Moving from continuous to discrete symmetry in the 2D XY model

James C. Osborn
ANL

Collaborators
Nouman Butt (UIUC)
Xiao-Yong Jin (ANL)
Zain Saleem (ANL)

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INT workshop Tensor Networks in Many Body and Quantum Field Theory
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Outline

- Examine effects of discretization and perturbations phase structure of 2d classical XY model
- Compare discretizations of XY model using HOTRG
  - Angular discretization (Z_N clock model)
  - U(1) character expansion
- Structure of TN formulations
  - Role of core and interaction tensors
- Perturbed XY model
  - Interpolates between XY and Z_N models
  - Study effect of small perturbation on phase structure
  - How large is the effect of small perturbation?

XY model discretizations

- 2d XY model

$$H = -\sum_{<xy>} \cos(\theta_x - \theta_y) - h \sum_x \cos(\theta_x)$$

- Angular discretization (N states) → Z_N clock model

$$\theta_x \in \left\{ \frac{2\pi k}{N} ; k = 0 \dots N-1 \right\}$$

- U(1) character expansion discretization (D = 2S+1 states)

$$e^{\beta \cos(\theta_x - \theta_y)} \approx \sum_{k=-S}^{S} I_k(\beta) e^{ik(\theta_x - \theta_y)}$$

Tensor network simulations

- HOTRG method (Z. Y. Xie, J. Chen, M. P. Qin, J. W. Zhu, L. P. Yang, T. Xiang 2012)
  - D (N for Z_N model) = initial number of states
  - D_{cut} = maximum states (after SVD)

- Observables
  - Specific heat

$$C_V = -T\partial_T^2 F$$

$$F = -(\ln Z)/(\beta V)$$

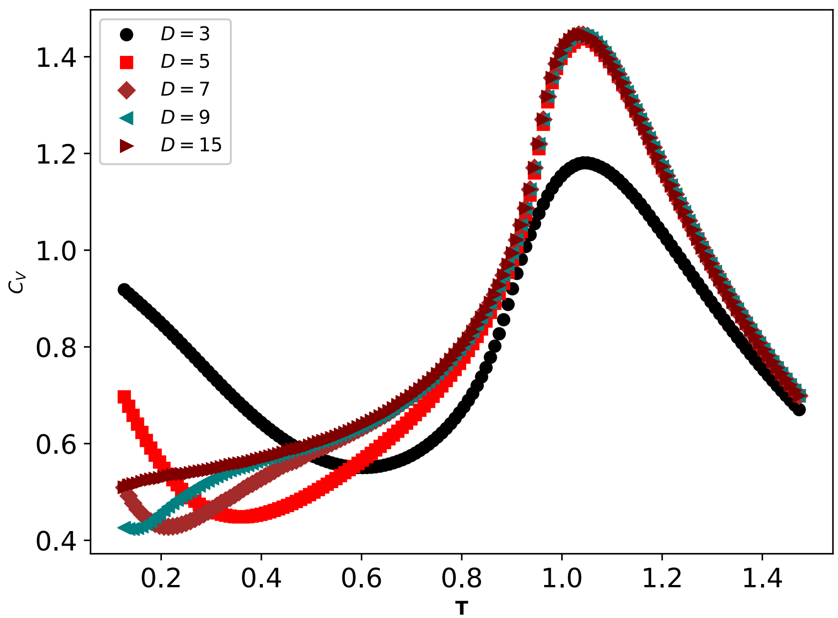
  - Second derivative from 7-point polynomial fit

  - Magnetization using impurity tensor method

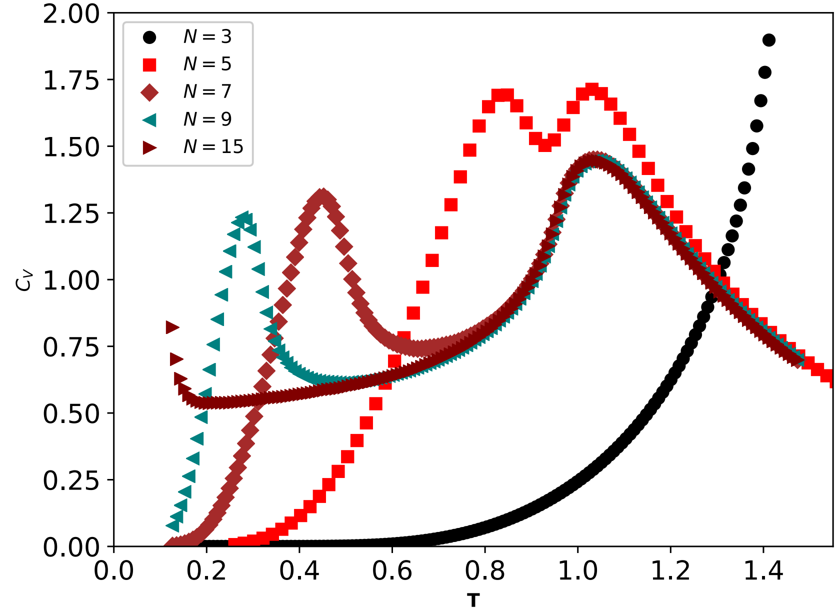
$$\langle m \rangle = \frac{1}{V}\langle \textstyle\sum_x \cos(\theta_x) \rangle$$

Specific heat of discretized XY model

$D_{cut} = 40$



U(1) character expansion



Z_N clock discretization

- Peak converges well for D ≥ 5
- Low temperature converges well for D ≥ 15
- Specific heat peak doesn't match phase transition (T_c), but similar

- One peak for N ≤ 4
- Two peaks for N ≥ 5
- Upper peak converges well for N ≥ 7
- Larger difference from XY at low T for same number of states (D=N)

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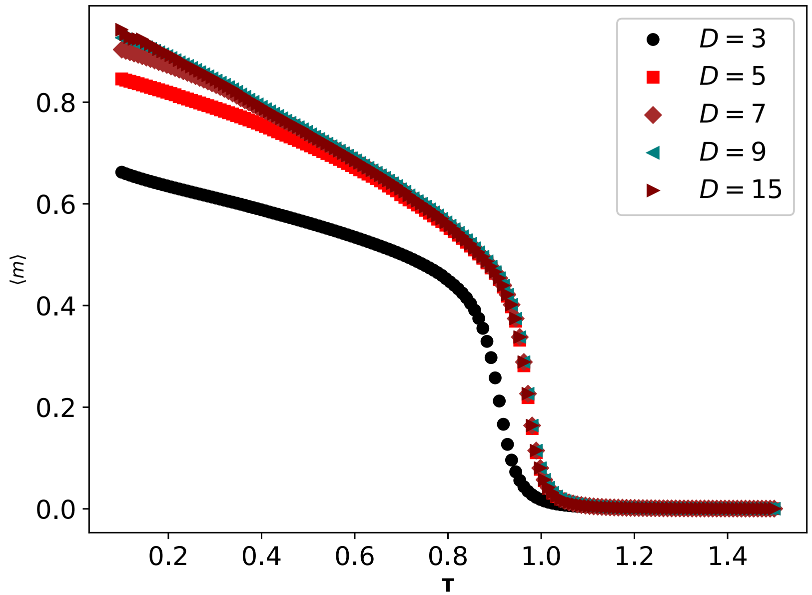
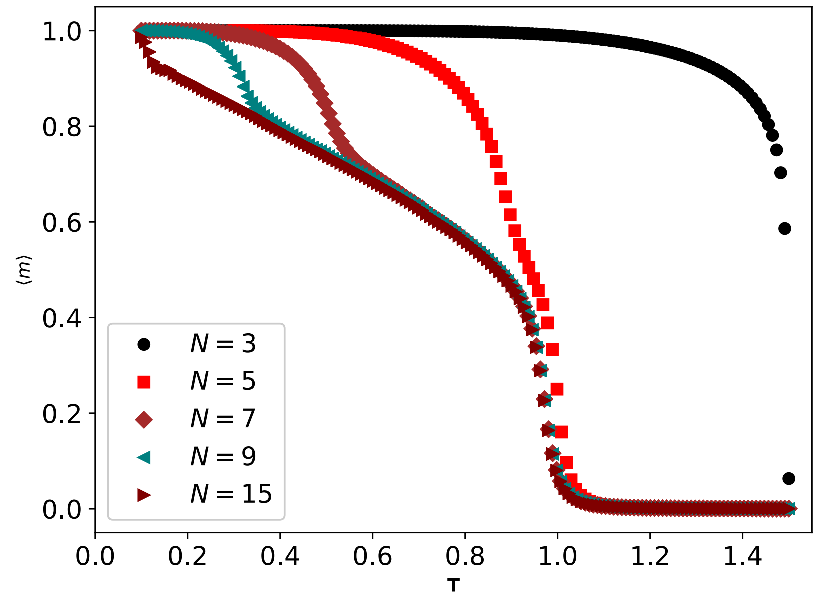
Magnetization of discretized XY model

$D_{cut} = 40$ $h = 10^{-4}$



U(1) character expansion



Z_N clock discretization

- Ordered phase appears in Z_N model, but not XY

Finding phase transitions

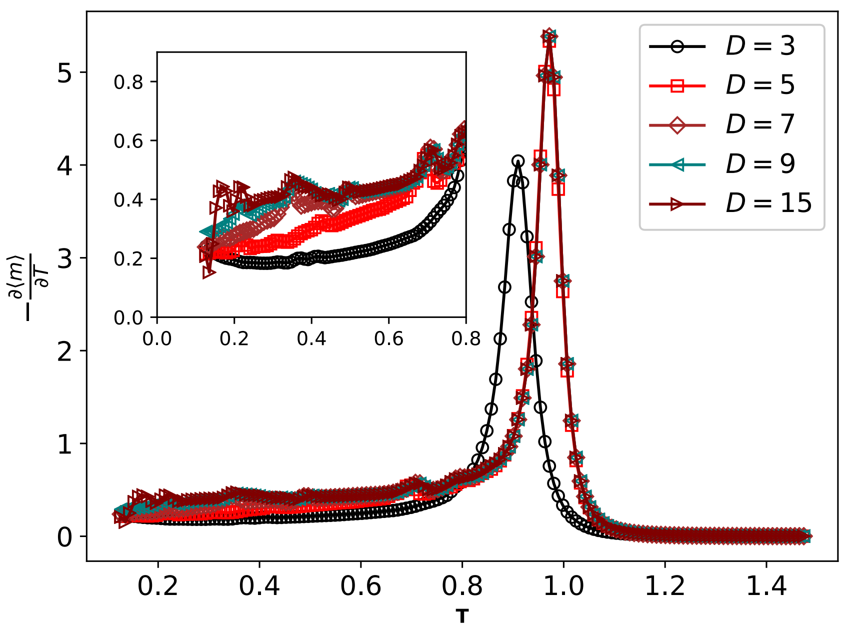
- Specific heat, magnetic susceptibility not reliable indicators of lower phase transition in Z_N ($N \geq 5$) model
- Cross derivative (Y. Chen, K. Ji, Z. Y. Xie, J. F. Yu 2020)

$$\frac{\partial^2 F}{\partial h \partial T} = -\frac{\partial \langle m \rangle}{\partial T}$$

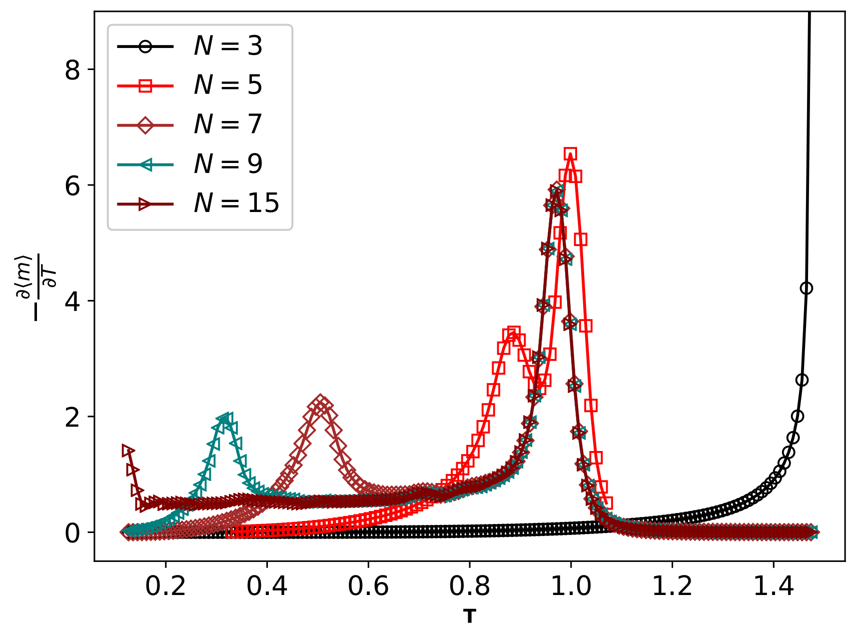
- Seems to be good indicator of phase transitions
- Obtained from fit of magnetization to polynomial in T

Cross derivative of discretized XY model

$D_{cut} = 40 \quad h = 10^{-4}$



U(1) character expansion



Z_N clock discretization

- Gives clear peaks at phase transitions

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Tensor network components

Spin models

- Typically start with interaction and local terms

$$H = \sum_{\langle xy \rangle} H_I(\theta_x, \theta_y) + \sum_x H_L(\theta_x)$$

- Expand interaction in some basis

$$e^{-\beta H_I(\theta_x,\theta_y)} \approx \sum_{a,b} M_{ab} f_a(\theta_x) g_b(\theta_y)$$

- Gives a core tensor of (for 2d)

$$C_{abcd} = \int d\theta e^{-\beta H_L(\theta)} f_a(\theta) g_b(\theta) f_c(\theta) g_d(\theta)$$

Tensor network components

- XY model tensor components (U(1) character basis)

$$M_{ab} = \delta_{ab} I_a(\beta)$$

$$C_{abcd} = I_{a-b+c-d}(\beta h) \xrightarrow{h\to 0} \delta_{a-b+c-d}$$

- Clock model tensor components (Z_N character basis)

$$M_{ab} = \delta_{ab} \sum_k I_{a+kN}(\beta)$$

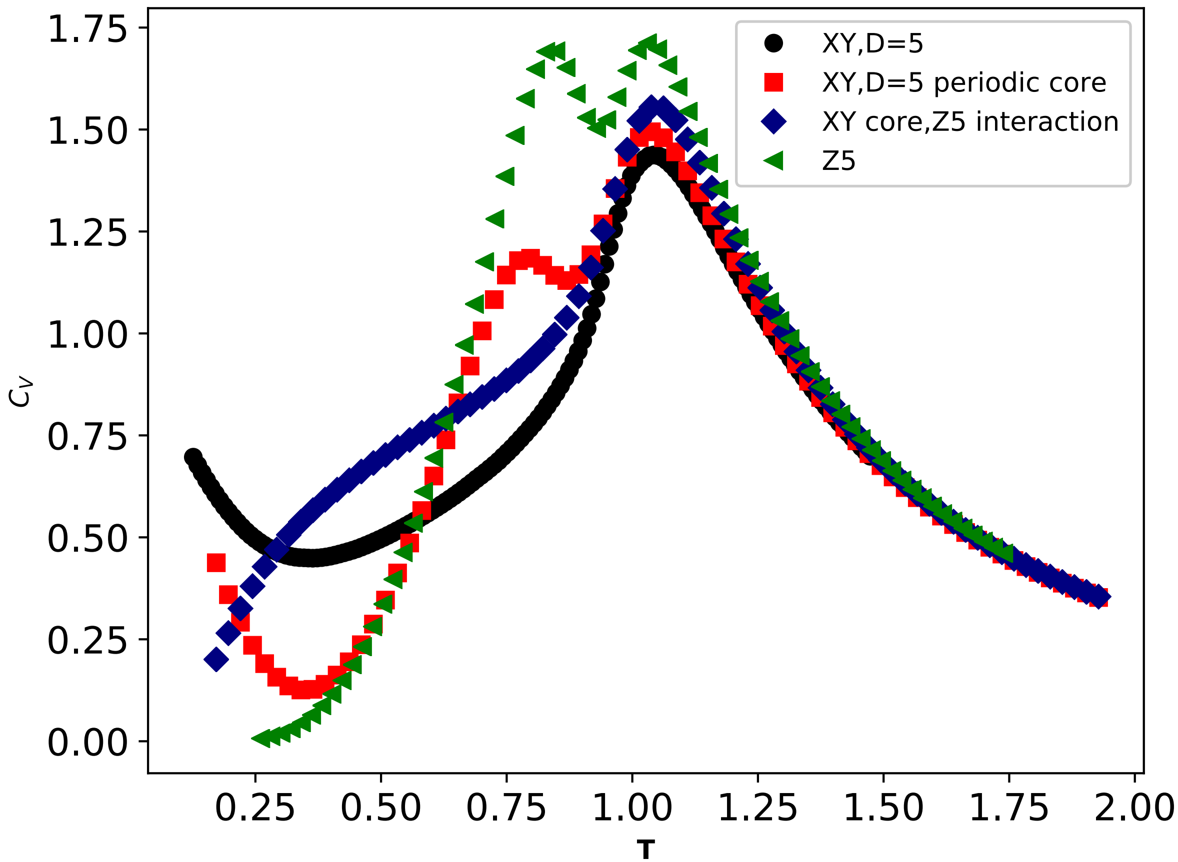
$$C_{abcd} = \sum_k I_{a-b+c-d+kN}(\beta h) \xrightarrow{h\to 0} \sum_k \delta_{a-b+c-d+kN}$$

- Can mix core and interaction tensors from different models

Specific heat for mixed XY/Z_N models

$D_{cut} = 40$

- 2 peaks appear with periodic core (Z_5 and XY+periodic core)
- Only one peak apparent with XY core
- Phase structure determined by symmetry of core tensor (Y. Meurice 2019)



Discretization as a choice of basis

- Core, interaction decomposition not unique
- For example, can move local term into interaction

$$H_I'(\theta_x,\theta_y) = H_I(\theta_x,\theta_y) + \frac{1}{2d}\left[H_L(\theta_x) + H_L(\theta_y)\right] \qquad e^{-\beta H_I'(\theta_x,\theta_y)} \approx \sum_{a,b} M_{ab}' f_a(\theta_x) g_b(\theta_y)$$

- Core tensor then determined only by basis

$$C_{abcd}' = \int d\theta f_a(\theta) g_b(\theta) f_c(\theta) g_d(\theta)$$

- Z_N clock model as approximation to interaction term using a rectangular step function basis, each over distinct angular range of $2\pi/N$, zero elsewhere
  - Evaluating interaction at center of each interval reproduces clock model

- Could choose other sets of basis functions, expansion approximations
- Evaluating error in discrete approximation to interaction term may help determine which basis is best (does this consider symmetries?)
  - RMS error for Z_N basis is larger than U(1) character basis
- Could use this to optimize basis for given model

Moving between XY and Z_N models

- Previously looked at Z_N for increasing N
- Instead consider fixed N
  - interpolate between periodic, non-periodic core tensor

$$C_{abcd} = \sum_k \alpha_k(h_N) I_{a-b+c-d+kN}(\beta h)$$

- Perturbed XY model (Jose, Kadanoff, Kirkpatrick, Nelson 1977)

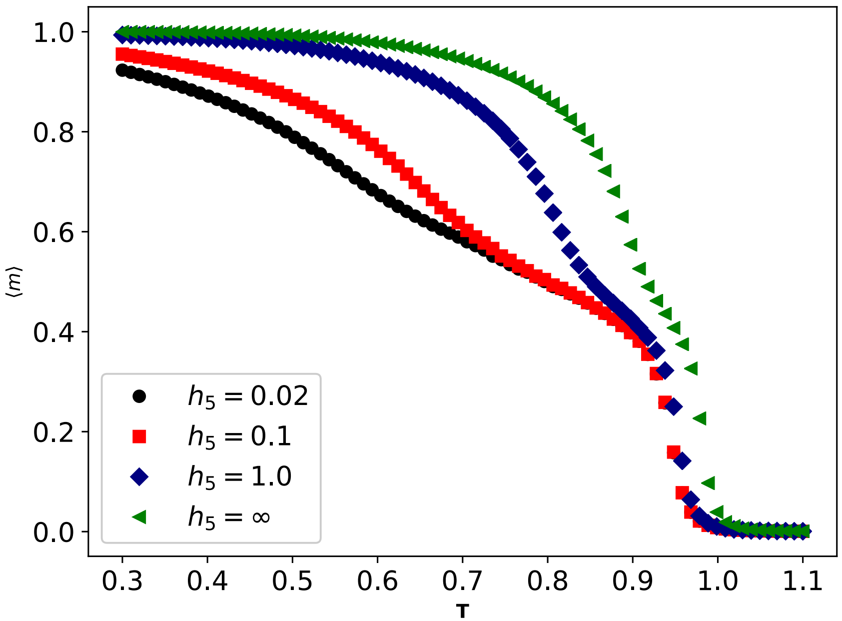
$$H = -\sum_{<xy>} \cos(\theta_x - \theta_y) - h \sum_x \cos(\theta_x) - h_N \sum_x \cos(N\theta_x)$$

$$\alpha_k(h_N) \propto \frac{I_k(\beta h_N)}{I_0(\beta h_N)} = \begin{cases} \delta_k : h_N = 0 \ (XY) \\ 1 : h_N \to \infty \ (ZN) \end{cases}$$

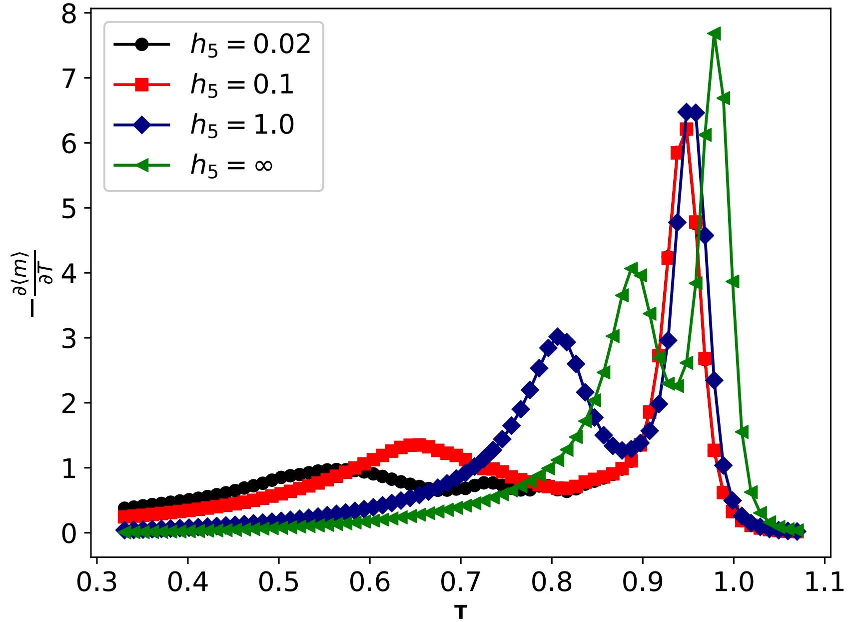
- Interested in behavior of lower phase transition for small h_N

Perturbed model N=5

D_{cut} = 40   h = 1e-5



Magnetization



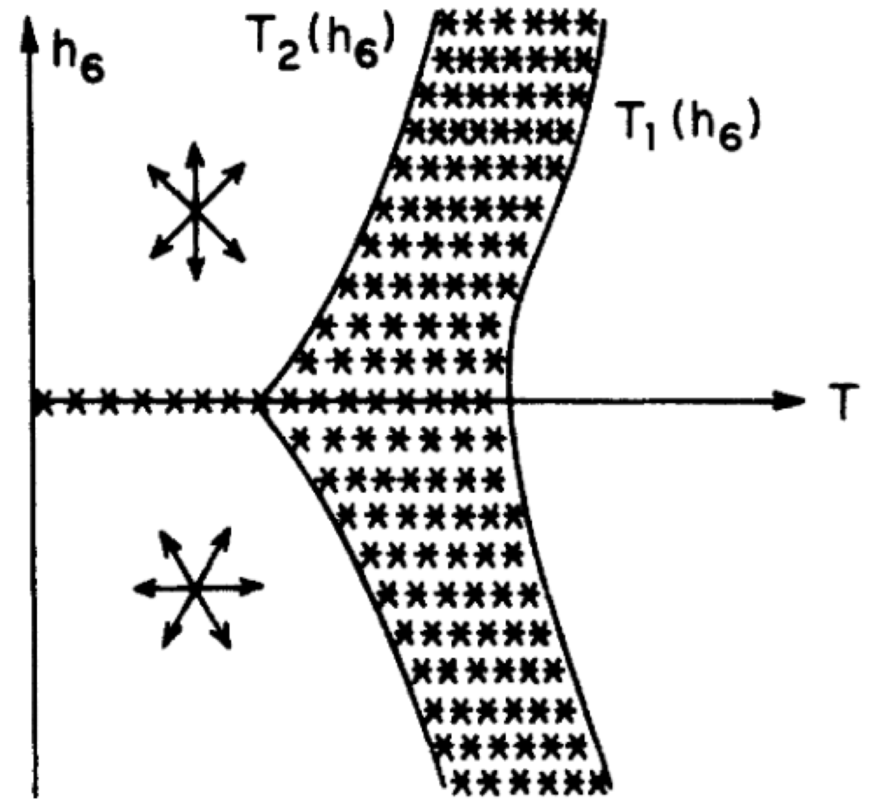
Cross derivative

Perturbed XY model RG analysis

- RG analysis using Villain approximation in JKKN paper

$$1 - \cos(\theta_x - \theta_y) \approx (\theta_x - \theta_y)^2/2$$

- For fixed $h_N > 0$, found
  - 2 critical temperatures for $N > 4$
  - At $N=4$ collapse to single T_c
- Value of $T_2(h_N \rightarrow 0)$ not clear from RG analysis
- Interested if non-zero, and how large
- If large, infinitesimal perturbation has large effect on phase diagram



(Jose, Kadanoff, Kirkpatrick, Nelson 1977)

Determining peaks of cross derivative

- Calculated <m> with spacing $\Delta T = 0.004$

- Fit 11 consecutive points to 4^{th} order polynomial
  - Plots of cross derivative from derivative of polynomial at center of interval
  - Peak location from zero of 2^{nd} derivative of polynomial

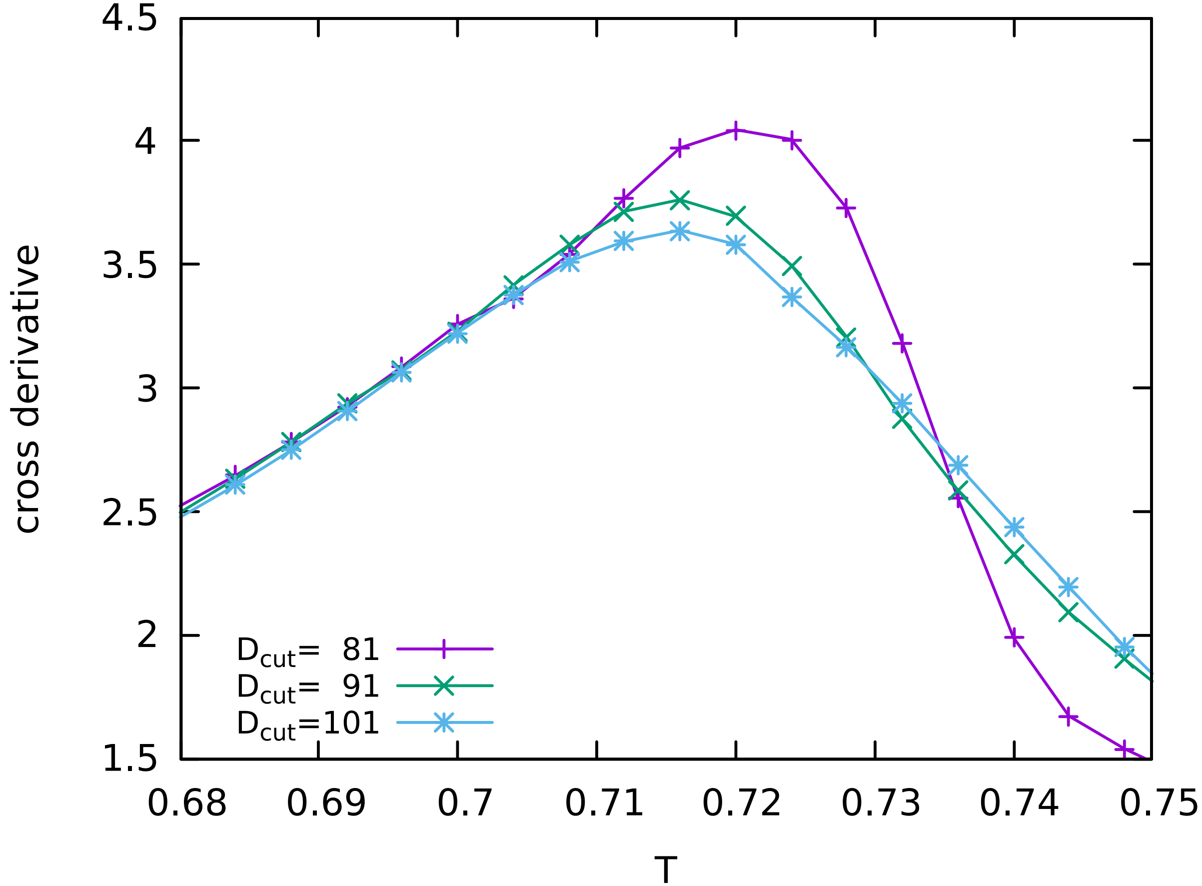
- Consider only peak fits where peak is well within interval

- Peak temperature and height averaged over fit intervals

Cross derivative peak example

$h_5 = 0.2$ $h = 2e\text{-}7$

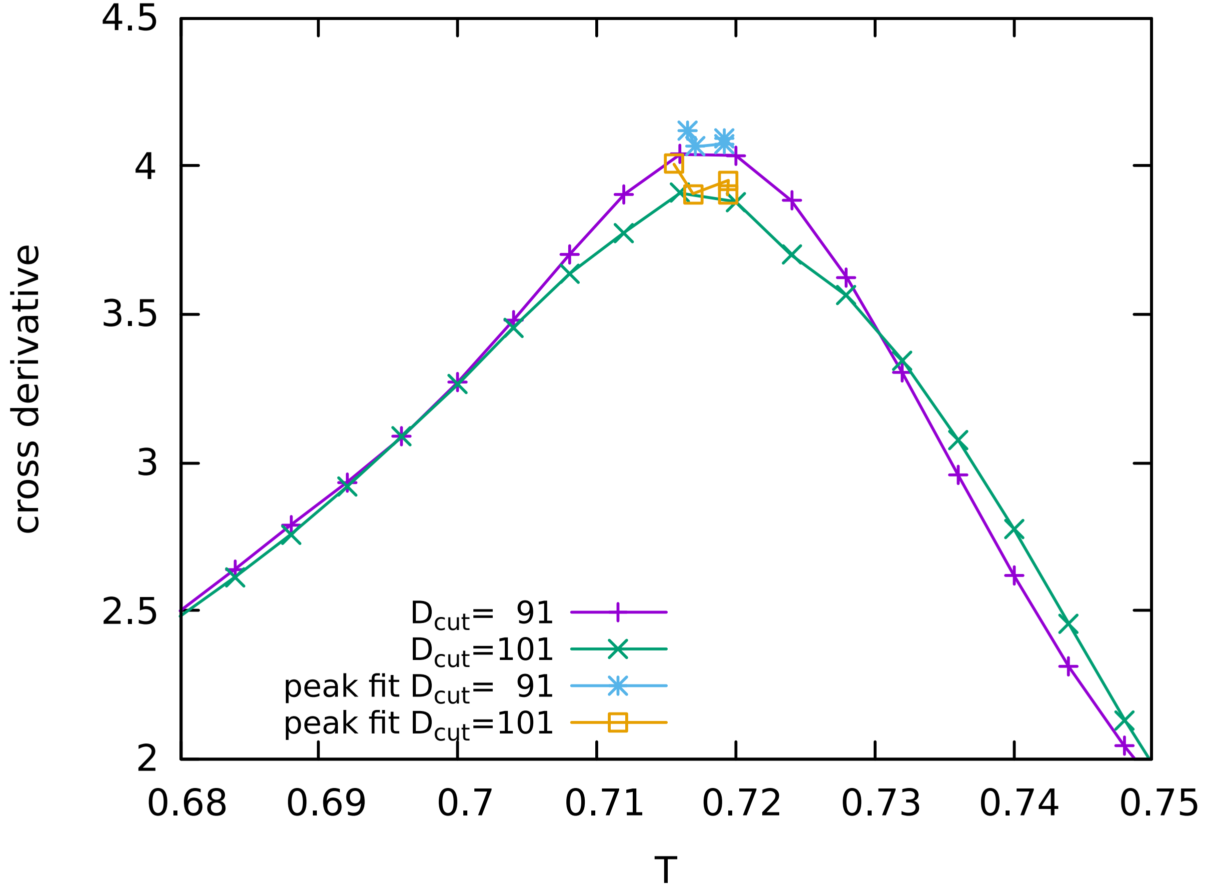
- D_{cut}=81 not enough
- Peak temperature consistent for D_{cut}=91, 101
- D_{cut}=91 seems sufficient for T_c
- Peak height still moving lower a bit
- Lowest $h_5 = 0.2$, simulations became unstable around h=1e-7 for $h_5 < 0.2$



Cross derivative peak fits

$h_5 = 0.2$ $h = 1e\text{-}7$

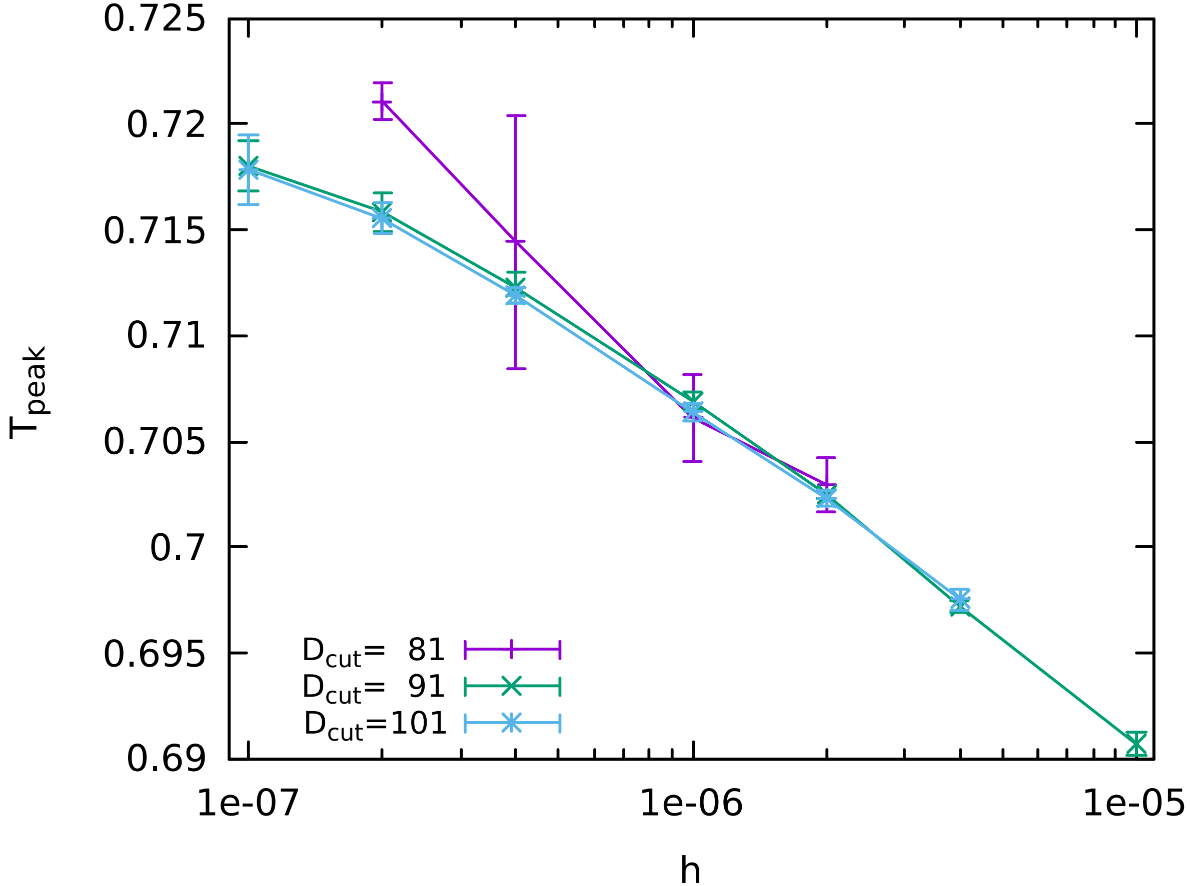
- ~4 points fell well within fit interval
- Peak determined from average, error from variance
- Error is systematic error of fit
- For peak height additional error from D_{cut}



Lower peak temperature

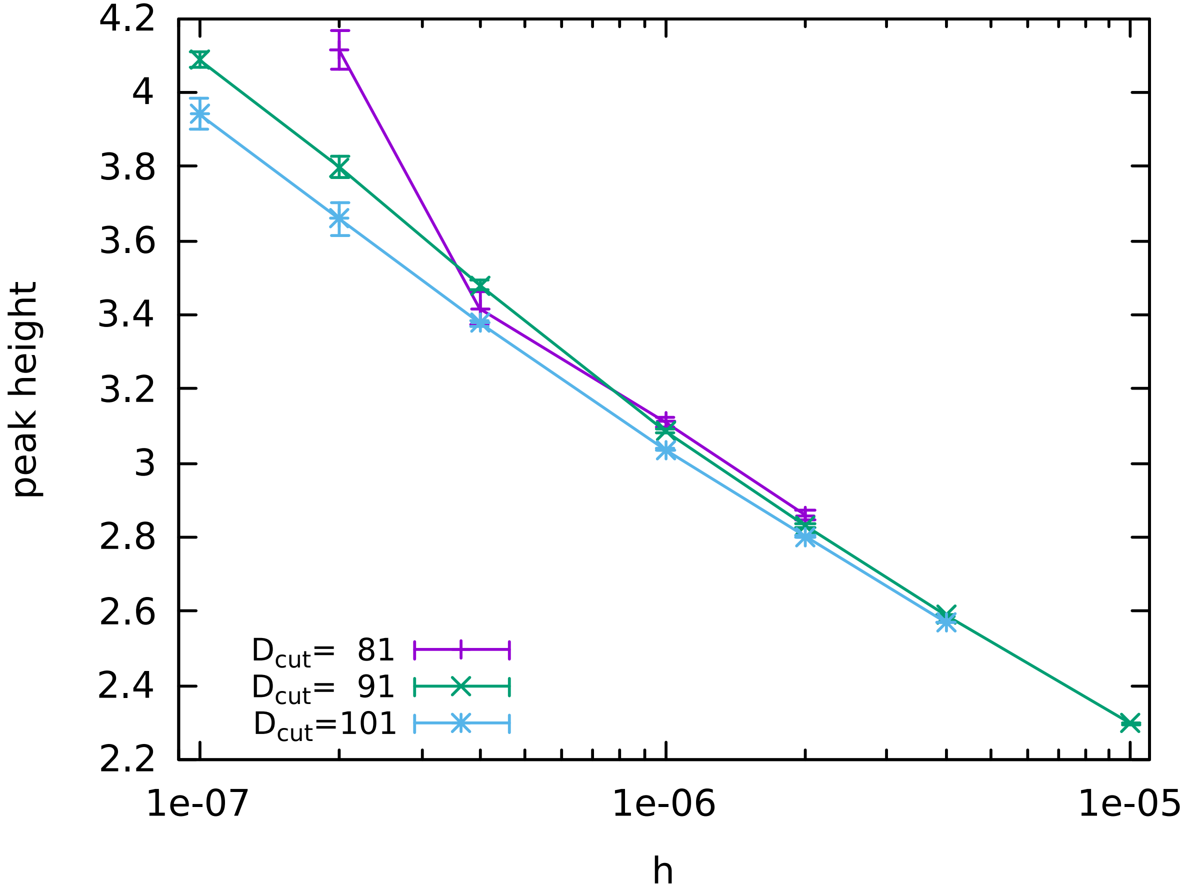
$h_5 = 0.2$

- D_{cut} = 81 not accurate enough at small h
- D_{cut} = 91 and 101 agree within errors
- Used D_{cut} = 91 for most of remaining results



Lower peak height

$h_5 = 0.2$



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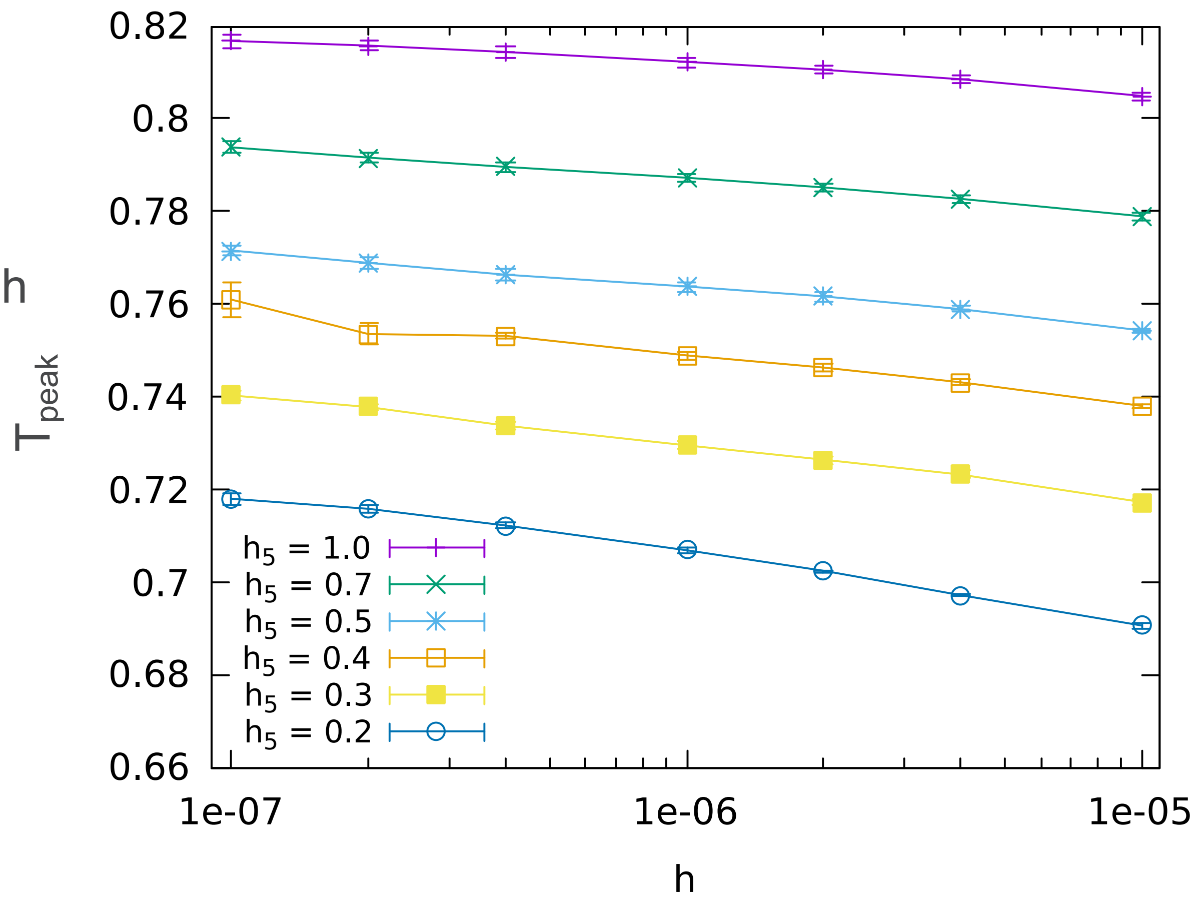
Lower peak temperature versus h

- T_{peak} rises at lower h, falls at lower h_5
- Fit T_{peak} to power-law in h

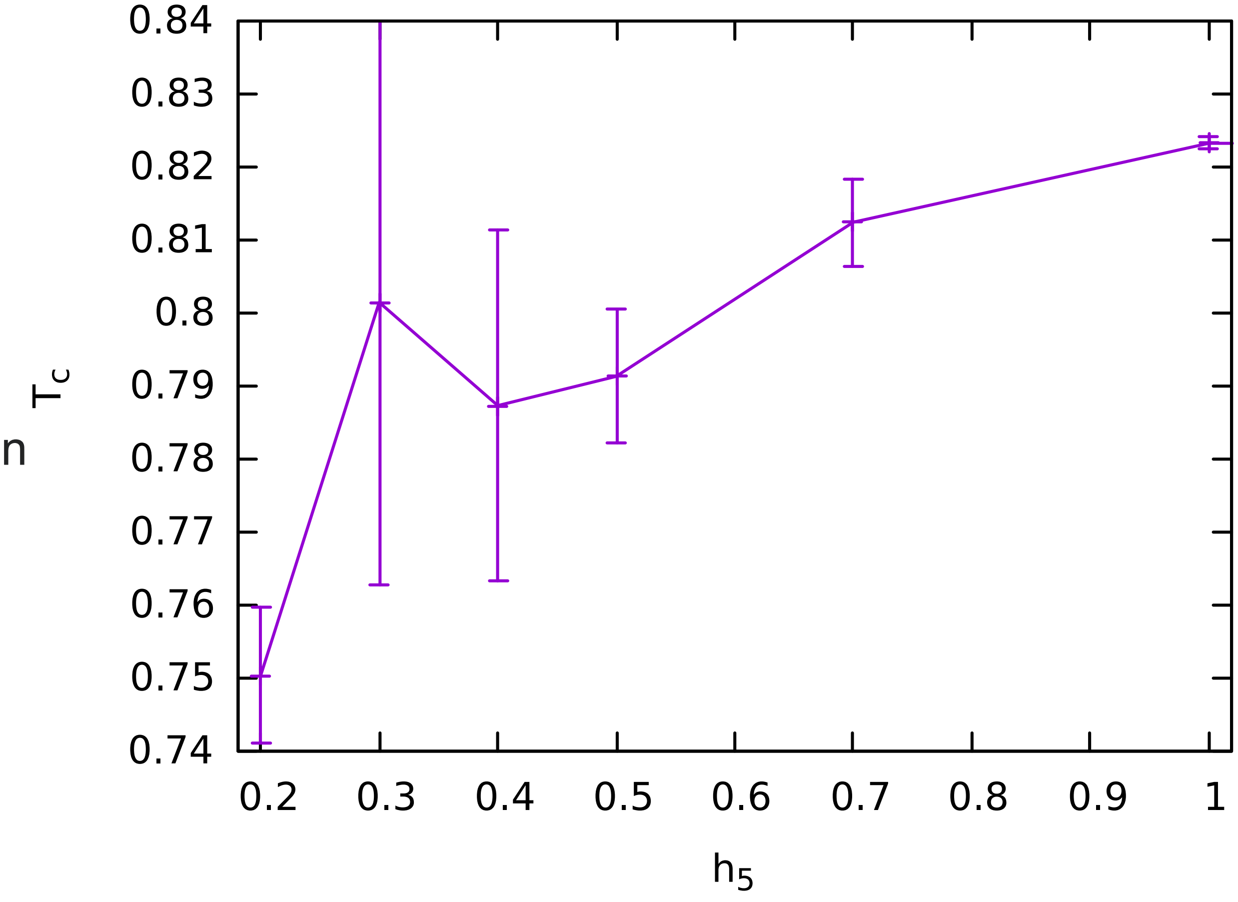
$$T_c(h_5) - T_{peak} \propto h^{\alpha}$$

to extract $T_c(h_5)$



Lower critical temperature versus h_5

- Error bars too large for power-law fit
- Trending down at lower h_5
- Still far from 0, but unable to do reliable extrapolation with this approach

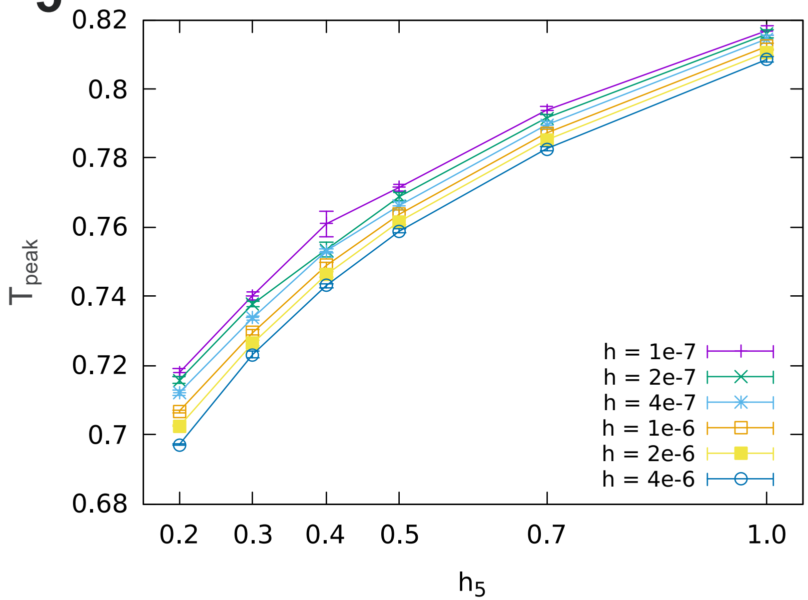


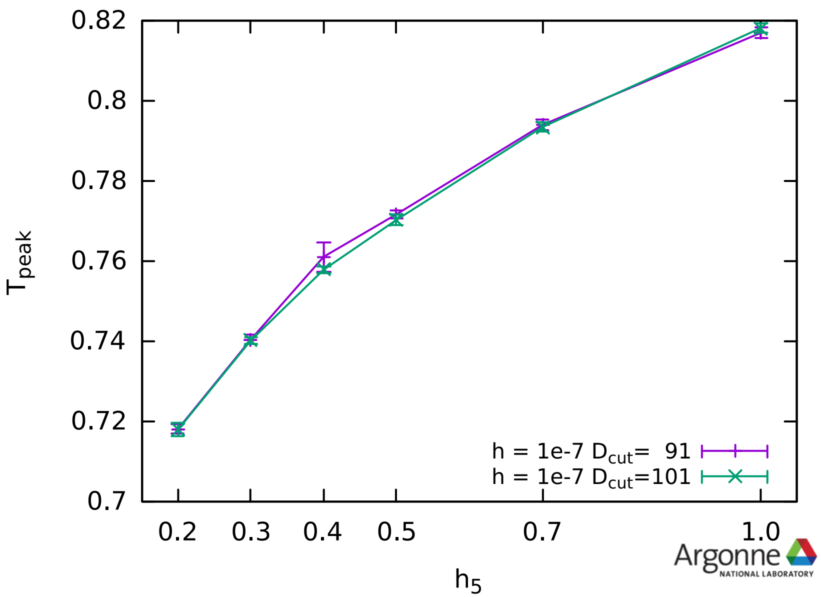
Peak temperature versus h_5

- First fit to h_5 at fixed h
- power-law in h_5

$$T_{peak}(h_5) - T_{peak}(0) \propto h_5^{\alpha}$$

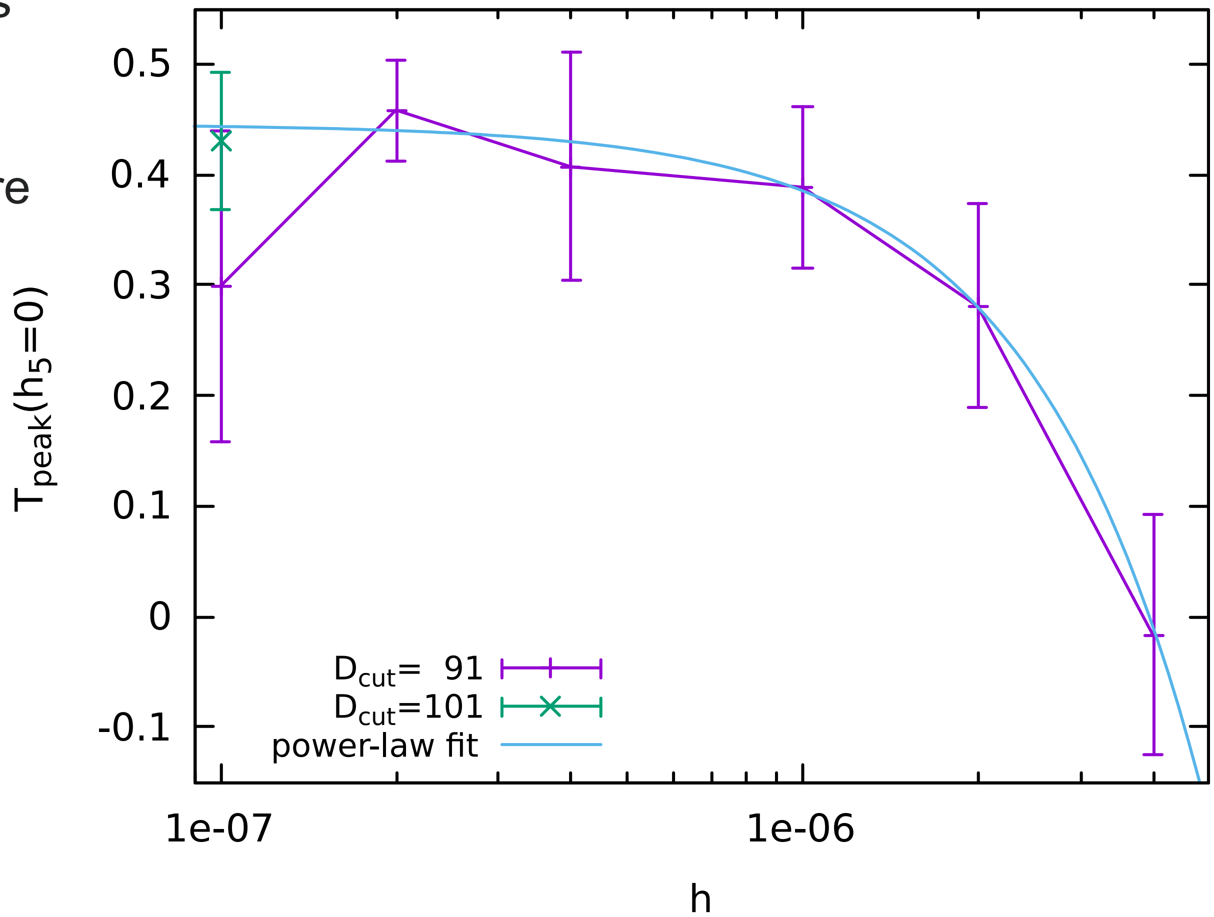
- Then extrapolate
  $T_{peak}(h_5 \rightarrow 0)$ to h = 0





Extrapolated peak temperature versus h

- Power-law fit extrapolates to $T_{peak}(h_5 \rightarrow 0) = 0.44(3)$
- About half the temperature of XY $T_c \sim 0.89$
- Small perturbation seems to have large effect on phase diagram

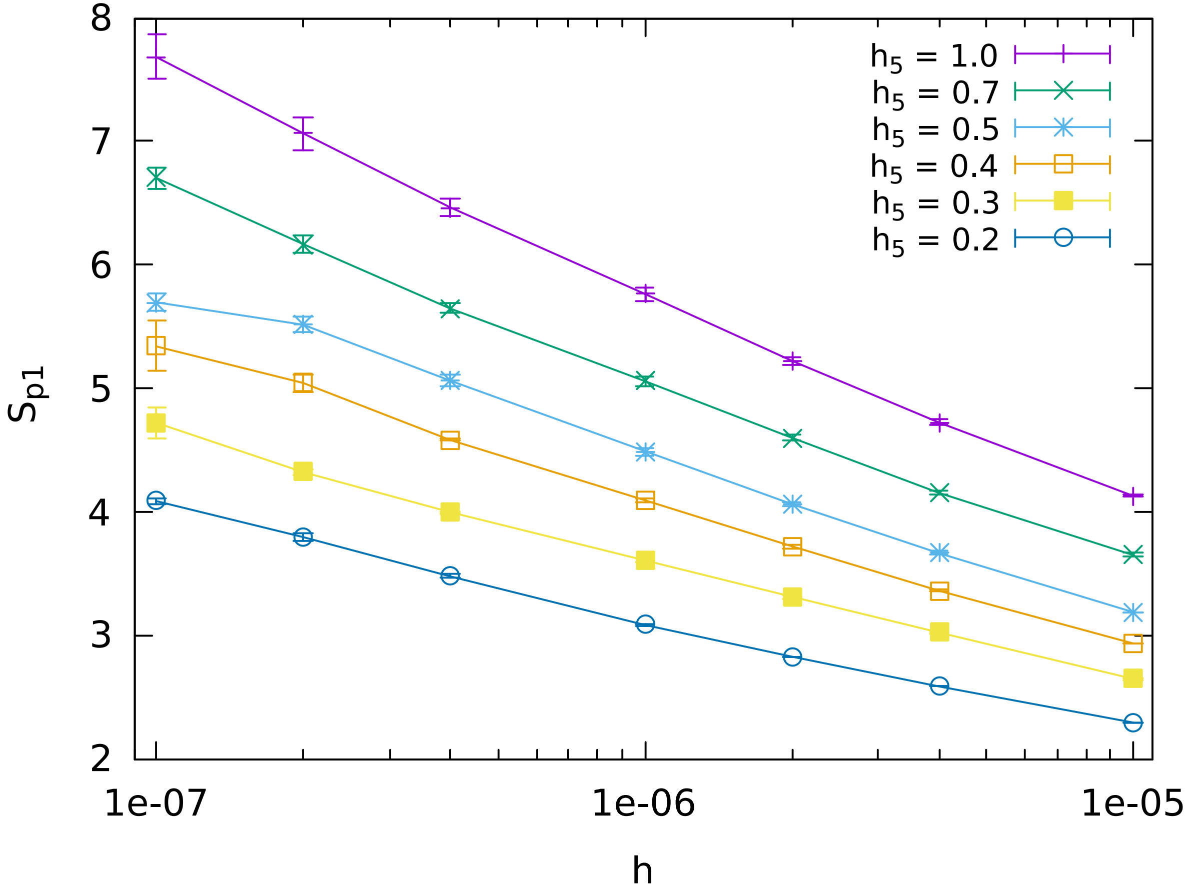


Peak height

- Need to check that system is critical (peak height diverges) as $h \to 0$
- Fit peak heights to power-law

$$height \propto h^{-\gamma}$$

- Check if gamma is nonzero as $h_5 \to 0$



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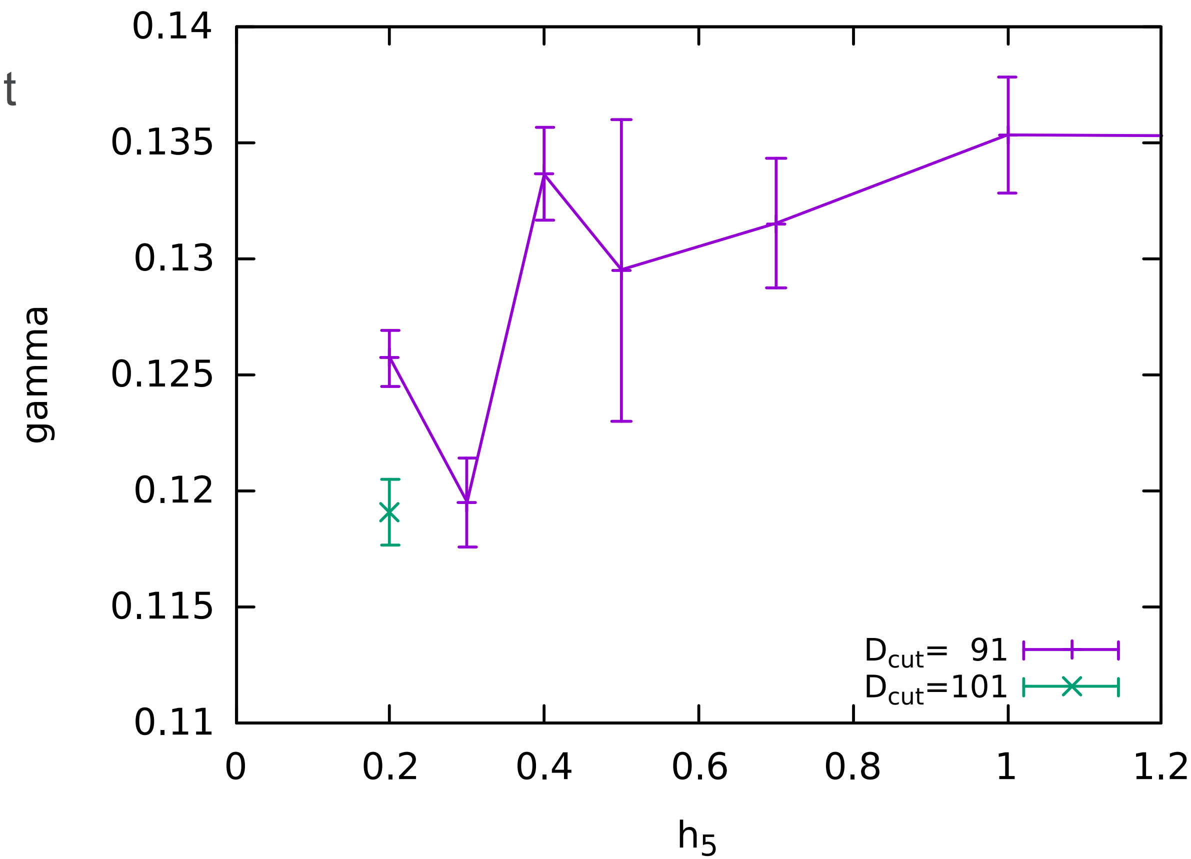
Peak height scaling

- Power-law exponent

  $height \propto h^{-\gamma}$

- Trending down as h_5 decreases, but still far from 0
- Consistent with critical transition as $h_5 \to 0$
- Need smaller h_5 to verify



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Summary

- XY truncation using U(1) character expansion has less effect on low energy behavior than angular discretization
- Expansion basis for TN construction determines core tensor symmetries, which in turn has large effect on phase structure
- Choosing basis that preserves symmetries
  - Does that also minimize error in truncated interaction expansion?
- Studied location of lower-T phase transition in perturbed XY model as function of h_5
  - consistent with non-zero T_c ($\sim 0.5\, T_{XY}$) as $h_5 \to 0$
- Small symmetry-breaking perturbations can have large effect on phase structure