0$$

$$\det_{k\ell m}\left[\mathcal{K}_{df,3} + F_3^{-1}\right] = 0$$

Matrix indices describe three on-shell particles

Fernando Romero-López, MIT

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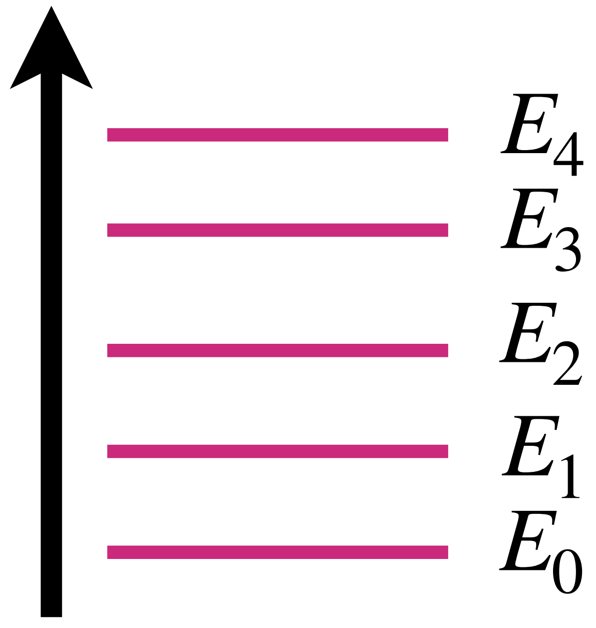

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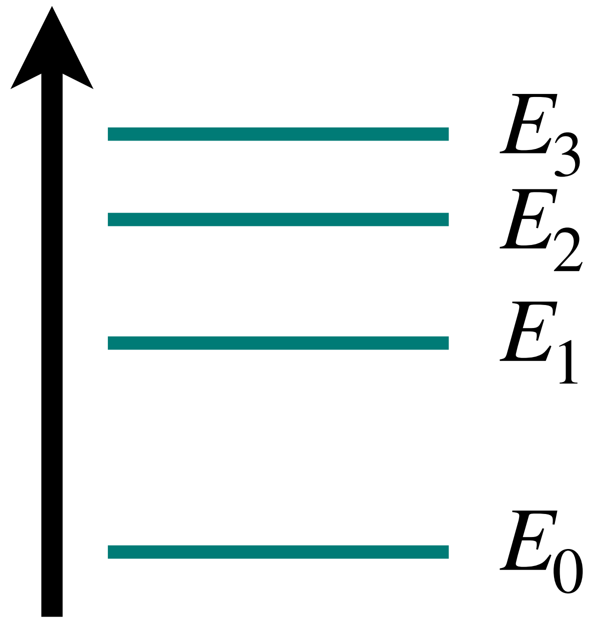

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Fit

K-matrices

$$\mathcal{K}_2$$

$$\mathcal{K}_{df,3}$$

Parametrize:

$$\mathcal{K}_2 = c_0 + c_1 k^2 + \ldots$$

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\left(\frac{s - 9m^2}{9m^2}\right) + \ldots$$

[Blanton, FRL, Sharpe, JHEP 2019]

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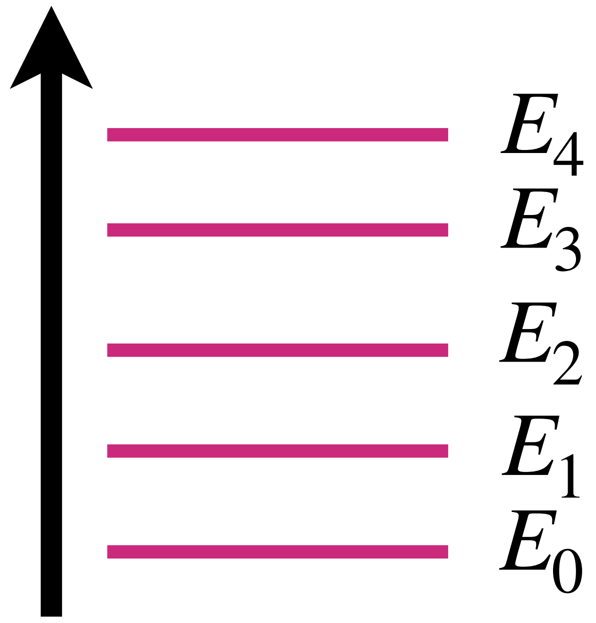

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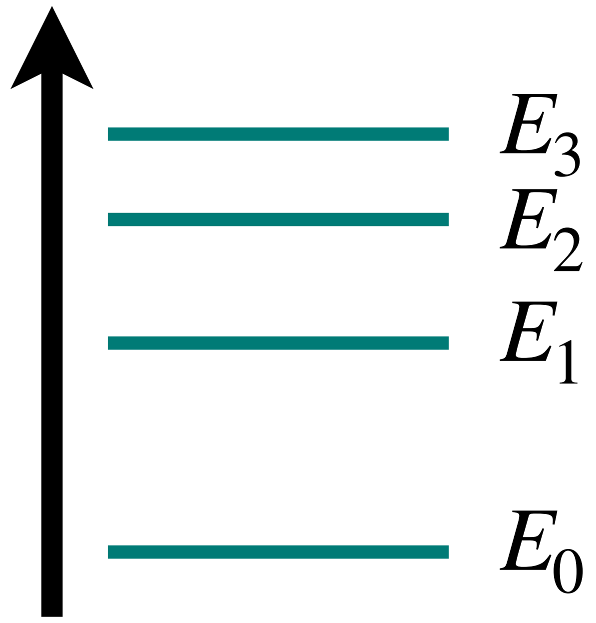

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[Blanton, FRL, Sharpe, JHEP 2019]

Unitarity relations →

Integral equations

[Briceño et al., PRD 2018]
[Hansen et al., PRL 2021]
[Jackura et al., PRD 2021]
[Dawid et al., 2303.04394]

[Talks by D. Islam & S. Dawid]

Scattering amplitudes

$\mathcal{M}_2$

$\mathcal{M}_3$

Relativistic three-particle formalism

[Hansen, Sharpe, PRD 2014 & 2015]

2π Spectrum

E_4, E_3, E_2, E_1, E_0

3π Spectrum

E_3, E_2, E_1, E_0

Quantization conditions

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Scattering amplitudes

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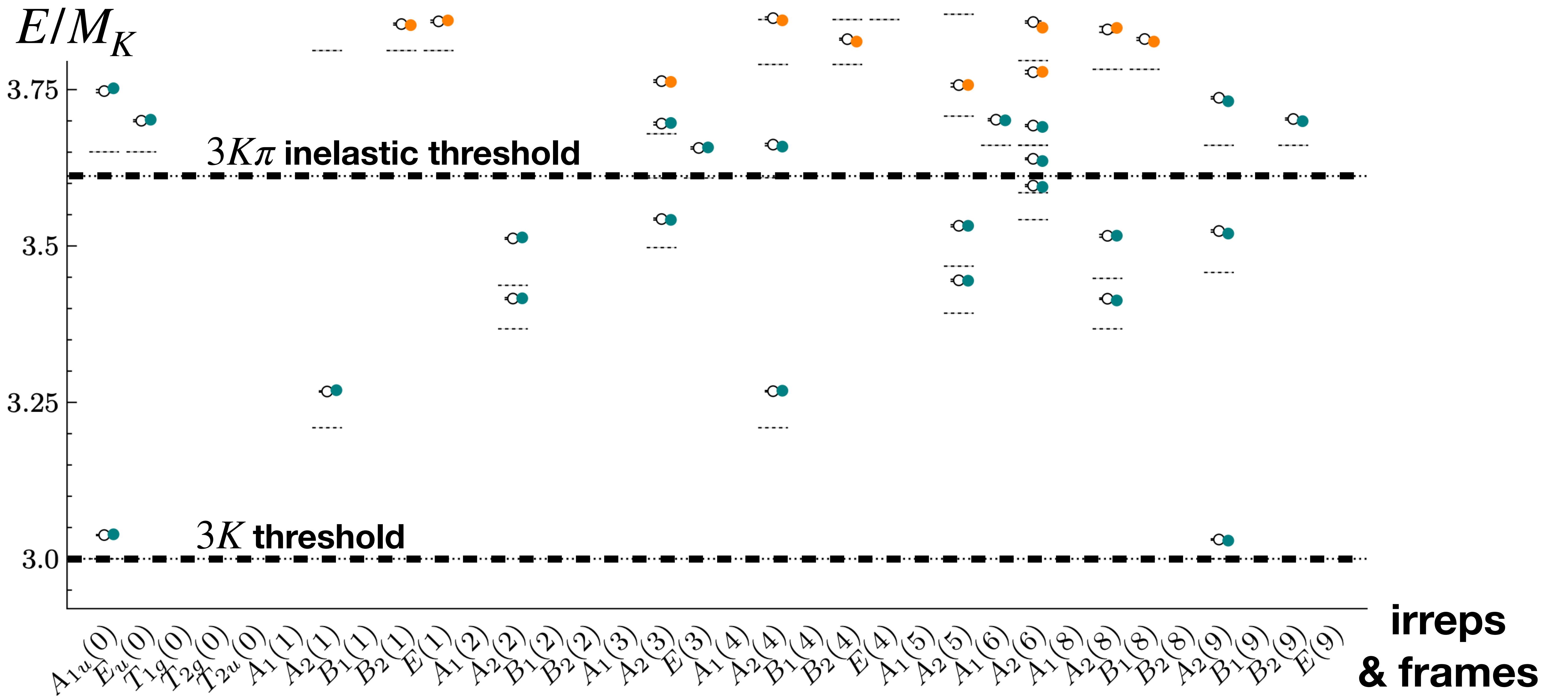
Experiments

Fernando Romero-López, MIT

The LQCD spectrum

- **Requires large sets of energy levels obtained using variational techniques**
  Using stochastic LapH method [Morningstar et al, PRD 2011]

$3K^+$ energy levels **O(30) energy levels!**



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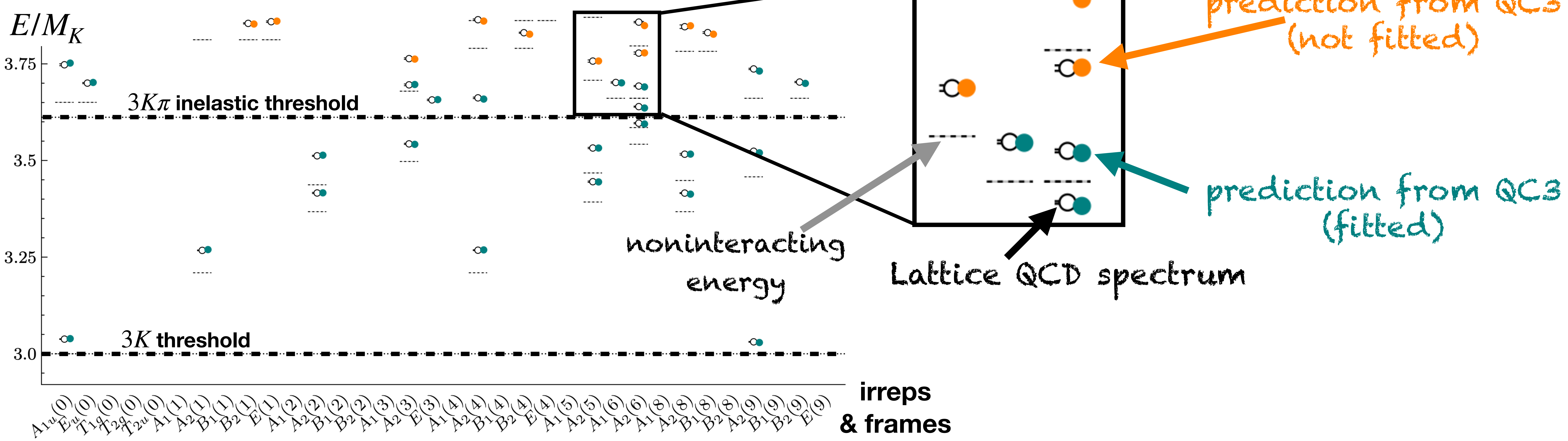

[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

Fitting the spectrum

- Extraction of the scattering parameters requires fitting spectral information.

"Spectrum method"

$$\chi^2(\vec{p}) = \sum_{ij} \left(E_{\mathrm{cm},i} - E^{\mathrm{QC}}_{\mathrm{cm},i}(\vec{p}) \right) \left(C^{-1} \right)_{ij} \left(E_{\mathrm{cm},j} - E^{\mathrm{QC}}_{\mathrm{cm},j}(\vec{p}) \right)$$

parameters in K-matrices

covariance matrix of energies

"predicted minus measured" center-of-mass energies

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"predicted minus measured" center-of-mass energies

D200, $a \simeq 0.063$ fm
$M_\pi = 200$ MeV
$M_K = 480$ MeV

$2\pi^+/3\pi^+$		$2K^+/3K^+$	
$\chi^2/\mathrm{dof} = 1.99$	(s wave)	$\chi^2/\mathrm{dof} = 1.84$	(s wave)
$\chi^2/\mathrm{dof} = 1.67$	(s and d waves)	$\chi^2/\mathrm{dof} = 1.34$	(s and d waves)

$3\pi^+$ $\mathcal{K}_{df,3}$ from lattice QCD

Depend on CM energy | **Angular dependence**

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,1}\Delta + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},\,2}\Delta^2 + \mathcal{K}_A\Delta_A + \mathcal{K}_B\Delta_B,$$

$$\Delta \equiv \frac{s-9m^2}{9m^2}$$

$$\Delta_B = \sum_{i,j=1}^{3} \widetilde{t}_{ij}^{\,2} - \Delta^2,$$

Fernando Romero-López, MIT

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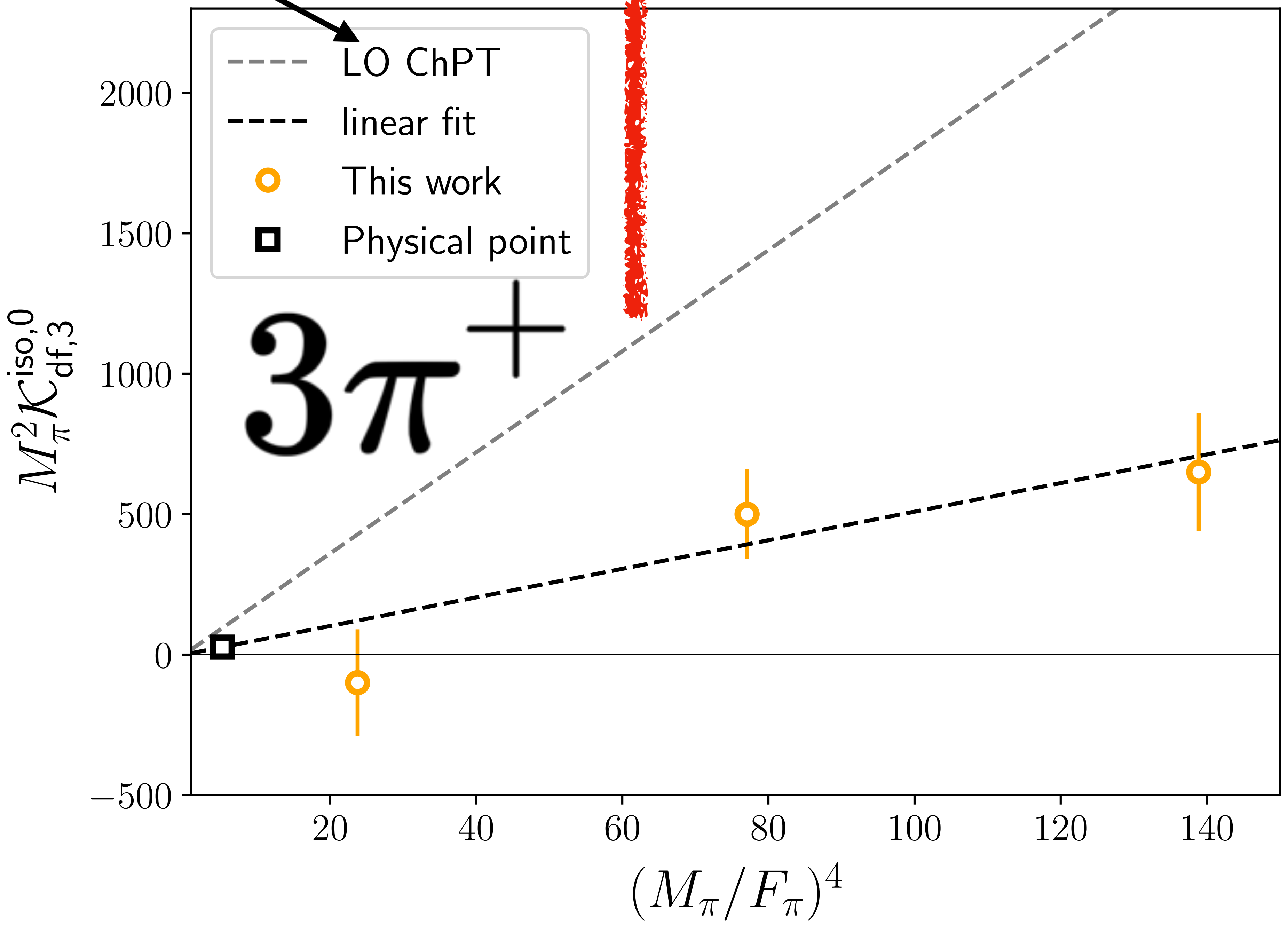

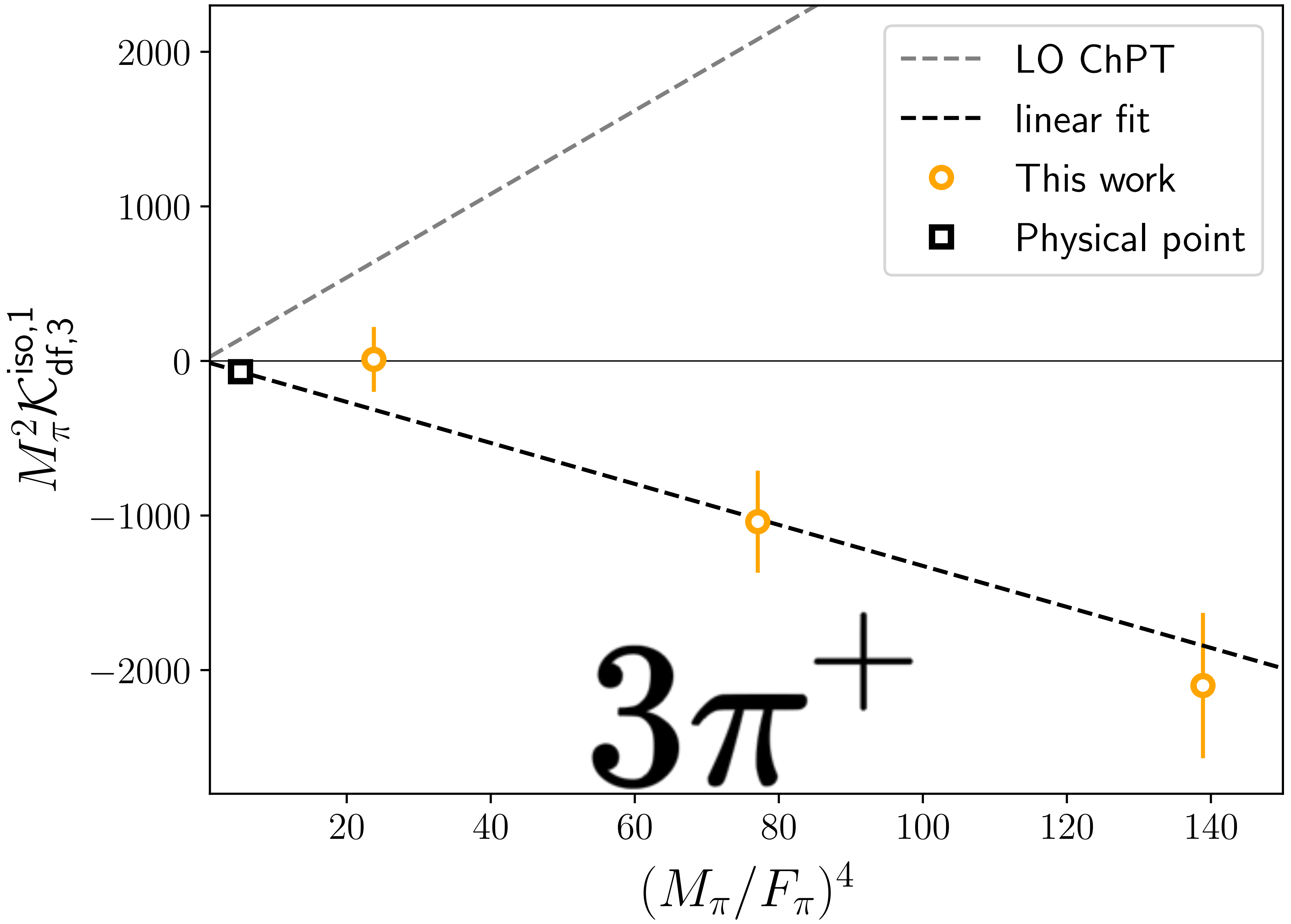

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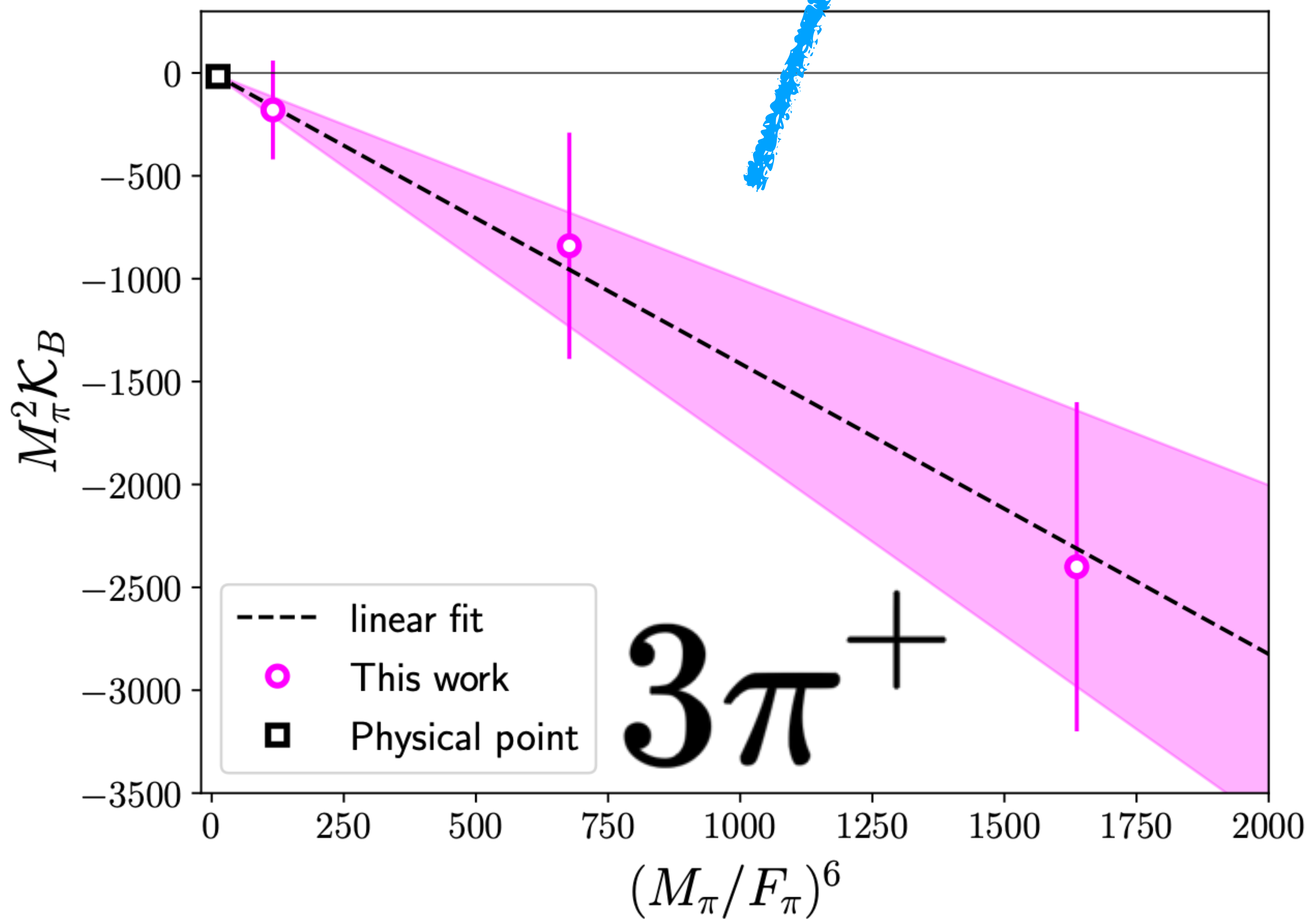
Successfully constrained subleading term including angular dependence!



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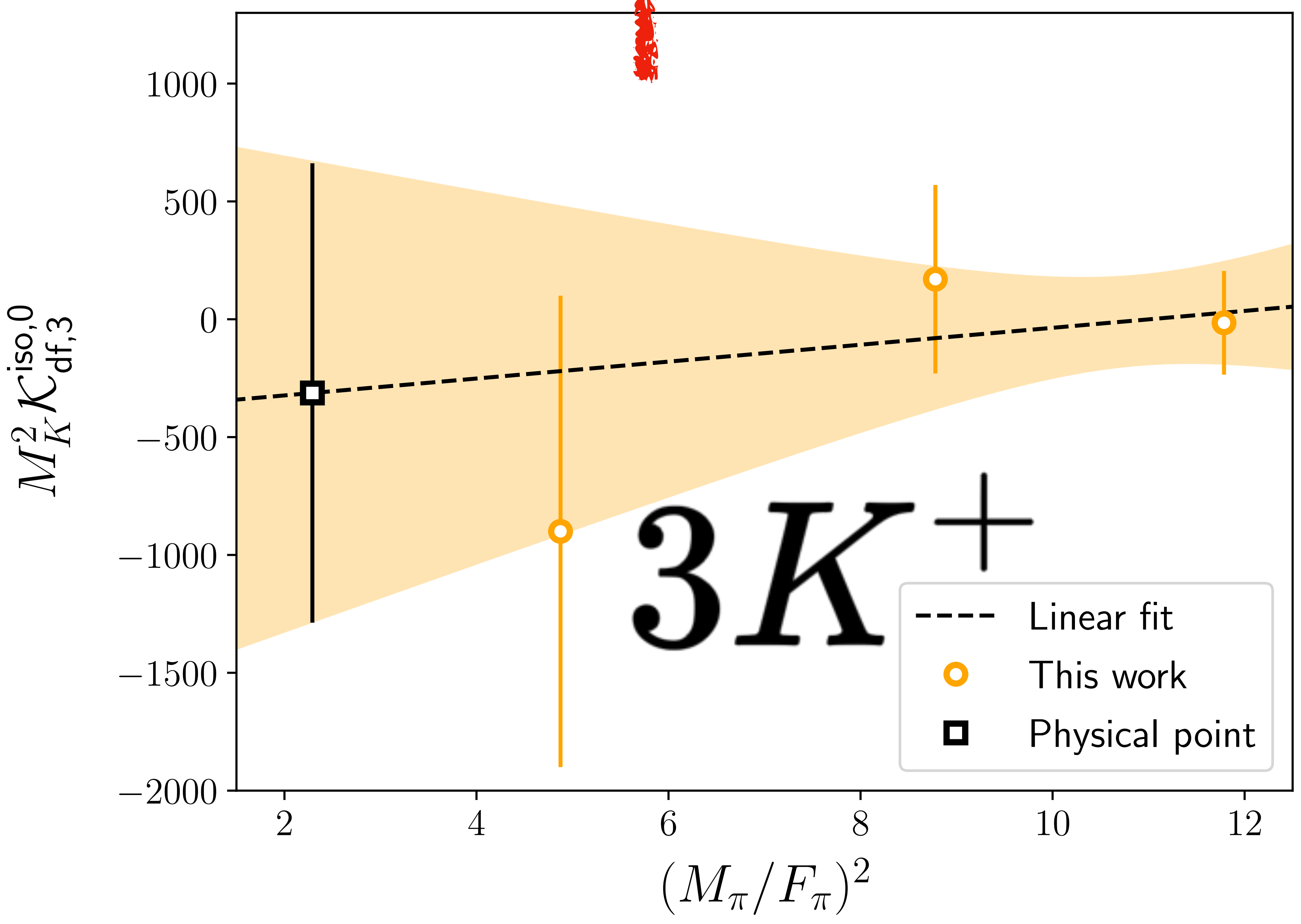

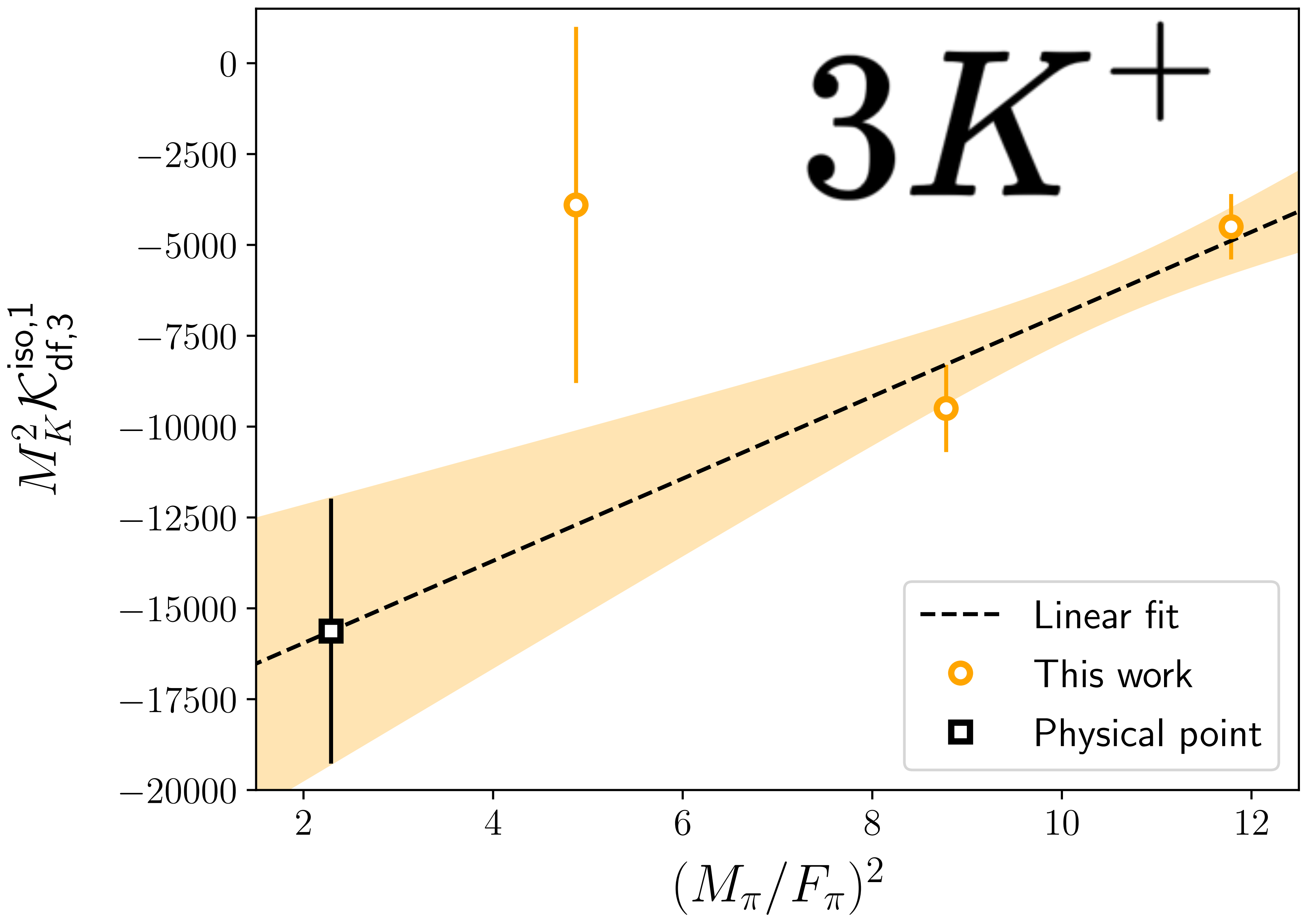

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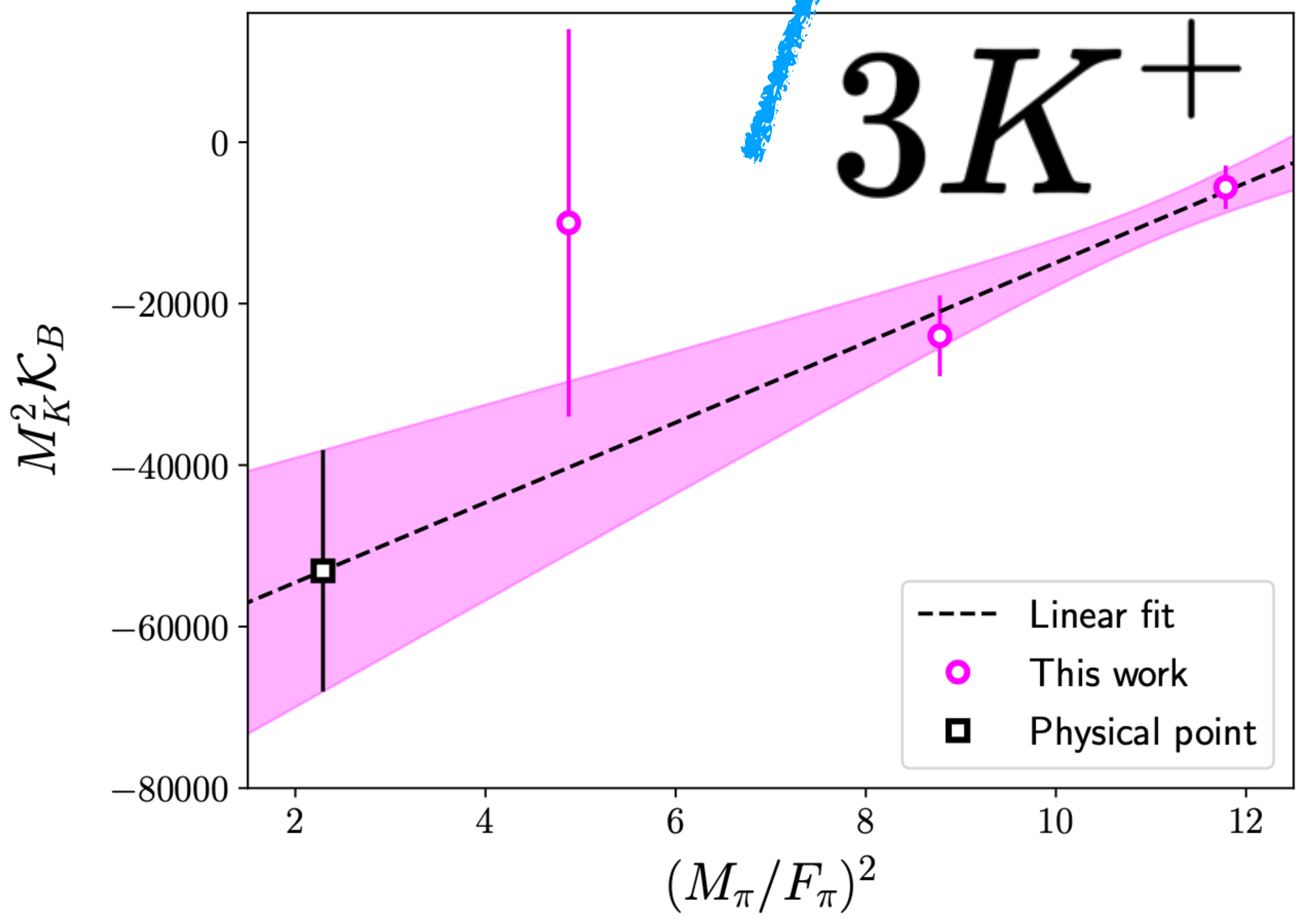
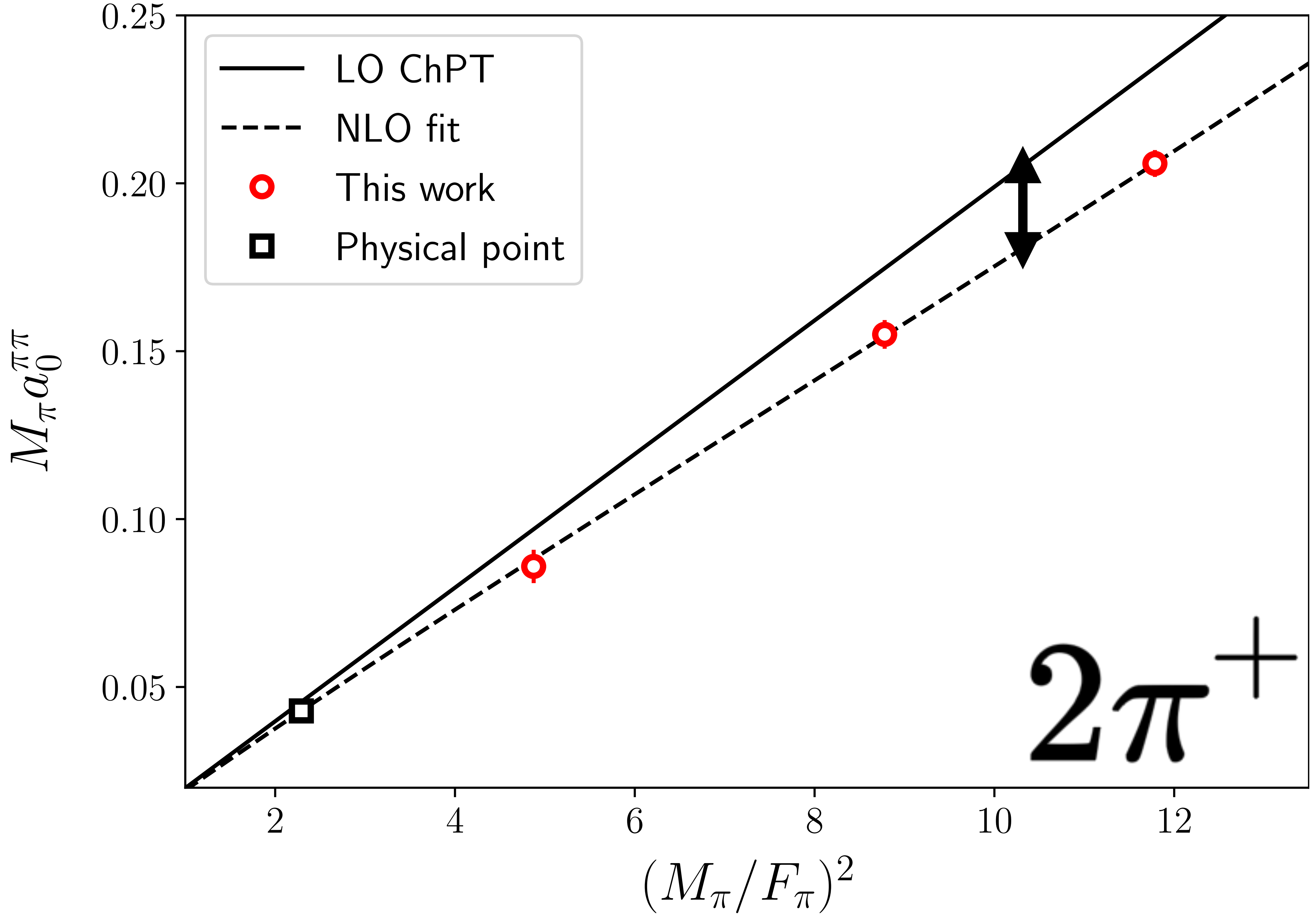


[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

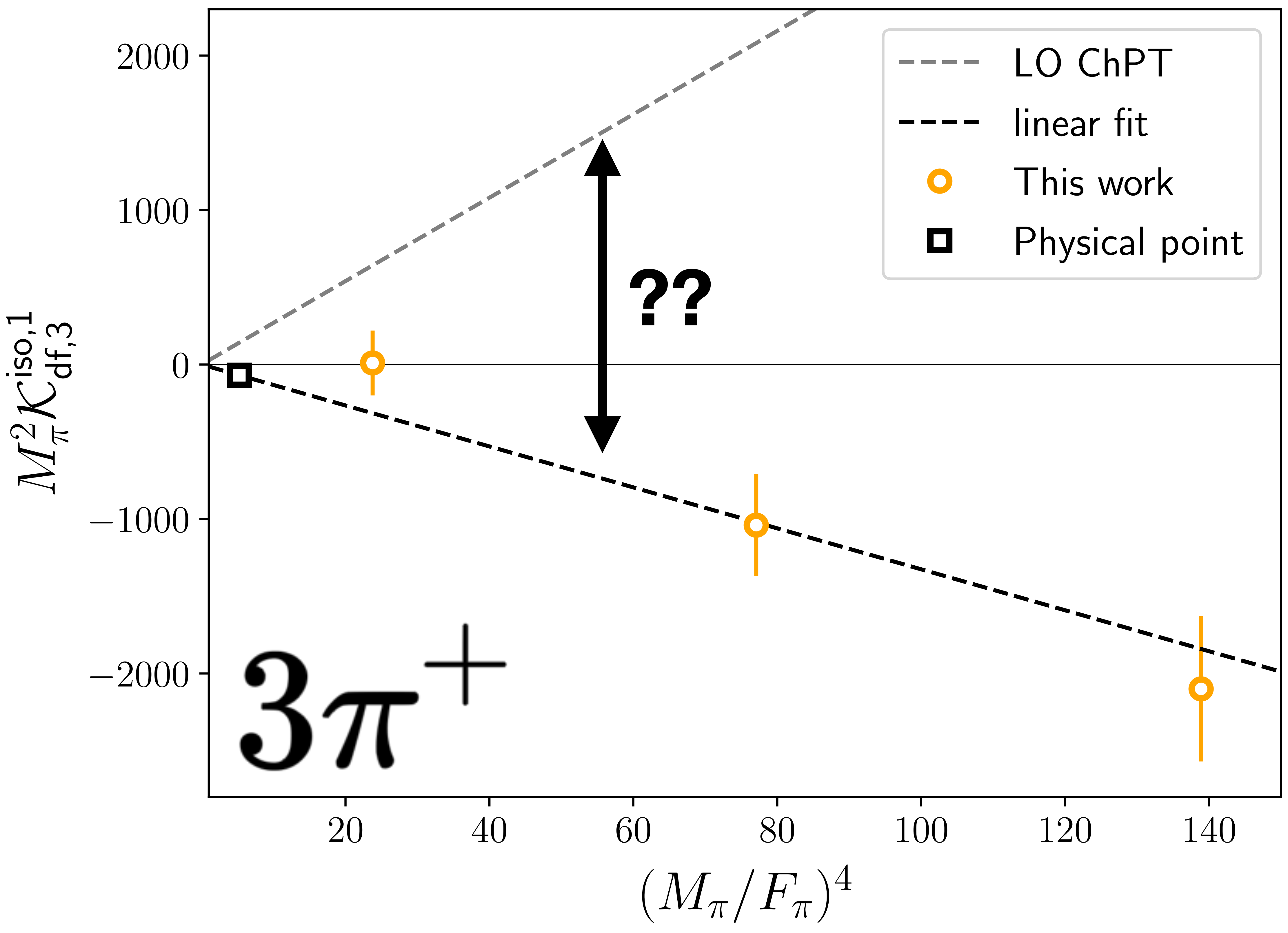
Three-pion scattering at NLO in ChPT

2π/3π interactions & ChPT

☑ **Two-pion sector is well described at LO**



❗ **Three-pion with large discrepancy**



[Blanton, Hanlon, Hörz, Morningstar, FRL, Sharpe, JHEP 2021]

$\mathcal{K}_{df,3}$ at NLO in ChPT

- Relation between amplitude and K matrix involves integral equations

$$\mathcal{M}_{\rm df,3}(\boldsymbol{p},\boldsymbol{k}) = \mathcal{M}_3 - \mathcal{S}\left(\mathcal{D}^{(u,u)}\right) = \mathcal{S}\left\{\int_s \int_r \mathcal{L}^{(u,u)}(\boldsymbol{p},\boldsymbol{s})\mathcal{T}(\boldsymbol{s},\boldsymbol{r})\mathcal{R}^{(u,u)}(\boldsymbol{r},\boldsymbol{k})\right\}$$

(cutoff dependent)

$$\mathcal{T}(\boldsymbol{p},\boldsymbol{k}) \equiv \mathcal{K}_{\rm df,3}(\boldsymbol{p},\boldsymbol{k}) - \int_{s,r} \mathcal{K}_{\rm df,3}(\boldsymbol{p},\boldsymbol{s})\rho(\boldsymbol{s})\mathcal{L}^{(u,u)}(\boldsymbol{s},\boldsymbol{r})\mathcal{T}(\boldsymbol{r},\boldsymbol{k})$$

$$\mathcal{L}^{(u,u)}(\boldsymbol{p},\boldsymbol{k}) \equiv \left(\frac{1}{3} - \mathcal{M}_2(\boldsymbol{p})\rho(\boldsymbol{p})\right)\bar{\delta}(\boldsymbol{p}-\boldsymbol{k}) - \mathcal{D}^{(u,u)}(\boldsymbol{p},\boldsymbol{k})\rho(\boldsymbol{k})$$

$$\mathcal{D}^{(u,u)}(\boldsymbol{p},\boldsymbol{k}) = -\mathcal{M}_2(\boldsymbol{p})G^{\infty}(\boldsymbol{p},\boldsymbol{k})\mathcal{M}_2(\boldsymbol{k}) + \int_r \mathcal{M}_2(\boldsymbol{p})G^{\infty}(\boldsymbol{p},\boldsymbol{r})\mathcal{D}^{(u,u)}(\boldsymbol{r},\boldsymbol{k})$$

$\mathcal{K}_{df,3}$ at NLO in ChPT

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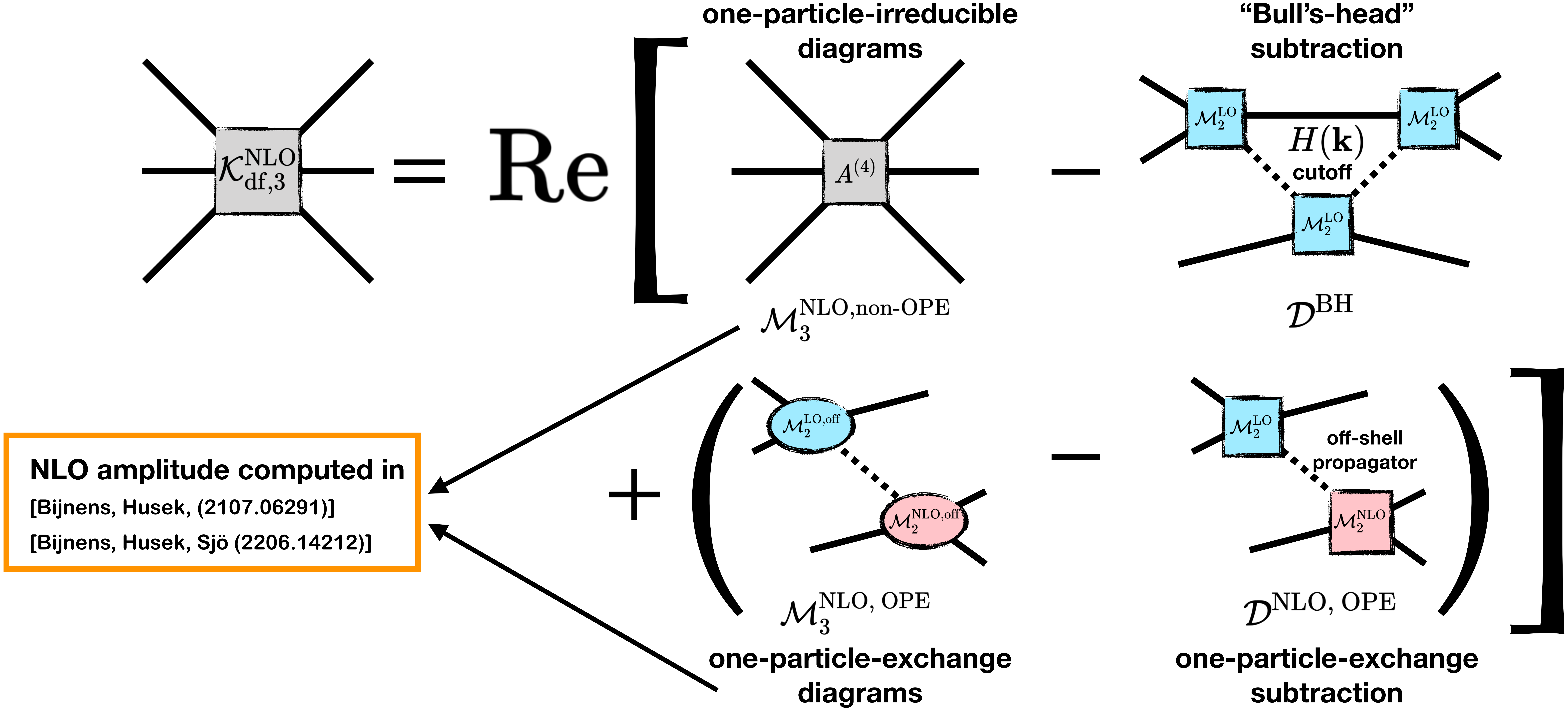
- Chiral counting at NLO leads to a very simple relation:

$$\mathcal{M}_2 \propto \frac{1}{F^2} \qquad \mathcal{M}_3 \sim \mathcal{K}_{\text{df},3} \sim \frac{1}{F^4}$$

$$\longrightarrow \quad \mathcal{L}^{(u,u)} = \frac{1}{3} - \rho M_2 + \mathcal{O}\left(\frac{1}{F^4}\right) \qquad \mathcal{T} = \mathcal{K}_{\text{df},3} + \mathcal{O}\left(\frac{1}{F^4}\right)$$

$$\longrightarrow \quad \boxed{\mathcal{K}_{\text{df},3}^{\text{NLO}} = \text{Re}\,\mathcal{M}_{\text{df},3}^{\text{NLO}}}$$

Computing $\mathcal{K}_{df,3}$ at NLO



Results for $\mathcal{K}_{df,3}$ at NLO

$\kappa = 1/(16\pi^2)$

$$\mathcal{K}_0 = \left(\frac{M_\pi}{F_\pi}\right)^4 18 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-3\kappa(35 + 12\log 3) - \mathcal{D}_0 + 111L + \ell^{\rm r}_{(0)}\right],$$

$$\mathcal{K}_1 = \left(\frac{M_\pi}{F_\pi}\right)^4 27 + \left(\frac{M_\pi}{F_\pi}\right)^6 \left[-\frac{\kappa}{20}\left(1999 + 1920\log 3\right) - \mathcal{D}_1 + 384L + \ell^{\rm r}_{(1)}\right],$$

$$\mathcal{K}_2 = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{207\kappa}{1400}(2923 - 420\log 3) - \mathcal{D}_2 + 360L + \ell^{\rm r}_{(2)}\right],$$

$$\mathcal{K}_{\rm A} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{9\kappa}{560}(21809 - 1050\log 3) - \mathcal{D}_{\rm A} - 9L + \ell^{\rm r}_{(\rm A)}\right],$$

$$\mathcal{K}_{\rm B} = \left(\frac{M_\pi}{F_\pi}\right)^6 \left[\frac{27\kappa}{1400}(6698 - 245\log 3) - \mathcal{D}_{\rm B} + 54L + \ell^{\rm r}_{(\rm B)}\right].$$

[Baeza-Ballesteros, Bijnens, Husek ,FRL, Sharpe, Sjö (to appear)]

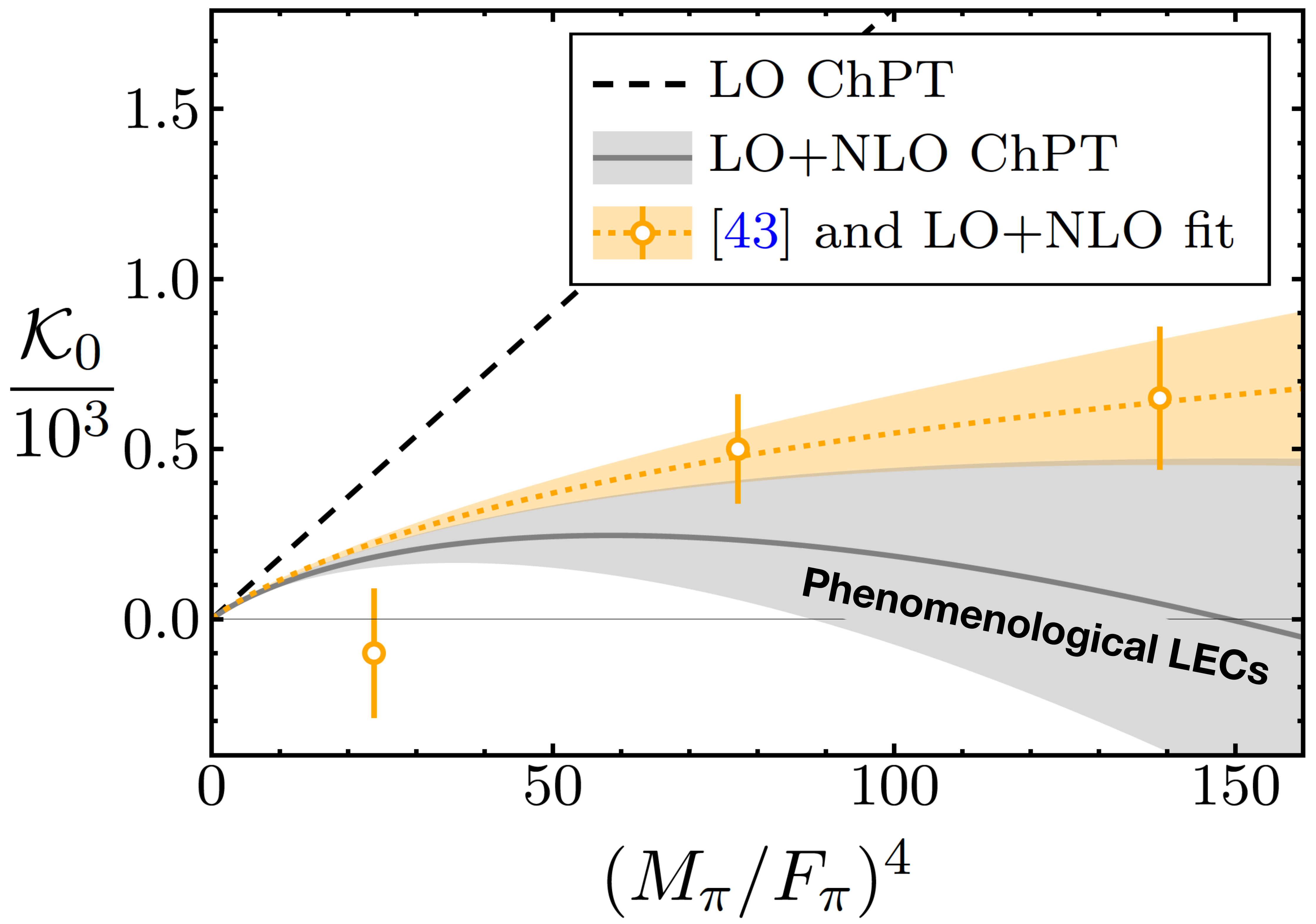
numerical coefficients (cutoff dependent)

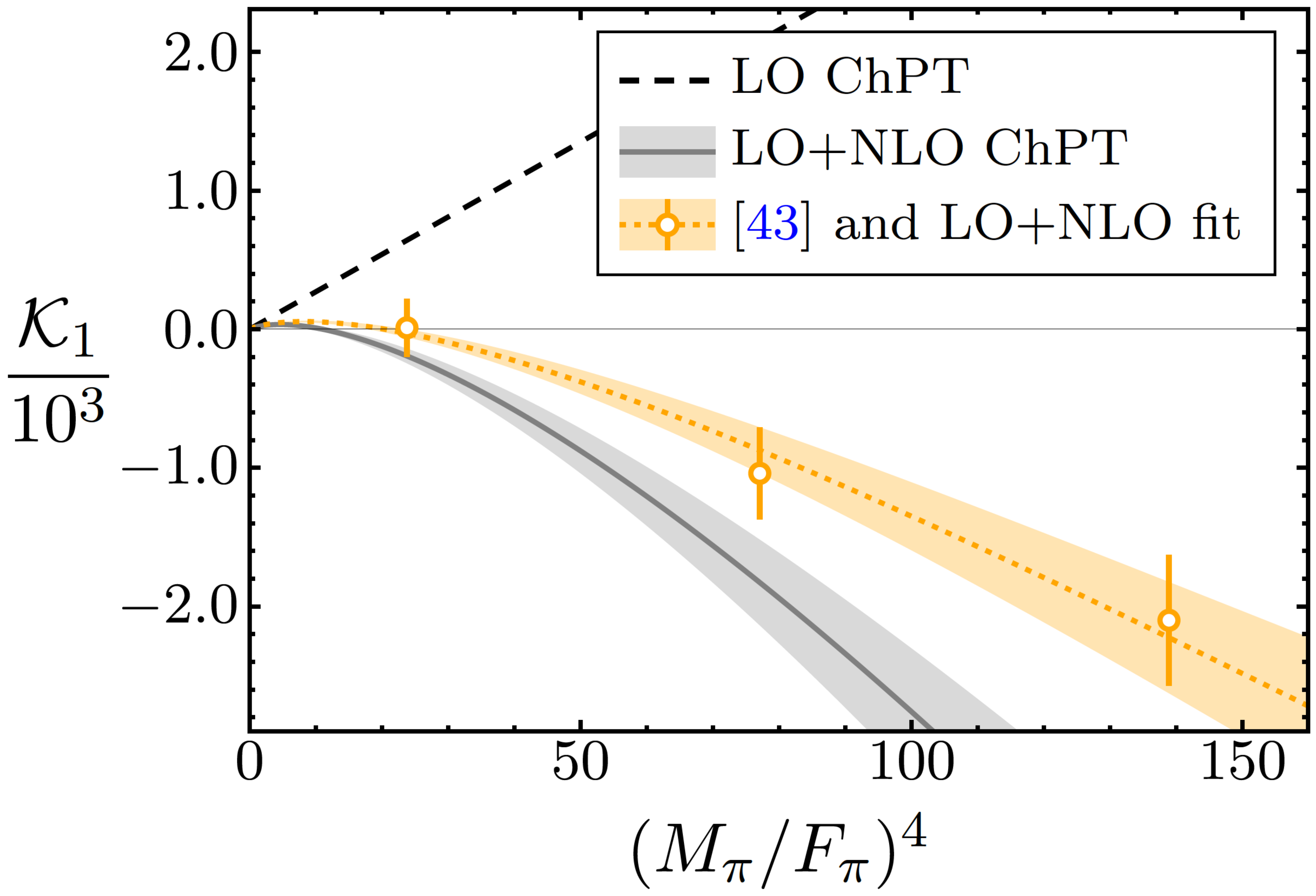
$L \equiv \kappa \log(M_\pi^2/\mu^2)$

LECs

Comparison to LQCD

[Baeza-Ballesteros, Bijnens, Husek ,FRL, Sharpe, Sjö (to appear)]



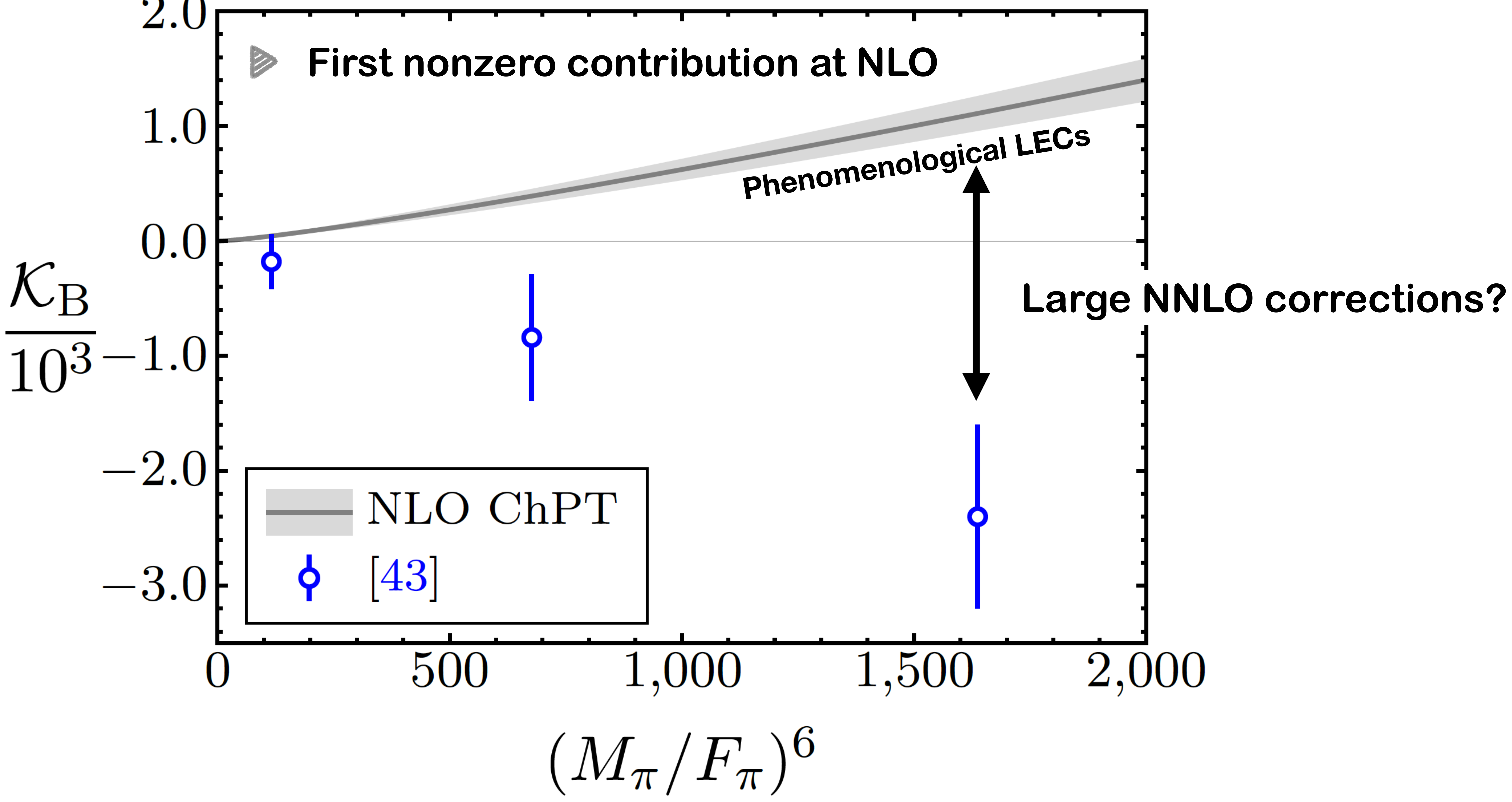


☑ **NLO effects can accommodate discrepancy!**

! **Large NLO corrections.**

Comparison to LQCD

[Baeza-Ballesteros, Bijnens, Husek ,FRL, Sharpe, Sjö (to appear)]



term with angular dependence

Mixed meson systems: πK, $\pi\pi K$, $K\pi\pi$

Nondegenerate systems

- Relevant three-body systems involve nonidentical particles ($\pi\pi N$)
- First step: RFT formalism for three different scalars
  [Blanton, Sharpe, PRD 2021] x 2, [Hansen, FRL, Sharpe, JHEP 2021]

e.g. $\pi^+\pi^0\pi^-,\ K^+K^+\pi^+,\ D_s^+D^0\pi^-$

$$\det_{k,\ell,m,\mathbf{f}}[1-\mathbf{K}_{\mathrm{df},3}(E^\star)\mathbf{F}_3(E,\boldsymbol{P},L)]=0$$

Implementation: github.com/ferolo2/QC3_release

determinant runs over an additional "flavor" index

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determinant runs over an additional "flavor" index

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- Less symmetry in the K-matrix, leading to more coefficients in the expansion:

Example: $\pi^+\pi^+K^+$ scattering

$$\mathcal{K}_{\mathrm{df},3}=\mathcal{K}_0+\mathcal{K}_1\Delta+\mathcal{K}_B\left(\Delta_2+\Delta_2'\right)+\mathcal{K}_E\tilde{t}_{22}$$

$$\Delta=\frac{s-M}{M^2}\qquad \tilde{t}_{22}=\frac{(p_K-p_K')^2}{M^2}\qquad \Delta_2=\frac{(p_\pi+p_{\pi'})^2-4m_3^2}{M^2}$$

Fitting "2+1" systems

- Requires fitting three systems at one. For instance: $\pi\pi K + \pi\pi + \pi K$
    - Perform a simultaneous (correlated) fit

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$$\chi^2(\vec{p}) = \sum_{ij} \Big(\Delta E_{\text{lab},i} - \Delta E^{\text{QC}}_{\text{lab},i}(\vec{p})\Big) \left(C^{-1}\right)_{ij} \Big(\Delta E_{\text{lab},j} - \Delta E^{\text{QC}}_{\text{lab},j}(\vec{p})\Big)$$

parameters in K-matrices

covariance matrix of lab-shifts

"predicted minus measured" lab-frame energy shifts

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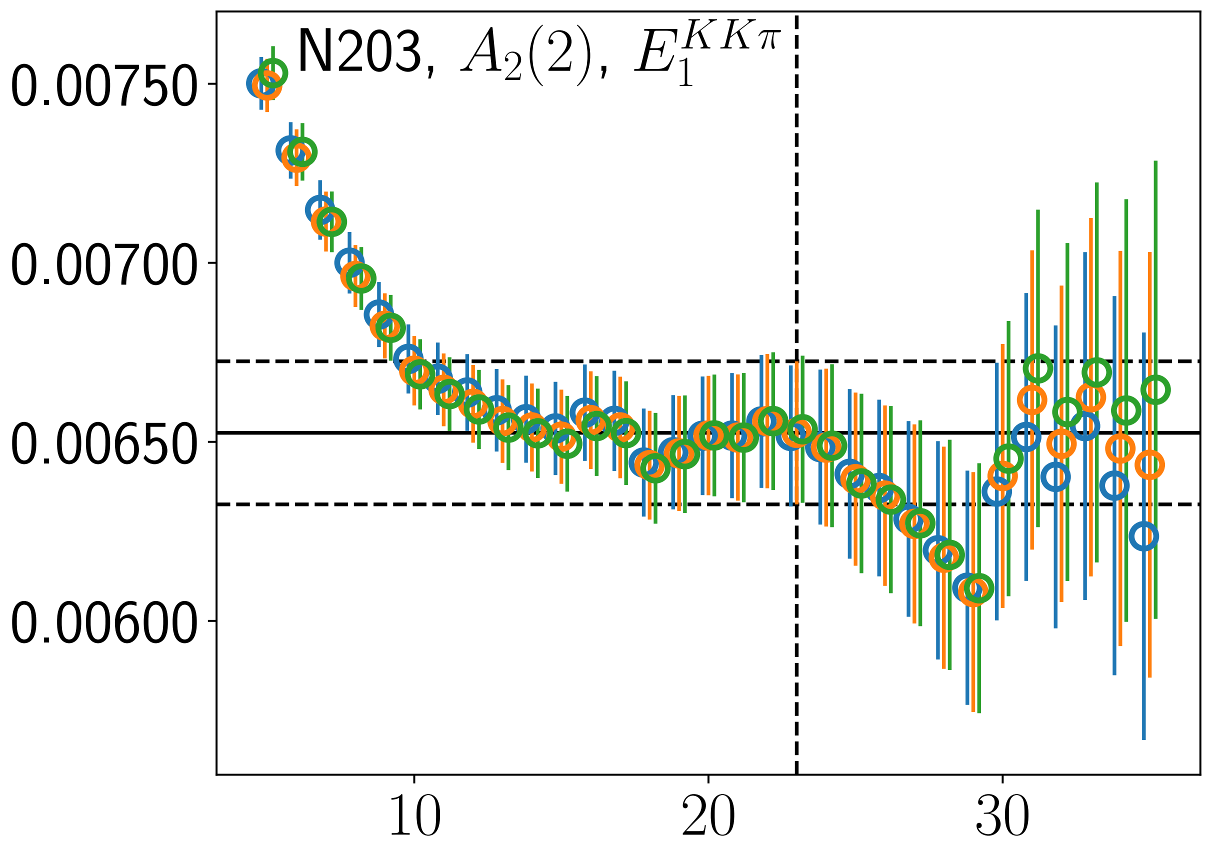
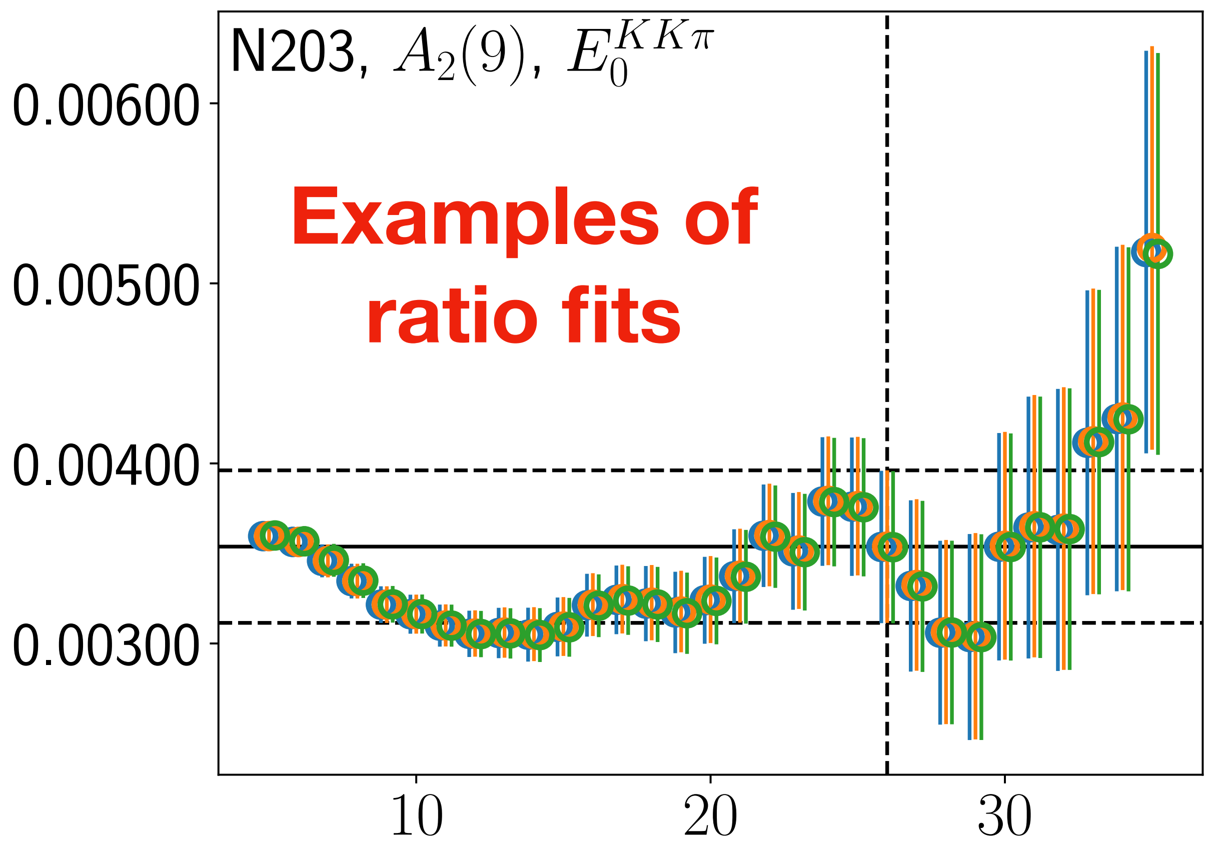
Motivations:

- Direct output of Lattice QCD
- Better conditioned covariance matrix
- More constrained fits (lower errors)





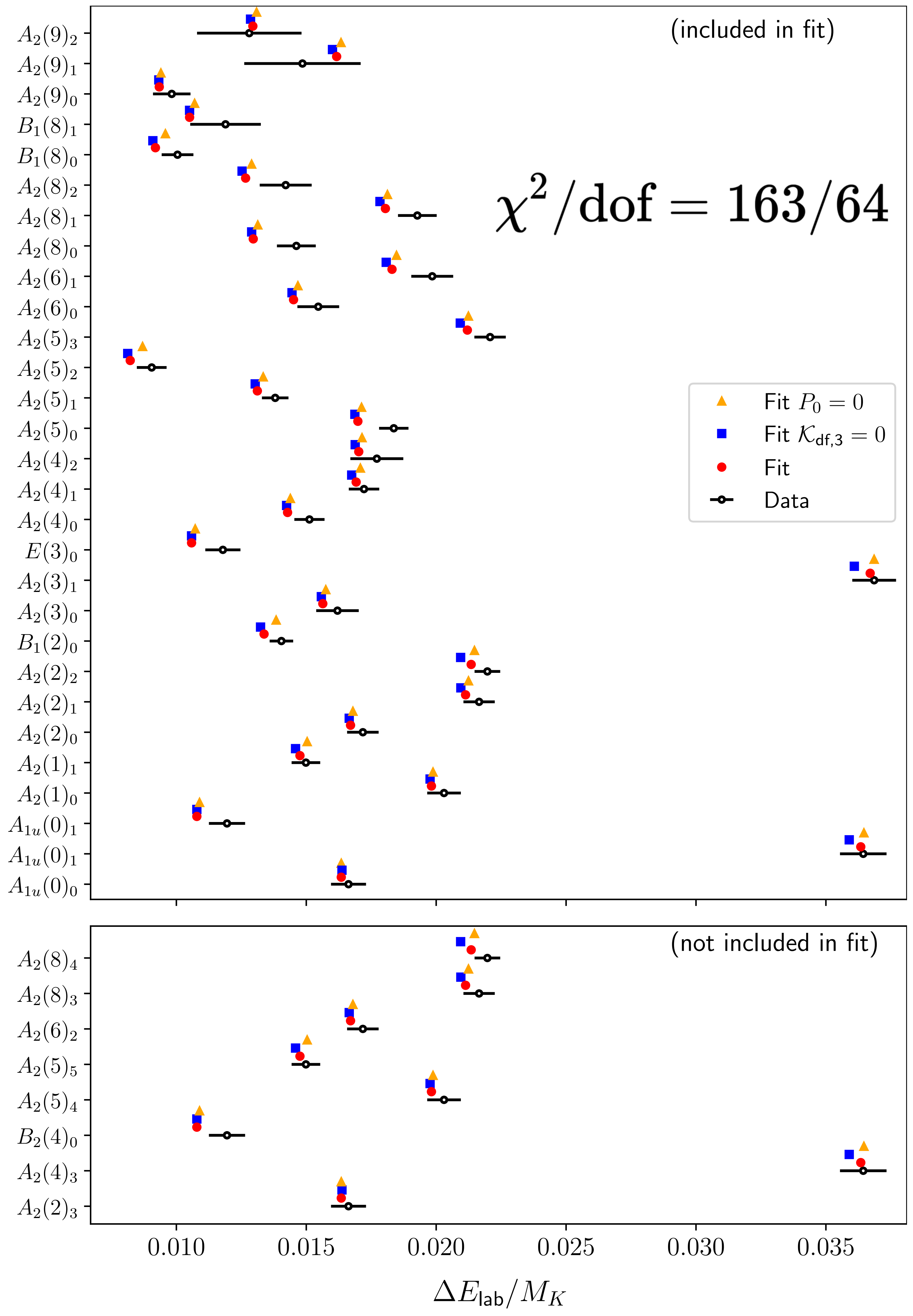
Fernando Romero-López, MIT

Lattice QCD spectrum

Fernando Romero-López, MIT

Visualization of fits

$KK\pi$, D200 $M_\pi = 200$ MeV
$M_K = 480$ MeV



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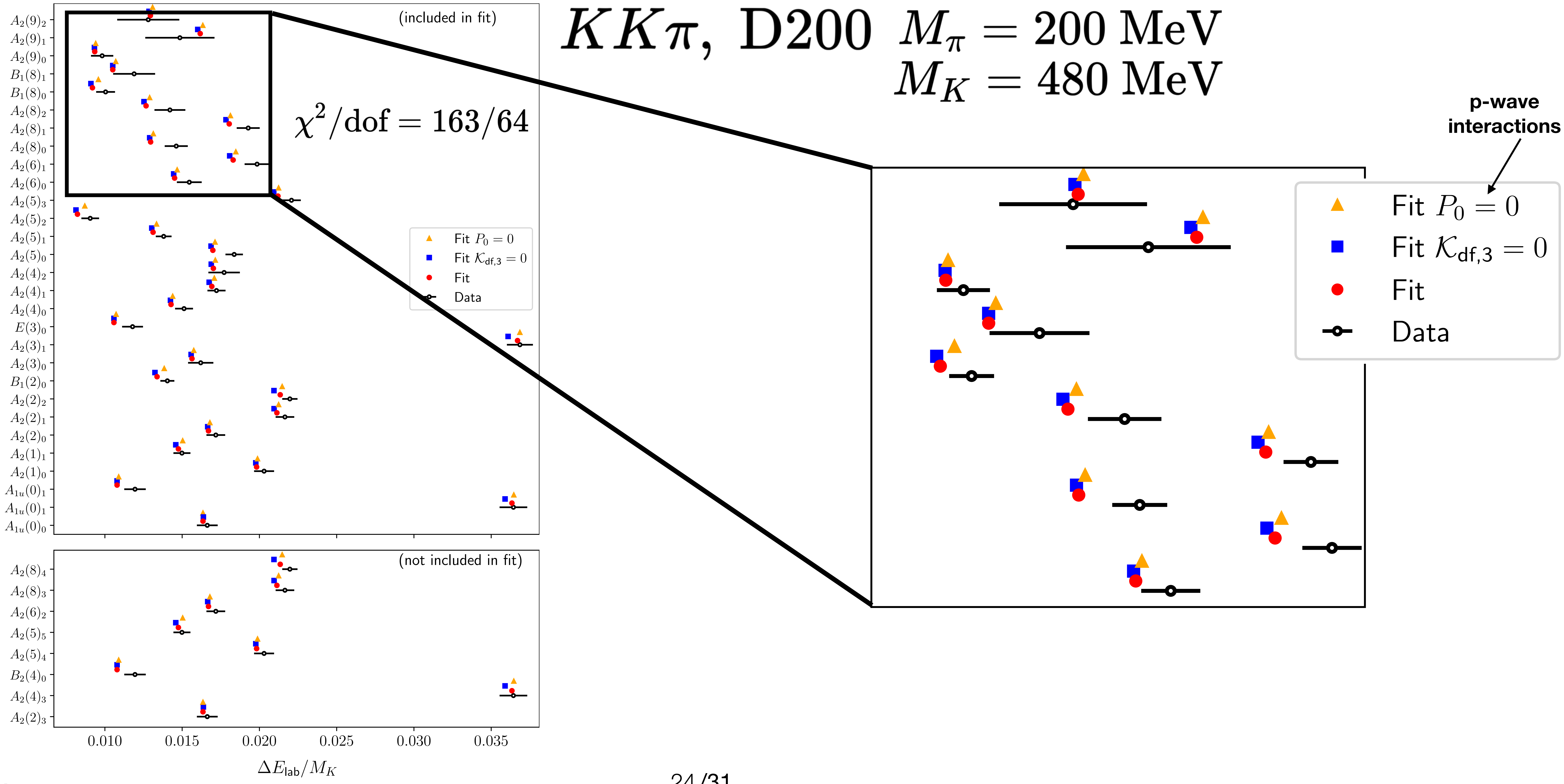

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Scattering lengths

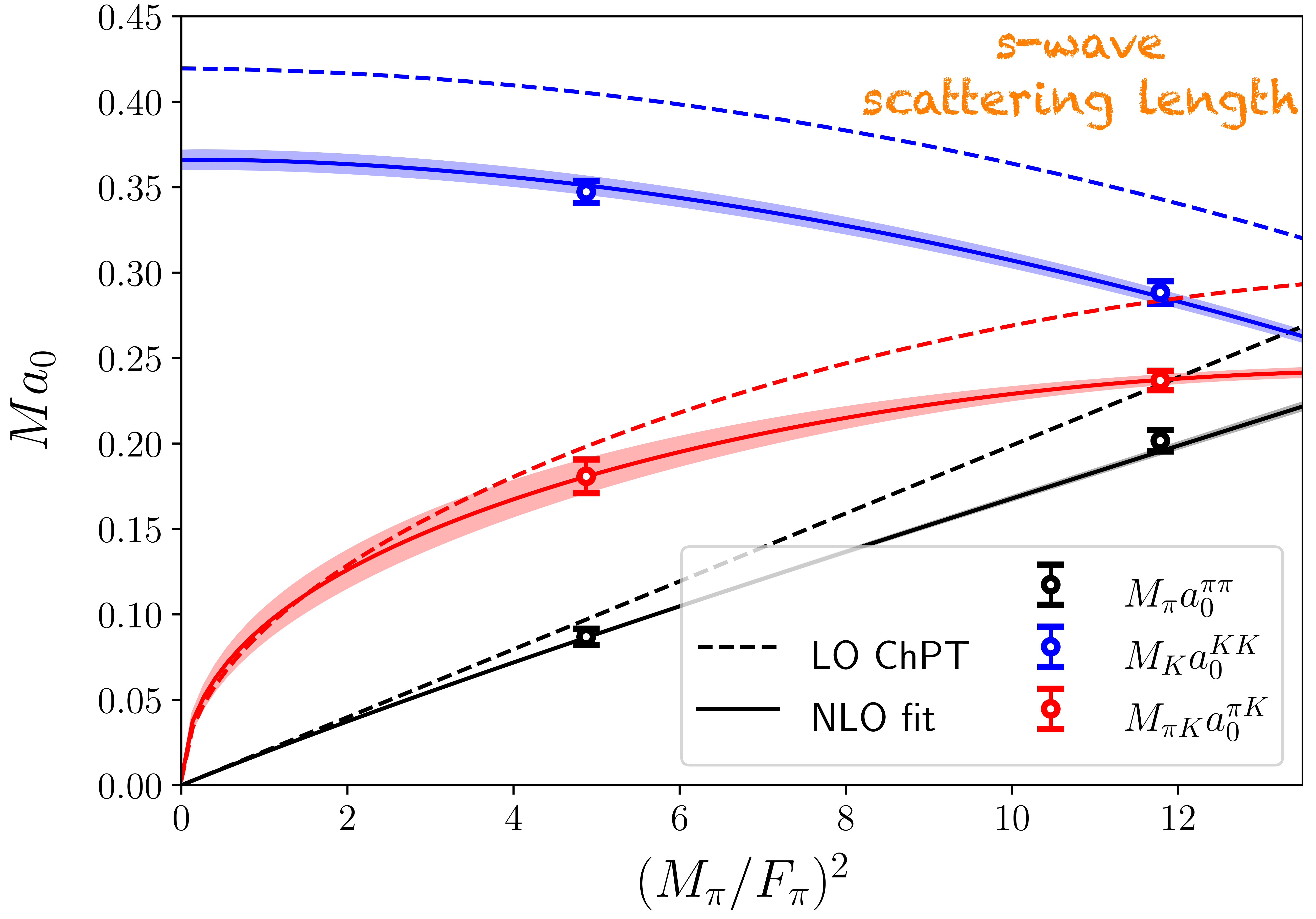
- By-product of three-body studies are two-meson amplitudes

- LECs from SU(3) NLO ChPT:

$$L_{\pi\pi} = -8.77(36) \times 10^{-4}$$

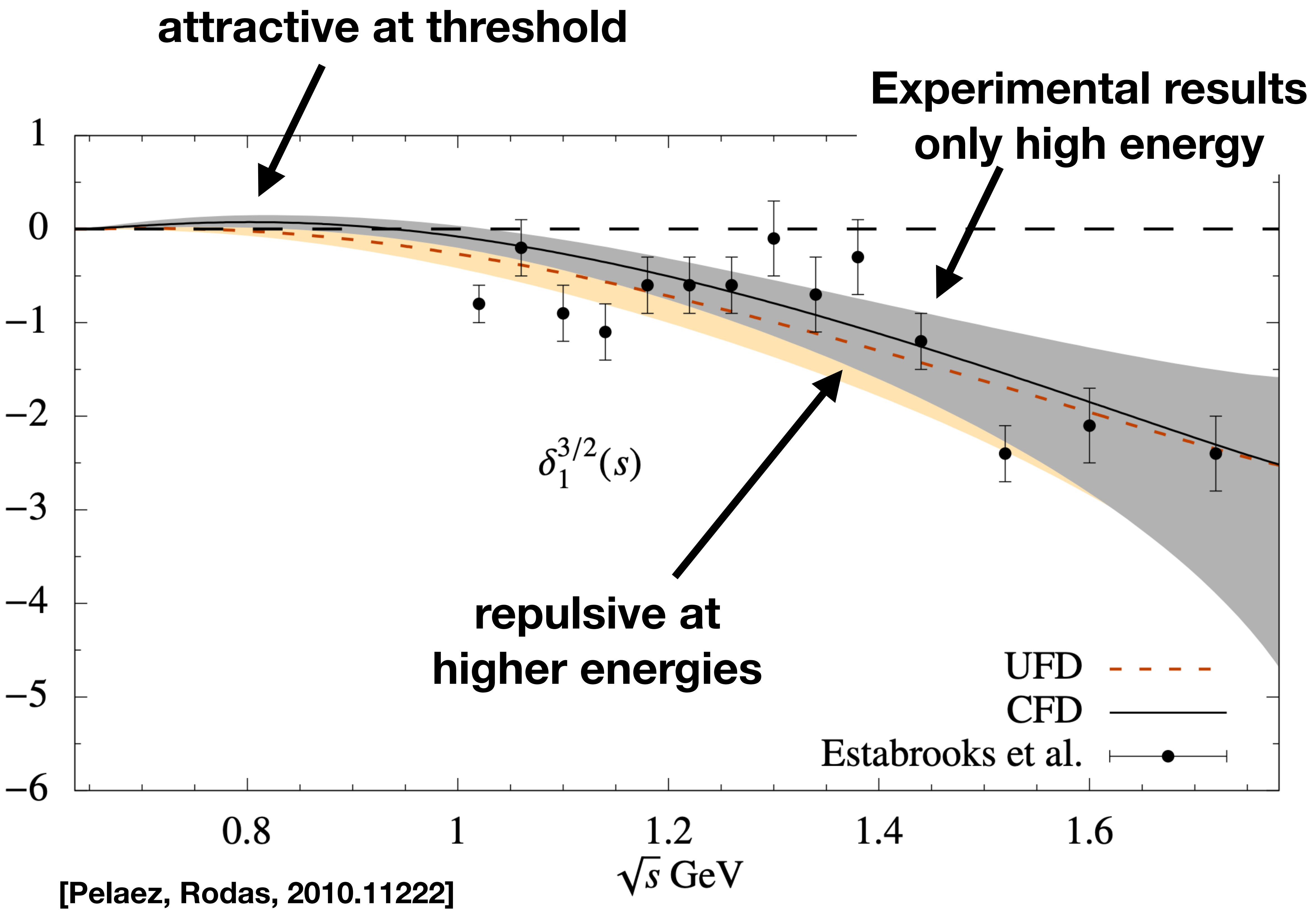
$$L_5 = 0.0(1.5) \times 10^{-3}$$

$$\chi^2/\text{dof} = 0.4$$

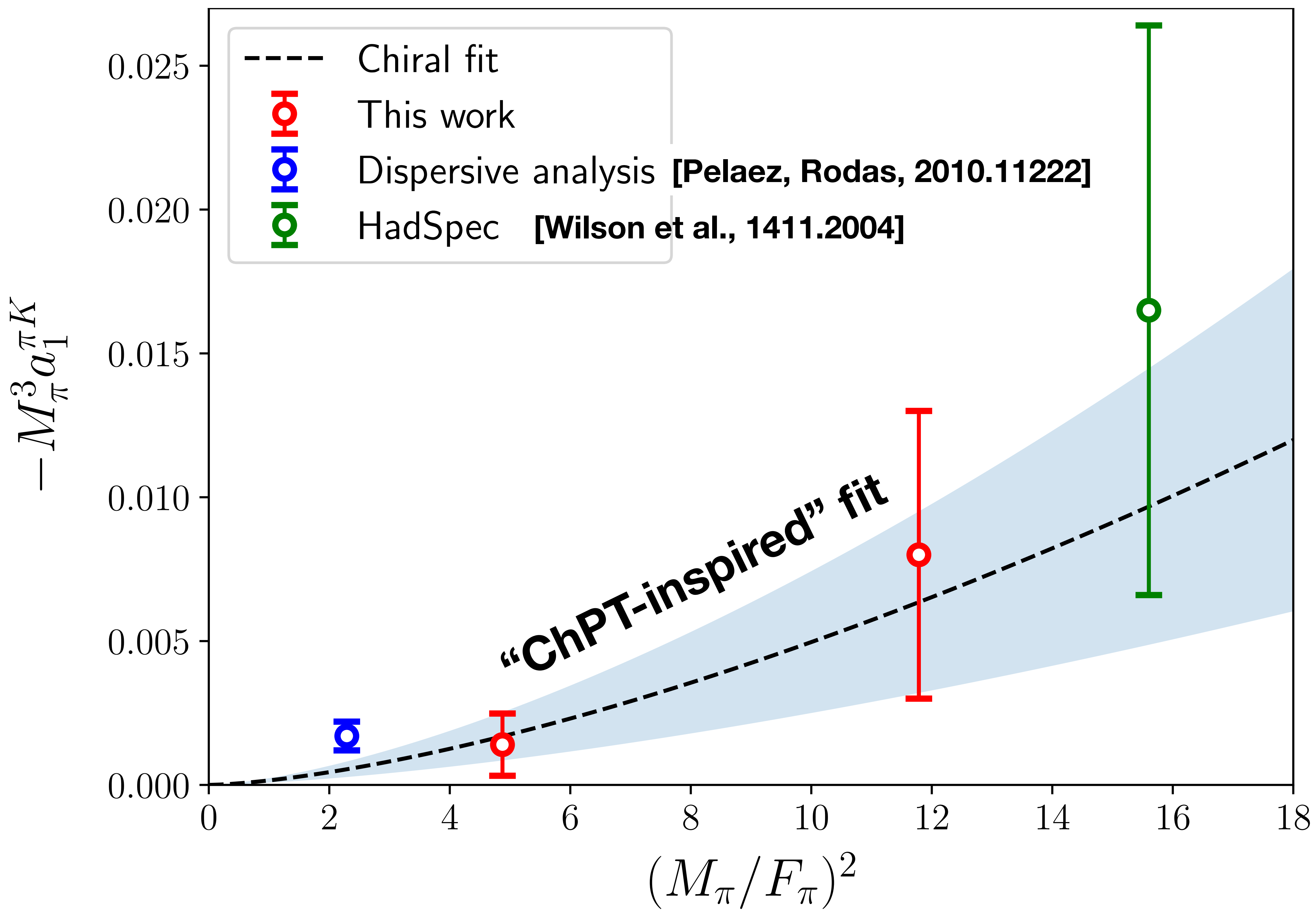


p-wave πK scattering

- Access to higher partial waves in the near threshold region



[Pelaez, Rodas, 2010.11222]

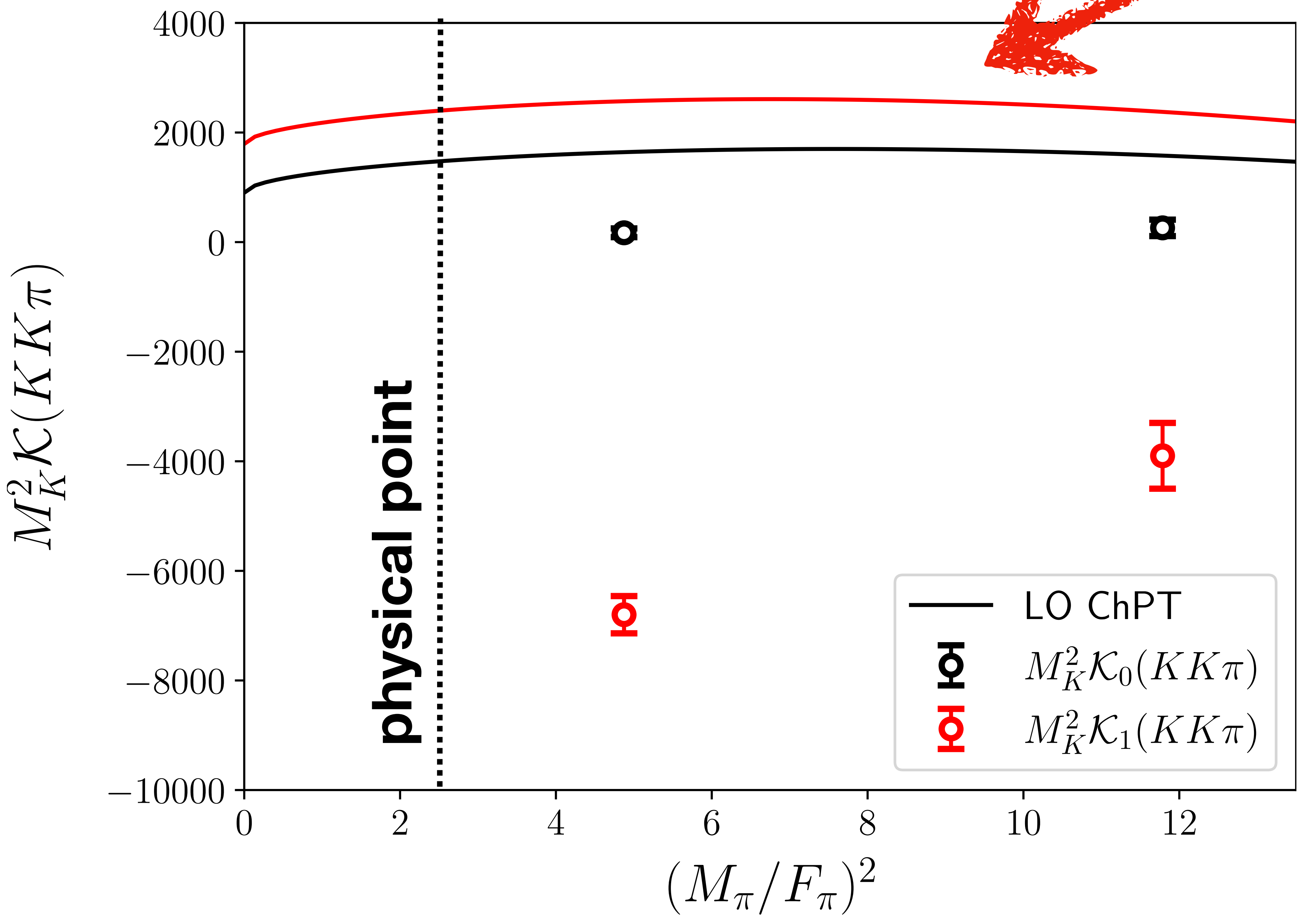


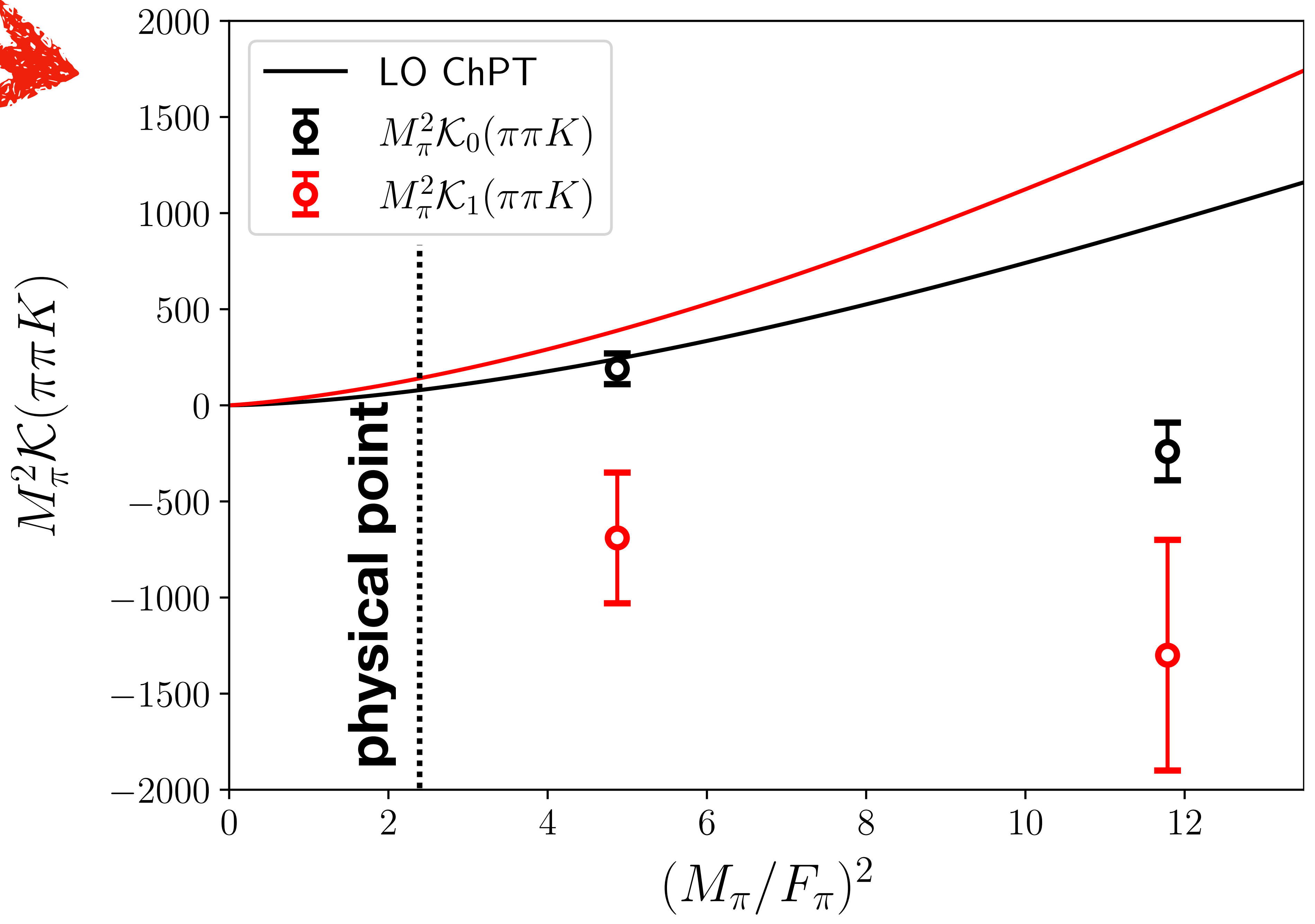
Kdf,3 results (I)

Depend on CM energy

Angular dependence

$$\mathcal{K}_{\mathrm{df},3} = \mathcal{K}_0 + \mathcal{K}_1\Delta + \mathcal{K}_B(\Delta_2 + \Delta_2') + \mathcal{K}_E\tilde{t}_{11}$$





- **Statistically significant different from zero**
- **Disagreement with LO ChPT. NLO effects?**

$\mathcal{K}_{df,3}$ results (II)

Depend on CM energy

Angular dependence

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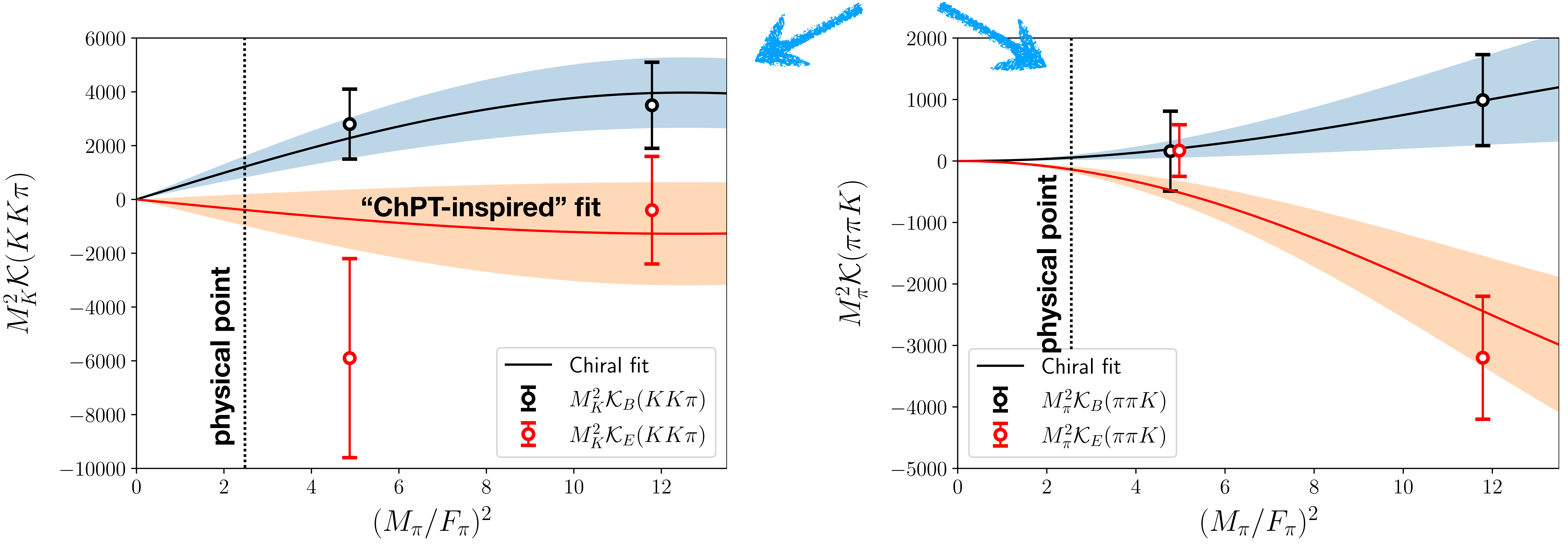

Discretization effects

- So far, single- attice spacing calculations. What about discretization effects?

Wilson-ChPT = ChPT + discretization effects

$$M_{\pi K} a_0^{\pi K} = M_{\pi K} a_0^{\pi K}\Big|_{a=0} - \frac{(2w_6' + w_8')}{16\pi}$$

$$M_\pi^2 \mathcal{K}_{\text{df},3}^{\pi\pi K} = M_\pi^2 \mathcal{K}_{\text{df},3}^{\pi\pi K}\Big|_{a=0} - 6x_\pi^2 (2w_6' + w_8')$$

Same combination of LECs in two- and three-body quantities

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$$2w_6' + w_8' = -0.14(23)$$

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Ensemble	$\mathcal{K}_0$	$\delta_a(\mathcal{K}_0)$
$\pi\pi + \pi K + \pi\pi K$ fits		
D200	190(80)	4(7)
N203	−240(150)	10(16)
$KK + \pi K + KK\pi$ fits		
D200	170(270)	17(27)
N203	260(310)	14(23)

Summary & Outlook

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- Significant progress both in the theoretical formalism and applications
  - Three mesons at maximal isospin as a benchmark system
- Determination of the three-particle K matrix in 3π, 3K, $\pi\pi$K, KKπ systems
  - $\mathcal{K}_{\mathrm{df},3} \neq 0$ with statistical significance (several "sigmas")
- Two-particle amplitudes as a by-product
  - LQCD determination of p-wave π^+K^+ scattering
- Qualitative agreement between NLO ChPT & Lattice QCD results increase confidence in results
  - Another example of synergy between lattice QCD and EFTs!
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  - Roper resonance, doubly-charmed tetraquark…

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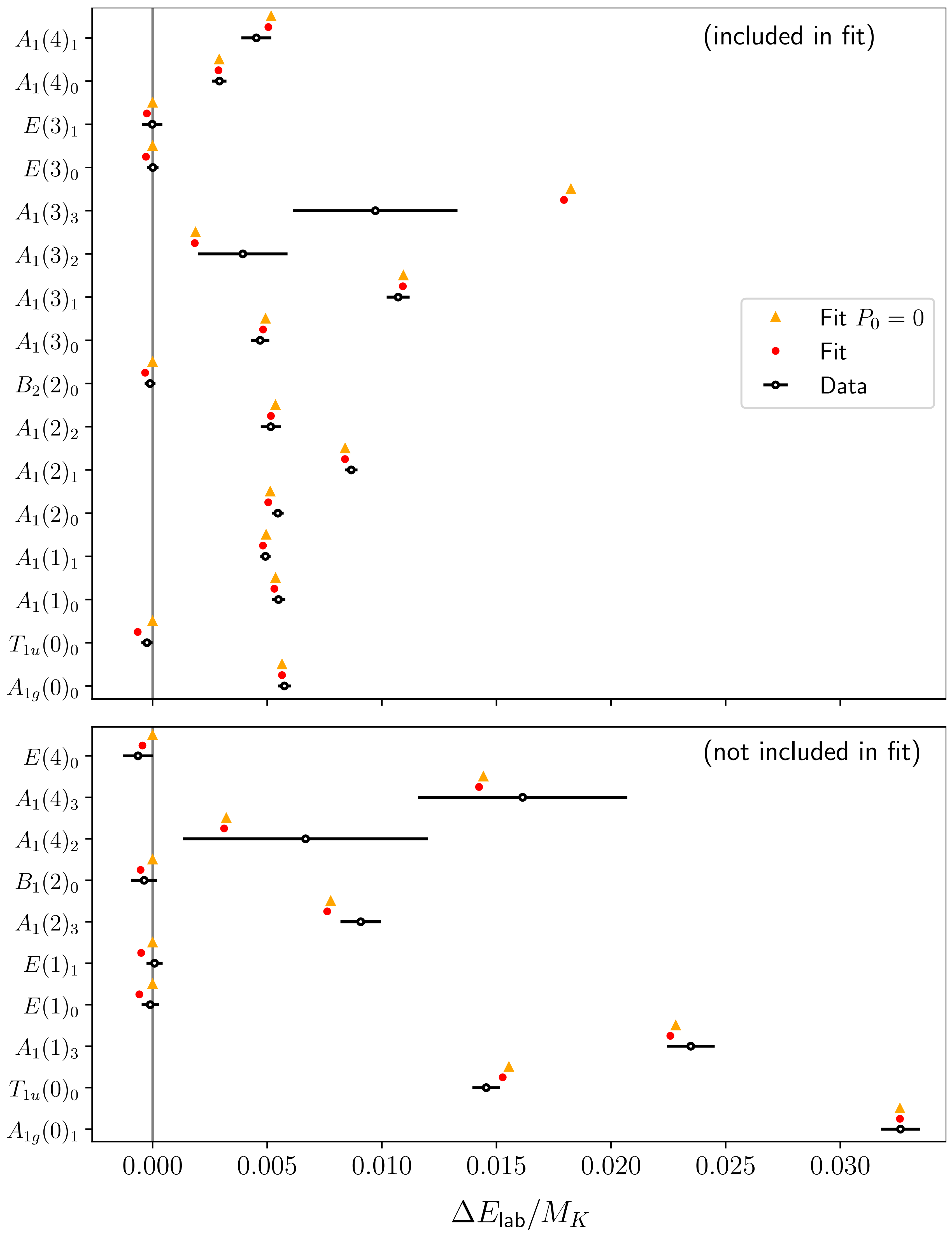
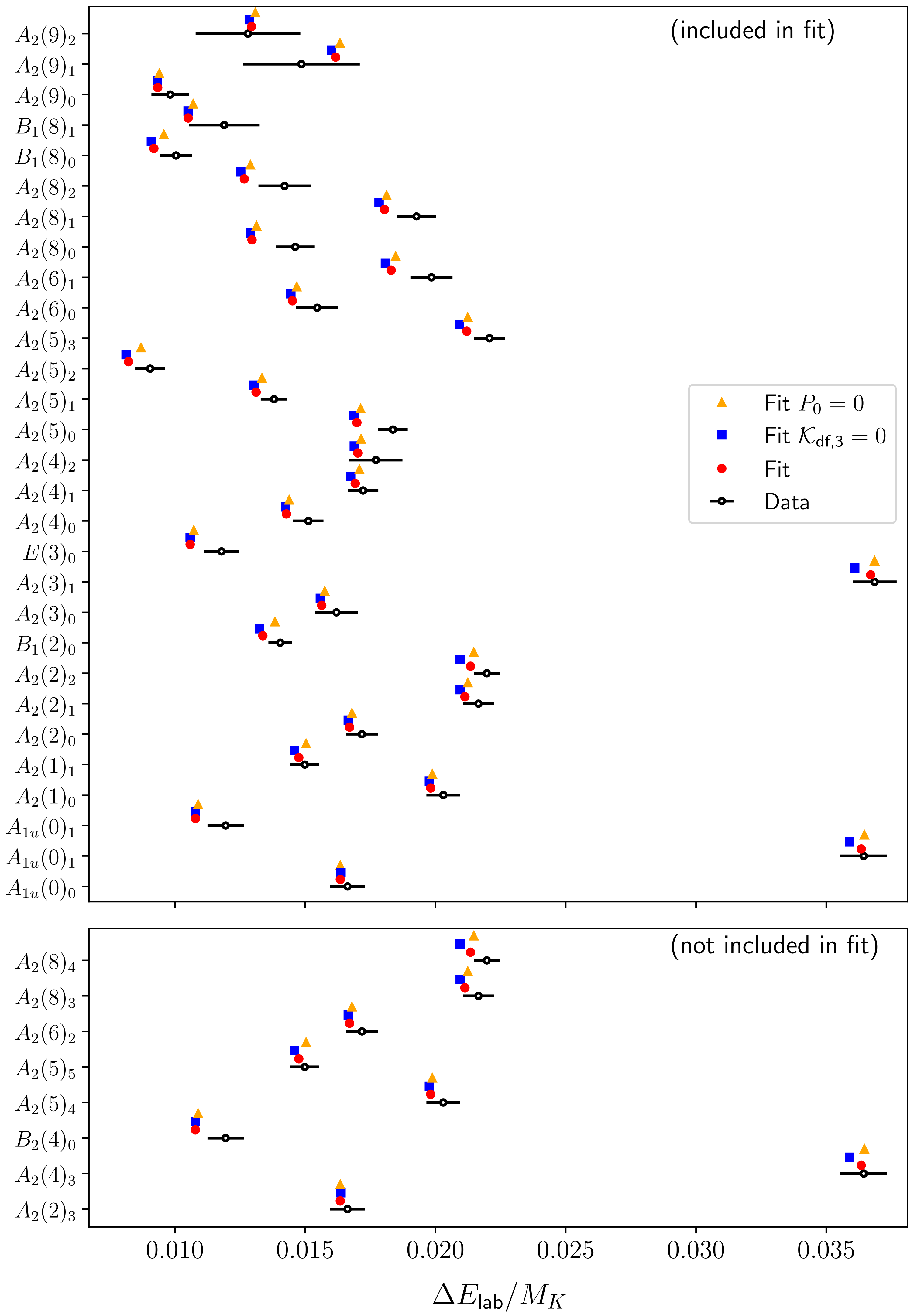
Thanks!

Back-up

Visualization of fits





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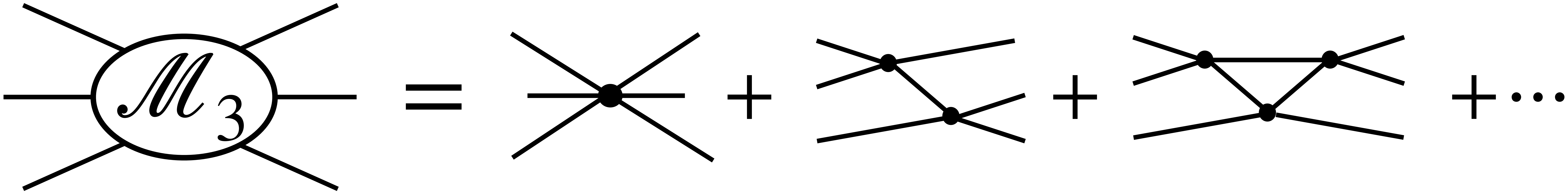
Three-particle scattering

Qualitatively more complicated than the two-particle case!



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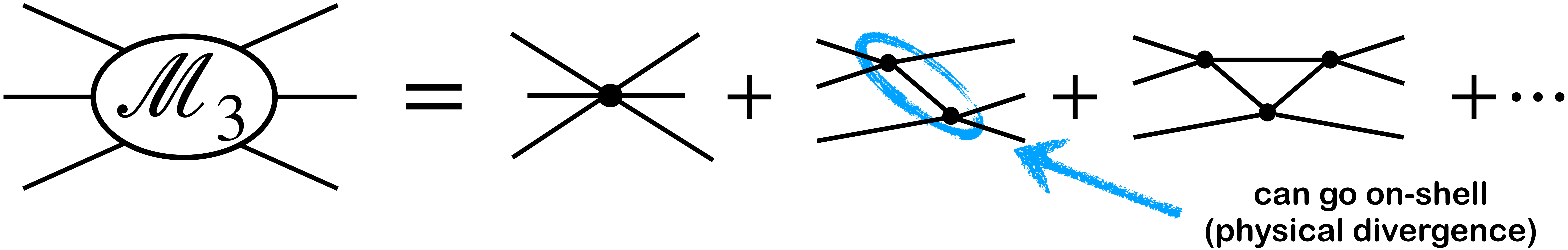

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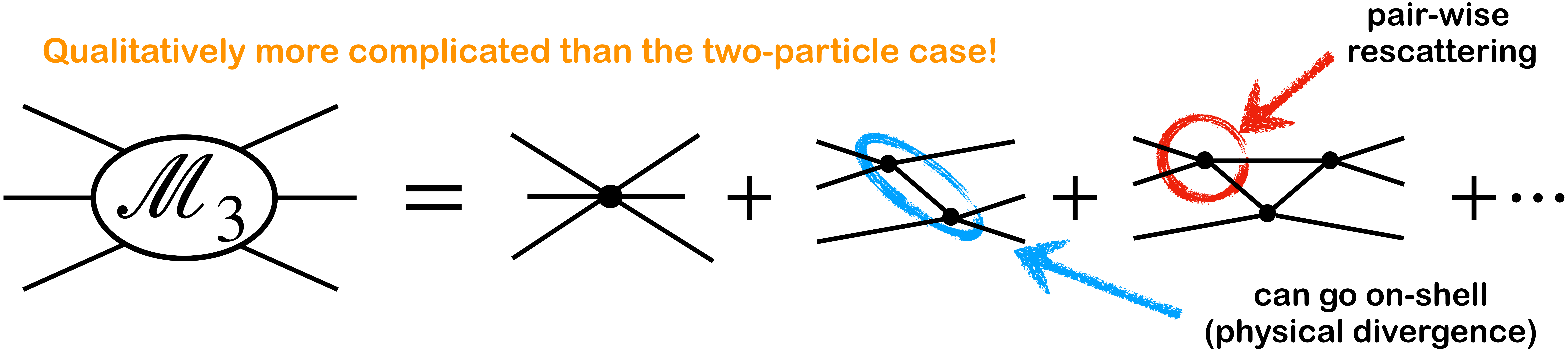

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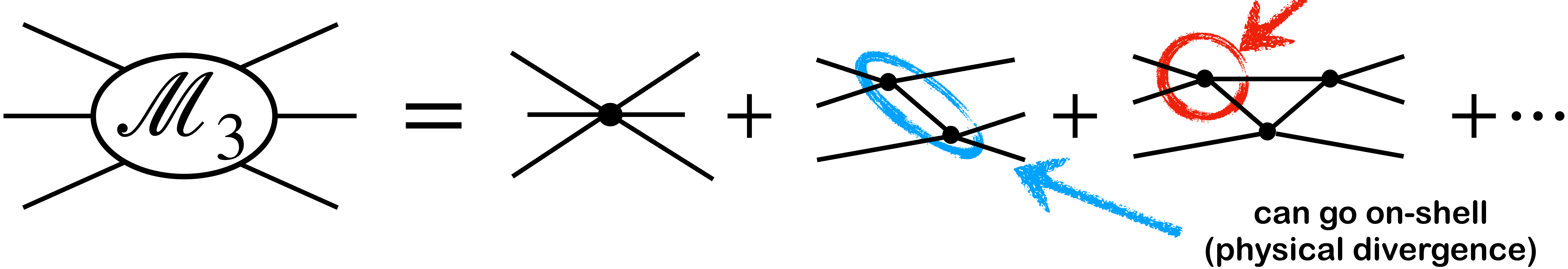

- Finite-volume formalism has been developed independently by three groups
  - Generic Relativistic Field Theory (RFT) [Hansen, Sharpe, PRD 2014 & 2015]
    [Talks by J. Baeza-Ballesteros, M. Hansen, S. Sharpe]
  - Non-Relativistic EFT (NREFT) [Hammer, Pang, Rusetsky, JHEP 2017] x 2
    [Talk by F. Müller]
  - Finite-Volume Unitarity (FVU) [Mai, Döring, EPJA 2017]
    [Talk by M. Mai]

Equivalence has been established
[Jackura et al. PRD 2019],[Blanton, Sharpe, PRD 2020],
[Jackura (to appear)]

Quantization Condition

$$\det_{k\ell m}\left[\mathcal{K}_{\mathrm{df},3} + F_3^{-1}\right] = 0$$

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Finite-volume information & two-body interactions

$$F_3 = \frac{1}{L^3}\left[\frac{F}{3} - F\frac{1}{(\mathcal{K}_2)^{-1} + F + G}F\right]$$

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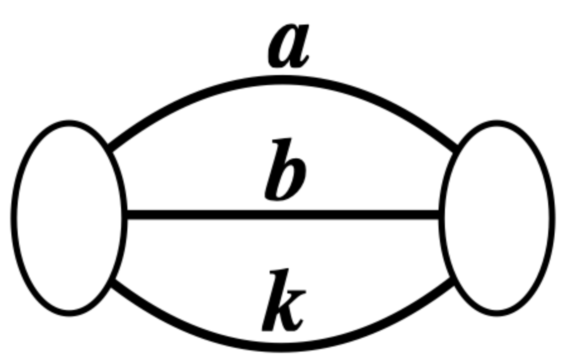
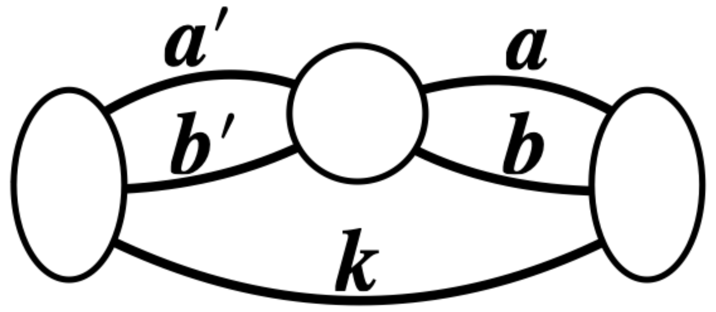
(a)



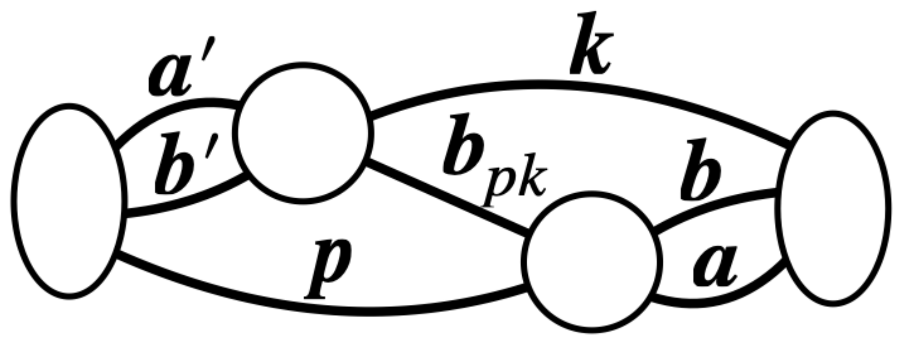
F

(b)



$\mathcal{K}_2$

(c)



G

$$F_{00}(q^2) \sim \left[\frac{1}{L^3}\sum_{\vec{k}} - \int\frac{d^3k}{(2\pi)^3}\right]\frac{1}{k^2-q^2}$$

$$\mathcal{K}_2^{\ell} = \frac{16\pi\sqrt{s}}{q^{2\ell+1}\cot\delta_\ell}$$

$$G_{p00;k00} \equiv \frac{1}{L^3}\frac{1}{2\omega_p}\frac{H(\vec{p})H(\vec{k})}{b_{pk}^2 - m^2}\frac{1}{2\omega_k}$$

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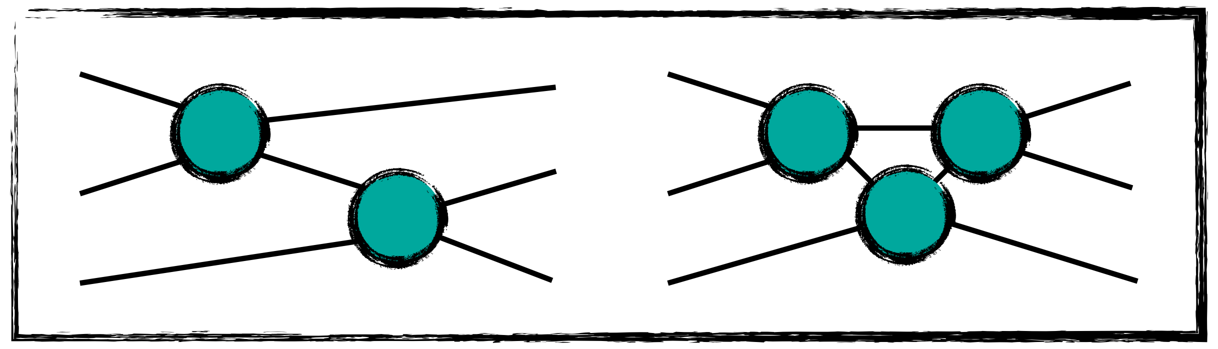
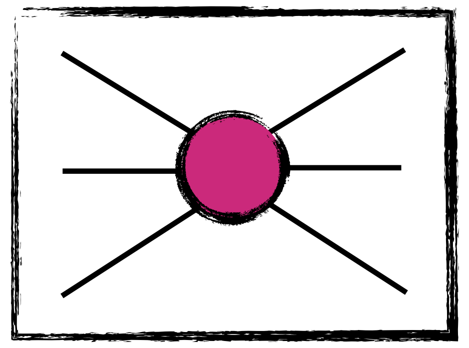
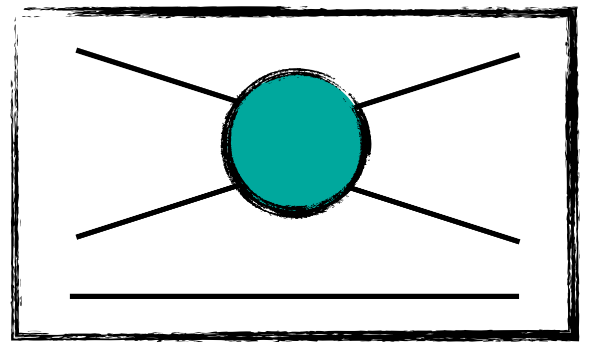
Integral equations

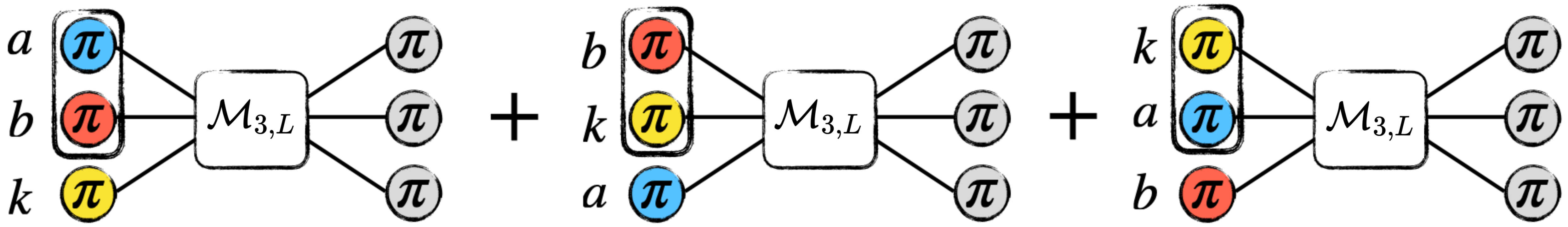
- Construct a "finite-volume amplitude" with spectator singled out

$$\mathcal{M}_{3,L}^{(u,u)}(P) \equiv \mathcal{D}^{(u,u)} + \mathcal{L}_L^{(u)} \frac{1}{1+\mathcal{K}_{\text{df},3} F_3} \mathcal{K}_{\text{df},3} \mathcal{R}_L^{(u)}$$

- Symmetrize (each particle gets a turn to be the spectator)

$$\mathcal{M}_{3,L}(P) \equiv \mathcal{S}\left[\mathcal{M}_{3,L}^{(u,u)}(P)\right] =$$



- Infinite-volume limit

$$\mathcal{M}_3(E, \boldsymbol{P}) = \lim_{\epsilon \to 0^+} \lim_{L \to \infty} \mathcal{M}_{3,L}(E + i\epsilon, \boldsymbol{P})$$

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Parametrizing $K_{df,3}$

- The three-particle K-matrix has the same symmetries as the scattering amplitude
- For identical particles:
  - Particle exchange symmetry: $\mathcal{K}_{\text{df},3}(p_1, p_2, p_3; k_i) = \mathcal{K}_{\text{df},3}(p_2, p_1, p_3; k_i) = \dots$
  - Time reversal: $\mathcal{K}_{\text{df},3}(p_i, k_i) = \mathcal{K}_{\text{df},3}(k_i, p_i)$
- For there-pions with definite isospin, need to consider flavor indices. Example:
  - I=0 flavor wave function is fully antisymmetric
    $|3\pi, I=0\rangle \sim |+0-\rangle - |0+-\rangle + |0-+\rangle + |-0+\rangle - |+-0\rangle + |-+0\rangle$
  - Therefore, the K-matrix must also be antisymmetric

$$t_{ij} \equiv (p_i - k_j)^2$$

$$\mathbf{K}_{\text{df},3}^{[I=0]} \supset \mathcal{K}_{\text{df},3}^{\text{AS}} \sum_{\substack{ijk \\ mnr}} \epsilon_{ijk}\epsilon_{mnr} t_{im} t_{jn}$$

  - Minimal parametrization for resonances:

$$\mathbf{K}_{\text{df},3}^{[I=0,\omega(780)]} = \frac{c_X V_\alpha V'^\alpha}{s - M_\omega^2}$$

$$V^\alpha = P_\mu \sum_{ijk} \epsilon_{ijk}\, p_j^\mu p_k^\alpha$$

The Lüscher method

- Indirect connection between the spectrum and the two-particle scattering amplitude [Lüscher 89']

Two-particle Quantization Condition

$$\det_{\ell m}\left[\mathcal{K}_2(E_n) + F^{-1}(E_n, \vec{P}, L)\right]\Big|_{E=E_n} = 0$$

Scattering K-Matrix

Known kinematic function

"QC2"

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- Fully general formalism (nondegenerate, spin, multi-channel):

[Rummukainen, Gottlieb, NPB 1995] [Kim, Sachrajda and Sharpe, NPB 2005]
[Bernard et al, JHEP 2008] [Davoudi, Savage, PRD 2011]
[Leskovec, Prevlosek, PRD 2012] [Göckeler et al, PRD 2012]
[Briceño, PRD 2014]

- See also the HALQCD method: