The EoS of dense nuclear matter from Bayesian analysis of heavy-ion collision data

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A density dependent potential enters the QMD equations: 

\[ \dot{r}_i = \frac{\partial H}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial H}{\partial r_i}. \]

- The potential energy term in the systems Hamiltonian \( H \) is density dependent.

The potential energy \( V(n_B) \) is related to the pressure as:

\[
P(n_B) = P_{id}(n_B) + \int_0^{n_B} n' \frac{\partial U(n')}{\partial n'} dn', \quad U(n_B) = \frac{\partial (n_B \cdot V(n_B))}{\partial n_B}
\]

- \( U(n_B) \Rightarrow \) single particle potential, \( P_{id}(n_B) \Rightarrow \) pressure of an ideal Fermi gas of baryons

Constraining the potential energy \( V(n_B) \Rightarrow \) constraining the EoS
Parameterisation of the potential energy

\[ V(n_B) = \sum_{i=1}^{7} \theta_i \left( \frac{n_B}{n_0} - 2 \right)^i + h \]

- Upto $2n_0$, EoS reasonably constrained by
  - nuclear incompressibility data  Y. Wang, Et al. PLB 778, 207 (2018)
  - bayesian analysis  S. Huth et al., Nature 606, 276 (2022)

- Upto $2n_0$, CMF model-fit is used  A. Motornenko et al., PRC 103.5 (2021)
  - reproduces nuclear matter properties
    - $E_0 \approx -15.2\ MeV$, $K_0 \approx 267\ MeV$, $S_0 \approx 31.9\ MeV$

- We constrain the $V(n_B) > 2n_0$
  - 7th degree polynomial is used
    - $h = -22.07\ MeV$ to match CMF at $2n_0$

We constrain $\theta = \{\theta_1, \theta_2, \ldots, \theta_7\}$

Potentials generated using the polynomial parameterization
The experimental data

- Proton observables (mid rapidity)
  - Elliptic flow: 10 data points
    - E895, CERES, FOPI, STAR, HADES
    - Mid-central collisions
  - Transverse kinetic energy: 5 data points
    - E802, NA49, STAR
    - Central collisions

The data, \( D = \{ v_2^{exp}, \langle m_T^{exp} \rangle - m_0 \} \), is used to constrain the parameters \( \theta \).
Gaussian Process models

- Bayesian inference involves numerous UrQMD simulations
  - UrQMD ~80 s/event
    - $v_2 \sim 12000$ events, 10 energies, $<m_T>-m_0$ 1000 events, 5 energies
    - For a parameter set ~125000 events, 80 s = ~2700 hrs
  - MCMC requires random walk through 1000s of parameter sets
    - not feasible to run UrQMD for MCMC

- Gaussian Process models are trained as fast emulators
  - trained on 200 randomly generated EoSs
  - tested on 50 EoS
  - validation $r^2 \sim 0.9$
Bayesian inference

- **Posterior**: probability that the parameters $\theta$ explains the data $D$

$$P(\theta | D) \propto P(D | \theta) P(\theta)$$

**Log-likelihood**

$$\ln P(D | \theta) = -\frac{1}{2} \sum_i \left[ \frac{(x_i^\theta - d_i)^2}{\sigma_i^2} + \ln(2\pi\sigma_i^2) \right]$$

- Includes uncertainty from experiment & GP model
- Gaussian priors are used
- $\mu, \sigma$ from GP training data
Closure tests

1. Consider a random EoS as “ground-truth”
2. UrQMD observables for this EoS is the “data”
   a. uncertainty similar to experimental data
3. Construct the posterior
4. Compare with the “ground-truth”

- Tight constraints up to $4n_0$
  ➢ large uncertainty above $4n_0$
    ■ yet mean closely follows “ground-truth”
- Two curves extracted:
  ➢ “MAP”: mode of posterior
  ➢ “MEAN”: mean of posterior
- Both MEAN and MAP closely follows “ground-truth” up to $6n_0$
Result from experimental data

- Posterior from experimental data
  - 10 measurements of $v_2$
  - 5 measurements of $<m_T> - m_0$

- Tight constraints up to $4n_0$
  - MEAN, MAP suggests stiff EoS
  - No phase transition
Sensitivity to choice of observables

- Only 13 data points are used
  - $<m_t>-m_0$ at 3.83, 4.29 GeV not used

- Significant differences in posterior
  - softening at $3-5n_0$
  - phase transition

- Beyond $3n_0$ strong dependence to choice of observables
Reconstructed EoSs: $v_2', <m_T>_0$-m_0

- better $v_2$ predictions at high energies when 2 data points are removed
  - but also results in lower $<m_T>_0$
- large $<m_T>_0$ values for the stiff EoS (extracted using all data points)
- possible tension in data at ~4 GeV!
  - Measurement uncertainty? or limitation of the model?
Reconstructed EoSs: $dv_1/dy$, $c_s^2$

- $dv_1/dy$ data was not used for inference
  - yet consistent with reconstructed EoSs
    - especially with all 15 data points

- 15 points, predicts a stiff EoS
  - consistent with astrophysical constraints
    - broad peak structure

- 13 points, drastic drop in $c_s^2$
  - first order phase transition
Summary

- Can we reconcile data from current and previous experiments?
  - Bayesian inference on data from several experiments

- Can we find a flexible common parametrization of the EOS, applicable to neutron star and HIC simulations?
  - A polynomial parameterization of the density dependence of EoS is used

- What other observables could enable the extraction of the EOS?
  - $v_2$, $<m_1>-m_0$ of protons are used for inference
  - any observable that can be calculated using the model could be used
  - Use more observables in future
Summary

Are the nuclear matter EOSs from astrophysics consistent with HIC observables in the range $\rho < 4.0\rho_0$?

- inference using all 15 data points:
  - constraints the EoS up to $4n_0$
  - stiff EoS up to $4n_0$, no phase transition
  - consistent with BNSM constraint, $dv_1/dy$ data

- strong dependence on choice of observables for $> 3n_0$

- tension in data at ~4 GeV
  - measurement uncertainty or model limitation?

What improvements on the constraints on the EOS can we expect from future heavy-ion experiments?

*For stricter, robust constraints on the EoS below $4n_0$, significant improvements and consistency in flow measurements are necessary for $E_{\text{lab}} = 2-10$ A GeV*
Backup slides
Potentials for training GP models
GP models: performance
GP models: performance

![Graphs showing the performance of GP models at different energy scales.](image-url)
Closure tests
Microscopic transport with density dependent potential

- Non-equilibrium MD part of UrQMD is used
- UrQMD:
  - Propagation of hadrons on classic trajectories
    - stochastic binary scattering, color string formation, resonance excitation and decays
  - Imaginary part of interactions:
    - geometric interpretation of cross section
      - Experiment, detailed balance
  - Hadronic cascade
    - effective EoS of HRG with respective dof
- Real part of interactions in UrQMD
  - QMD + density dependent potential
    - Unlike other mean field models, QMD is an n-body theory of interactions between n nucleons
Microscopic transport with density dependent potential

A density dependent potential enters QMD equations

\[ \dot{\mathbf{r}}_i = \frac{\partial H}{\partial \mathbf{p}_i}, \quad \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{r}_i}. \]

The total hamiltonian function is sum over all hamiltonians of the i baryons

\[ H = \sum_i H_i, \quad H_i = E_i^{\text{kin}} + V_i \]

This include KE and total potential energy \( V \)

\[ V = \sum_i V_i \equiv \sum_i V(n_B(r_i)) \]

The change in momentum for baryon ‘i’ is then

\[ \dot{\mathbf{p}}_i = -\frac{\partial H}{\partial \mathbf{r}_i} = -\frac{\partial V}{\partial \mathbf{r}_i} n_{\{i,j\}} \equiv n_B(r_{\{i,j\}}) \]

\[ = -\left( \frac{\partial V_i}{\partial n_i} \cdot \frac{\partial n_i}{\partial \mathbf{r}_i} \right) - \left( \sum_{j \neq i} \frac{\partial V_j}{\partial n_j} \cdot \frac{\partial n_j}{\partial \mathbf{r}_i} \right) \]

- solved in timestep 0.2fm/c

The local interaction density \( n_B \) at \( r_k \) is calc by assuming each particle as gaussian wave packet

\[ n_B(r_k) = n_k = \sum_{j,j \neq k} n_{j,k} \]

\[ = \left( \frac{a}{\pi} \right)^{3/2} \sum_{j,j \neq k} B_j \exp \left( -\alpha(r_k - r_j)^2 \right) \]

\( \alpha = 1/2L, \quad L = 2 \text{ fm}^2 \)

Force on \( i^{\text{th}} \) baryon depends on change in potential energy at point \( r_i \) due to local gradient of \( n_B(r_i) \) and change in potential at positions \( r_j \) of all baryons \( j \) due to change in \( r_i \)
\[ P(n_B) = P_{id}(n_B) + \int_0^{n_B} n' \frac{\partial U(n')}{\partial n'} dn', \quad U(n_B) = \frac{\partial (n_B \cdot V(n_B))}{\partial n_B} \]

\[ \mu'_B(n_B) = \mu^id_B(n_B) + U(n_B) \]

\[ \epsilon(n_B) = -P(n_B) + \mu'_B n_B + sT \]