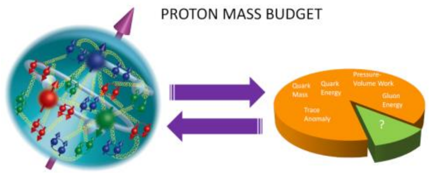
INSTITUTE FOR NUCLEAR THEORY

INT Workshop INT-20R-77
Origin of the Visible Universe: Unraveling the Proton Mass
June 13-17, 2022



Virial theorem and nucleon mass structure

Based on **[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]**

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IP PARIS

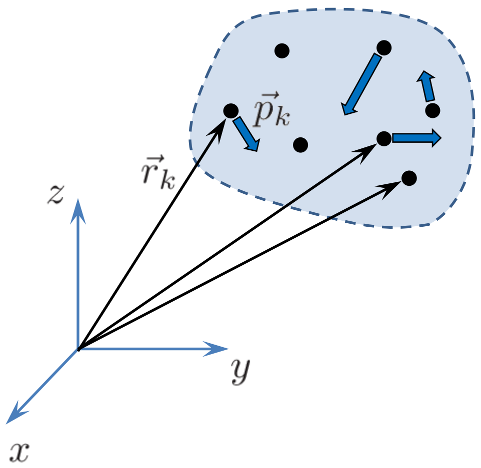
June 14, 2022, INT, Seattle, USA

Outline

- **What is the virial theorem ?**
- **What is its physical meaning ?**
- **What is its relation to mass ?**

Virial theorem (classical point mechanics)

[Clausius, PM40 (1870) 122]
[Goldstein, *Classical Mechanics* (1980)]



$$\vec{P} = \sum_k \vec{p}_k \qquad G = \sum_k \vec{p}_k \cdot \vec{r}_k$$

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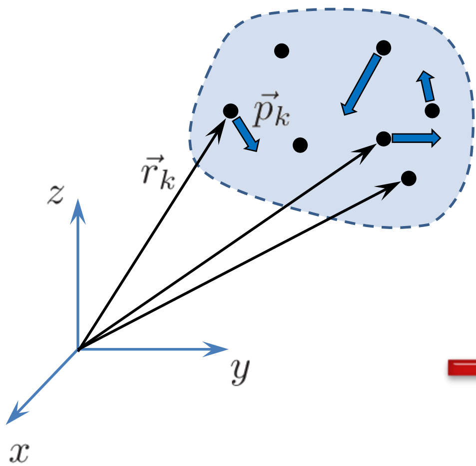

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$$\Rightarrow \quad \frac{\mathrm{d}G}{\mathrm{d}t} = \sum_k \left(\vec{p}_k \cdot \vec{v}_k + \vec{F}_k \cdot \vec{r}_k \right)$$

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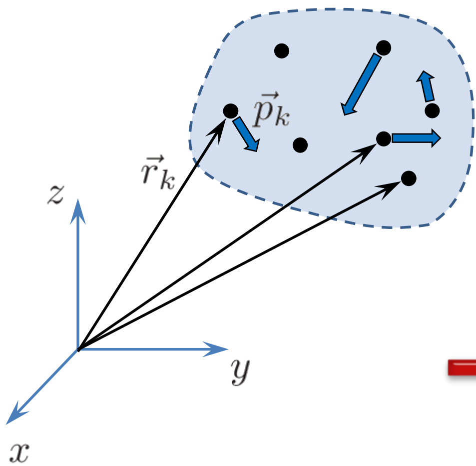

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Time average

$$[\![O]\!] \equiv \lim_{\tau\to\infty} \frac{1}{\tau} \int_0^\tau \mathrm{d}t\, O(t)$$

$$\left[\!\!\left[\frac{\mathrm{d}G}{\mathrm{d}t}\right]\!\!\right] = \lim_{\tau\to\infty} \frac{G(\tau) - G(0)}{\tau} \underbrace{= 0}_{\text{Bound state at rest}}$$

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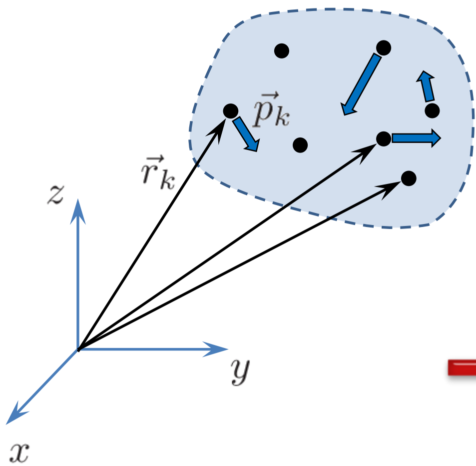

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$$\Rightarrow \boxed{[\![\mathcal{T}]\!] = -\frac{1}{2} \underbrace{\sum_k [\![\vec{F}_k \cdot \vec{r}_k]\!]}_{\text{Virial}}}$$

$$\mathcal{T} = \frac{1}{2} \sum_k m_k \vec{v}_k^{\,2}$$

Vis (latin): force, energy, power

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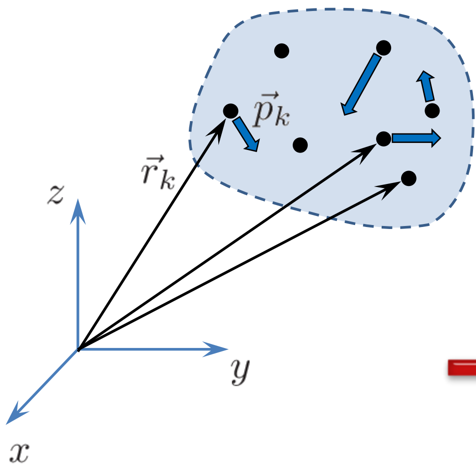

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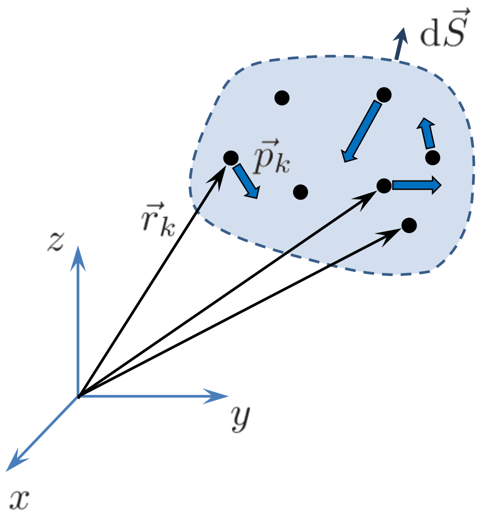
Virial theorem (classical point mechanics)

[Schectman , Good, AJP25 (1957) 219]



Ideal gas

$$-\frac{1}{2}\sum_k \underbrace{[\![\vec{F}_k \cdot \vec{r}_k]\!]}_{\text{Container}} = \frac{1}{2}\int p\,\mathrm{d}\vec{S}\cdot\vec{r}$$

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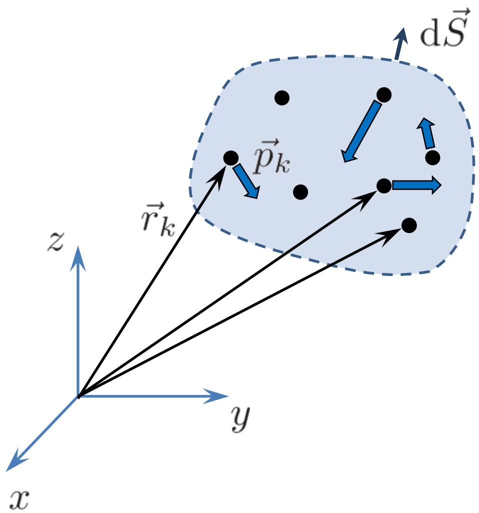

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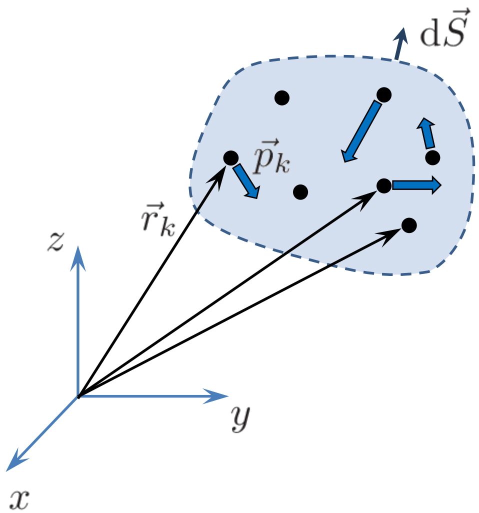

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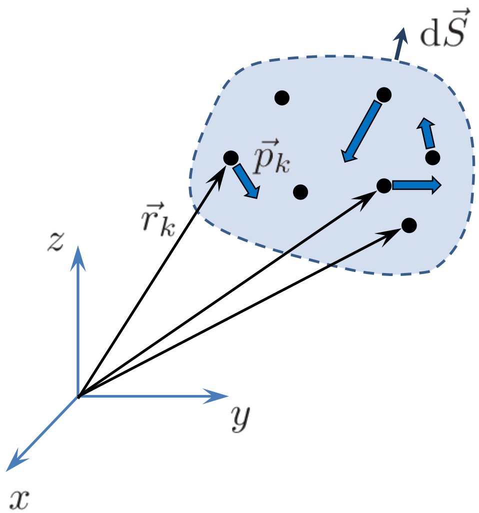

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Virial theorem $$[\![\mathcal{T}]\!] = \frac{3}{2}\,pV$$

Equipartition theorem $$[\![\mathcal{T}]\!] = \frac{3}{2}\,Nk_BT$$

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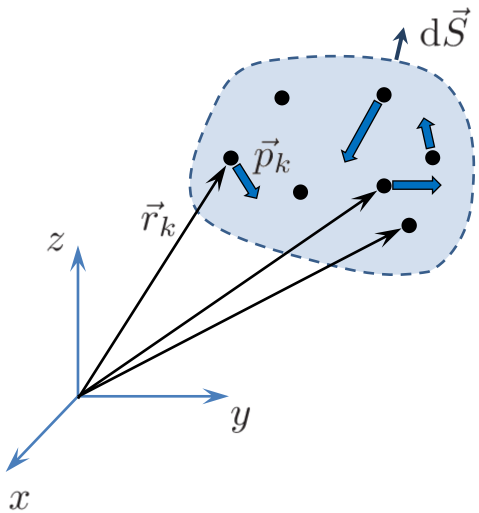

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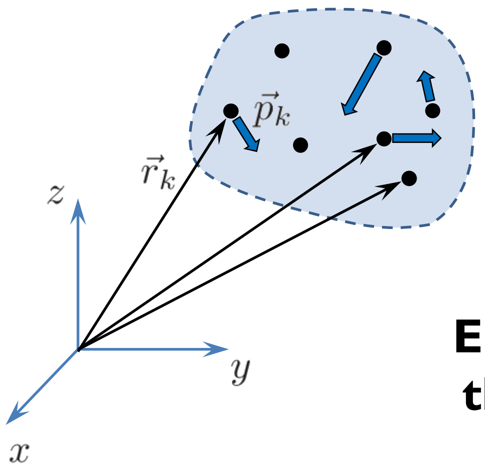
$\boxed{pV = Nk_BT}$

Virial theorem (quantum mechanics)

[Born, Heisenberg, Jordan, ZP35 (1926) 557]
[Killingbeck, AJP38 (1970) 590]



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Erhenfest theorem

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle G\rangle = \frac{1}{i}\langle [G, H]\rangle$$

$$H = \sum_k \frac{\vec{p}_k^{\,2}}{2m_k} + \mathcal{V}(\{\vec{r}_k\})$$

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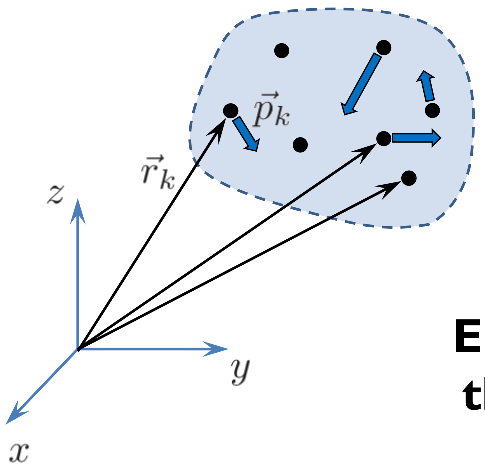

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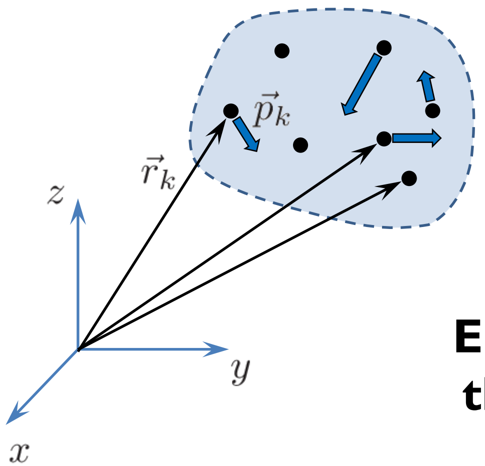

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Normalizable stationary state

$$H|\Psi\rangle = E|\Psi\rangle \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}t}\langle G\rangle = 0$$

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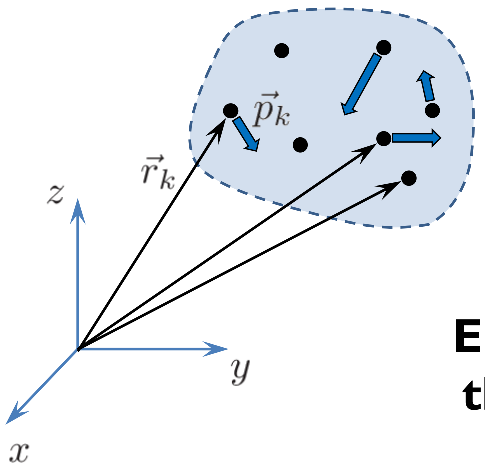

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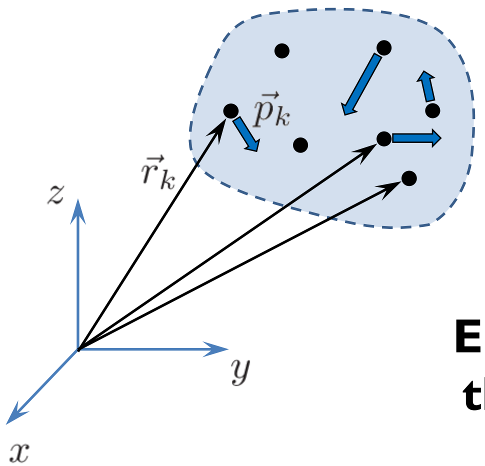

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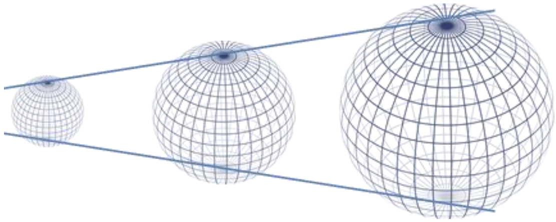
NB:
- For a *superposition* of stationary states, one also averages over time $\left[\!\left[\frac{\mathrm{d}\langle G\rangle}{\mathrm{d}t}\right]\!\right] = 0$
- The system is *at rest* since $\frac{\mathrm{d}}{\mathrm{d}t}\langle\vec{R}_{\mathrm{CM}}\rangle = \vec{0}$

Virial theorem (quantum mechanics)

[Fock, ZP63 (1930) 855]
[Lucha, Schöberl, PRL64 (1990) 2733]



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$$U_D^{-1}\vec{r}_k U_D = \lambda \vec{r}_k, \qquad U_D^{-1}\vec{p}_k U_D = \lambda^{-1}\vec{p}_k$$

$$U_D = e^{-i\kappa G}$$
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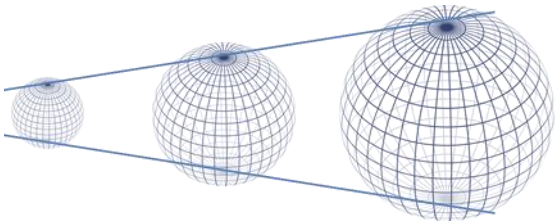

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Variational approach

$$E(\kappa) \equiv \langle \Psi_\kappa | H | \Psi_\kappa \rangle = \langle \Psi | U_D^{-1} H U_D | \Psi \rangle \qquad |\Psi_\kappa\rangle = U_D |\Psi\rangle$$

$|\Psi\rangle$ **is a stationary state with eigenvalue** $E = E(0)$ **if**

$$\frac{\partial E}{\partial \kappa}(0) = \langle \Psi | \frac{\partial (U_D^{-1} H U_D)}{\partial \kappa} \Big|_{\kappa=0} |\Psi\rangle = 0$$

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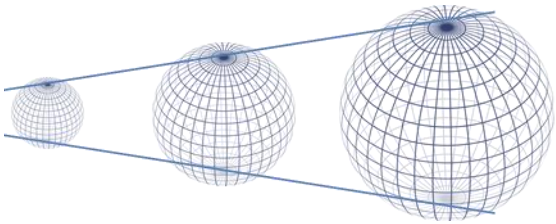

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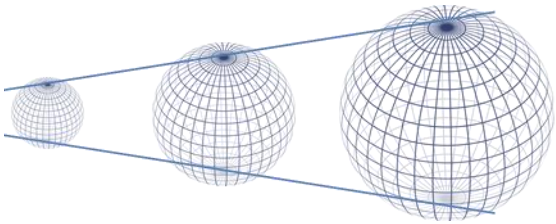

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<u>NB:</u> Bag-like models use this variational approach to determine the mass of the system

Virial theorem (field theory)

[Landau, Lifshitz, *The Classical Theory of Fields* (1951)]
[Deser, PLB64 (1976) 463]
[Dudas, Pirjol, PLB260 (1991) 186]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

$$G = \sum_k \vec{p}_k \cdot \vec{r}_k \quad \Rightarrow \quad \int \mathrm{d}^3x \, T^{0i} x^i$$

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&= \int \mathrm{d}^3x\, (\partial_\mu T^{\mu i}) x^i - \int \mathrm{d}^3x\, (\partial_k T^{ki}) x^i \\
&= \int \mathrm{d}^3x\, \vec{\mathcal{F}} \cdot \vec{x} + \sum_i \int \mathrm{d}^3x\, T^{ii}
\end{aligned}$$

Four-force density $\mathcal{F}^\nu \equiv \partial_\mu T^{\mu\nu}$

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[Dudas, Pirjol, PLB260 (1991) 186]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

$$G = \sum_k \vec{p}_k \cdot \vec{r}_k \quad \rightarrow \quad \int \mathrm{d}^3x\, T^{0i} x^i$$

$$\begin{aligned}\frac{\mathrm{d}G}{\mathrm{d}t} &= \int \mathrm{d}^3x\, \partial_0 T^{0i} x^i \\ &= \int \mathrm{d}^3x\, (\partial_\mu T^{\mu i}) x^i - \int \mathrm{d}^3x\, (\partial_k T^{ki}) x^i \\ &= \int \mathrm{d}^3x\, \vec{\mathcal{F}} \cdot \vec{x} + \sum_i \int \mathrm{d}^3x\, T^{ii}\end{aligned}$$

Four-force density $\quad \mathcal{F}^\nu \equiv \partial_\mu T^{\mu\nu}$

Classical

$$\boxed{[\![\sum_i \int \mathrm{d}^3x\, T^{ii}]\!] = -[\![\int \mathrm{d}^3x\, \vec{\mathcal{F}} \cdot \vec{x}]\!]}$$

Quantum
(stationary state)

$$\boxed{\langle \sum_i \int \mathrm{d}^3x\, T^{ii} \rangle = -\langle \int \mathrm{d}^3x\, \vec{\mathcal{F}} \cdot \vec{x} \rangle}$$

Virial theorem (field theory)

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Isolated system
(von Laue condition)

Quantum
(stationary state)

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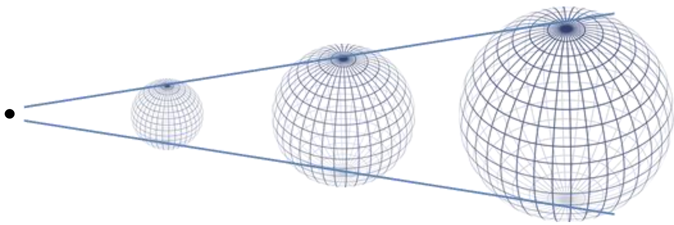
Virial theorem (field theory)

Physical interpretation $\vec{x} \mapsto (1 + \delta\kappa)\vec{x}$



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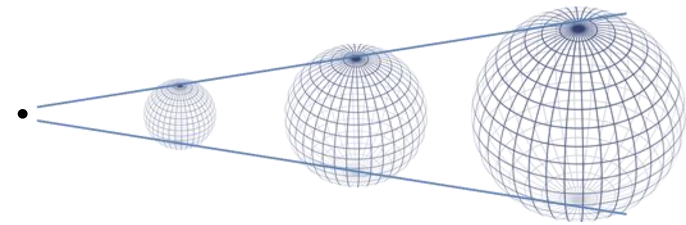

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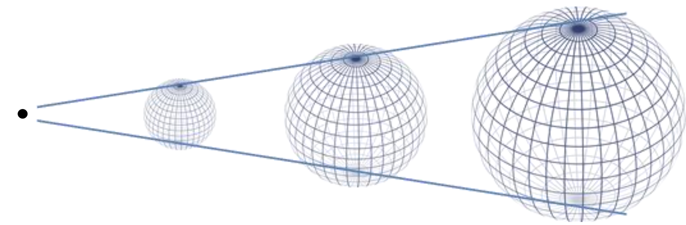

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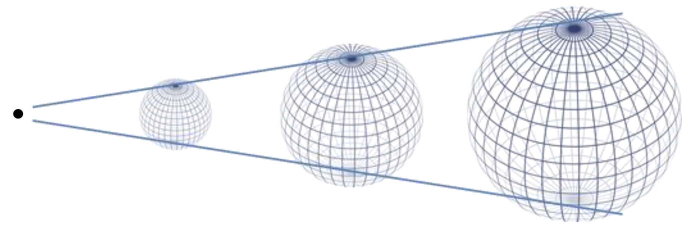

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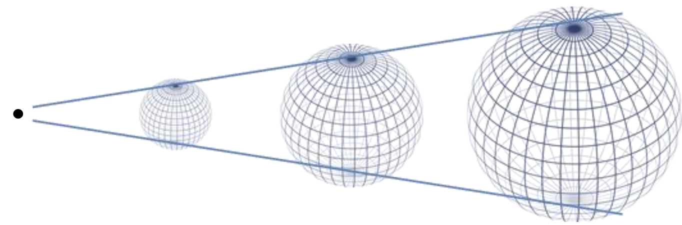

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Energy density | Momentum density

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix}$$

Shear stress

Normal stress (pressure)

Energy flux | Momentum flux

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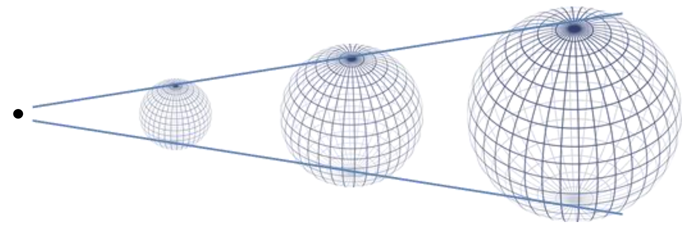

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Energy density (T^{00})

Momentum density (T^{01}, T^{02}, T^{03})

Energy flux (T^{10}, T^{20}, T^{30})

Momentum flux (T^{ij})

Shear stress

Normal stress (pressure)

Virial theorem $\boxed{\langle \delta W \rangle = 0}$ Mechanical equilibrium !

Virial theorem (field theory)

Generalization to spacetime dilatations $\quad j^{\mu}_{D} = T^{\mu\nu} x_{\nu}$

$$D = \int \mathrm{d}^3 x\, j^0_D = Ht - G$$

$$e^{i\kappa D}\phi(x)e^{-i\kappa D} = e^{\kappa d_\phi}\phi(e^{\kappa}x)$$

$$\Leftrightarrow \quad \frac{1}{i}\,[\phi(x), D] = (x_\mu \partial^\mu + d_\phi)\phi(x)$$

$$H = \int \mathrm{d}^3 x\, T^{00}$$

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No new information from temporal dilatations !

Virial theorem (field theory)

To sum up, the virial theorem

1. is a statement about mechanical equilibrium of stationary states under dilatations
2. provides an equality between average values of different quantities at rest
3. does not require isolation of the system

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Generalization to arbitrary spatial deformations **[Parker, PR96 (1954) 1686]**

$$G^{ij} = \int \mathrm{d}^3x\, T^{0i} x^j \quad \Longrightarrow \quad \boxed{\langle \int \mathrm{d}^3x\, T^{ij} \rangle = -\langle \int \mathrm{d}^3x\, \mathcal{F}^i x^j \rangle}$$

Tensor virial theorem

What is mass ?

Formal operator definition $P^\mu P_\mu = M^2$

⚠ Not additive and hence not suited for a mass decomposition

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➡ A mass decomposition is fundamentally an energy decomposition $H = \sum_a H_a$

Consequences of the virial theorem

[Rafelski, PRD16 (1977) 1890]

Example: **Dirac theory with external potential**

$$E = \underbrace{\int \mathrm{d}^3r\, \psi^\dagger \vec{\alpha} \cdot \vec{p}\, \psi}_{\text{Kinetic energy}} + \underbrace{m \int \mathrm{d}^3r\, \psi^\dagger \beta \psi}_{\text{Rest mass energy}} + \underbrace{\int \mathrm{d}^3r\, \psi^\dagger \mathcal{V}(\vec{r}) \psi}_{\text{Potential energy}}$$

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$\mathcal{V}(\vec{r}) \propto r^n$

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Numerical equality between different physical quantities

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The virial theorem simplifies the calculations
BUT spoils the physical picture !

Consequences of the virial theorem

[C.L., EPJC78 (2018) 120]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

For an isolated system in a stationary state (at rest)

Definition of mass $$\langle H\rangle = M$$

Virial theorem $$\langle H\rangle = \left\langle \frac{\mathrm{d}D}{\mathrm{d}t} \right\rangle = \langle \int \mathrm{d}^3x\, T^\mu{}_\mu \rangle$$

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Starting point of trace decomposition

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Decomposition into irreducible representations **of Poincaré group**

$$T^{\mu\nu} = (T^{\mu\nu} - \tfrac{1}{4}\, g^{\mu\nu} T^\alpha{}_\alpha) + \tfrac{1}{4}\, g^{\mu\nu} T^\alpha{}_\alpha$$

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Starting point of Ji's decomposition

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Decomposition into irreducible representations **of Poincaré group**

$$T^{\mu\nu} = (T^{\mu\nu} - \tfrac{1}{4}\, g^{\mu\nu}T^\alpha{}_\alpha) + \tfrac{1}{4}\, g^{\mu\nu}T^\alpha{}_\alpha$$

$$\Rightarrow \underbrace{\langle \int \mathrm{d}^3x\, T^{00}\rangle}_{=M} = \underbrace{\langle \int \mathrm{d}^3x\,(T^{00} - \tfrac{1}{4}\, T^\alpha{}_\alpha)\rangle}_{=\frac{3}{4}M} + \underbrace{\langle \int \mathrm{d}^3x\,\tfrac{1}{4}\, T^\alpha{}_\alpha\rangle}_{=\frac{1}{4}M}$$

Starting point of Ji's decomposition

The virial theorem introduces the stress tensor T^{ij} in the mass decomposition even though it has nothing to do with the notion of mass

$$P^\mu = \int \mathrm{d}^3x\, T^{0\mu}(x)$$

Mass decompositions

Trace decomposition

[Shifman, Vainshtein, Zakharov, PL78B (1978) 443]
[Donoghue, Golowich, Holstein, *Dynamics of the Standard Model* (1992)]
[Kharzeev, PISPF130 (1996) 105]

$$M = \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x \left(\frac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi \right) \rangle}_{M - \sigma_q}$$

Requires 1 independent input

Mass decompositions

Trace decomposition

[Shifman, Vainshtein, Zakharov, PL78B (1978) 443]
[Donoghue, Golowich, Holstein, *Dynamics of the Standard Model* (1992)]
[Kharzeev, PISPF130 (1996) 105]

$$M = \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \big(\tfrac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi\big) \rangle}_{M - \sigma_q}$$

Requires 1 independent input

Energy decomposition

[C.L., EPJC78 (2018) 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]

$$M = \underbrace{\langle \int \mathrm{d}^3x\, (T^{00}_q - \overline{\psi} m \psi) \rangle}_{[A_q(0) + \bar{C}_q(0)]\, M - \sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, T^{00}_g \rangle}_{[A_g(0) + \bar{C}_g(0)]\, M}$$

$$A_q(0) + A_g(0) = 1$$
$$\bar{C}_q(0) + \bar{C}_g(0) = 0$$

$$A_q(0) = \langle x \rangle_q$$
$$A_q(0) + 4\bar{C}_q(0) = c_1 + c_2\, \frac{\sigma_q}{M}$$

Requires 2 independent inputs

Scheme and scale-dependent !

Mass decompositions

Trace decomposition

[Shifman, Vainshtein, Zakharov, PL78B (1978) 443]
[Donoghue, Golowich, Holstein, *Dynamics of the Standard Model* (1992)]
[Kharzeev, PISPF130 (1996) 105]

$$M = \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x \left(\frac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi \right) \rangle}_{M - \sigma_q}$$

Requires 1 independent input

Energy decomposition

[C.L., EPJC78 (2018) 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]

$$M = \underbrace{\langle \int \mathrm{d}^3x\, (T^{00}_q - \overline{\psi} m \psi) \rangle}_{[A_q(0) + \bar{C}_q(0)]\, M - \sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, T^{00}_g \rangle}_{[A_g(0) + \bar{C}_g(0)]\, M}$$

$$A_q(0) + A_g(0) = 1$$
$$\bar{C}_q(0) + \bar{C}_g(0) = 0$$

$$A_q(0) = \langle x \rangle_q$$
$$A_q(0) + 4\bar{C}_q(0) = c_1 + c_2\, \frac{\sigma_q}{M}$$

Scheme and scale-dependent !

Requires 2 independent inputs

Ji's decomposition

[Ji, PRL74 (1995) 1071]
[Ji, PRD52 (1995) 271]

$$M = \underbrace{\langle \int \mathrm{d}^3x\, (\bar{T}^{00}_q - \tfrac{3}{4}\, \overline{\psi} m \psi) \rangle}_{\frac{3}{4}[A_q(0)\, M - \sigma_q]} + \underbrace{\langle \int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle}_{\sigma_q} + \underbrace{\langle \int \mathrm{d}^3x\, \bar{T}^{00}_g \rangle}_{\frac{3}{4} A_g(0)\, M} + \underbrace{\langle \int \mathrm{d}^3x\, \tfrac{1}{4} \left(\frac{\beta(g)}{2g}\, G^2 + \gamma_m \overline{\psi} m \psi \right) \rangle}_{\frac{1}{4}(M - \sigma_q)}$$

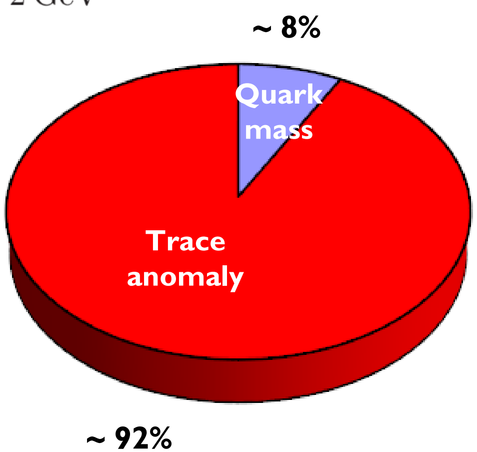
Requires 2 independent inputs

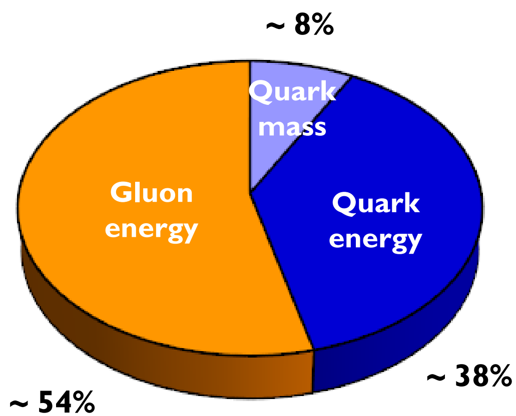
$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha}$$

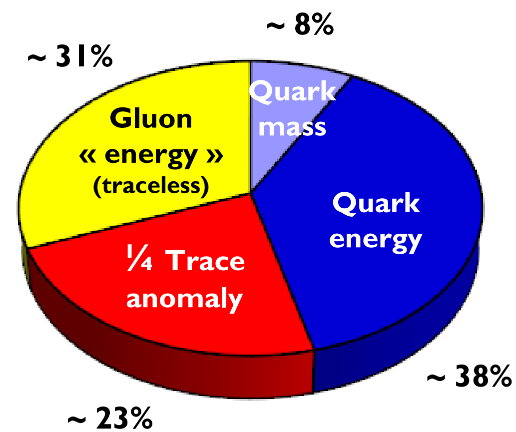
Mass decompositions (in D2 scheme)

Trace decomposition | **Energy decomposition** | **Ji's decomposition**

$\mu = 2\,\mathrm{GeV}$







Relies on virial theorem

Independent of virial theorem

Motivated by virial theorem

Conclusions

- **What is the virial theorem ?**
  - $\langle \sum_i \int d^3x\, T^{ii} \rangle = -\langle \int d^3x\, \vec{\mathcal{F}} \cdot \vec{x} \rangle$
- **What is its physical meaning ?**
  - **Statement about mechanical equilibrium under spatial dilatations**
- **What is its relation to mass ?**
  - **It has nothing to do with mass** (i.e. rest-frame energy)
    but it can be used to create « new » sum rules

Mass (energy) decomposition

$$M = \langle \int d^3x\, T^{00} \rangle = \sum_a \langle \int d^3x\, T^{00}_a \rangle$$

Virial theorem

$$0 = \langle \sum_i \int d^3x\, T^{ii} \rangle = \sum_a \langle \sum_i \int d^3x\, T^{ii}_a \rangle$$

$$M = \sum_a \langle \int d^3x\, [\alpha T^{00}_a + \beta \sum_i T^{ii}_a] \rangle + \sum_a \langle \int d^3x\, [(1-\alpha) T^{00}_a + \gamma \sum_i T^{ii}_a] \rangle$$

Trace decomposition $\alpha = 1, \quad \beta = -1, \quad \gamma = 0$

Ji's decomposition $\alpha = \frac{3}{4}, \quad \beta = \frac{1}{4}, \quad \gamma = -\frac{1}{4}$

Backup slides

Energy-momentum tensor (EMT)

The definition of the EMT is not unique

Canonical EMT (Noether's theorem)

$$\boxed{T^{\mu\nu}_{\text{can}} = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\,\partial^\nu\phi_a - g^{\mu\nu}\mathcal{L}}$$

Usually neither symmetric (for fields with non-zero spin)

nor gauge invariant (when field gradients do not transform covariantly under gauge transformations)

However, P^μ **is unique and gauge invariant !**

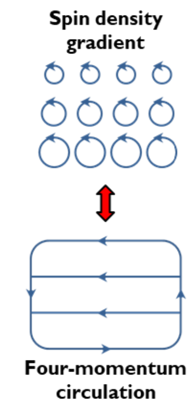
« Improved » EMT (relocalization of energy and momentum)

$$\boxed{T^{\mu\nu}_{\text{new}} = T^{\mu\nu}_{\text{can}} + \partial_\alpha G^{\alpha\mu\nu}} \qquad G^{\alpha\mu\nu} = -G^{\mu\alpha\nu}$$

$$\Rightarrow \quad P^\mu = \int \mathrm{d}^3x\, T^{0\mu}_{\text{new}} = \int \mathrm{d}^3x\, T^{0\mu}_{\text{can}}$$

Energy-momentum tensor (EMT)

	Gauge invariant	Symmetric
Canonical	✗	✗
Kinetic	✓	✗
Belinfante	✓	✓



Gauge invariance is a necessity

Symmetry under exchange of indices is only motivated by GR …

NB: In GR one *assumes* the symmetry of the metric
or the absence of torsion
or the purely orbital form of AM

Kinetic EMT (most natural one in QFT)

$$J^{\mu\alpha\beta} = \underbrace{x^\alpha T^{\mu\beta} - x^\beta T^{\mu\alpha}}_{\text{Orbital}} + \underbrace{S^{\mu\alpha\beta}}_{\text{Intrinsic}}$$

$$\begin{aligned} \partial_\mu T^{\mu\nu} &= 0 \\ \partial_\mu J^{\mu\alpha\beta} &= 0 \end{aligned} \quad \Rightarrow \quad \boxed{T^{\alpha\beta} - T^{\beta\alpha} = -\partial_\mu S^{\mu\alpha\beta}}$$

Four-momentum conservation

Expectation value

$$\langle P^\mu_a\rangle = \frac{\langle p|P^\mu_a|p\rangle}{\langle p|p\rangle} = \underbrace{\frac{\int \mathrm{d}^3r}{(2\pi)^3\delta^{(3)}(\mathbf{0})}}_{=1}\,\frac{\langle p|T^{0\mu}_a(0)|p\rangle}{2p^0}$$

$$\langle p|T^{\mu\nu}_a(0)|p\rangle = 2p^\mu p^\nu A_a(0) + 2M^2 g^{\mu\nu}\bar{C}_a(0)$$

$$\Rightarrow \quad \langle P^\mu_a\rangle = p^\mu A_a(0) + \frac{M^2}{p^0}\,g^{0\mu}\bar{C}_a(0)$$

Not a four-vector !
(unless state is massless)

Light-front version

$p^\pm = (p^0 \pm p^3)/\sqrt{2}$

$$\langle P^\mu_{a,\mathrm{LF}}\rangle = p^\mu A_a(0) + \frac{M^2}{p^+}\,g^{+\mu}\bar{C}_a(0) \quad \underset{g^{++}=0}{\Rightarrow} \quad \langle P^+_{a,\mathrm{LF}}\rangle = \underbrace{A_a(0)}_{=\langle x\rangle_a}\,p^+$$

Deep-inelastic scattering

Four-momentum sum rules

$$p^\mu = \sum_a \langle P^\mu_a\rangle \quad \Rightarrow \quad \boxed{\begin{aligned}\sum_a A_a(0) &= 1\\ \sum_a \bar{C}_a(0) &= 0\end{aligned}}$$

Why two sum rules ?

What is the meaning of $\bar{C}_a(0)$?

Mechanical equilibrium

Physical interpretation is simpler in target rest frame

$$\frac{\langle p_{\text{rest}}|\int \mathrm{d}^3x\, T_a^{\mu\nu}(x)|p_{\text{rest}}\rangle}{\langle p_{\text{rest}}|p_{\text{rest}}\rangle} = M\left(\begin{array}{c|ccc} A_a(0)+\bar{C}_a(0) & 0 & 0 & 0 \\ \hline 0 & -\bar{C}_a(0) & 0 & 0 \\ 0 & 0 & -\bar{C}_a(0) & 0 \\ 0 & 0 & 0 & -\bar{C}_a(0) \end{array}\right)$$

$$\Leftrightarrow \quad \left(\begin{array}{c|ccc} \varepsilon_a & 0 & 0 & 0 \\ \hline 0 & p_a & 0 & 0 \\ 0 & 0 & p_a & 0 \\ 0 & 0 & 0 & p_a \end{array}\right) V$$

$-\bar{C}_a(0)$ **measures the average stress (or pressure) exerted by subsystem** a

Mechanical equilibrium implies $\sum_a p_a = 0 \quad \Rightarrow \quad \sum_a \bar{C}_a(0) = 0$

Mass decomposition

[Shifman, Vainshtein, Zakharov, PL78B (1978) 443]
[Donoghue, Golowich, Holstein, *Dynamics of the Standard Model* (1992)]
[Kharzeev, PISPF130 (1996) 105]

Trace decomposition

Poincaré symmetry tells us that $\langle p|T^{\mu\nu}(0)|p\rangle = 2p^\mu p^\nu$ $\quad \langle p'|p\rangle = 2p^0(2\pi)^3\delta^{(3)}(\vec{p}\,'-\vec{p})$

$$\Rightarrow \boxed{\langle p|T^\mu{}_\mu(0)|p\rangle = 2M^2}$$

or $\dfrac{\langle p|\int \mathrm{d}\mathcal{V}\, T^\mu{}_\mu(x)|p\rangle}{\langle p|p\rangle} = M,\qquad \mathrm{d}\mathcal{V} = u^0\,\mathrm{d}^3x$

Proper volume element

Quark mass and quantum corrections break conformal symmetry

$$T^\mu{}_\mu = \underbrace{\left[\frac{\beta(g)}{2g}\,G^2 + \gamma_m\,\overline{\psi}m\psi\right]}_{\text{Trace anomaly}} + \underbrace{\overline{\psi}m\psi}_{\text{Quark mass matrix}}$$

$$\Rightarrow \underbrace{\langle \textstyle\int \mathrm{d}\mathcal{V}\,\frac{\beta(g)}{2g}\,G^2\rangle}_{\sim 90\%} + \underbrace{\langle \textstyle\int \mathrm{d}\mathcal{V}\,(1+\gamma_m)\,\overline{\psi}m\psi\rangle}_{\sim 10\%} = M$$

$\mu = 2\,\mathrm{GeV}$

~ 90% (to be measured)

~ 10% (measurement to be improved)

$\langle\overline{\psi}m\psi\rangle$ **Nucleon-meson scattering**

$\langle G^2\rangle$ **Near-threshold heavy meson production**

Mass decomposition

Trace decomposition

The physical interpretation of the « quark » and « gluon » contributions is however not so clear …

It is clearer to work in the rest frame

$$\underbrace{\langle \int \mathrm{d}^3x\, T^{\mu}{}_{\mu} \rangle}_{=M} = \underbrace{\langle \int \mathrm{d}^3x\, T^{00} \rangle}_{=M} - \sum_i \underbrace{\langle \int \mathrm{d}^3x\, T^{ii} \rangle}_{=0}$$

Mechanical equilibrium

$$T^{\mu\nu} = \sum_a T_a^{\mu\nu}$$

$$\underbrace{\langle \int \mathrm{d}^3x\, T^{\mu}_{a\mu} \rangle}_{\text{Can be negative !}} = \underbrace{\langle \int \mathrm{d}^3x\, T^{00}_a \rangle}_{=\langle H_a \rangle} - \sum_i \underbrace{\langle \int \mathrm{d}^3x\, T^{ii}_a \rangle}_{\neq 0} \qquad a = q, g$$

Partial pressure-volume work

The « gluon » contribution is enhanced because the gluon pressure-volume work is negative (attractive forces)

Mass decomposition

[C.L., EPJC78 (2018) 120]
[Metz, Pasquini, Rodini, PRD102 (2020) 114042]
[C.L., Metz, Pasquini, Rodini, JHEP11 (2021) 121]

Energy decomposition

Renormalized QCD operators $T^{\mu\nu} = T_q^{\mu\nu} + T_g^{\mu\nu}$

$$T_q^{\mu\nu} = \overline{\psi}\gamma^\mu \tfrac{i}{2}\overleftrightarrow{D}^\nu\psi$$
$$T_g^{\mu\nu} = -G^{\mu\lambda}G^\nu{}_\lambda + \tfrac{1}{4}\,g^{\mu\nu}\,G^2$$

Rest-frame energy

$$\begin{aligned} M &= \langle H_q\rangle + \langle H_g\rangle \\ &= \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\overline{\psi}\gamma^0 iD^0\psi\rangle}_{[A_q(0)+\bar{C}_q(0)]\,M} + \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\rangle}_{[A_g(0)+\bar{C}_g(0)]\,M} \end{aligned}$$

$$A_a(0) = \langle x\rangle_a$$
$$\bar{C}_a(0) = f_a(\langle x\rangle_q, \langle\overline{\psi}m\psi\rangle)$$

Known but scheme and scale-dependent !

Refinement

$$M = \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\overline{\psi}\vec{\gamma}\cdot i\vec{D}\psi\rangle}_{\sim 21\%\,(\overline{\mathrm{MS}}),\ \sim 38\%\,(\mathrm{D2})} + \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\overline{\psi}m\psi\rangle}_{\sim 8\%\,(\overline{\mathrm{MS}}),\ \sim 8\%\,(\mathrm{D2})} + \underbrace{\langle \textstyle\int \mathrm{d}^3x\,\tfrac{1}{2}(\vec{E}^2+\vec{B}^2)\rangle}_{\sim 71\%\,(\overline{\mathrm{MS}}),\ \sim 54\%\,(\mathrm{D2})}$$

$\mu = 2\,\mathrm{GeV}$

Mass decomposition

[Ji, PRL74 (1995)]
[Ji, PRD52 (1995)]

Ji's decomposition

It combines features from both trace and energy decompositions

Step 1 $T^{\mu\nu} = \bar{T}^{\mu\nu} + \hat{T}^{\mu\nu}$

$$\bar{T}^{\mu\nu} = T^{\mu\nu} - \tfrac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha}$$ Twist-2

$$\hat{T}^{\mu\nu} = \tfrac{1}{4}\, g^{\mu\nu} T^{\alpha}{}_{\alpha}$$ Twist-4

Poincaré symmetry ensures that this separation is scheme and scale-independent !

Step 2

$$\bar{T}^{\mu\nu} = \bar{T}^{\mu\nu}_q + \bar{T}^{\mu\nu}_g$$

$$\hat{T}^{\mu\nu} = \hat{T}^{\mu\nu}_m + \hat{T}^{\mu\nu}_\mathrm{a}$$

$$\hat{T}^{\mu\nu}_m = \tfrac{1}{4}\, g^{\mu\nu}\, \overline{\psi} m \psi$$

$$\hat{T}^{\mu\nu}_\mathrm{a} = \tfrac{1}{4}\, g^{\mu\nu} \left[\frac{\beta(g)}{2g}\, G^2 + \gamma_m\, \overline{\psi} m \psi\right]$$

Rest-frame energy

$$\Rightarrow \quad M = \langle \textstyle\int \mathrm{d}^3x\, (\bar{T}^{00}_q - \tfrac{3}{4}\, \overline{\psi} m \psi)\rangle + \langle \textstyle\int \mathrm{d}^3x\, \overline{\psi} m \psi \rangle + \langle \textstyle\int \mathrm{d}^3x\, \bar{T}^{00}_g \rangle + \langle \textstyle\int \mathrm{d}^3x\, \hat{T}^{00}_\mathrm{a} \rangle$$

« Quantum anomalous energy »

Mass decomposition

Ji's decomposition

The physical interpretation of the « quark » and « gluon » contributions is however not so clear …

$$T^{00}_a = \underbrace{\bar{T}^{00}_a}_{= \frac{3}{4}T^{00}_a + \frac{1}{4}\sum_i T^{ii}_a} + \underbrace{\hat{T}^{00}_a}_{= \frac{1}{4}T^{00}_a - \frac{1}{4}\sum_i T^{ii}_a} \qquad a = q, g$$

$$\sum_i \langle \textstyle\int \mathrm{d}^3x\, T^{ii}_a \rangle \neq 0$$

Partial pressure-volume work

Also, it is tempting to write

$$\bar{T}^{00}_q - \frac{3}{4}\,\overline{\psi} m \psi \overset{?}{=} \overline{\psi}\vec{\gamma} \cdot i\vec{D}\psi$$

$$\bar{T}^{00}_g \overset{?}{=} \frac{1}{2}(\vec{E}^2 + \vec{B}^2)$$

… but there is no scheme where both are simultaneously justified !