# Phases of 2d QCD from Qubit Regularization

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# Overview

- Goal: study QFTs through lattice models with finite dimensional local Hilbert space
- Our model: *qubit regularization*[1] of traditional lattice models
- This talk: reproduce the physics of 2d QCD using qubit regularization
  - The critical theory: Wess-Zumino-Witten (WZW) model
  - Phase diagram: gapped/gapless
  - Confinement properties: confined/deconfined
- Methods: strong coupling analysis and tensor network

[1] H. Liu and S. Chandrasekharan, 2022, *Symmetry* arXiv: `2112.02090` (hep-lat)

# 2d QCD Lagrangian

## SU($N$) Yang-Mills theory coupled to $N$ massless Dirac fermions

$$\mathcal{L} = -\frac{1}{2g^2}\,\mathrm{tr}\,F^2 + \bar{\psi}^\alpha \mathrm{i}\not{D}\psi^\alpha + 2\lambda\,\mathrm{tr}(J_L \cdot J_R)$$

$\lambda = 0 \implies$ full chiral symmetry. Generically, $\lambda \neq 0$ on the lattice. For now, we assume $\lambda = 0$.

## Phases of 2d QCD

- Pure Yang-Mills: gapped
- Large quark number: gapless
- Intermediate: bosonization analysis

# Bosonization of 2d QCD

$N$ Dirac fermions without gauge fields: $\mathrm{SO}(2N)$ global symmetry.

Low-energy physics: $\mathrm{SO}(2N)_1$ WZW model.

Gauge $\mathrm{SU}(N)$ symmetry, low-energy physics: $\mathrm{SO}(2N)_1/\mathrm{SU}(N)_1$ coset WZW model.

Coset WZW model is gapped if and only if $c = 0$.[2]

- $G_k$ WZW model central charge $c(G_k) = \frac{k\,\dim(G)}{k+h^\vee}$
- $G_k/H_{k'}$ WZW model central charge $c = c(G_k) - c(H_{k'})$

[2] D. Delmastro et al., 2023, *JHEP* arXiv: `2108.02202` (hep-th)

# Examples

- $\mathrm{SU}(N)$ gauge theory with $N$ massless Dirac fermions:

$$c = c(\mathrm{SO}(2N)_1) - c(\mathrm{SU}(N)_1) = N - (N-1) = 1$$

For $N = 2$, $\mathrm{SO}(4) \cong \mathrm{SU}(2)_s \times \mathrm{SU}(2)_c$, coset is $\mathrm{SU}(2)_1$ WZW model in the charge sector.

- $\mathrm{U}(1)$ gauge theory with a charge $q$ Dirac fermion:

$$c = c(\mathrm{U}(1)_1) - c(\mathrm{U}(1)_{q^2}) = 1 - 1 = 0$$

- $\mathrm{SU}(N)$ gauge theory with $N^2 - 1$ massless adjoint Majorana fermions:

$$c = c(\mathrm{SO}(N^2-1)_1) - c(\mathrm{SU}(N)_N) = \frac{1}{2}(N^2-1) - \frac{1}{2}(N^2-1) = 0$$

# Traditional Hamiltonian lattice gauge theory

## Kogut-Susskind Hamiltonian

$$H = t \sum_{\langle i,j \rangle} (c_i^{\alpha\dagger} U_{ij}^{\alpha\beta} c_j^{\beta} + \text{H.c.}) + \frac{g^2}{2} \sum_{\langle i,j \rangle} (L_{ij}^{a2} + R_{ij}^{a2}) - \frac{1}{4g^2} \sum_{\square} (W_\square + W_\square^\dagger)$$

- Gauge generator: $G_i^a = c_i^{\alpha\dagger} T_{\alpha\beta}^a c_i^\beta + L_{i,i+1}^a + R_{i-1,i}^a$
- Gauge invariant states $|\psi\rangle$ satisfy $G_i^a |\psi\rangle = 0 \; \forall i, a \iff \sum_a (G_i^a)^2 |\psi\rangle = 0 \; \forall i.$
- Single gauge link Hilbert space: $\mathcal{H}_{ij} \cong L^2(G)$ for gauge group $G$

# Qubit regularization

- $|g\rangle \in L^2(G)$, where $g \in G$
- Under $G_L \times G_R$, $|g\rangle \mapsto |h_L^{-1} g h_R\rangle \in L^2(G)$
- As a representation of $G_L \times G_R$, $L^2(G)$ decomposes into:

$$L^2(G) = \bigoplus_{\lambda \in \hat{G}} V_\lambda \otimes V_\lambda^* \quad \text{(Peter-Weyl theorem)}$$

- Qubit regularization: project to $Q \subseteq \hat{G}$ irreps with projector $\Pi_Q$:

$$\mathcal{H}_Q := \bigoplus_{\lambda \in Q} V_\lambda \otimes V_\lambda^*$$

# Regularization schemes for $G = \mathrm{SU}(N)$

$\widehat{\mathrm{SU}(N)}$: Young diagrams with at most $N-1$ rows.

$$Q = \left\{\circ,\ \Box,\ \begin{array}{c}\Box\\[-2pt]\Box\end{array},\ \begin{array}{c}\Box\\[-2pt]\Box\\[-2pt]\Box\end{array},\ \cdots,\ \overline{\begin{array}{c}\Box\\[-2pt]\Box\end{array}},\ \overline{\Box}\right\}$$

Reasons

- Single flavor fermion representations: easily form singlets with fermions
- Contains all $N$-ality: string tensions at large distance are dictated by $N$-ality
- Smallest quadratic Casimir among each $N$-ality: minimize $\frac{g^2}{2}\left(L^{a2} + R^{a2}\right)$
- In $g^2 \to \infty$ limit: same physics as traditional theory

If we are only interested in deep IR physics of fundamental quarks in 2d:

$$\bar{Q} = \{\circ,\ \Box,\ \overline{\Box}\} = \{\mathbf{1},\ \mathbf{N},\ \bar{\mathbf{N}}\}$$

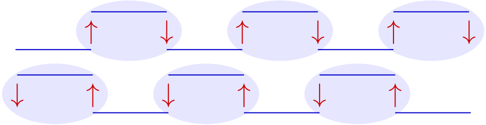
# $\mathrm{SU}(2)$

$$H = \frac{g^2}{2}\sum_{\langle i,j\rangle}(L_{ij}^{a2} + R_{ij}^{a2}) + t\sum_{\langle i,j\rangle}(c_i^{\alpha\dagger}U_{ij}^{\alpha\beta}c_j^{\beta} + \mathrm{H.c.}) + U(n_\uparrow - \frac{1}{2})(n_\downarrow - \frac{1}{2})$$

- Link Hilbert space: $\circ \oplus \square$
- $g^2 > 0$: prefer $\circ$ link over $\square$ link
- $U > 0$: prefer spin states $\square$ ($\uparrow$ and $\downarrow$) over charge states $\circ$ ($\uparrow\downarrow$ and empty)

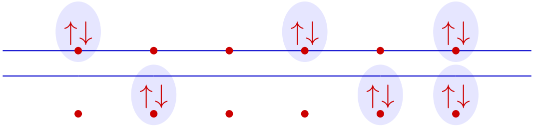
# Phase analysis

- $U \to \infty$: Only spin sector survives, only two states, gapped

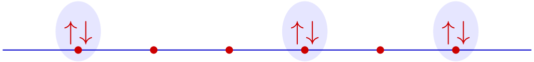


  (Cannot be trivially gapped due to 't Hooft anomaly)

- $U \to -\infty$: Only charge sector survives, spin chain physics, gapless



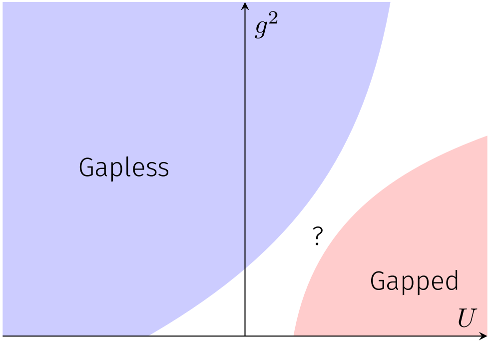
- $g^2 \to \infty$: Only charge sector survives, gapless



Traditional theory:

- Infinitely many states, but when $g^2 > 0$, all other states are separated with a finite gap
- Infinitely many link sectors, but when $g^2 > 0$, all other link sectors are separated with a finite gap
- Same

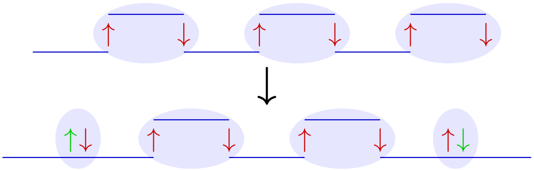
# Phase diagram



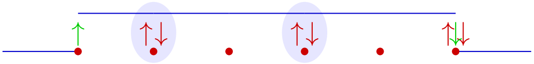
# Confinement analysis

Put two test quarks and pull them apart, see how the energy changes:

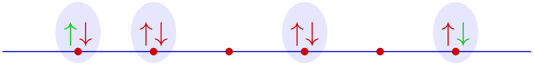
- $U \to \infty$: exchange singlet link with doublet link, deconfined



- $U \to -\infty$: Raise all links in-between from singlet to doublet, confined

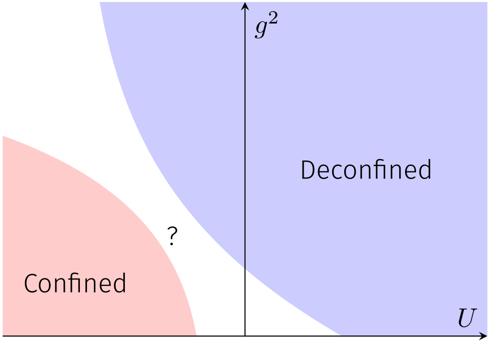


- $g^2 \to \infty$: Screened by quarks, deconfined

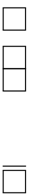


Traditional theory: same analysis, same results.

# Confinement diagram



Fermion local Hilbert space:



| State | | Young tableau | $k_F$($N$-ality) |
|---|---|---|---|
| $\lvert 0\rangle$ | $\longleftrightarrow$ | $\circ$ | $0$ |
| $c^{\alpha_1\dagger}\lvert 0\rangle$ | $\longleftrightarrow$ | | $1$ |
| $c^{\alpha_1\dagger}c^{\alpha_2\dagger}\lvert 0\rangle$ | $\longleftrightarrow$ | | $2$ |
| $\vdots$ | | | |
| $c^{\alpha_1\dagger}\cdots c^{\alpha_{N-1}\dagger}\lvert 0\rangle$ | $\longleftrightarrow$ | | $N-1$ |
| $c^{\alpha_1\dagger}\cdots c^{\alpha_N\dagger}\lvert 0\rangle$ | $\longleftrightarrow$ | $\bar{\circ}$ | $N\equiv 0$ |

Gauss' law on each site:

$$-k_L + k_F + k_R \equiv 0 \mod N$$

Quadratic Casimir for $N$-ality $k$ link:

$$c_2(k) = \frac{N+1}{N}k(N-k)$$

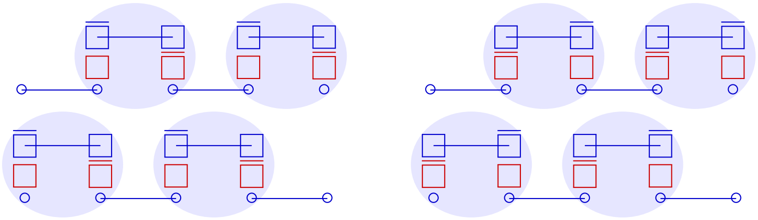
Gauge invariant configurations for any $k_F$ that minimize $c_2(k_L) + c_2(k_R)$:

$$k_L = 0 \text{ or } k_R = 0$$

# Phase analysis

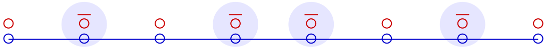
$U$ coupling gaps two fermion singlets. Analysis is similar, for example,

- $U \to \infty$: Ground state is four-fold degenerate



  When $g^2 > 0$, all other states will be separated by a finite energy gap.

- $U \to -\infty$ or $g^2 \to \infty$: Only two fermion singlet survives, spin chain physics, gapless

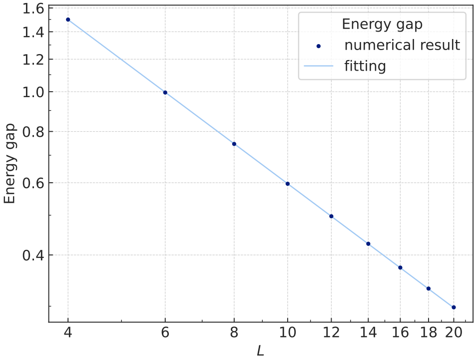


# Tensor network implementation

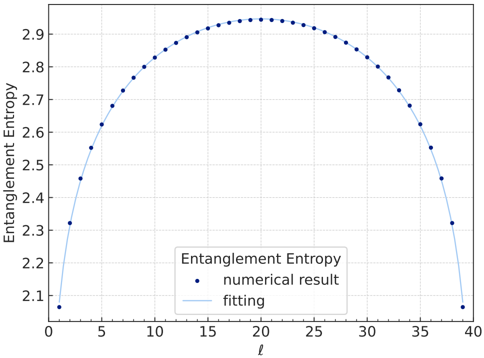
- ITensor[3]
- Gauge invariance: Penalty on $\sum_i \langle\psi| \sum_a (G_i^a)^2 |\psi\rangle$
- Test quark on site $i$: Penalty on $\left(\langle\psi| \sum_a (G_i^a)^2 |\psi\rangle - \langle q_i| \sum_a (G_i^a)^2 |q_i\rangle\right)^2$

[3] M. Fishman et al., 2022, *SciPost Phys. Codebases*

# Gapless at $g^2 = 0, U = 0$




$$\Delta E = \frac{6.03}{L}$$




$$S = \frac{c}{3} \log\left(\frac{L}{\pi} \sin\frac{\pi \ell}{L}\right) + \text{const.} \quad (c \cong 1.02)$$

# Marginal operator, level crossing and critical point

- $\mathrm{SU}(2)_1$ WZW has $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry
  Lowest 5 states: $(s_L, s_R) = (0,0)$ and $(\frac{1}{2}, \frac{1}{2})$
- On the lattice: chiral symmetry is broken
  $\lambda J_L \cdot J_R$ is allowed, can be tuned by $U$

$$\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R \xrightarrow{\text{broken}} \mathrm{SU}(2)_{\mathrm{diag}}$$

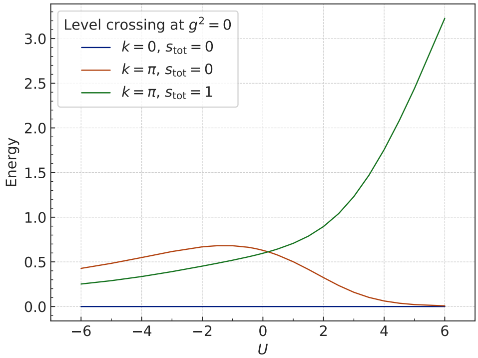
$$(s_L, s_R) = (\frac{1}{2}, \frac{1}{2}) \longrightarrow s_{\mathrm{tot}} = 1, 0$$

$$\langle J_L \cdot J_R \rangle = \frac{1}{2}\langle (J_L + J_R)^2 - J_L^2 - J_R^2 \rangle$$

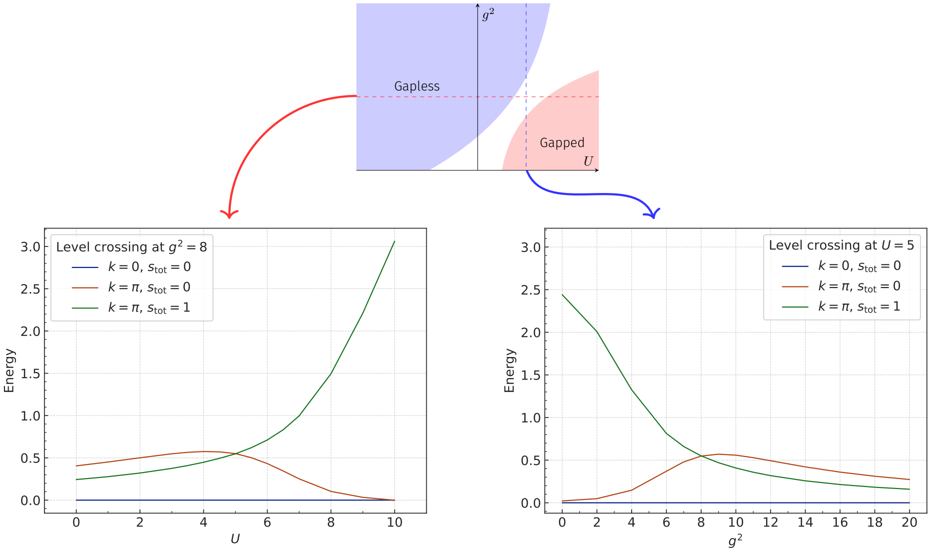
$$= \frac{1}{2}\left(s_{\mathrm{tot}}(s_{\mathrm{tot}}+1) - s_L(s_L+1) - s_R(s_R+1)\right)$$

$\lambda$ is marginal, $\beta$-function:

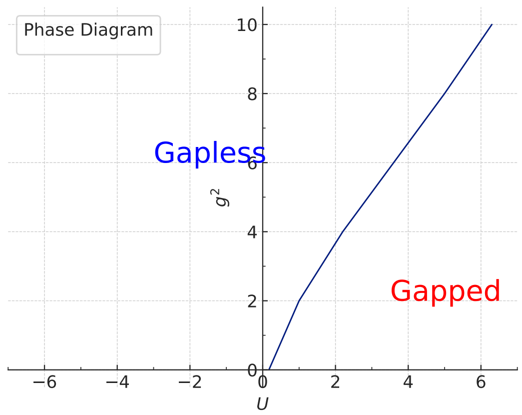
$$\frac{\mathrm{d}\lambda}{\mathrm{d}\mu} = -\frac{1}{2\pi}\lambda^2$$



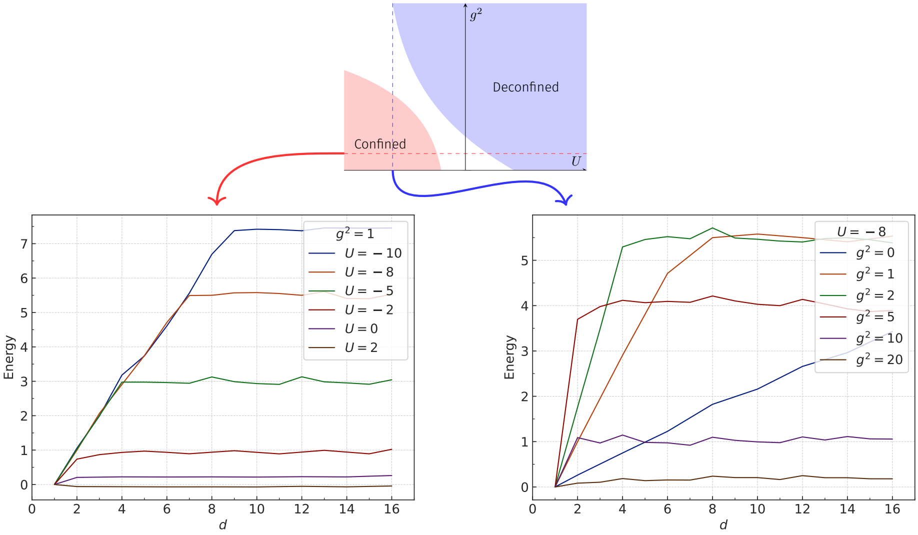
# Phase diagram



# Phase boundary



# Confinement diagram



# Conclusions

- Reproduced the 2d QCD physics (gapped/gapless, confined/deconfined) using finite-dimensional local Hilbert space
  - At $g^2 = 0$ and $U = 0$, the low-energy physics is $\mathrm{SU}(2)_1$ WZW model perturbed by a tiny marginally irrelevant operator
  - $g^2$ term (electric field term) is marginally irrelevant; $g^2 = 0$ exhibits weak confinement $\implies$ Lattice $g^2 = 0$ corresponds to continuum $g^2 \gtrsim 0$
- Suggests qubit regularization as a promising method even in higher dimension and for more complex gauge theories

Thanks for attention!