Fractional Topological Charge

RDP & V.P. Nair, 2206.11284

In SU(N) gauge theories withOUT dynamical quarks,
at large N, need objects not $\bar{c}$ $Q_{top} = \pm 1, \pm 2 \ldots$
but

$$Q_{top} = \pm \frac{1}{N}, \pm \frac{2}{N} \ldots$$

Arise immediately $\bar{c}$ $Z(N)$ twisted b.c.'s
't Hooft '80 + ...

On "femto-slab": width $L \ll \Lambda^{-1}_{QCD}$
Unsal 2007.03880; Poppitz 2111.10423

Today: "quantum" instantons $\bar{c}$ size $\sim 1/\Lambda^{-1}_{QCD}$
measurable on lattice with<u>out</u> cooling

Instantons

Topological charge $Q \sim \int d^4x \, \mathrm{tr}\, G_{\mu\nu} \tilde{G}^{\mu\nu}$ $\quad \varepsilon^{\mu\nu\alpha\beta} G^{\alpha\beta}$

Instantons – solutions to classical equations of motion

self-dual, $G_{\mu\nu} = \pm \tilde{G}_{\mu\nu}$

As classical eqs. scale invariant, instantons come in all scale sizes, $\rho : 0 \to \infty$

For $Q = n =$ integer, $S = \int \mathrm{tr}\, G_{\mu\nu}^2 = \frac{8\pi^2}{g^2} n$

Construction of all instantons known – collective coor.'s (moduli space) involved

Atiyah, Hitchin, Drinfeld, Manin Phys. Lett. A65 '78

Instantons @ $T \neq 0$

Inst.'s valid semi-classically, when $g^2 \ll 1$.

E.g., temperature $T \gg \Lambda_{QCD}$.

By asymptotic freedom, $g^2(T) \sim \#/\ln T$

$$\Rightarrow Z_{inst.} \sim e^{-8\pi^2/g^2(T)} \sim \#'/T^{c-4} \qquad c = \frac{11N_c - 2N_f}{3}$$

$\underline{T \gg \Lambda_{QCD}}$

Lattice, $N_c = 3$:

Instantons dominate down to $T \sim 300$ MeV $\sim \Lambda_{QCD}$!

$\#' \sim 10 * $ 1-loop result

$\Rightarrow$ need 2-loop

Borsanyi+..., 1606.07494

Petreczky+..., 1606.03145

For pressure, c/o resummation, pert. thy. fails for $T < 100$ GeV

Lattice - pure glue

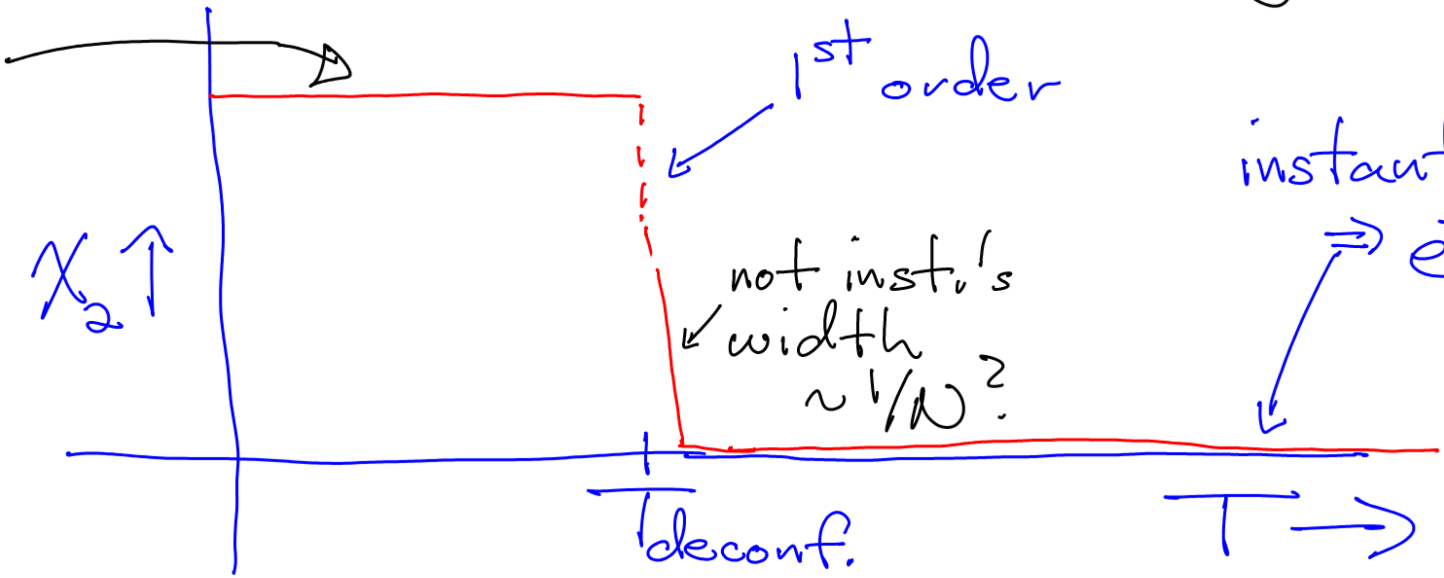
Compute top. susceptibility:

$$S_\theta = S_0 + i\theta Q \qquad \chi_2 = \frac{\partial^2 \ln Z}{\partial \theta^2} \sim \langle Q^2 \rangle$$

For pure $SU(N)$ glue, for $N \geq 3$, weak dependence on N. First order transition @ $T_{deconfinement} \sim 270$ MeV

E.g., $T_c/\sqrt{\sigma} \sim$ const. $\approx N$ ($\sigma = T=0$ string tension)

$\approx$ const.



See:
Bonati + ...,
1301.7640

Lattice – $\bar{c}$ quarks

$T > 300$ MeV – instantons

$300 > T > 155$ $(= T_\chi)$ – not instantons, slower fall off

$155 > T$ – $\chi_2 \approx$ const.

Recently: Boccaletti + Nogradi 2001.03383

Gross, RPY Yaffe '81

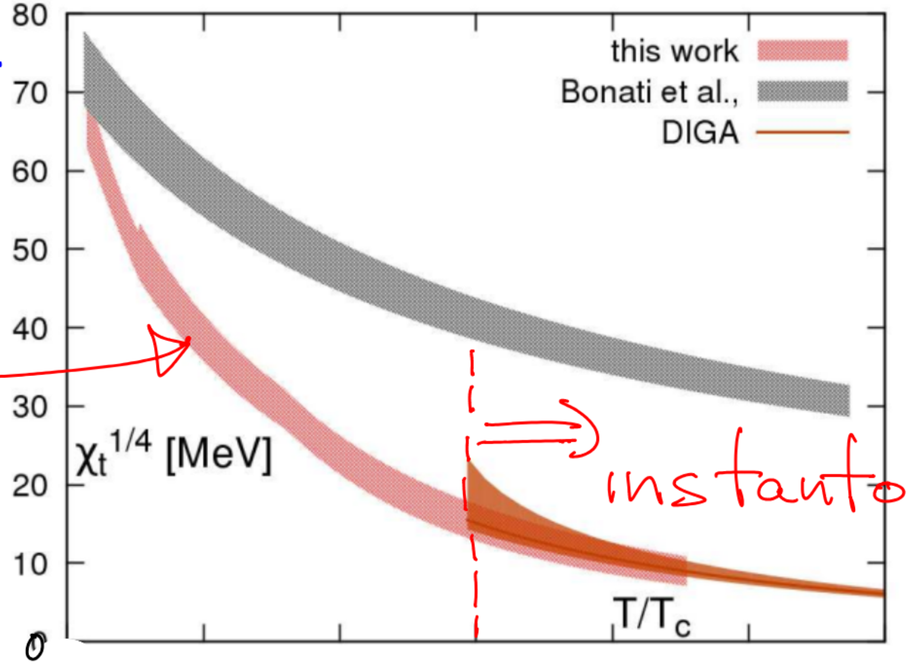
Lattice: Athenodorou+... 2208.08921

Petreczky + ... 1606.03145

$\chi^{1/4}$ ↑

NOT instantons

instantons



T_χ

$T/T_\chi \rightarrow$

Instantons @ large N

Veneziano '79, Witten '79

As $N \to \infty$, hold $g^2 N$ fixed as $N \to \infty$

$$\chi_2 \sim e^{-8\pi^2/g^2} \sim e^{\left(-8\pi^2/g^2 N\right) N}$$

Even so, argued $\chi_2 \sim 1$ @ $T=0$, $N=\infty$

Instanton liquid?

At $N=\infty$, instantons of <u>one</u> scale will dominate

$$S(g^2) = 8\pi^2 N \left(\frac{1}{g^2 N} + n \ln g^2 N + O(g^2 N) \right)$$

1 loop   2 loop

$$\left.\frac{\partial S}{\partial g^2}\right|_{g_*^2} = 0 \Rightarrow g_*^2 N \sim \frac{1}{n}$$

Instantons of one size dominate as $N \to \infty$,

But

$$S(g_*^2) \sim N \neq 0$$

but <u>need</u> the action $\sim N = 0$!

N.B. - Liu, Shuryak, Zahed 1802.00540

"Fractional" instantons

If there are objects $\bar{c}$ $Q_{top} \sim 1/N$, then no problem!

$$\chi_2 \sim e^{-8\pi^2/(g^2 N)} \sim 1 \quad \text{as} \quad N \to \infty$$

't Hooft '80, van Baal '82, Sedlacek CMP 86 '82

$Q_{top} \sim 1/N$ $\bar{c}$ $Z(N)$ twisted boundary conditions

manifestly finite volume

only analytic soln's for finite box

$\bar{c}$ sizes in certain ratio, etc.

Do they persist in infinite volume?

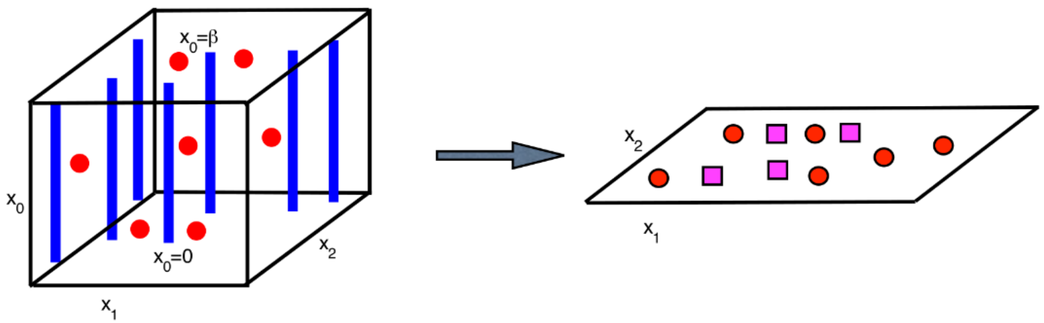
Femto-slab

Unsal, Poppitz, Anber ....

Take <u>one</u> spatial dimension, L, $\ll \Lambda_{QCD}^{-1}$, so $g^2(L\Lambda_{QCD}) \ll 1$. Using results in 2+1 dim.'s, over large distances, confining thy $\bar{c}$

"monopole-instantons", $Q_{top} \sim 1/N$

Poppitz, 2111.10423



Size of monopole-inst.'s $\sim L$

What happens as $L \sim \Lambda_{QCD}^{-1}$?

Punchline

Size of monopole-instanton gets
stuck at $L \sim \Lambda_{QCD}^{-1}$

Dominant configurations $\approx$ one size

Measurable on lattice $\bar{c}$ adjoint (not fund.)
quark prop. as external probe
not dynamical

CP^{N-1} model

In 1+1 dim.'s : N component field z^i, $\bar{z}^i z^i = 1$, inv. :

$$z^i(x) \to e^{i\alpha(x)} z^i(x)$$

local $U(1)$

$$\mathcal{L} = \frac{1}{g^2}\int d^2x \, |D_\mu z^i|^2 \qquad D_\mu = \partial_\mu - iA_\mu$$

Global sym. : $z^i \to U^{ij} z^j \Rightarrow SU(N)$

But: if $U = \omega_1 = e^{2\pi i/N}$, $z^i \to \underbrace{e^{2\pi i/N}}_{\text{part of } U(1)} z^i$

$\Rightarrow$ global sym. $SU(N)/Z(N)$

g^2 asymptotically free, soluble as $N \to \infty$

Witten '79 d'Adda, Lüscher, Di Vecchia '78

CP^{N-1} inst.'s

Topological chg.

$$Q_{top} = \frac{1}{2\pi} \int d^2x \; \varepsilon^{\mu\nu} \partial_\mu A_\nu$$

<u>All</u> classical soln's known; self-dual,

$$D_\mu z = \varepsilon^{\mu\nu} D_\nu z \qquad z^i \sim \frac{(x+iy)\, v^i}{\sqrt{x^2+y^2+\rho^2}}$$

Can compute 1-loop fluc's about all instantons

$$S_{top} \sim \frac{1}{g^2} \sim \Big(\underbrace{\frac{1}{g^2 N}}_{\text{fixed}}\Big) N \;\Rightarrow\; e^{-S_{top}} \sim e^{-\# N}$$

But: large N soln. shows that

$$\chi \sim \langle Q_{top}^2 \rangle \sim \frac{1}{N} \quad \underline{\text{not}} \quad e^{-N} \; !$$

Frac. inst.'s in CP^{N-1}

Consider

$$z^1(r,\theta) = e^{i\varphi/N}\, h(r) \qquad z^{2,\ldots} = 0$$

Not single-valued:

$$z^1(r,2\pi) = e^{2\pi i/N}\, h(r) \sim z^1(r,0)$$

by $Z(N)$ sym. Then the corresponding

$$A_\varphi \sim \frac{1}{rN}, \quad r\to\infty \qquad \Rightarrow \qquad Q_{top} = \frac{1}{N}$$

To obtain frac. Q, require multi-valued soln.'s allowed by $Z(N)$

CP^N @ $N \to \infty$

Introduce a constraint field $\lambda(|z|^2 - 1)$, integrate out z's

$$S_{eff} = N \,\mathrm{tr} \ln(-D_\mu^2 + i\lambda) - i \int \frac{\lambda}{g^2} d^2x$$

Vacuum: $i\lambda = m^2$ (dim. trans.), $A_\mu = 0$

Frac. inst.: action non-local, only limiting behavior

But: scale sym. of classical action lost

frac. inst. has <u>one</u> size $\sim 1/m$ = confinement distance

SU(N) gauge, NO quarks

Back to 3+1 dim.'s, no quarks, $A_0 = 0$ gauge

Parametrize gauge field as function of arbitrary parameter ξ,

$$A_i(\bar{x}, \xi) = (1-\xi) A_i(\bar{x}) + \xi A_i^{\Omega}(\bar{x})$$

gauge transf. of A_i

If $\Omega \to \mathbb{1}$ @ $\bar{x} \to \infty$,

$$Q_{top} = \frac{1}{8\pi^2} \int tr\, G_{\mu\nu} \tilde{G}^{\mu\nu} = \frac{1}{24\pi^2} \int d^3x \; tr\left(\Omega^{\dagger} \partial_i \Omega\right)^3$$

$$= \text{integer}$$

Above is not soln. $\bar{c}$ minimal action, but gets Q_{top} right

$Z(N)$ vacua

But alternately, we can choose $\Omega_\infty = \omega_j = e^{2\pi i j/N}\mathbb{1}$

$$\Omega_\infty(\xi) = e^{i\chi^j \xi} \qquad \chi^j = \frac{2\pi j}{N} t_N, \quad t_N = \begin{pmatrix} \mathbb{1}_{N-1} & \\ & -(N-1) \end{pmatrix}$$

For finite $r = \sqrt{x^2+y^2+z^2}$, need more involved ansatz to embed $SU(2)$-ish instanton in $SU(N)$, connect $\bar{c}$ $Z(N)$:

$$Y_{ij} = \frac{\sigma\cdot\hat{x}}{2} + \frac{1}{N} - \frac{1}{2} \quad i,j=1,2 \qquad Y_{ij} = \frac{\delta^{ij}}{N} \quad i,j=3\ldots N$$

$$\Omega(\bar{x},\xi) = e^{iY\theta(r,\xi)}$$

$\theta = 0$ @ $r=0$, $\xi=0$

$= 2\pi\xi$ @ $r=\infty$

Above illustrative, <u>not</u> minimal action

Topological Chg.

$$G_{\mu\nu} \to \Omega^{+} G\, \Omega + \frac{\partial}{\partial \xi} A^{\Omega} = \Omega^{+} (G - Da)\, \Omega$$

Variation in ξ like "time" $\quad a = \frac{d\Omega}{d\xi}\, \Omega^{-1}$

"a" analogous to A_0

$$Q_{top} = \frac{1}{8\pi^2} \int tr (G - Da)^2\, d^4x$$

$$= \frac{1}{4\pi^2} \int_{x=\infty} d^2 S^i \int dx\; tr\,(a B^i)$$

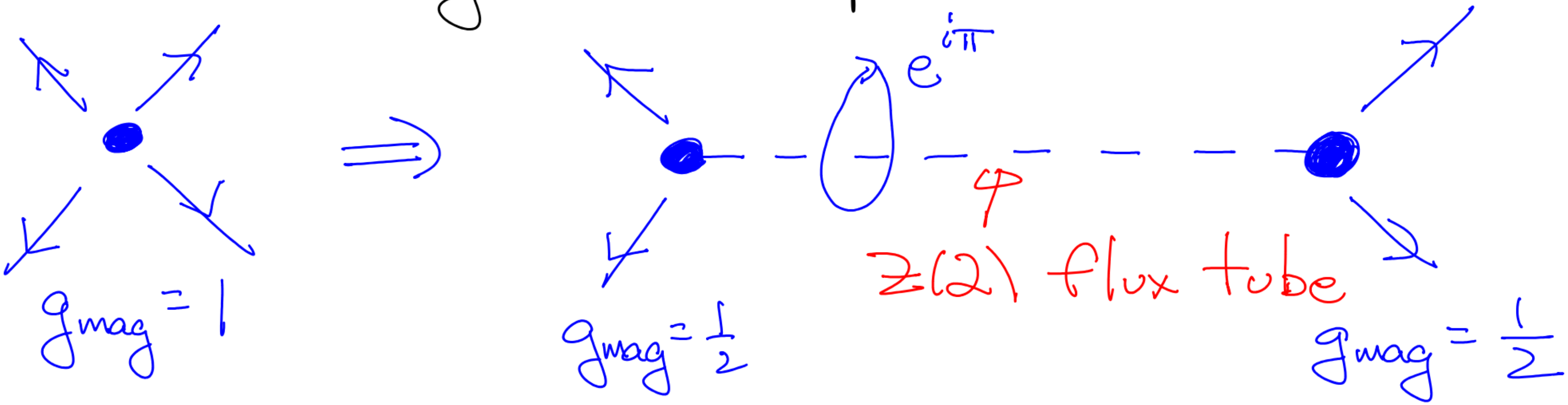
<u>Need</u> magnetic chg. For SU(2):

$$G_{ij} \sim -\frac{i}{2}\, \bar{\sigma} \cdot \hat{x}\; \varepsilon_{ijk} \frac{\hat{x}^k}{r^2} \quad \Rightarrow \quad Q_{top} = 1$$

S_0?

"Split" Z_2 monopole

For a SU(2) magnetic monopole



Without quarks, Z(2) flux tube invisible

Each end has $Q_{top} = +\frac{1}{2}$.

To observe each end seperately, do need Z(2) twisted boundary conditions

Vacuum of frac. inst.'s

For $SU(N)$, instanton $\bar{c}$ $Q_{top} = 1$ $= N$ frac.'s

Natural length scale of $\mathbb{Z}(N)$ flux tubes $\sim \Lambda^{-1}_{QCD}$

Manifestly <u>non</u>-pert.

Unsal, 2007.03880 - on femto-slab,

θ-dependence $= f(\theta/N)$, not $f(\theta)$.

Presumably carries over

$Z(N)$ dyon

An explicit construction @ $T > T_{deconf}$. Now $A_0 \neq 0$

$$A_0^\infty = \frac{2\pi T}{gN} k, \qquad k = t_N = \begin{pmatrix} \mathbb{1}_{N-1} & \\ & -(N-1) \end{pmatrix}$$

$$\mathbb{L} = e^{ig\int_0^{1/T} A_0 dx} = e^{\frac{2\pi i}{N} k} \qquad \text{or} \qquad t_N' = \begin{pmatrix} -(N-1) & \\ & \mathbb{1}_{N-1} \end{pmatrix}$$

k = nontrivial holonomy. Above value is a minimum of the holonomous potential

$$A_0 = \frac{2\pi T}{gN} q\, k \Rightarrow V_{holonomous} \sim T^4 q^2 (1-q)^2$$

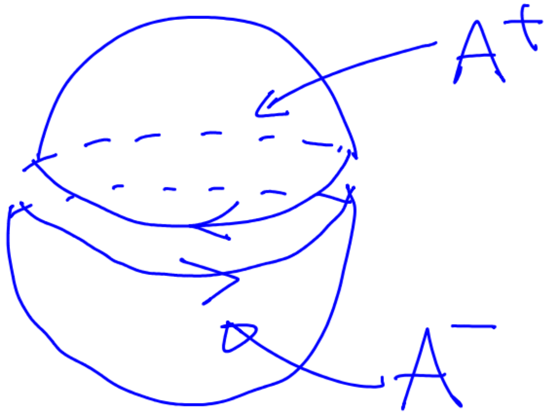
$|q| \bmod 1$

$Z(N)$ monopole

At spatial ∞,

$$A_\varphi^{\pm} = \frac{1}{N\,2r}\, m \frac{(\pm 1 - \cos\theta)}{\sin\theta} = \frac{1}{N} * \text{Dirac monopole}$$

m = magnetic chg, $\sim t_N$ or t_N'



$$e^{i\oint A^+} e^{-i\oint A^-} = e^{\frac{2\pi i}{N} m}$$

Above A_φ at spatial ∞. For A_0

$$A_0(r) \underset{r\to\infty}{\sim} \frac{2\pi T}{N} k - \frac{1}{2Nr} m + \ldots$$

Only have asymptotic soln @ $r \to 0, \infty$

$Z(N)$ dyon

Assume there is a regular solution for all r, esp. $r=0$. For simplicity, take it to be static

$$Q_{top} = \frac{1}{4\pi^2} \int \partial_i \, tr \, A_0 B_i = \frac{1}{N^2} tr(mk)$$

$m=k$: $Q_{top} = \frac{N-1}{N}$ $\qquad$ $m \neq k$: $Q_{top} = \frac{1}{N}$

't Hooft '81, van Baal '82, Sedlacek '82:

$Z(N)$ flux in a box $\bar{c}$ $Z(N)$ twisted boundary conditions

Here just put the twist in the radial direction, between $r=0$ & ∞

Dyons vs calorons

Lee & Lu: th/9802108
Kraan & van Baal: th/9805168 } KvBLL

Show instanton @ $T \neq 0$

= N constituents $\bar{c}$ $Q_{top} = 1/N$

	KvBLL	Z(N) dyon
magnetic chg	integer	1/N
$V_{hol}(q)$	maximum ~ 1/N	minimum ⇒ integer

But max. of $V_{hol.}$ ⇒ @ 1-loop, free energy

~ + volume of space!

$Z(N)$ dyons

$T > T_d$: elec. chg. unconfined, but mag. chg. confined

$\Rightarrow$ $Z(N)$ dyons only relevant for $T: T_d \to T_{inst}$

Instantons dominate $T > T_{inst}$.

$T < T_d$: $Z(N)$ dyons do not propagate straight in time

Equivalent to vortices?

Tangled over size $\sim \Lambda_{QCD}$

Entropy of config.'s $\bar{c}$ $Q_{top} = \pm 1/N \ldots > Q_{top} = \pm 1 \ldots$

Lattice - pure glue

Edwards, Heller, Narayanan lat/9806011

To measure $Q_{top} \sim 1/N$, use χ-symmetric qk. prop.
in adjoint, <u>not</u> fund., representation

Fund. rep.: 2 zero modes for $Q_{top} = 1$

Adj. rep.: 2N " " " "

2 zero modes for $Q_{top} = 1/N$

From eigenvector, estimate size

If $Q_{top} \sim 1/N$, are they dilute or densely packed?

probably

Lattice & Top. Chg.

Bonanno, Bonati & d'Elia (BBdE) 2012.14000

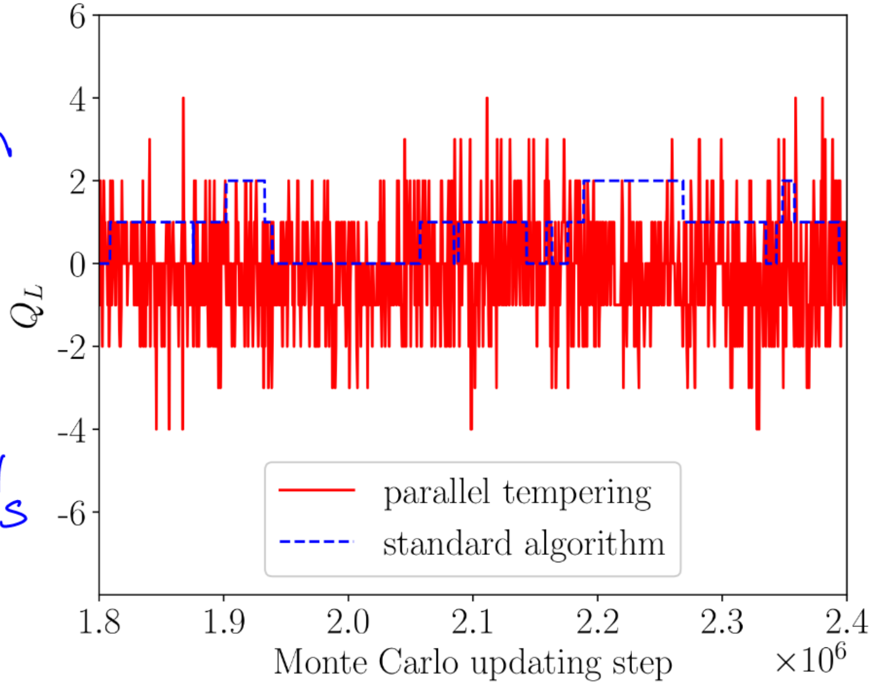
SU(N), N = 3, 4 & 6, <u>no</u> quarks

Top. chg. c̄ "parallel tempering"

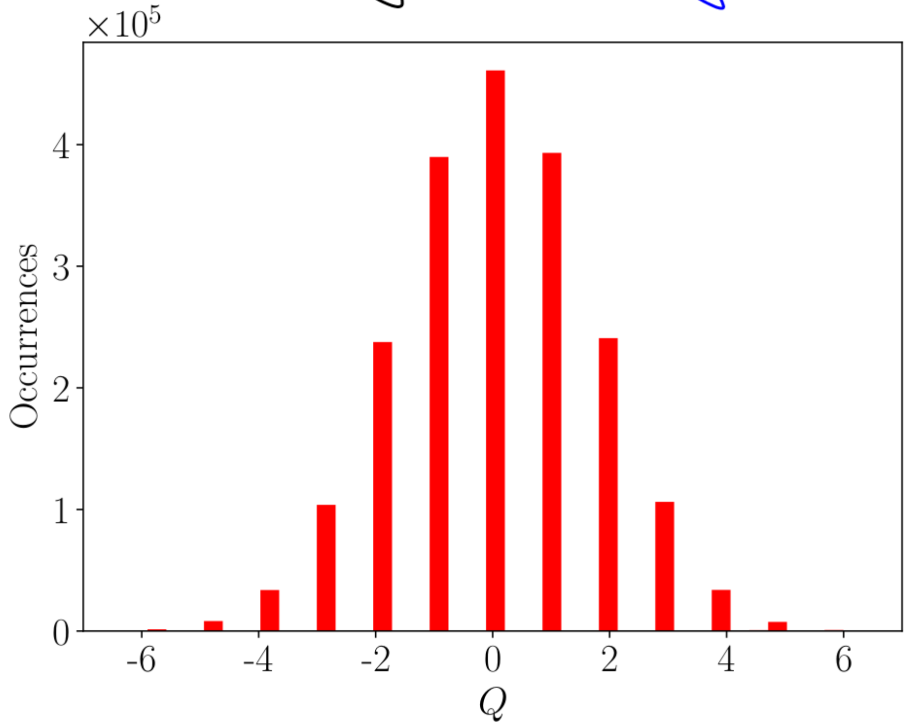
Total top. chg. after cooling = <u>integer</u>

Q ↑

BIG fluc.'s



Monte Carlo "time"



← Q →

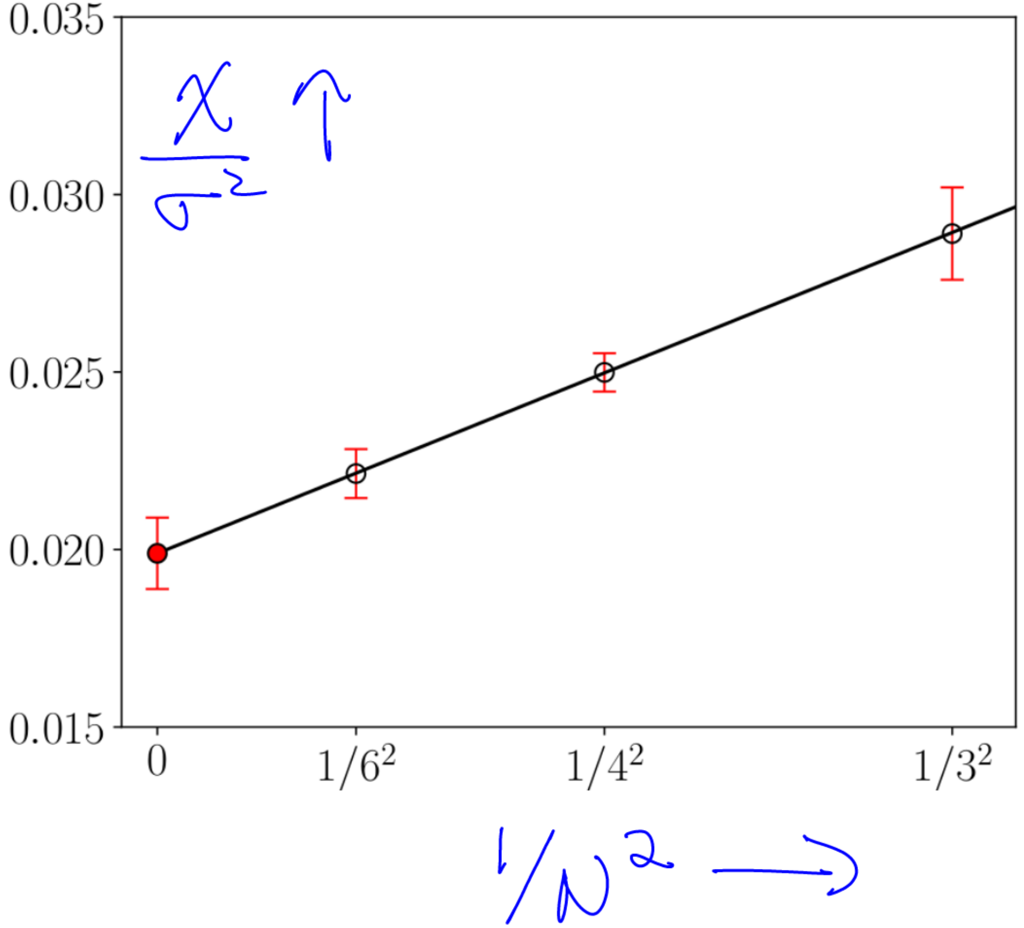
$$\langle Q^2 \rangle$$

$\chi = \langle Q^2 \rangle$ = topological sucep,

σ = string tension

BBdE

Weak N-dependence



$$\frac{\chi}{\sigma^2} = 0.19 + \frac{.08}{N^2}$$

Smooth limit as $N \to \infty$

as expected

$$\langle Q^4 \rangle$$

Kurtosis for top. chg.: $b_2 = -\frac{1}{12}\,\frac{\langle Q^4\rangle - 3\langle Q^2\rangle^2}{\langle Q^2\rangle}$

Dilute gas of $Q = 1$:

$b_2 = -1/12$ ind. of N

Dilute gas, $Q = 1/N$:

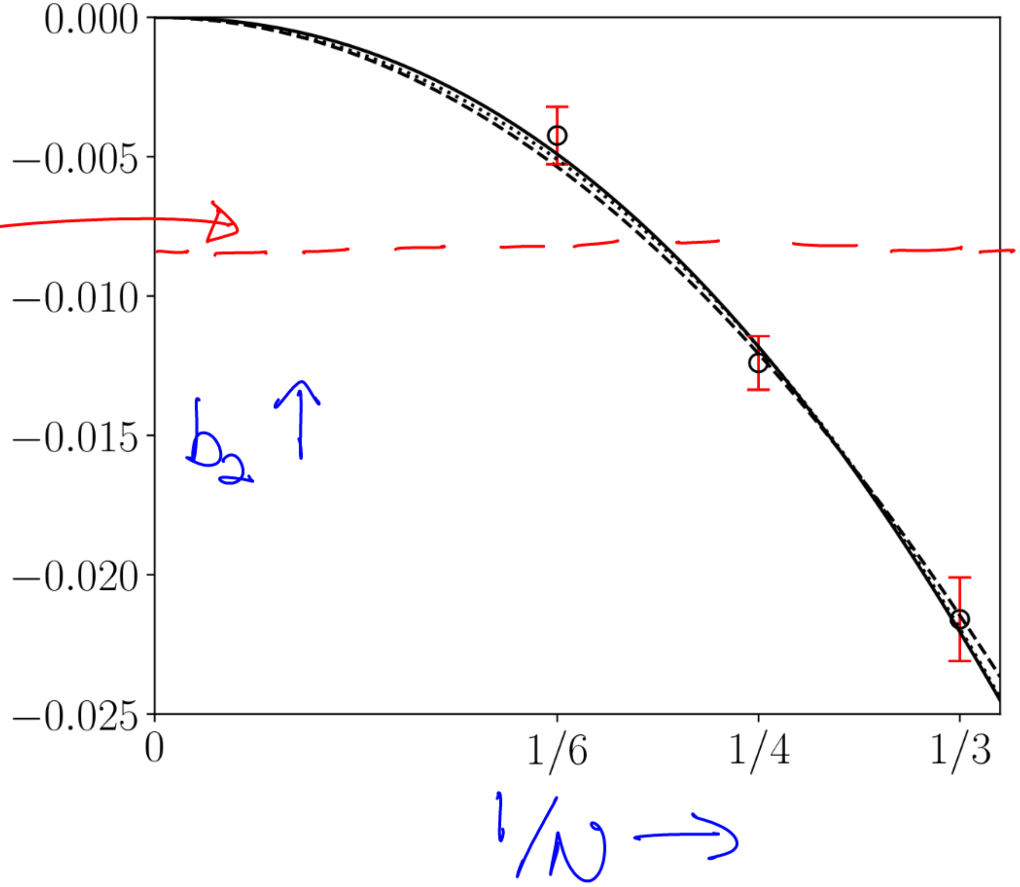
$$b_2 = -\frac{1}{12}\frac{1}{N^2} \sim \frac{-.08}{N^2}$$

Lattice:

$$b_2 \sim \frac{-.19}{N^2}$$

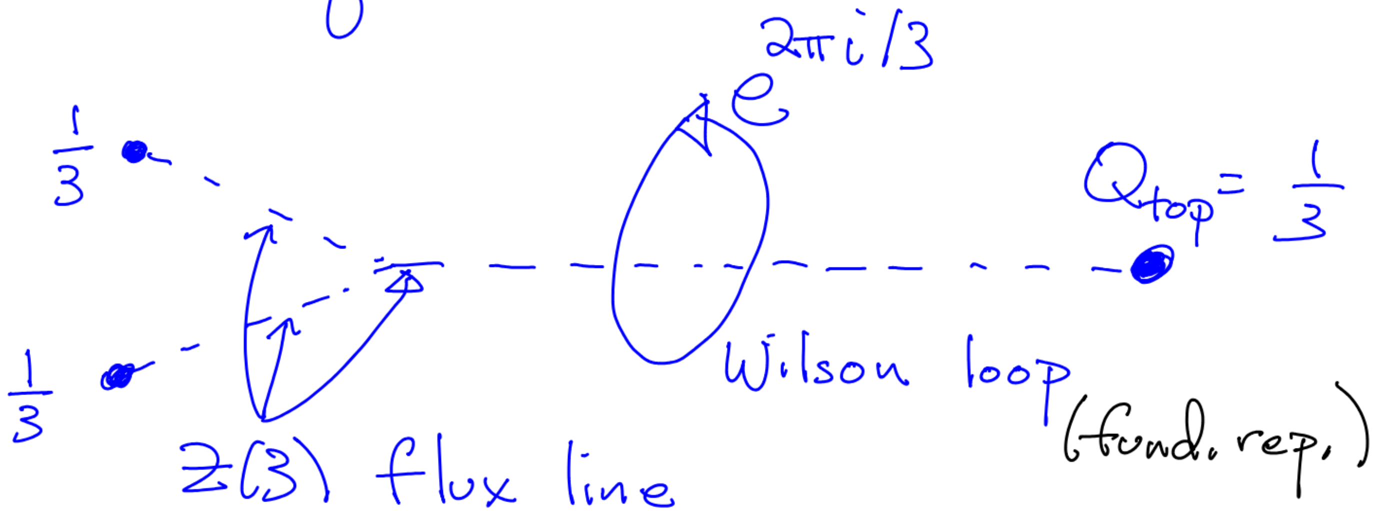
Dense system of objects $\bar{c}$ frac. top. chg.

BBdE



+ Dynamical quarks

"Split" monopoles in SU(3):



Z(3) flux tube: invisible to gluons, not to quarks in fundamental rep.

$\Rightarrow$ Dynamical quarks confine objects $\bar{c}$ $Q_{top} = \pm \frac{1}{N}$

Complicated interactions between quarks & dyons

Expect -?- quarks suppress Z(3) monopoles (& dyons)

Z(N) vortices on the lattice, in vacuum (T=0)

Take gauge link on the lattice, $U_\mu(x)$, in SU(3)

Go to maximal centre gauge,

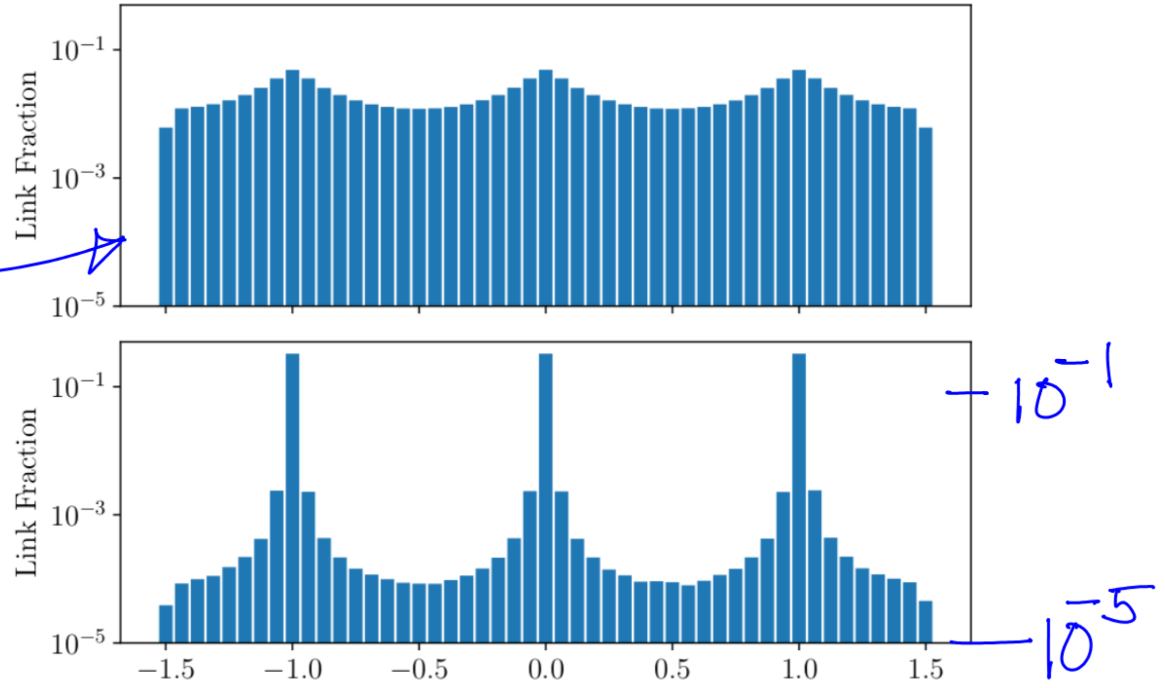
minimize $\sum_{x,\mu} |\mathrm{tr}\, U_\mu(x)|^2$

$$\mathrm{tr}\, U_\mu(x) = \underbrace{r_\mu}_{\text{real}} e^{\frac{2\pi i}{3}\overbrace{\phi_\mu(x)}^{\text{phase}}}$$

$= 0, 1, 2$

ϕ_μ before gauge fixing

after



Biddle, Kamleh & Leinweber (BKL) 2302.05897

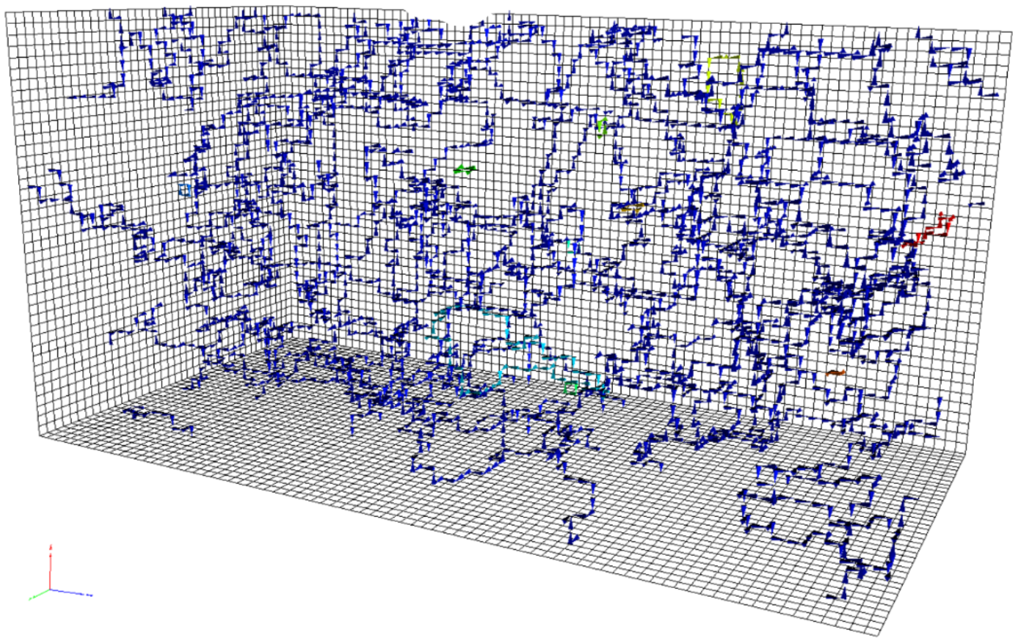
Must gauge fix, but one way of extracting density of Z(N) vortices (= worldline of Z(N) monopole)

Z(N) vortices, pure gauge vs + dynamical quarks

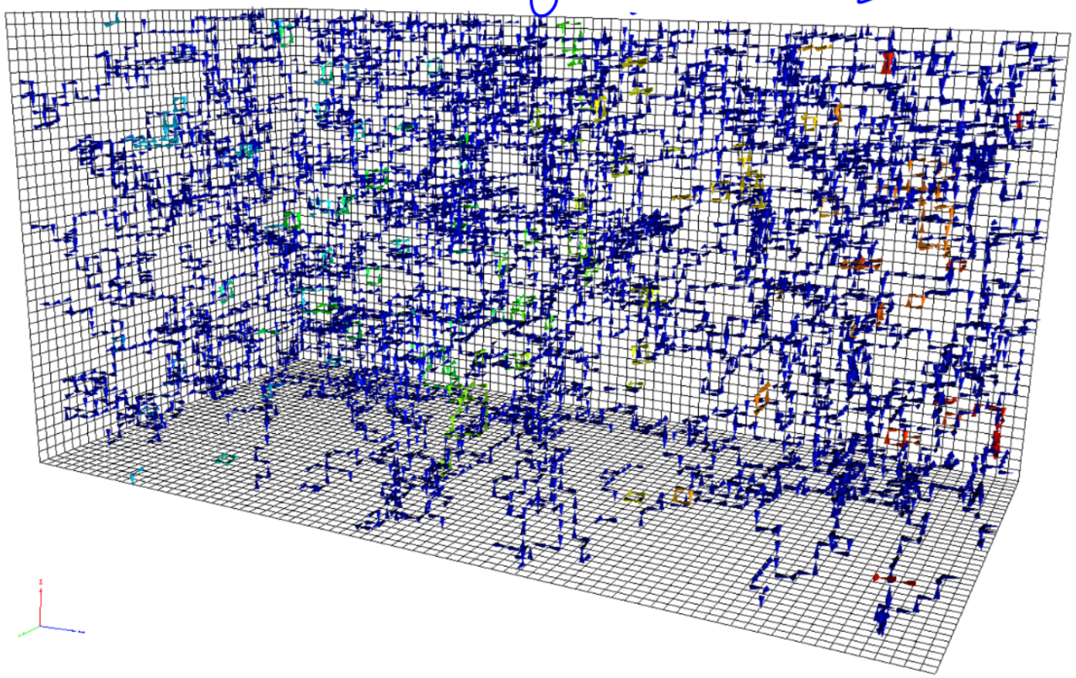
BKL:

SU(3) gauge, $32^3 \times 64$. No qks vs 2+1 flavors, $m_\pi = 156$ MeV

Pure gauge



With quarks



vortices increases $\bar{c}$ dynamical quarks $\approx \times 2$

goes ↑ as m_π goes ↓

Astonishing

Z(N) vortices on the lattice

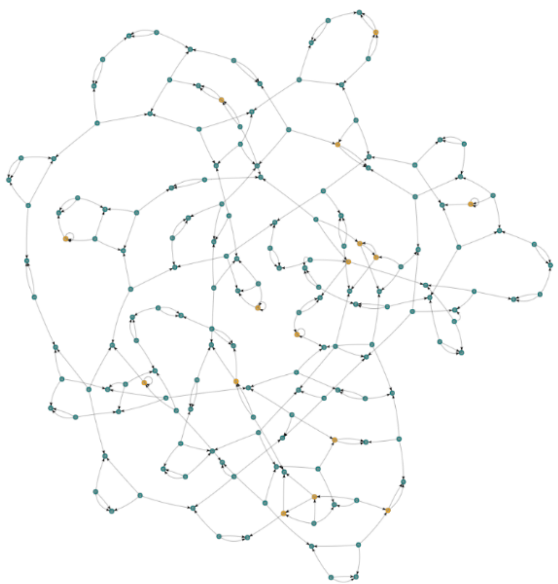
Vortex vacuum dominated by single large cluster

Secondary loops also enhanced

Also: directed graphs, abstract cluster ind. of position,
just connectivity

BKL:

pure gauge



2+1 flavors

Lattice: dynamical quarks don't suppress Z(N) vortices,
they enhance them.

Back to $T \neq 0$

Lattice: c/o qks, $T_{deconf} \sim 270$ MeV

2+1 flavors, $T_\chi \sim 155$ "

Three regimes:

$T > 300$: instanton dominated
(Z(3) dyons confined)

$300 > T > T_\chi$: Z(3) dyons + $\approx$ massless qks

$T_\chi > T$: Z(3) dyons + massive qks

Worldline of Z(N) dyon = Z(N) vortex + ...

Finite μ vs finite T

Instantons have color $\vec{E}$ & $\vec{B}$.

Suppressed by Debye mass:

$$m^2_{Debye} = g^2\left(\underbrace{\frac{1}{3}\left(N_c+\frac{N_f}{2}\right)T^2}_{\text{big}} + \underbrace{2N_f\left(\frac{\mu_{qk}}{2\pi}\right)^2}_{\text{small}}\right)$$

Because of # d.o.f., Debye suppression is <u>much</u> larger @ $T\neq 0$ than $T=0$, $\mu\neq 0$

Three regimes @ $\mu \neq 0$, $T = 0$

RDP & Rennecke: 1910.14052 –

instantons @ $T \sim 150$ MeV $\approx$ $\mu_{qk} \sim 2$ GeV!

deep in perturbative regime

Perhaps - again three regimes

$\mu > \mu_{qk} > 2$ GeV – instantons

$\mu_\chi = \chi$-transition

$\mu_{qk} > \mu > \mu_\chi$ – $Z(N)$ dyons + massless qks

$\mu_\chi > \mu > 313$ MeV – " + massive qks

Three regimes @ $\mu \neq 0$, $T = 0$

RDP & Pennecke, 1910.14052. With approx. inst. def. @ $\mu \neq 0$,
instantons @ $T \sim 150$ MeV, $\mu = 0 \Longleftrightarrow$ $\mu_{qk} \sim 2$ GeV, $T = 0$

Gorda+..., 2103.07427: $\mu_{pert.} \sim 1$ GeV !
$\tilde{\ }$pressure(μ) @ 4 loop order $\sim g^6$. where pert. thy works

$\Rightarrow$ Three regimes?

$\mu > \mu_{pert} \sim 1$ GeV Both pert. thy. & semi-cl. ok

$\mu_{pert} > \mu > \mu_\chi$ chiral trans. Z(N) dyons + massless quarks

$\mu_\chi > \underbrace{\mu > \ }\gtrsim 313$ MeV Z(N) dyons + massive quarks

Two sub-regimes: nuclear matter, quarks confined as baryons
Quarkyonic matter, quarks confined as baryons only near the Fermi surface

Qualification!

At $T \neq 0$ and $\mu = 0$, analytic computation of
1-loop fluctuations about $Q = 1$ instanton
Gross, RP, Yaffe '81

At $T = 0$ and $\mu \neq 0$, at 1-loop order, only
limits as $\rho \to \infty$ ($\sim m^2_{Debye}$) & $\rho \to 0$ ($\sim \ln(\rho\mu)$)
ρ = instanton scale size de Carvalho '81

<u>Need</u> to compute 1-loop determinant @ $\mu \neq 0$, $T = 0$
= function $(\rho\mu)$. $\mu \neq 0$ breaks $O(4)$ sym. $\to O(3)$

Perhaps? Enhancement of instanton density for $\rho\mu \sim 1$?

Not accessible by lattice, until quantum computers for QCD

Summary

For pure gauge, two sources of fluc's in topological charge

instantons in weak coupling – <u>all</u> sizes

$Z(N)$ dyons " strong " – <u>one</u> size

→ confine?

With quarks: <u>much</u> more complicated

$\mu = 0$, $T > 300$ MeV – dyons confined into inst.'s

$T <$ " " – dyons & quarks int.'g

$T = 0$, $\mu \neq 0$: dyons & qks relevant for <u>all</u> densities in neutron stars

instantons never matter