

# Quarkonium beyond the E/T expansion: Non-Lindblad master equations

Tom Magorsch

in collaboration with

Nora Brambilla, Arthur Lin and Antonio Vairo

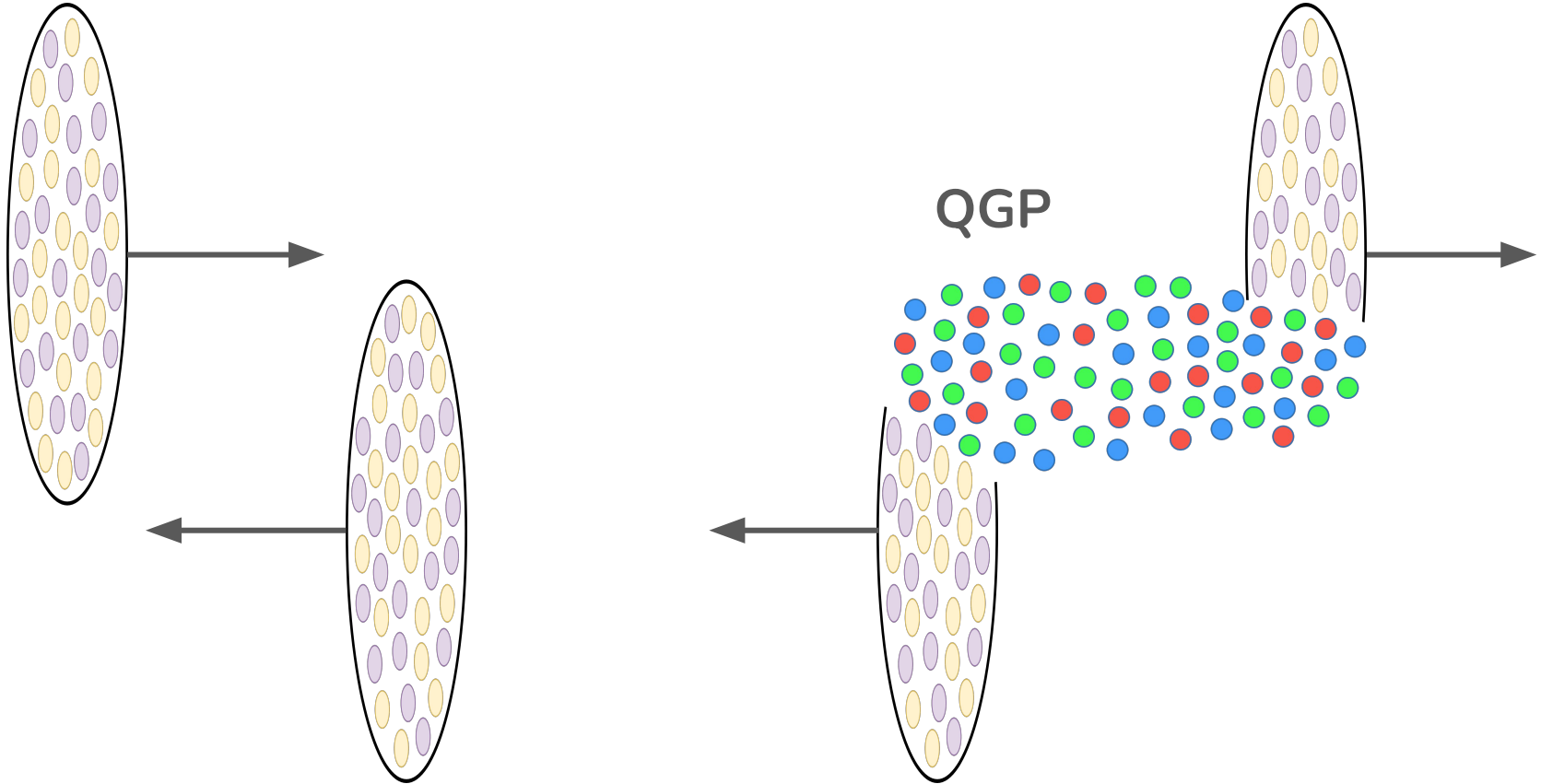
*Open Quantum Systems:*

*Dissipative Dynamics from Quarks to the Cosmos*

03.12.25



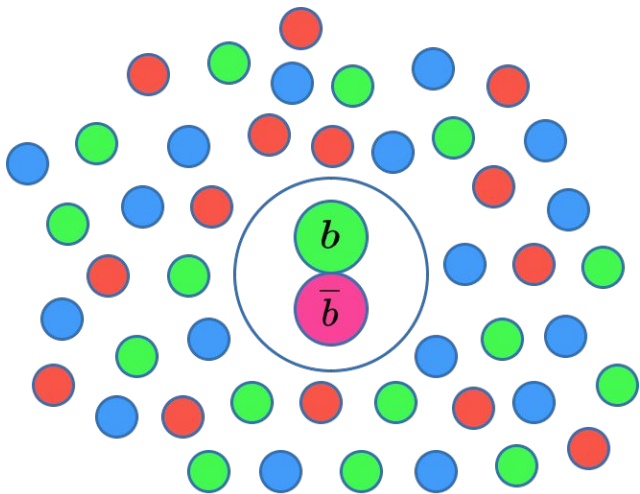
# Quarkonium suppression



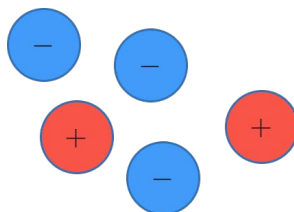
# Quarkonium suppression

Veres, Gábor I. "Heavy ion physics at CMS and ATLAS: hard probes." *arXiv preprint arXiv:1905.10461* (2019).

Propagation through QGP



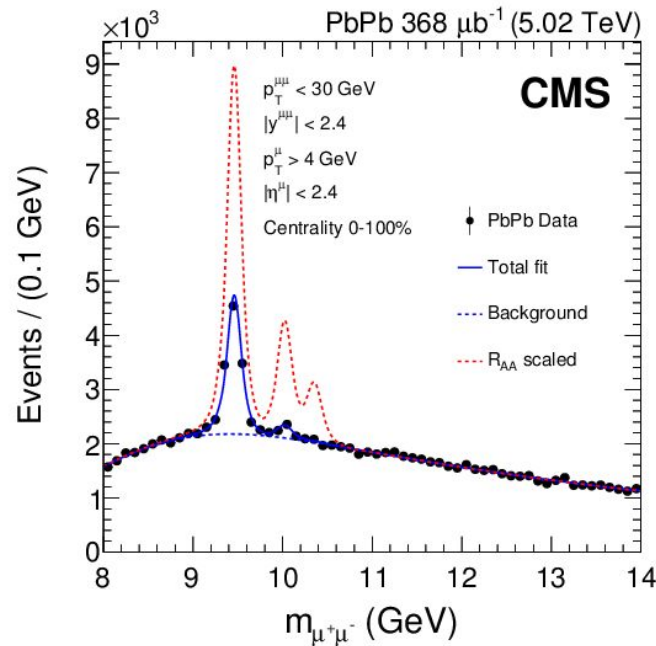
T. Matsui, H. Satz, Phys. Lett. B 178 (1986) 416



$$V(r) = -\frac{\alpha}{r}$$

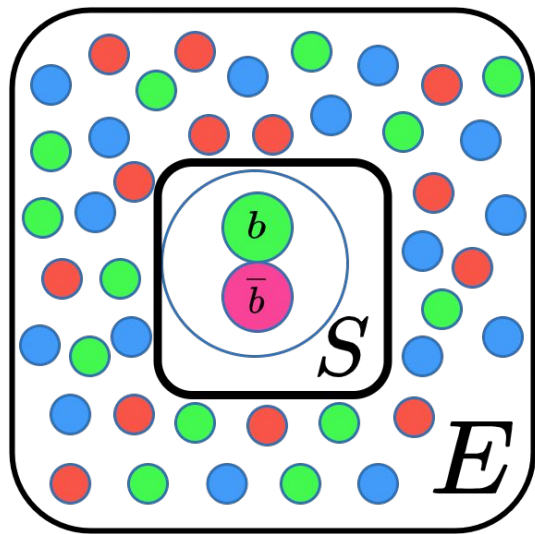
$$V(r) = -\frac{\alpha}{r} e^{-m_D r}$$

Debye-screening in medium

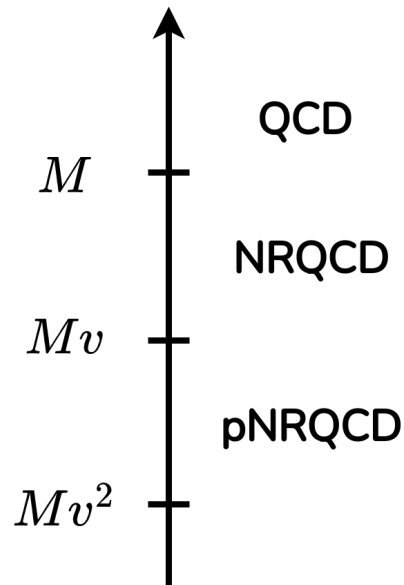


# Quarkonium suppression

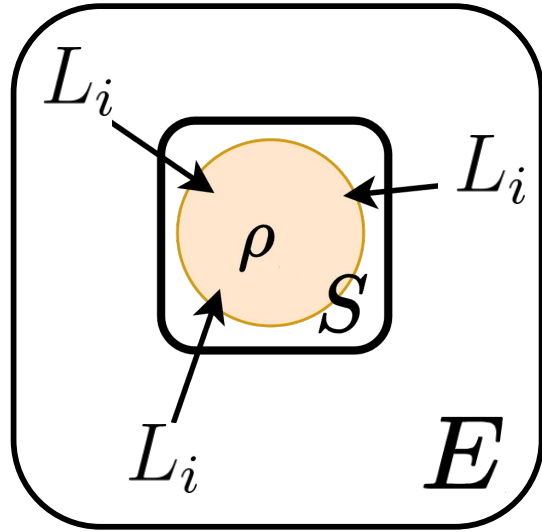
Yukinao Akamatsu, Prog.Part.Nucl.Phys. 123 (2022), 103932  
Xiaojun Yao, Int.J.Mod.Phys.A 36 (2021) 20, 2130010



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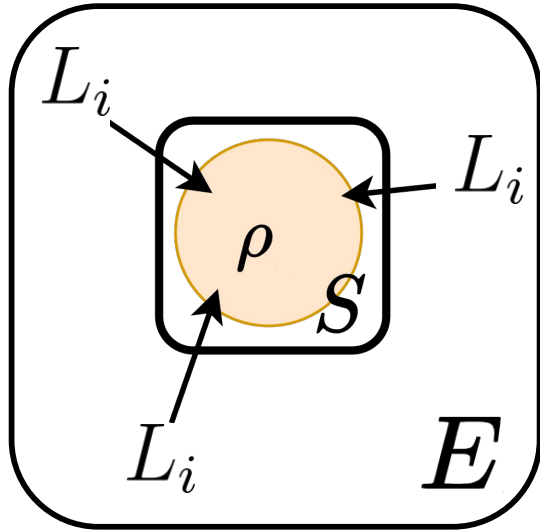


# Quarkonium suppression



$$\frac{d\rho(t)}{dt} = -i [H, \rho(t)] + \sum_{nm} h_{nm} \left( L_i^n \rho(t) L_i^{m\dagger} - \frac{1}{2} \left\{ L_i^{m\dagger} L_i^n, \rho(t) \right\} \right)$$

# Quarkonium suppression



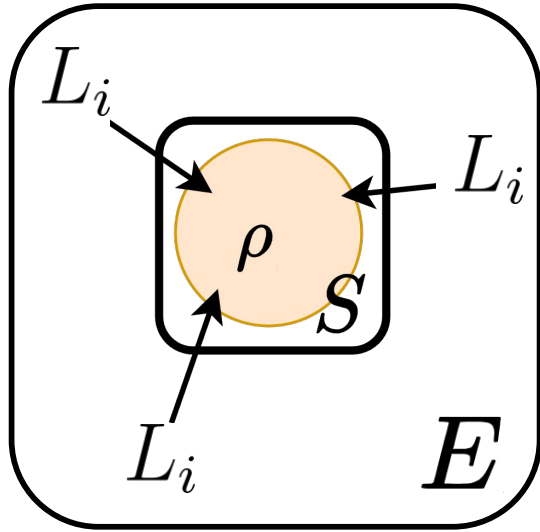
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$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$L_i^0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} r_i, \quad L_i^1 = \begin{pmatrix} 0 & 0 \\ 0 & \frac{N_c^2 - 4}{2(N_c^2 - 1)} A_i^{oo\dagger} \end{pmatrix}$$

$$L_i^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} r_i, \quad L_i^3 = \begin{pmatrix} 0 & \frac{1}{N_c^2 - 1} A_i^{os\dagger} \\ A_i^{so\dagger} & 0 \end{pmatrix}$$

# Quarkonium suppression



$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_{nm} h_{nm} \left( L_i^n \rho(t) L_i^{m\dagger} - \frac{1}{2} \{ L_i^{m\dagger} L_i^n, \rho(t) \} \right)$$

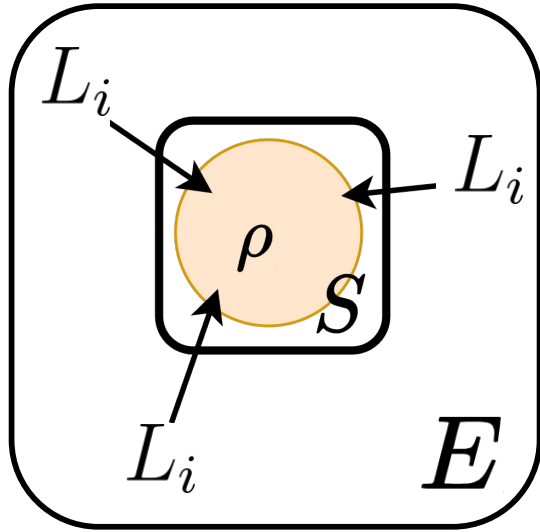
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$$A_i^{uv} = \frac{g^2}{6N_c} \int_0^\infty ds e^{-ih_u s} r_i e^{ih_v s} \langle E_j^a(s) \Omega(s, 0)^{ab} E_j^b(0) \rangle$$

# Quarkonium suppression



$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_{nm} h_{nm} \left( L_i^n \rho(t) L_i^{m\dagger} - \frac{1}{2} \left\{ L_i^{m\dagger} L_i^n, \rho(t) \right\} \right)$$

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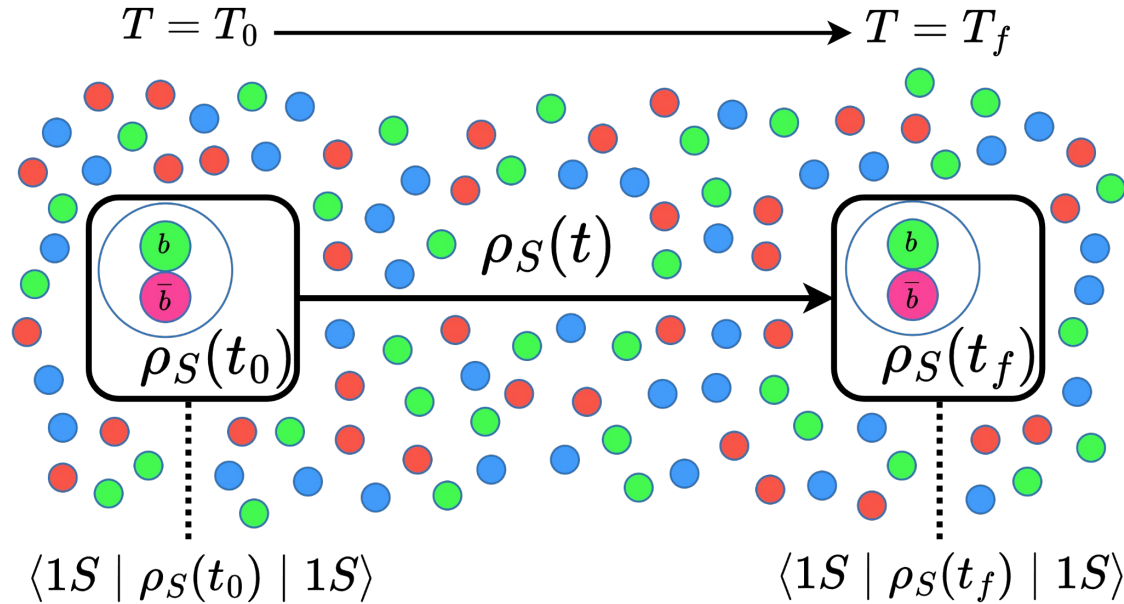
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$$h_u = \frac{p^2}{M} + V_u(r)$$

# Solving the master equation gives the evolution of the bottomonium



$$P_{\text{survival}}(1S) = \frac{\langle 1S | \rho(t_f) | 1S \rangle}{\langle 1S | \rho(t_0) | 1S \rangle}$$

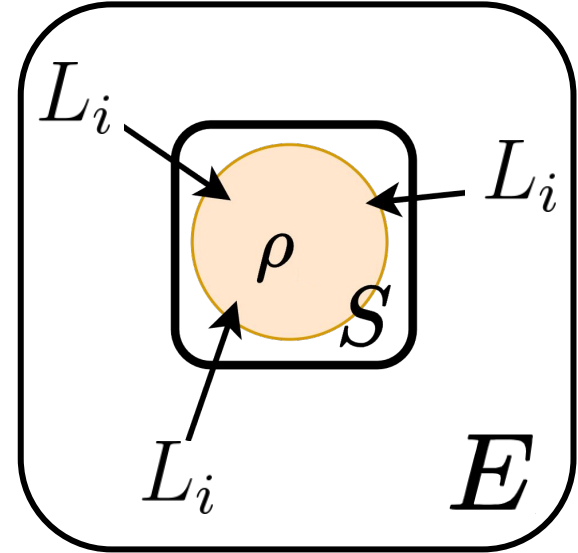
To simplify the equation one can expand in  $E/T$

$$A_i^{uv} = \int_0^\infty ds e^{-ih_us} r_i e^{ih_vs} \langle E_j^a(s) \Omega^{ab}(s, 0) E_j^b(0) \rangle$$

$$1 - ih_us + \mathcal{O}(E^2/T^2) \quad 1 - ih_vs + \mathcal{O}(E^2/T^2)$$

Next-to-Leading order in  $E/T$

Leading order in  $E/T$



# To simplify the equation one can expand in E/T

Transport coefficients

$$A_i^{uv} = \underbrace{\frac{r_i}{2}(\kappa - i\gamma)}_{\text{LO}} + \underbrace{\kappa \left( -\frac{ip_i}{2MT} + \frac{\Delta V_{uv}}{4T} r_i \right)}_{\text{NLO}}$$

# The master equation is not positive

$$\frac{d\rho(t)}{dt} = -i [H, \rho(t)] + \sum_{nm} h_{nm} \left( L_i^n \rho(t) L_i^{m\dagger} - \frac{1}{2} \left\{ L_i^{m\dagger} L_i^n, \rho(t) \right\} \right)$$

$$L_i \longrightarrow L'_i$$

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \longrightarrow h' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

# The master equation is not positive

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$$\begin{array}{ccc} L_i & \longrightarrow & L'_i \\ h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} & \longrightarrow & h' = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

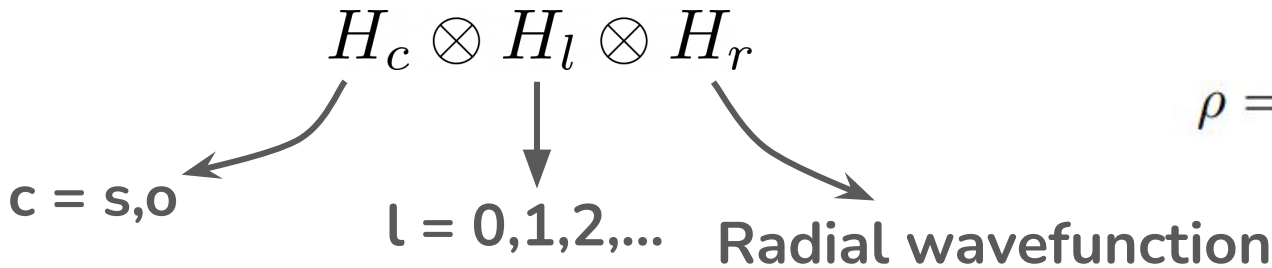
$\mathcal{O}(E^2/T^2)$

Neglect negative terms as they are suppressed!

# Projection on spherical harmonics

$$\frac{d\rho(t)}{dt} = -i[H, \rho(t)] + \sum_{n=0}^{\cancel{1}^6} \left( \cancel{L_i^n} \rho(t) \cancel{L_i^{n\dagger}} - \frac{1}{2} \{ \cancel{L_i^{n\dagger}} \cancel{L_i^n}, \rho(t) \} \right)$$

Hilbert space:



$$\rho = \begin{pmatrix} \rho_s^0 & 0 & \dots & 0 & 0 & \dots \\ 0 & \rho_s^1 & \dots & 0 & 0 & \dots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \\ 0 & 0 & \dots & \rho_o^0 & 0 & \dots \\ 0 & 0 & \dots & 0 & \rho_o^1 & \dots \\ \vdots & \vdots & & \vdots & \vdots & \ddots \end{pmatrix}$$

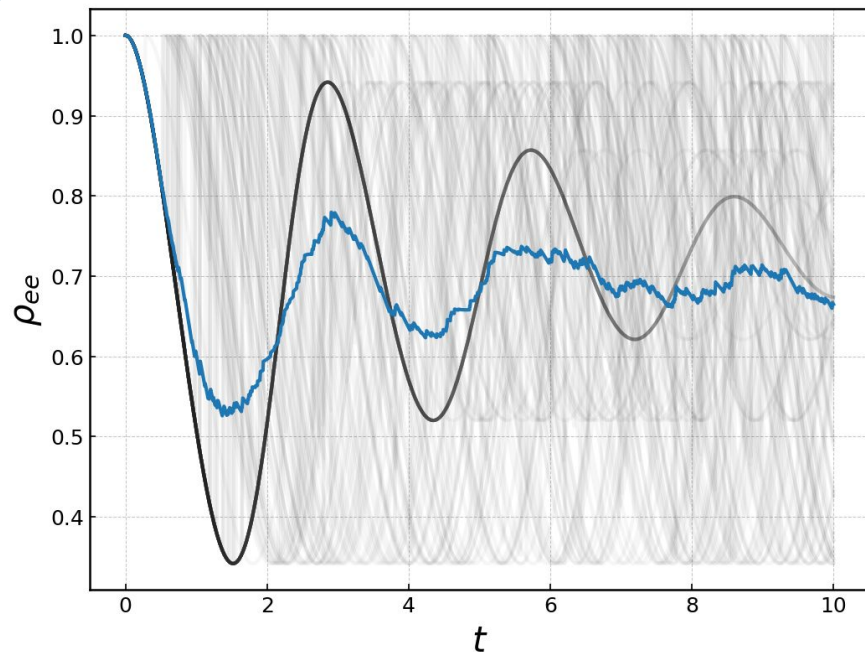
# Quantum trajectory algorithm

J. Dalibard, Y. Castin, and K. Mølmer, Wave-function approach to dissipative processes in quantum optics, Phys. Rev. Lett. 68 (1992), pp. 580–583.

- Idea:

1. Evolve individual trajectories  $|\psi(t)\rangle$  **stochastically**
2. Calculate observables by averaging over trajectories

$$\langle A \rangle = \mathbb{E} [\langle \psi(t) | A | \psi(t) \rangle]$$



# Quantum Trajectories

e.g.:  $U = 1 - iH_{\text{eff}}\delta t$

$\psi_0$   
↓

1. Evolve state  $|\psi(t)\rangle$  with  $H_{\text{eff}} = H - \frac{i}{2}C^\dagger C$

$$|\psi'(t + \delta t)\rangle = U|\psi(t)\rangle$$

# Quantum Trajectories

e.g.:  $U = 1 - iH_{\text{eff}}\delta t$

$$\psi_0$$


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$$|\psi'(t + \delta t)\rangle = U|\psi(t)\rangle$$

2. Compute norm

$$\langle \psi'(t + \delta t) | \psi'(t + \delta t) \rangle = 1 - \delta p < 1$$

# Quantum Trajectories

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$$\langle\psi'(t + \delta t)|\psi'(t + \delta t)\rangle = 1 - \delta p < 1$$

3. Apply jump operator  $C$  with probability  $\delta p$

$$P(|\psi(t + \delta t)\rangle = C|\psi(t)\rangle) = \delta p$$

$$P(|\psi(t + \delta t)\rangle = |\psi'(t + \delta t)\rangle) = 1 - \delta p$$

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e.g.:  $U = 1 - iH_{\text{eff}}\delta t$

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$$|\psi'(t + \delta t)\rangle = U|\psi(t)\rangle$$

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$$\langle\psi'(t + \delta t)|\psi'(t + \delta t)\rangle = 1 - \delta p < 1$$

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4. Normalize  $|\psi(t + \delta t)\rangle$  —

# Quantum trajectory algorithm

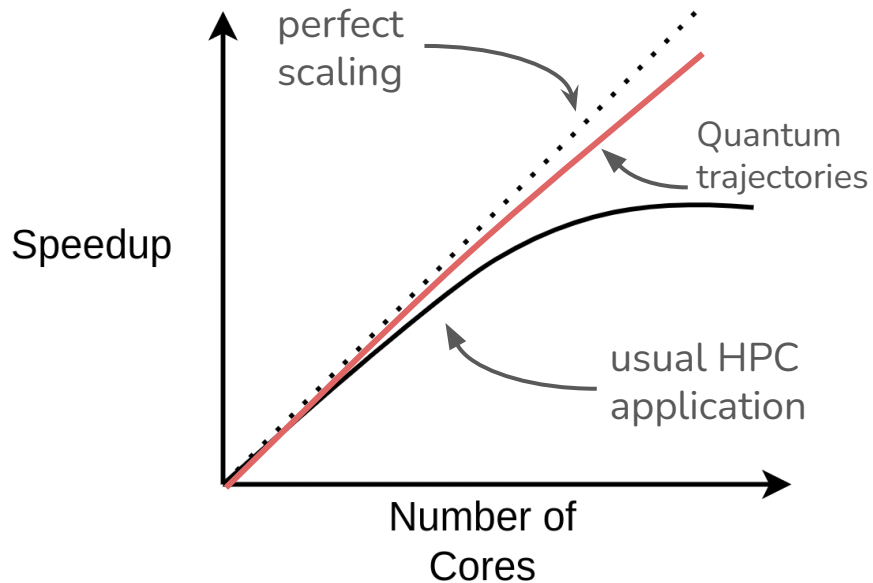
J. Dalibard, Y. Castin, and K. Mølmer, Wave-function approach to dissipative processes in quantum optics, Phys. Rev. Lett. 68 (1992), pp. 580–583.

- Idea:
  1. Evolve individual trajectories **stochastically**
  2. Calculate observables by averaging over trajectories

$$\langle A \rangle = \mathbb{E} [\langle \psi(t) | A | \psi(t) \rangle]$$

## Advantages:

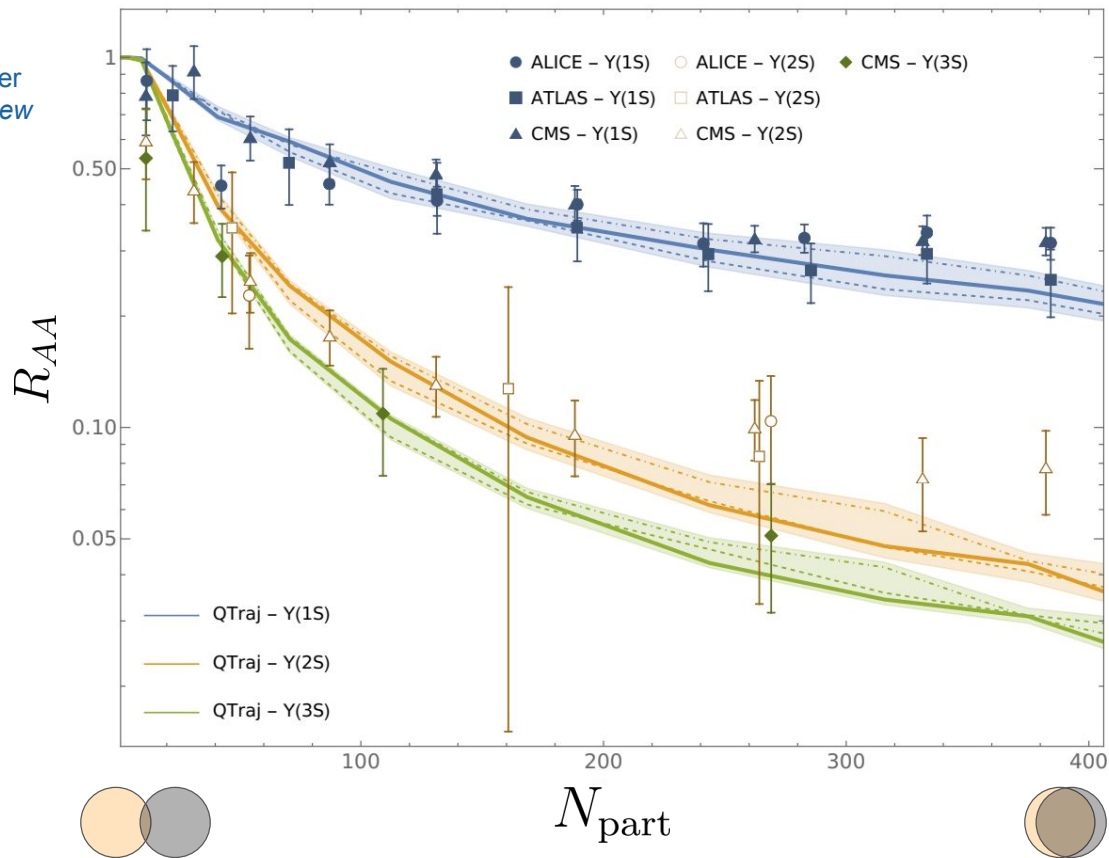
- Evolve vector of size  $N_H$  instead  $N_H^2$  density matrix
- Simulation of individual trajectories is **embarrassingly parallel**



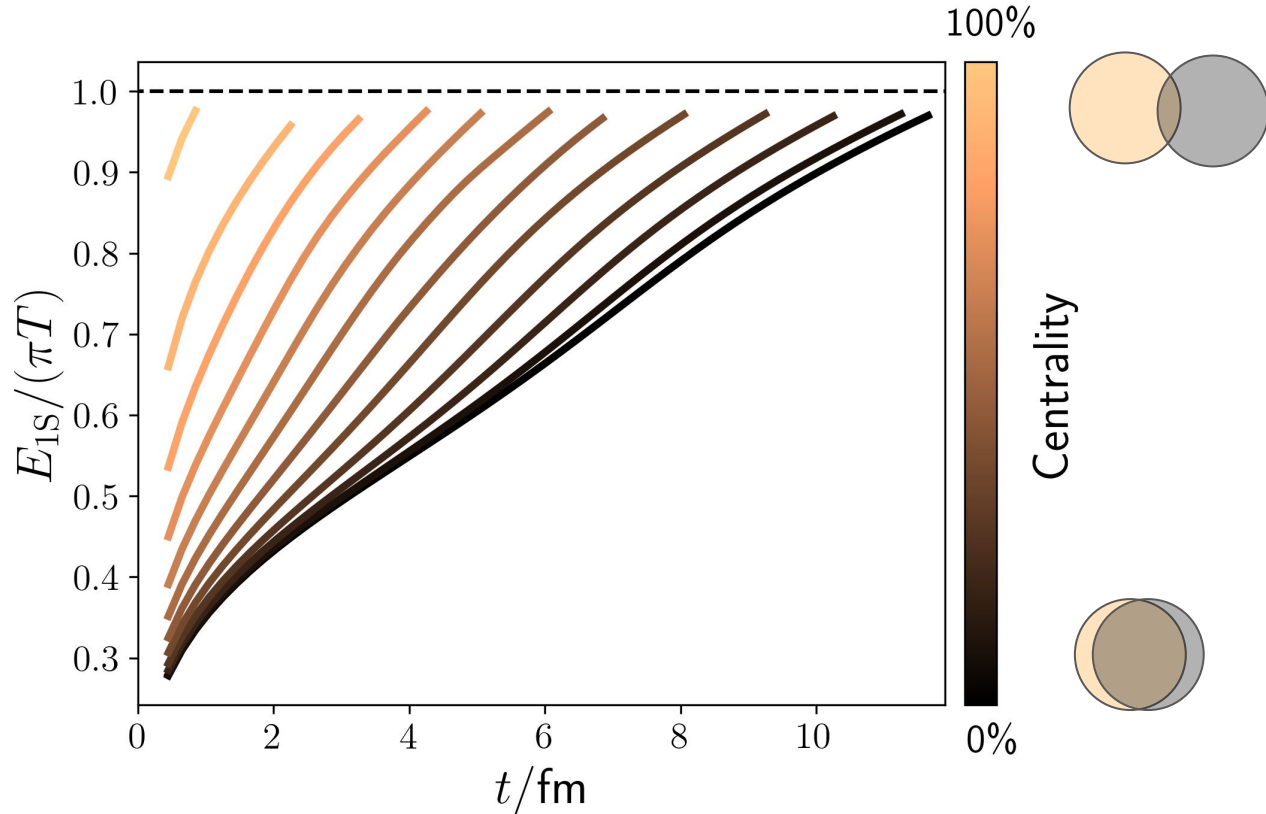
Omar, H. B., Escobedo, M. Á., Islam, A., Strickland, M., Thapa, S., Vander Griend, P., & Weber, J. H. (2022). *Computer Physics Communications*, 273, 108266.

# Overlaps lead to phenomenological predictions

Brambilla, N., Magorsch, T., Strickland, M., Vairo, A., & Vander Griend, P. (2024). *Physical Review D*, 109(11), 114016.



At low temperatures the E/T expansion converges slowly



# The original master equation is not positive

$$\frac{d\rho(t)}{dt} = -i [H, \rho(t)] + \sum_{nm} h_{nm} \left( L_i^n \rho(t) L_i^{m\dagger} - \frac{1}{2} \left\{ L_i^{m\dagger} L_i^n, \rho(t) \right\} \right)$$

$$h = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$L_i^n \propto \frac{g^2}{6N_c} \int_0^\infty ds e^{-ih_us} r_i e^{ih_vs} \langle E_j^a(s) \Omega(s, 0)^{ab} E_j^b(0) \rangle$$

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Non-positive master equation

# Pseudo Lindblad Quantum Trajectories

Becker, T., Netzer, C., &  
Eckardt, A. (2023).  
*Physical Review Letters*,  
131(16), 160401.

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \left( L_+ \rho L_+^\dagger - \frac{1}{2} \{ L_+^\dagger L_+, \rho(t) \} \right) - \left( L_- \rho L_-^\dagger - \frac{1}{2} \{ L_-^\dagger L_-, \rho(t) \} \right)$$

# Pseudo Lindblad Quantum Trajectories

$$\psi_0 \quad s(0) = 1$$



Calculate rates  $r_\sigma$

$$r_\sigma = \frac{||L_\sigma|\psi(t)\rangle||^2}{||\psi(t)\rangle||^2}$$

two jump operators

$$L_+, L_-$$

Becker, T., Netzer, C., &  
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**1.** Calculate rates  $r_\sigma$

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1. Calculate rates  $r_\sigma$

$$r_\sigma = \frac{||L_\sigma|\psi(t)\rangle||^2}{||\psi(t)\rangle||^2}$$

2. With probability  $r_+\delta t$

$$|\psi(t + \delta t)\rangle = \frac{L_+|\psi(t)\rangle}{\sqrt{r_+(t)}}$$

$$s(t + \delta t) = s(t)$$

With probability  $r_-\delta t$

$$|\psi(t + \delta t)\rangle = \frac{L_-|\psi(t)\rangle}{\sqrt{r_-(t)}}$$

$$s(t + \delta t) = -s(t)$$

# Pseudo Lindblad Quantum Trajectories

two jump operators

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$$s(t + \delta t) = -s(t)$$

With probability  $1 - \sum_\sigma r_\sigma \delta t$

$$|\psi(t + \delta t)\rangle = \frac{(1 - i\delta t H_{\text{Heff}})|\psi(t)\rangle}{\sqrt{1 - \sum_\sigma r_\sigma(t)\delta t}}$$

$$s(t + \delta t) = s(t)$$



# Pseudo Lindblad Quantum Trajectories

Becker, T., Netzer, C., &  
Eckardt, A. (2023).  
*Physical Review Letters*,  
131(16), 160401.

$$\rho(t) = \mathbb{E} [s(t) |\psi(t)\rangle \langle \psi(t)|]$$

# Pseudo Lindblad Quantum Trajectories

$$\psi_0 \quad s(0) = 1$$



1. Draw  $p_1 \in [0, 1]$ . While  $R(t) > p_1$  evolve

$$R(t') = \exp \left( - \int_t^{t'} \sum_i r_i(s) ds \right) R(t) \quad |\psi(t')\rangle = \frac{\exp \left( -i \int_t^{t'} H_{\text{eff}}(s) ds \right)}{\sqrt{\exp \left( - \int_t^{t'} \sum_i r_i(s) ds \right)}} |\psi(t)\rangle$$

$$r_i(t) = \frac{||L_i|\psi(t)\rangle||^2}{|||\psi(t)\rangle||^2} \quad s(t') = s(t)$$

2. Draw Jump operator  $i$  with probability  $\propto r_i(t)$  and perform jump

$$|\psi(t)\rangle \leftarrow \frac{L_i|\psi(t)\rangle}{\sqrt{r_i(t)}}$$

3. Flip the sign bit **if the applied jump operator is a negative one**

$$s(t) \leftarrow -s(t)$$

# Pseudo Lindblad Quantum Trajectories

$$\rho(t) = \mathbb{E} [s(t)|\psi(t)\rangle\langle\psi(t)|]$$

$$\bar{s}(t) \rightarrow 0 \quad \text{Sign problem}$$

$$\bar{s}(t) = \exp \left[ -2 \int_0^t \sum_i r_{i,-}(s) ds \right]$$

Minimize negative  
rates

$$r_i(t) = \frac{||L_i|\psi(t)\rangle||^2}{|||\psi(t)\rangle||^2}$$

# Operator optimizations

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \left( L_+\rho L_+^\dagger - \frac{1}{2}\{L_+^\dagger L_+, \rho(t)\} \right) - \left( L_-\rho L_-^\dagger - \frac{1}{2}\{L_-^\dagger L_-, \rho(t)\} \right)$$

# Operator optimizations

$$\begin{pmatrix} L'_+ \\ L'_- \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} L_+ \\ L_- \end{pmatrix}$$

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \left( L_+ \rho L_+^\dagger - \frac{1}{2} \{L_+^\dagger L_+, \rho(t)\} \right) - \left( L_- \rho L_-^\dagger - \frac{1}{2} \{L_-^\dagger L_-, \rho(t)\} \right)$$

# Operator optimizations

$$\begin{pmatrix} L'_+ \\ L'_- \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} L_+ \\ L_- \end{pmatrix}$$

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \left( L'_+ \rho L'^{\dagger}_+ - \frac{1}{2}\{L'^{\dagger}_+ L'_+, \rho(t)\} \right) - \left( L'_- \rho L'^{\dagger}_- - \frac{1}{2}\{L'^{\dagger}_- L'_-, \rho(t)\} \right)$$

$$L'_+(w, \phi) = e^{i\phi/2} \cosh(w) L_+ + e^{-i\phi/2} \sinh(w) L_-$$

$$L'_-(w, \phi) = e^{i\phi/2} \sinh(w) L_+ + e^{-i\phi/2} \cosh(w) L_-$$

# Operator optimizations

Global optimization

$$\operatorname{argmin}_{\omega, \phi} ||L'_-||^2$$

$$||L_\sigma||^2 = \operatorname{Tr} [L_\sigma^\dagger L_\sigma]$$

Local optimization

$$\operatorname{argmin}_{\omega, \phi} \langle \psi | L'^\dagger_- L'_- | \psi \rangle$$

# Operator optimizations

## Global optimization

$$\operatorname{argmin}_{\omega, \phi} ||L'_-||^2$$

$$\phi = \pi - \arg \left[ \operatorname{Tr}(L_+ L_-^\dagger) \right]$$
$$w = \frac{1}{2} \operatorname{arctanh} \left( \frac{2 \operatorname{Tr}(L_+ L_-^\dagger)}{\operatorname{Tr}(L_+ L_+^\dagger) + \operatorname{Tr}(L_- L_-^\dagger)} \right)$$

## Local optimization

$$\operatorname{argmin}_{\omega, \phi} \langle \psi | L_-'^\dagger L'_- | \psi \rangle$$

$$\phi = \pi - \arg \left[ \langle \psi | L_-^\dagger L_+ | \psi \rangle \right]$$
$$w = \frac{1}{2} \operatorname{arctanh} \left( \frac{2 \left| \langle \psi | L_-^\dagger L_+ | \psi \rangle \right|}{||L_+ | \psi \rangle||^2 + ||L_- | \psi \rangle||^2} \right)$$

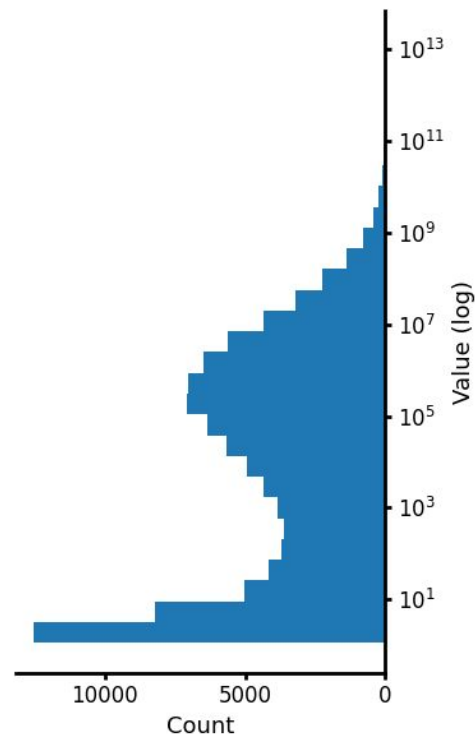
# Because trajectories can have large norm, we are sampling a heavy tailed distribution

$$|\psi(t')\rangle = \frac{\exp\left(-i \int_t^{t'} H_{\text{eff}}(s) ds\right)}{\sqrt{\exp\left(-\int_t^{t'} \sum_i r_i(s) ds\right)}}$$

Norm can grow

$$\text{Tr}[A\rho(t)] = \mathbb{E} [s(t) \langle \psi(t) | A | \psi(t) \rangle]$$

Tailed distribution: Takes many samples to converge



# Truncation

$$\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \left( L_+\rho L_+^\dagger - \frac{1}{2}\{L_+^\dagger L_+, \rho(t)\} \right) - \left( L_-\rho L_-^\dagger - \frac{1}{2}\{L_-^\dagger L_-, \rho(t)\} \right)$$

# Truncation

$$\underbrace{\frac{d}{dt}\rho(t) = -i[H, \rho(t)] \oplus \left( L_+ \rho L_+^\dagger - \frac{1}{2} \{ L_+^\dagger L_+, \rho(t) \} \right) \ominus \left( L_- \rho L_-^\dagger - \frac{1}{2} \{ L_-^\dagger L_-, \rho(t) \} \right)}_{\text{“Lindblad equation”}} \quad \mathcal{O}(E^2/T^2)$$

# Truncation

$$\begin{pmatrix} L'_+ \\ L'_- \end{pmatrix} = \begin{pmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{pmatrix} \begin{pmatrix} L_+ \\ L_- \end{pmatrix}$$

$$\underbrace{\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \left( L'_+ \rho L'^{\dagger}_+ - \frac{1}{2} \{ L'^{\dagger}_+ L'_+, \rho(t) \} \right)}_{\text{“Lindblad equation”}} - \underbrace{\left( L'_- \rho L'^{\dagger}_- - \frac{1}{2} \{ L'^{\dagger}_- L'_-, \rho(t) \} \right)}_{\text{minimal}}$$

“Lindblad equation”

minimal

# Truncation

Global optimization

$$\operatorname{argmin}_{\omega, \phi} ||L'_-||^2$$

Local optimization

$$\operatorname{argmin}_{\omega, \phi} \langle \psi | L'^{\dagger}_- L'_- | \psi \rangle$$

$$\underbrace{\frac{d}{dt}\rho(t) = -i[H, \rho(t)] + \left( L'_+ \rho L'^{\dagger}_+ - \frac{1}{2} \{ L'^{\dagger}_+ L'_+, \rho(t) \} \right)}_{\text{“Lindblad equation”}} - \underbrace{\left( L'_- \rho L'^{\dagger}_- - \frac{1}{2} \{ L'^{\dagger}_- L'_-, \rho(t) \} \right)}_{\text{minimal}}$$

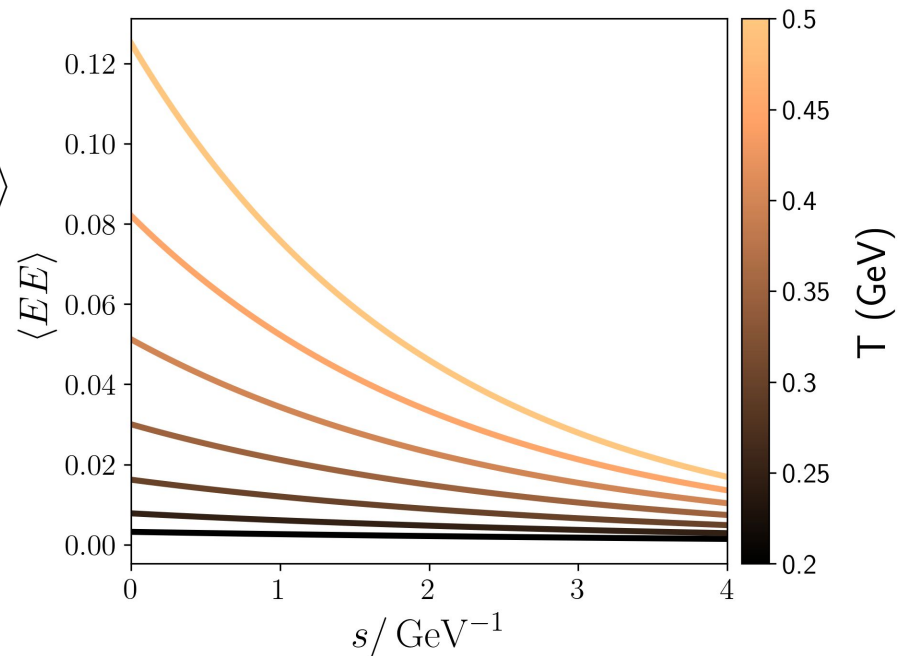
“Lindblad equation”

minimal

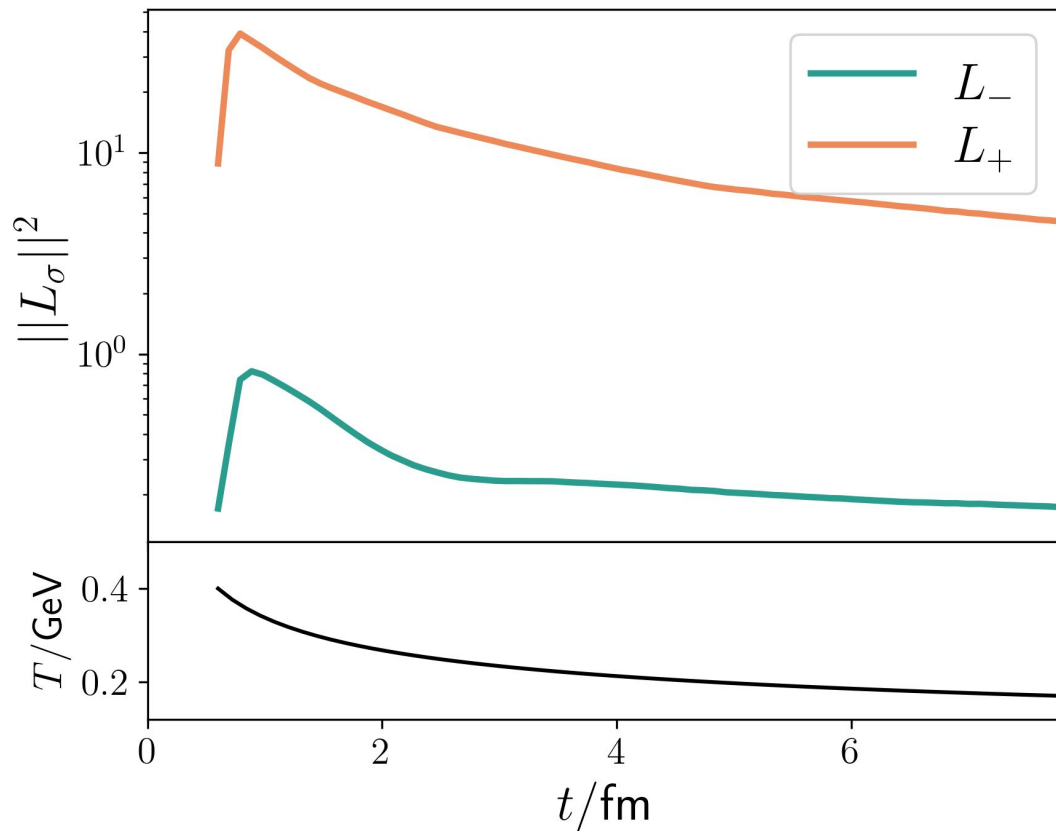
# Numerical study

$$L_i \propto \int_0^\infty ds e^{-ih_us} r_i e^{ih_vs} \langle E_j^a(s) \Omega^{ab}(s, 0) E_j^b(0) \rangle$$

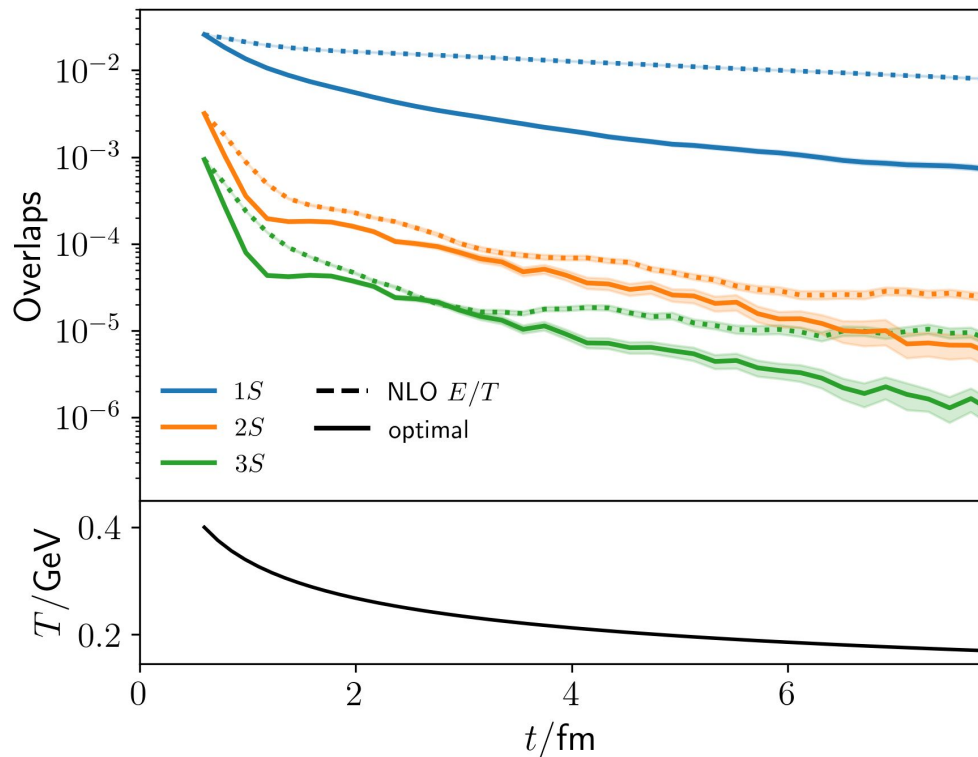
$$\langle E_j^a(s) \Omega^{ab}(s, 0) E_j^b(0) \rangle \propto \frac{\kappa T^4}{2} e^{-sT}$$



# The truncation to the optimal form is efficient

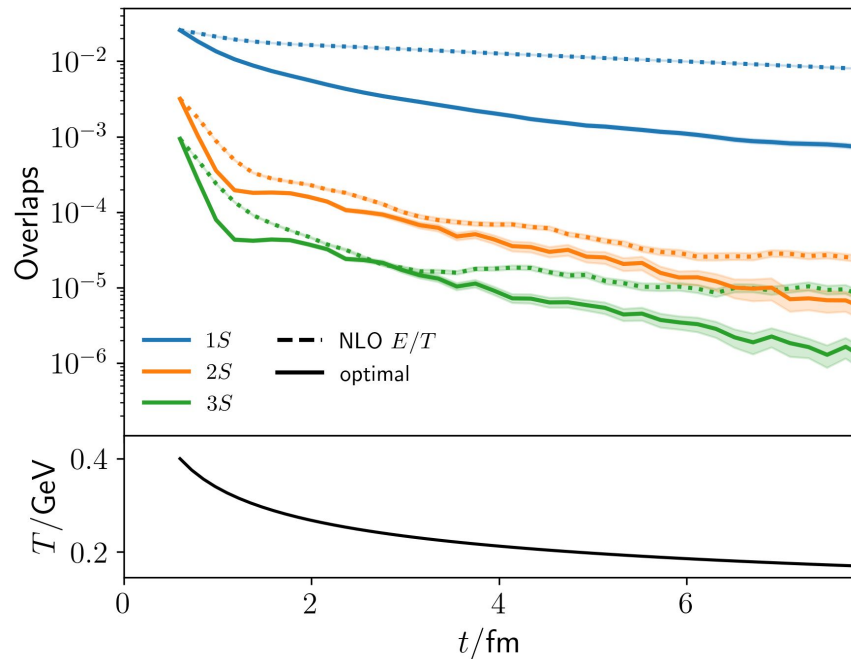


For high temperatures, we find agreement for the 2S and 3S and corrections for the 1S



# Summary

- At low temperatures our master equation is not positive
- Operator optimizations lead to efficient Lindblad truncation
- Efficient simulation of quarkonium dynamics at low temperatures



# Backup



# Connecting to phenomenology

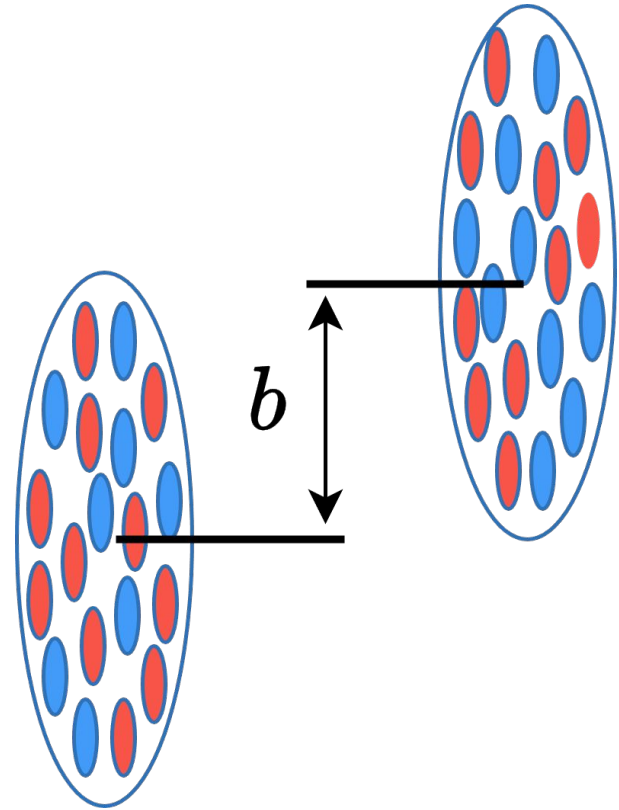
$$R_{AA}^i(c) = \frac{(F \cdot S(c) \cdot \vec{\sigma}_{\text{direct}})^i}{\sigma_{\text{exp}}^i}$$

$$S_{ii} = P_{\text{survival}}(i) = \frac{\langle i | \rho(t_f) | i \rangle}{\langle i | \rho(t_0) | i \rangle}$$

# Connecting to phenomenology

centrality

$$R_{AA}^i(c) = \frac{(F \cdot S(c) \cdot \vec{\sigma}_{\text{direct}})^i}{\sigma_{\text{exp}}^i}$$



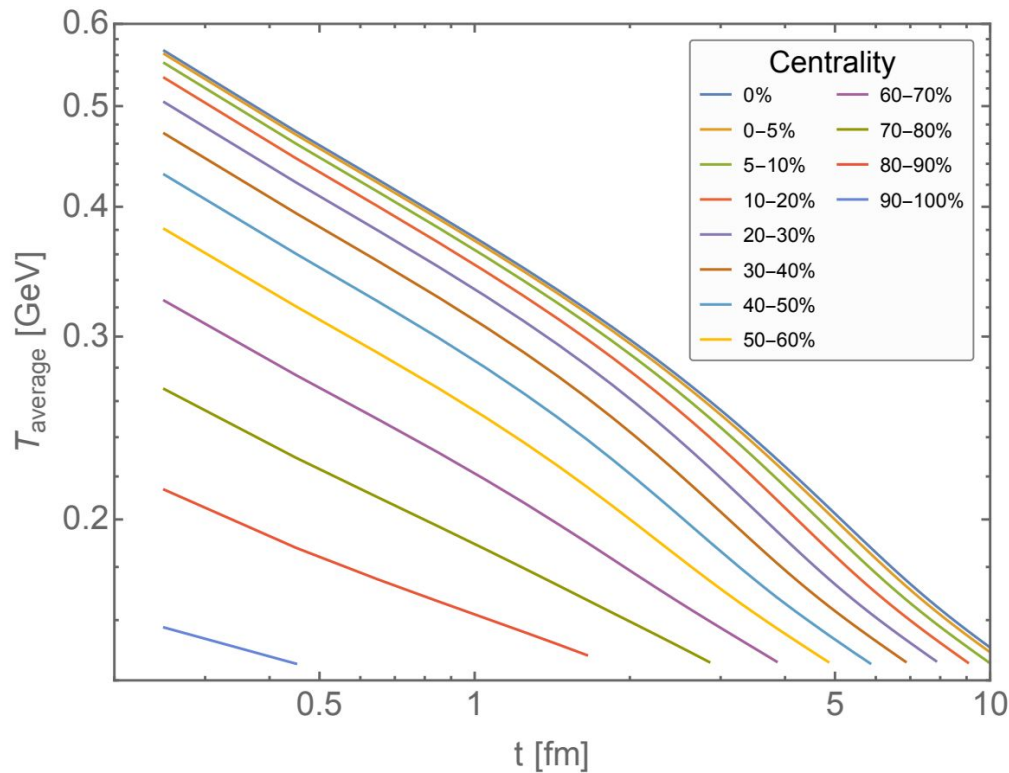
$b$ : Impact parameter

# Connecting to phenomenology

Brambilla, Nora, et al. *Journal of High Energy Physics*  
2021.5 (2021): 1-47.

centrality

$$R_{AA}^i(c) = \frac{(F \cdot S(c) \cdot \vec{\sigma}_{\text{direct}})^i}{\sigma_{\text{exp}}^i}$$



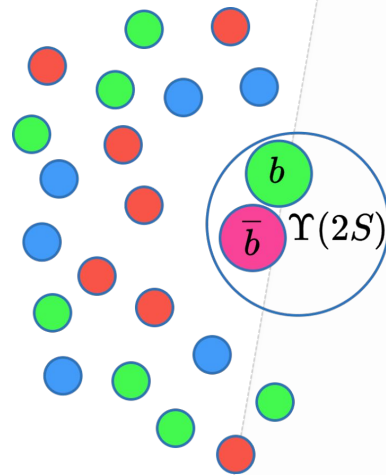
# Connecting to phenomenology

Feeddown matrix

$$R_{AA}^i(c) = \frac{(F \cdot S(c) \cdot \vec{\sigma}_{\text{direct}})^i}{\sigma_{\text{exp}}^i}$$

$$\vec{\sigma}_{\text{direct}} = F^{-1} \vec{\sigma}_{\text{exp}}$$

direct production



Detector

$\mu^+$

$\mu^-$

experimental  
observation

N=128

