D meson R_{AA} and v_2 in a Boltzmann transport approach

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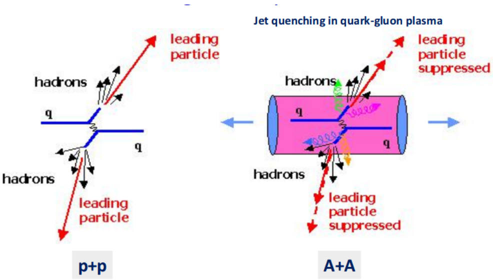
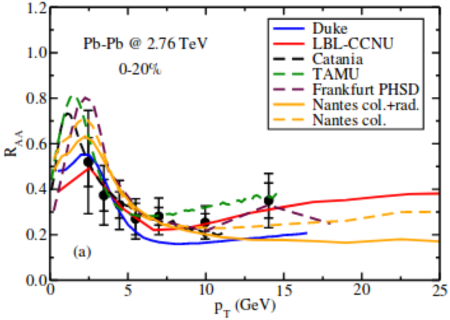
Heavy Qurak : Hard Probe in QGP



- Ideal probe of QGP.
  - $m_Q \gg T_{QGP}$
  - $m_Q \gg \Lambda_{\mathrm{QCD}}$
- Explore the transport properties of QGP through the energy loss of heavy quarks.
  - Describe D meason R_{AA} and v_2
  - Extract $\hat{q}$ and diffusion coefficient D_s

Heavy quark transport models



Shanshan Cao et.al Phys. Rev. C 99, 054907 (2019)

- Two common transport approaches: Langevin and Boltzmann transport models.
- LBT model
  - Elastic scattering: Leading order pQCD
  - Inelastic scattering: high twist model
- Catania QPM
  - Quasi Particle model
  - Elastic scattering

QLBT model motivation

- Our QLBT model combines the linear Boltzmann Transport (LBT) model and quasi particle model(QPM)
  - Elastic process: leading order pQCD amplitude for elastic process, the Quasi Particle model for equation of state of QGP.
  - Inelastic process: high twist model.

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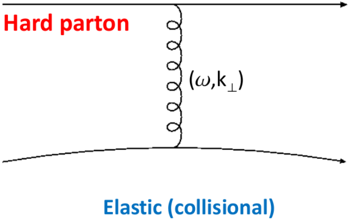
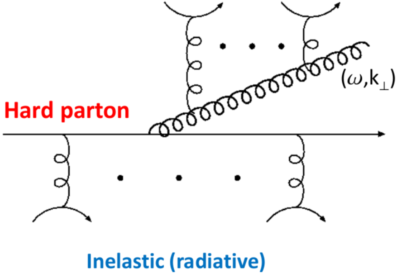
LBT model

In the LBT model [1], the evolution of the phase space distribution f_1 of a given parton (denoted as "1" below) is described using the Boltzmann equation as follows:

$$p_1 \cdot \partial f_1(x_1, p_1) = E_1 \left(C_{el}[f_1] + C_{inel}[f_1]\right),$$

where C_{el} and C_{inel} are collision integrals arising from elastic and inelastic processes.





[1]Zhu, Wang, PRL 2013; He, Luo, Wang, Zhu, PRC 2015; Cao, Tan, Qin, Wang, Phys.Rev.C 94 (2016) 1, 014909; Phys.Lett.B 777 (2018) 255-259

LBT model

$$C_{el}[f_1] \equiv \int d^3k \left[w\left(\vec{p}_1 + \vec{k}, \vec{k}\right) f_1\left(\vec{p}_1 + \vec{k}\right) - w\left(\vec{p}_1, \vec{k}\right) f_1(\vec{p}_1) \right]$$

where $w\left(\vec{p}_1, \vec{k}\right)$ denotes the transition rate for parton "1" from the momentum state $\vec{p}_1$ to $\vec{p}_1 - \vec{k}$. Elastic Scattering ($1 + 2 \rightarrow 3 + 4$ process):

$$w\left(\vec{p}_1, \vec{k}\right) \equiv \sum_{2,3,4} w_{12\rightarrow 34}\left(\vec{p}_1, \vec{k}\right)$$

$$w_{12\rightarrow 34}\left(\vec{p}_1, \vec{k}\right) = \gamma_2 \int \frac{d^3p_2}{(2\pi)^3} f_2(\vec{p}_2) \left[1 \pm f_3\left(\vec{p}_1 - \vec{k}\right)\right] \left[1 \pm f_4\left(\vec{p}_2 + \vec{k}\right)\right]$$

$$\times v_{\mathrm{rel}} d\sigma_{12\rightarrow 34}\left(\vec{p}_1, \vec{p}_2 \rightarrow \vec{p}_1 - \vec{k}, \vec{p}_2 + \vec{k}\right)$$

where the summation runs over all flavors in all possible "$1 + 2 \rightarrow 3 + 4$" channels,

LBT model

The elastic scattering rate for parton "1" through a given channel can be obtained by integrating the transition rate over the exchange momentum $\vec{k}$:

$$\Gamma_{12\to34}(\vec{p}_1) = \int d^3k w_{12\to34}\left(\vec{p}_1, \vec{k}\right) = \frac{\gamma_2}{2E_1}\int\frac{d^3p_2}{(2\pi)^3 2E_2}\int\frac{d^3p_3}{(2\pi)^3 2E_3}\int\frac{d^3p_4}{(2\pi)^3 2E_4}$$
$$\times f_2(\vec{p}_2)[1\pm f_3(\vec{p}_3)][1\pm f_4(\vec{p}_4)]S_2(s,t,u)$$
$$\times (2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)|M_{12\to34}|^2,$$

where γ_2 is the spin-color degeneracy factor of parton.

- The leading order pQCD matrix elements are taken for $|M_{12\to34}|^2$ (Qq $\to$ Qq, Qg $\to$ Qg).
- Kinematic cut: $S_2(s,t,u) = \theta(s \geq 2\mu_D^2)\theta(t \leq -\mu_D^2)\theta(u \leq -\mu_D^2)$

LBT model

Using the δ-function and the azimuthal angular symmetry with respect to the $\vec{p}_1$ direction, scattering rate can be reduced to:

$$\Gamma_{12\to34}(\vec{p}_1, T) = \frac{\gamma_2}{16E_1(2\pi)^4}\int dE_2 d\theta_2 d\theta_4 d$$

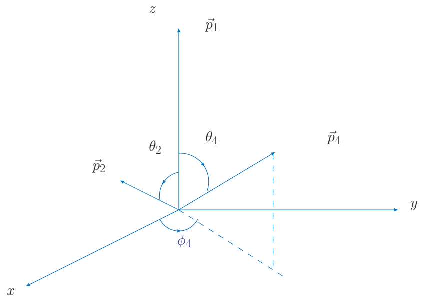
$$\times f_2(E_2, T)\left[1 \pm f_4(E_4, T)\right] S_2(s, t, u$$

$$\times \frac{E_2 E_4 \sin\theta_2 \sin\theta_4}{E_1 - |\vec{p}_1|\cos\theta_4 + E_2 - E_2\cos\theta_{24}}$$

where

$\cos\theta_{24} = \sin\theta_2 \sin\theta_4 \cos\phi_4 + \cos\theta_2 \cos\theta_4$

$$E_4 = \frac{E_1 E_2 - p_1 E_2 \cos\theta_2}{E_1 - p_1\cos\theta_4 + E_2 - E_2\cos\theta_{24}}.$$



LBT model

For convenience, we use more general form to define the following notation:

$$\begin{aligned}\langle\langle X(\vec{p}_1, T)\rangle\rangle = &\sum_{12\to34} \frac{\gamma_2}{16E_1(2\pi)^4} \int dE_2 d\theta_2 d\theta_4 d\phi_4 \\ &\times X(\vec{p}_1, T) f_2(E_2, T)\left[1 \pm f_4(E_4, T)\right] S_2(s,t,u) \\ &\times \left|\mathcal{M}_{12\to34}\right|^2 \frac{E_2 E_4 \sin\theta_2 \sin\theta_4}{E_1 - |\vec{p}_1|\cos\theta_4 + E_2 - E_2\cos\theta_{24}}.\end{aligned}$$

In particular, we have $\Gamma = \langle\langle 1\rangle\rangle$(scattering rate) and

$$\hat{q} = \left\langle\left\langle(\vec{p}_3 - \hat{p}_1\cdot\vec{p}_3)^2\right\rangle\right\rangle,$$

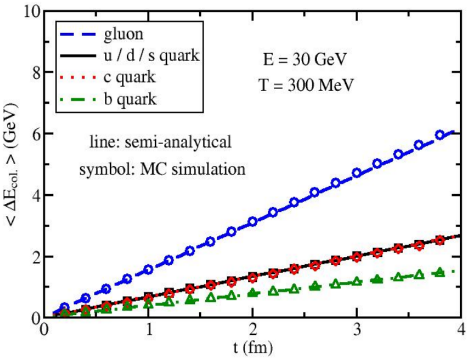
where $\hat{q}$ denotes the momentum broadening and energy loss of jet parton, respectively, per unit time due to elastic scattering.

LBT model

Based above equation, Monte Carlo Simulation:

- Use total rate $\Gamma = \sum_i \Gamma_i$ to determine the probability of elastic scattering $P_{\rm el} = 1 - e^{-\Gamma_{\rm el}\Delta t} \approx \Gamma\Delta t$ (Δt is small).
- Use branching ratios Γ_i/Γ to determine the scattering channel.
- Use the differential rate to sample the momentum space of the two outgoing partons.

LBT model



For Elastic Process: ΔE_{col} from our MC simulation agrees with the semi-analytical result.

LBT model

Inlastic Process:The distribution function of radiated gluon（hight twist） [2]

$$\frac{dN_g}{dxdk_\perp^2 dt} = \frac{2\alpha_s C_A P(x) k_\perp^4}{\pi(k_\perp^2 + x^2M^2)^4}\hat{q}\sin^2\left(\frac{t-t_i}{2\tau_f}\right),$$

where $k_\perp$ is the gluon transverse momentum with respect to the parent parton, α_s is the strong coupling.$P(x)$ is the vacuum splitting function, τ_f is the formation time of the radiated gluon in the form of $\tau_f = 2Ex(1-x)/(k_\perp^2 + x^2M^2)$ with M being the mass of the parent parton, and t_i denotes the initial time or the production time of the parent parton.

[2] Guo, Wang, PRL 2000, Majumder, PRC 2012; Zhang, Wang, Wang, PRL 2004

LBT model

The average number of radiated gluons from single heavy quark

$$\langle N_g\rangle(t,\Delta t) = \Gamma_{\rm inel}\Delta t = \Delta t \int dx dk_\perp^2 \frac{dN_g}{dx dk_\perp^2 dt},$$

Poisson distribution for the number n of radiated gluons during Δt

$$P(n) = \frac{\langle N_g\rangle^n}{n!} e^{-\langle N_g\rangle}$$

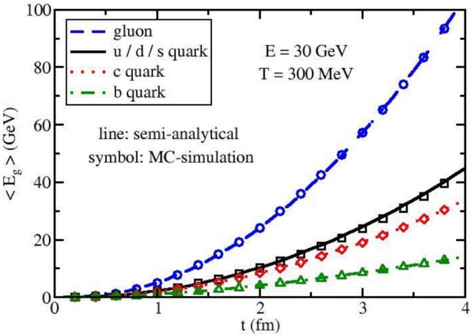
The inelastic scattering probability can be written as

$$P_{\rm inel} = 1 - e^{-\Gamma_{\rm inel}\Delta t}.$$

Monte Carlo Simulation:

- Calculate $< N_g >$ and P_{inel}
- If gluon radiation happens, sample n gluons from Poisson distribution
- Sample E and p of radiatied gluons using the differential radiation spectrum
- First do 2→2 process, then adjust E and p of 2 + n final partons to guarantee E and p conservation for 2→2+n process

LBT model



For Inlastic Process: $< E_g >$ from our MC simulation agrees with the semi-analytical result.

LBT model

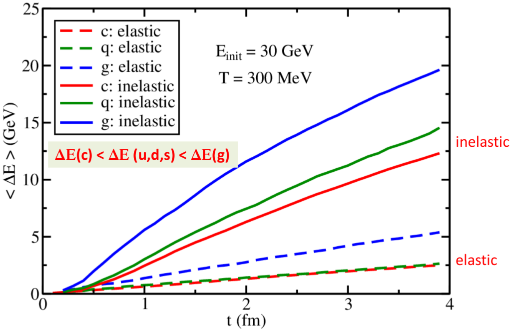
Combine elastic and inelastic:

- Total scattering probability:

$$P_{\mathrm{tot}} = 1 - e^{-(\Gamma_{\mathrm{el}}+\Gamma_{\mathrm{inel}})\Delta t} = P_{\mathrm{el}} + P_{\mathrm{inel}} - P_{\mathrm{el}} P_{\mathrm{inel}}.$$

  The above probability can be splitted into two parts: pure elastic scattering with probability $P_{\mathrm{el}}(1 - P_{\mathrm{inel}})$ and inelastic scattering with probability P_{inel}.
- Use P_{tot} to determine whether jet parton interact with thermal medium
- If jet-medium interaction happens, then determine whether it is pure elastic or inelastic
- Then simulate 2$\rightarrow$2 or 2 $\rightarrow$ 2 +n process

LBT model



The parton energy loss for elastic and inelastic process increases as time increases in a static medium.

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Quasi-Particle model

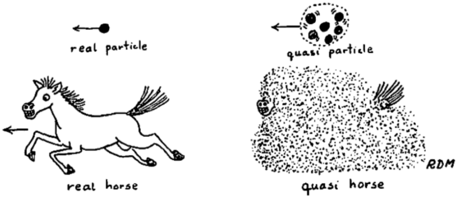


Figure by Mattuck

- The system of interacting particles is effectively represented non-interacting massive particles.
- The mass of quasi particles is viewed as interaction among particles in the system.

Real particle + 'cloud' of other particle = quasi particle

In quantum filed theory:
'bare' particle + 'cloud' = 'renormalized' particle A guide to Feynman diagrams in the many-body problem–Mattuck

Quasi Particle model： describe the equation of state

We utilize the following forms (motivated by perturative QCD calculation) for the temperature-dependent effective masses of quarks and gluons S. Plumari, W. M. Alberico, V. Greco, C. Ratti, Phys. Rev. D84 (2011)

$$
\begin{aligned}
m_g^2(T) &= \frac{1}{6}\left(N_c + \frac{1}{2}N_f\right) g^2(T)T^2, \\
m_{u,d,s}^2(T) &= \frac{N_c^2 - 1}{8N_c} g^2(T)T^2,
\end{aligned}
$$

The pressure can be calculated by summing over the contributions from different constituents:

$$
P(T) = \sum_i d_i \int \frac{d^3p}{(2\pi)^3} \frac{p^2}{3E_i(p,T)} f_i(p,T) - B(T),
$$

Quasi Particle model : describe the equation of state

The energy density of the system can be obtained as follows:

$$\epsilon(T) = \sum_i d_i \int \frac{d^3p}{(2\pi)^3} E_i(p,T) f_i(p,T) + B(T).$$

As for the entropy density, the bag constant $B(T)$ cancels:

$$s(T) = \frac{\epsilon(T) + P(T)}{T}.$$

Motivated by the perturbative QCD calculation, we use the following parametric form to model the temperature dependence of the coupling $g(T)$:

$$g^2(T) = \frac{48\pi^2}{(11N_c - 2N_f)\ln\left[\frac{(aT/T_{\rm c}+b)^2}{1+ce^{-d(T/T_{\rm c})^2}}\right]}$$

where a, b, c and d are parameters to be determined by the equation of state data from lattice QCD simulations.

Bayesian statistical analysis method

We employ the Bayesian statistical analysis method, which can be simply summarized as:

$$P(\theta|\mathrm{data}) = \frac{P(\theta)P(\mathrm{data}|\theta)}{P(\mathrm{data})}.$$

In the above equation, $P(\theta|\mathrm{data})$ is the posterior distribution of the parameter set given the experimental data, $P(\theta)$ is the prior distribution of the parameter set θ, and $P(\mathrm{data}|\theta)$ is the Gaussian likelihood between experimental data and the output for any given set of parameters θ:

$$P(\mathrm{data}|\theta) = \prod_i \frac{1}{\sqrt{2\pi}\sigma_i} e^{-\frac{(y_i - y_{\mathrm{exp}})^2}{2\sigma_i^2}},$$

where y_i denotes model calculation results, y_{exp} denote the experiment or lattice QCD data, σ_i denote the standard errors at each data point.

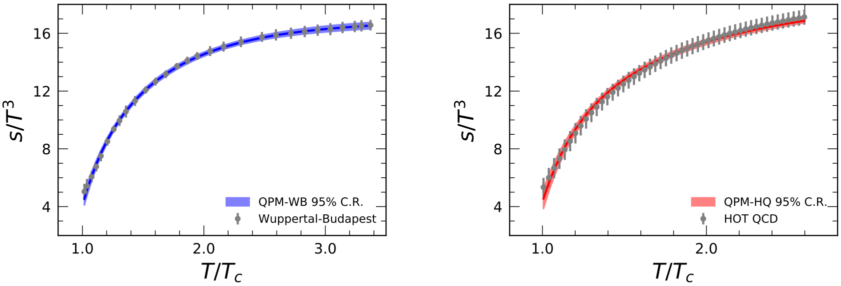
Extract g(T) coupling constant

Lattice Data	Parameters	Prior Range
WB	a	[0.1, 10]
	b	[0, 1]
	c	[0, 10]
	d	[0, 2]
HQ	a	[0, 10]
	b	[-1, 1]
	c	[10, 80]
	d	[0, 10]

Table: The ranges of model parameters (a, b, c, d) used in the prior distributions.

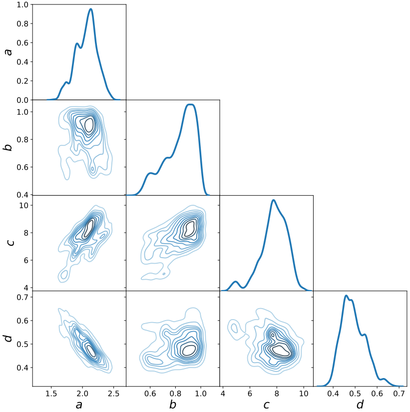
- WB: Wuppertal-Budapest lattice QCD group, HQ: HOT QCD lattice group.
- During the g(T) fitting process, the prior distributions of model parameters (a, b, c, d) are taken as the uniform distribution within given ranges as shown in above Table.

Extract g(T) coupling constant

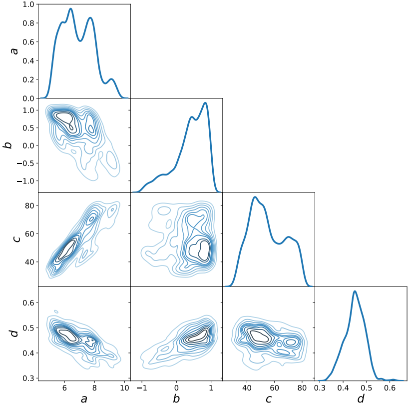


- Calibration of the entropy density s(T) as a function of temperature obtained from the quasi-particle model against the lattice QCD data from both the Wuppertal-Budapest (WB) and the Hot QCD (HQ) Collaborations.
- Note that for the two sets of lattice QCD data, the values of T_c are a little different: $T_c = 150$ MeV for WB and $T_c = 154$ MeV for HQ. To compare to different lattice results, we also use different T_c values in our quasi-particle model.
- The band show 95% confidence intervals for the calibrated parameter values.

Extract g(T) coupling constant



WB



HQ

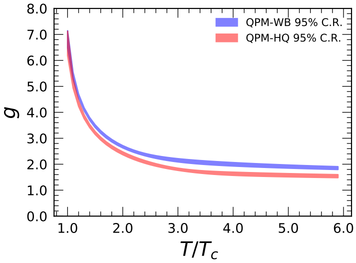
- Posterior distribution of parameters a, b, c,d as well as the correlations between them(left pandel: WB, right panel: HQ).

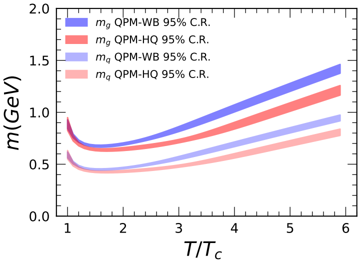
Mean and standard deviations of parameters (a, b, c, d)

Lattice Data	Parameters	Mean Values	Standard Errors
WB	a	2.063	0.1705
	b	0.836	0.1186
	c	7.792	0.9857
	d	0.492	0.054
HQ	a	6.8899	1.055
	b	0.398	0.462
	c	54.9825	12.76
	d	0.449	0.0432

Table: The mean and standard deviations of parameters (a, b, c, d) in the coupling $g(T)$ extracted from Wuppertal-Budapest and Hot QCD lattice QCD data.

$g(T)$ and Quasi Particle mass





- In the right panel, one can see that as T decreases, the quasi-particle masses first decrease and then increase; the transition is around $1.4T_\mathrm{c}$
- g(T) from WB is larger than HQ collaboration, which means that the interaction caused by temperature dependent vertices makes the former enhance the interaction between quasi pariticles and heavy quarks than the latter.
- m(T) comes from that WB is larger than HQ, and the distribution of quasi particles in the former is more dilute than that in the latter. This will reduce the scattering probability of quasi particles and heavy quarks.

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QLBT model

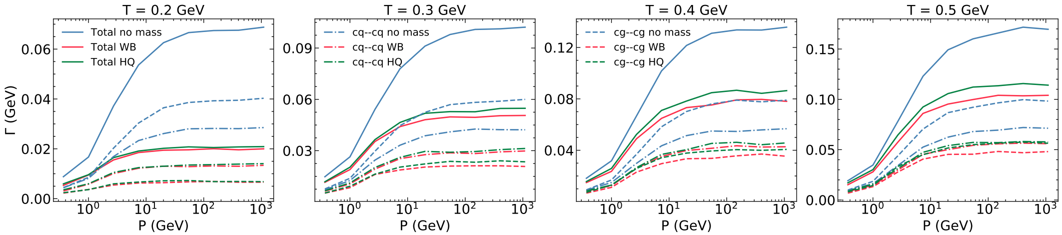
Elastic Scattering ($1 + 2 \to 3 + 4$ process), after introducing thermal mass, elastic scattering rate can be reduced to

$$\Gamma_{12\to34} = \frac{\gamma_2}{16E_1(2\pi)^4} \int dE_2 d\theta_2 d\theta_4 d\phi_4$$
$$\times f_2(E_2, T)(1 \mp f_4(E_4, T))S_2(s, t, u)\left|\mathcal{M}_{12\to34}\right|^2$$
$$\times \frac{p_2 p_4 \sin\theta_2 \sin\theta_4}{E_1 + E_2 - p_1 \cos\phi_4 \frac{E_4}{p_4} - p_2 \cos\theta_{24} \frac{E_4}{p_4}}$$

- The parton 1 and 3 are intial and final heavy quarks. parton 2 and 4 denote intial and final Quasi Particles.
- The leading order pQCD matrix elements are taken for $|M_{12\to34}|^2$.
- Kinematic cut: $S_2(s, t, u) = \theta(s \geq 2\mu_D^2)\theta(t \leq -\mu_D^2)\theta(u \leq -\mu_D^2)$
- Note that the quasi-particle masses of the thermal partons have been included in evaluating the Mandelstam variables s, t and u.

QLBT model

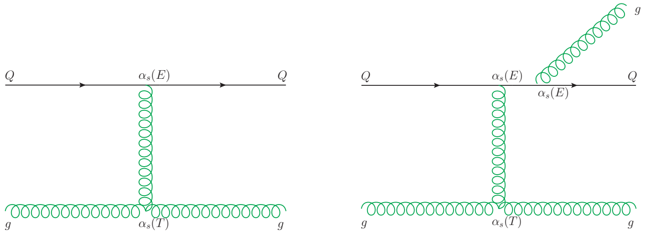
scattering rate (fixed $\alpha_s = 0.3$)



- α_s is fixed at 0.3.
- Compared with the mass of quasi particle to massless(red), scattering rate Γ_{tot} and branch ratio Γ_i/Γ_{tot} reduce about half after including effective quasi partilce mass (WB and HQ) when T equals 0.3 GeV.
- As temperature increase, the difference in scattering rate due to mass of quasi particle gradually decrease.

QLBT model: T and E dependent coupling

Inelastic Process: Except for α_s, the other implementations are same with LBT model as decrobed in above section.



- For vertices that connect to the jet partons (heavy quarks), we assume the following parametric form(energy dependent):

$$\alpha_s(E) = \frac{12\pi}{(11N_c - 2N_f)\log\left[(AE/T_c + B)^2\right]},$$

where A, B are parameters to be fiited by D meason R_{AA} and V_2.
- Vertices that connect to medium partons are temperature dependent .

Work Flow

- The spatial distributions of the heavy quark production vertices are calculated using the Monte-Carlo Glauber model, while their initial momentum distribution is taken from the LO perturbative QCD calculation.
- The QGP medium is simulated using the (3+1)-dimensional CLVisc hydrodynamic model.(τ_0=0.6 fm, η/s=0.08(LHC), 0.16(RHIC)).
- Before the QGP phase ($\tau < \tau_0$), heavy quarks are assumed to stream freely.
- The subsequent heavy quark interaction with the QGP is described using our QLBT model as described in the above content.
- At the chemical freeze-out hypersurface ($T_{\rm c}$), we convert heavy quarks into heavy flavor hadrons using a hybrid coalescence-fragmentation hadronization model.

Extract Energy dependent coupling constant

Lattice Data	Parameters	Prior Range
WB/HQ	A	[0.032, 0.16]
WB/HQ	B	[0.64, 15]

Table: The ranges of model parameters (A, B) used in the prior distributions.

- The prior distributions for two parameters A and B are taken as uniform within the ranges summarized in above Table.
- In this work, we use mean value of $g(T)$ fitted by lattice data for QLBT model caculations.
- We apply the QLBT calculations on 100 sets of (A, B) parameters that are sampled using the Latin-Hypercube algorithm, and then use the corresponding results to train the Gaussian process emulator.

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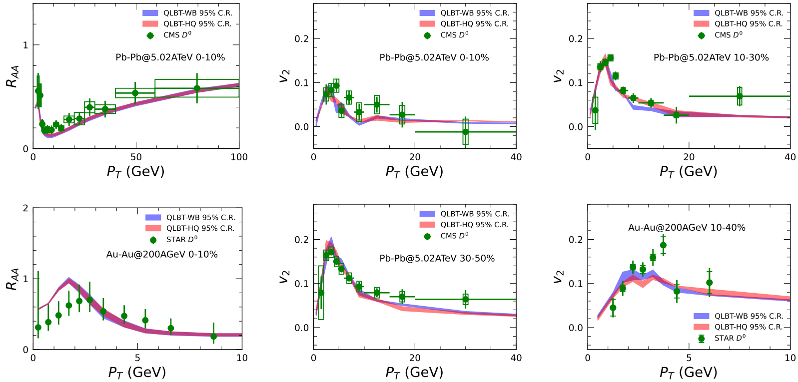
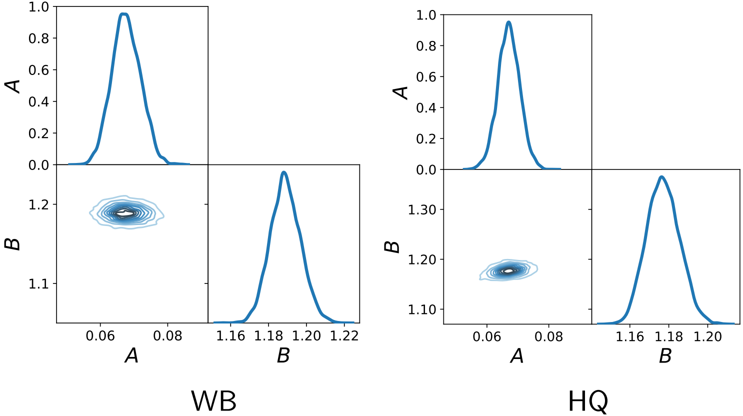
D meason R_{AA} and v_2



- Using the Bayesian analysis, the nuclear modification factor R_{AA} and the elliptic flow v_2 for D mesons at RHIC and the LHC are compared to the experimental data.
- No significant difference can be observed between applying the WB EoS and the HQ EoS.

Extract Energy dependent coupling



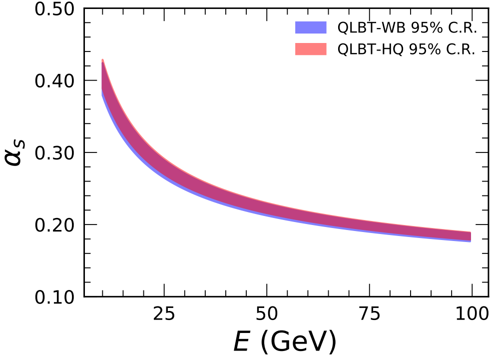
- Two different lattice EoS are used in our analysis, they lead to similar values of A and B for heavy-quark-medium interaction.

Mean and standard deviations of parameters (A, B)

Lattice Data	Parameters	Mean Values	Standard Errors
WB	A	0.067	0.004
	B	1.188	0.008
HQ	A	0.067	0.0035
	B	1.177	0.008

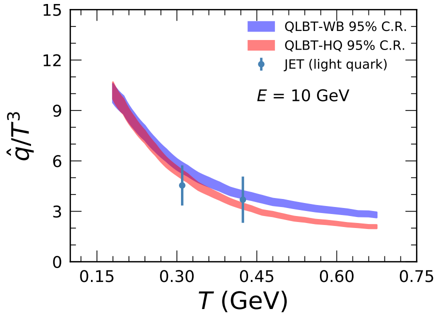
Table: The mean and standard deviations of parameters (A, B) in the coupling $\alpha_s(E_Q)$ extracted from R_{AA} and v_2 using Wuppertal-Budapest and Hot QCD EOS.

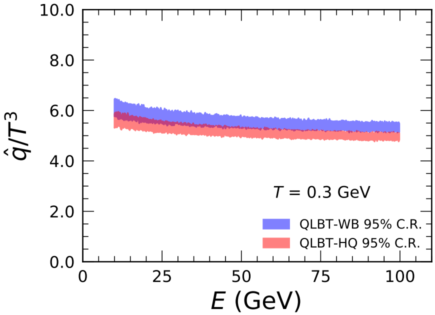
Energy dependent coupling



- The energy-dependent running coupling $\alpha_s(E)$ obtained from the Bayesian analysis of D meson R_{AA} and v_2.
- Runing coupling $\alpha_s(E)$ is insensitive to the choice of EoS due to the competing effects between the coupling strength $g(T)$ and the thermal parton mass $m(T)$.

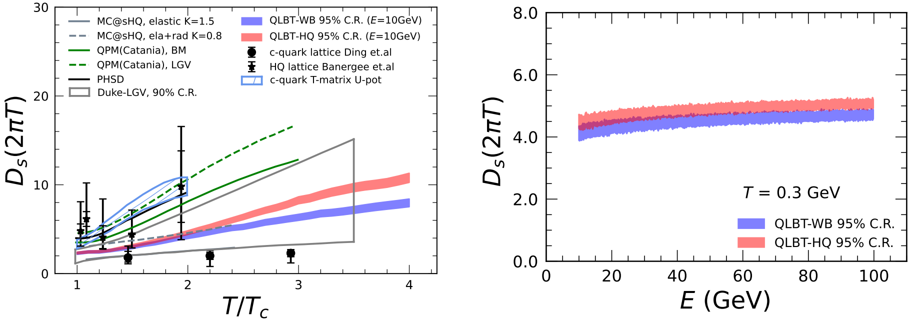
Transport coefficient $\hat{q}/T^3$





- $\hat{q}/T^3 \propto \alpha_s(T) \times \alpha_s(E)$, due to the elastic scattering process.
- The temperature-rescaled transport coefficient $\hat{q}/T^3$ decreases as the medium temperature T increases.
- $\hat{q}/T^3$ slightly decreases with the increase of the heavy quark energy E,

Diffusion coefficient $D_s(2\pi T)$

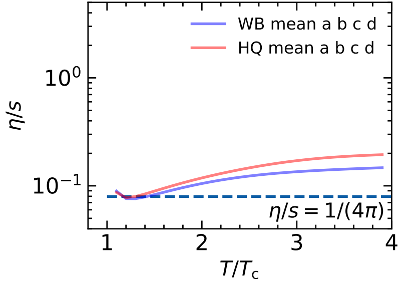


- Reasonable consistency between our results and other groups.
- $D_s(2\pi T)$ increases with the increase of temperature in system, but decreases with the increase of heavy quark energy.

Shear viscosity

The formulas for the viscosities η are derived for a quasi-particle description with bosonic and fermionic constituents S. Plumari, W. M. Alberico, V. Greco, C. Ratti, Phys. Rev. D84 (2011)

$$\eta = \frac{1}{15T}\sum_i d_i \int \frac{d^3p}{(2\pi)^3}\tau_i \frac{\vec{p}^4}{E_i^2} f_i\,(1 \mp f_i)$$



1. Background
2. LBT model: the Linear Boltzmann Transport model
3. Quasi-Particle model
4. QLBT model : improvet LBT model by modeling QGP as a collection of quasi-particles
5. Numerical Results
6. Summary

Summary

- We have developed a new QLBT model that improves the previous linear Boltzmann transport (LBT) model by treating the QGP as a collection of quasi-particles.
  - $g(\mathrm{T})$ have been obtained via calibrating the equation of state (EoS)
  - Comparing the QLBT model results to the experimental data, we are able to extract the heavy-quark-QGP coupling strength α_s with the Bayesian analysis method.
- Enhancement and attenuation effects of the energy loss caused by the coupling strength and the quasi-particle distribution cancel each other out. As a result, the energy loss of heavy quarks using $g(\mathrm{T})$ fiited two lattice data is almost the same.
- We extract the heavy quark transport parameter $\hat{q}$ and diffusion coefficient D_s in the temperature range of $1-4\ T_\mathrm{c}$ and compare to the lattice QCD results and other phenomenological studies.

Thanks

Back up: final energy of parton 4 for 1+2 $\rightarrow$ 3+4

After introducing thermal mass, elastic scattering rate can be reduced to

$$\Gamma_{12\to 34} = \frac{\gamma_2}{16E_1(2\pi)^4}\int dE_2 d\theta_2 d\theta_4 d\phi_4$$
$$\times f_2(E_2,T)(1\mp f_4(E_4,T))S_2(s,t,u)\left|\mathcal{M}_{12\to 34}\right|^2$$
$$\times \frac{p_2 p_4 \sin\theta_2 \sin\theta_4}{E_1+E_2-p_1\cos\phi_4\frac{E_4}{p_4}-p_2\cos\theta_{24}\frac{E_4}{p_4}}$$

$$\cos\theta_{24} = \sin\theta_2\sin\theta_4\cos\phi_4+\cos\theta_2\cos\theta_4$$

$$E_4 = \frac{(E_1+E_2)B\pm\sqrt{A^2(m_4^2A^2+B^2-m_4^2(E_1+E_2)^2)}}{(E_1+E_2)^2-A^2}$$

with $A = |\vec{p}_1|\cos\theta_4+|\vec{p}_2|\cos\theta_{24},\ B = p_1\cdot p_2+m_4^2$

Backup: Temperature dependent bag constant

The energy density ϵ of the system is obtained from the pressure through the thermodynamic relationship $\epsilon(T) = TdP(T)/dT - P(T)$, where pressure P of system only depend on T.

In order to have above thermodynamic consistency, the following relationship has to be satisfied

$$\left(\frac{\partial P_{qp}}{\partial m_i}\right)_{T,\mu} = 0, \quad i = u, d \ldots$$

which gives rise to a set of equations of the form

$$\frac{\partial B}{\partial m_i} + d_i \int \frac{d^3p}{(2\pi)^3} \frac{m_i}{E_i} f_i\left(E_i\right) = 0$$

Bag constant B is Temperature dependent due to Temperature dependent effective thermal mass of quasi particles.

Backup: Shear viscosity

The formulas for the viscosities η are derived for a quasi-particle description with bosonic and fermionic constituents S. Plumari, W. M. Alberico, V. Greco, C. Ratti, Phys. Rev. D84 (2011)

$$\eta = \frac{1}{15T} \sum_i d_i \int \frac{d^3p}{(2\pi)^3} \tau_i \frac{\vec{p}^4}{E_i^2} f_i \left(1 \mp f_i\right)$$

$d^3p = p^2 dp \sin\theta d\theta d\psi$.
In relaxtime approximation, shear viscosity depends on collision relax time τ_i given by (HTL):

$$\tau_q^{-1} = 2\frac{N_C^2 - 1}{2N_C} \frac{g^2 T}{8\pi} \ln\frac{2k}{g^2}, \quad \tau_g^{-1} = 2N_C \frac{g^2 T}{8\pi} \ln\frac{2k}{g^2},$$

where g is the coupling obtained and k is a parameter which is fixed by requiring that τ_i yields a minimum of one for the quantity $4\pi\eta/s$.

- k if fixed to have a minimum $\eta/s = 1/4\pi$. For WB: k=23.3 and For HQ: k=22.7

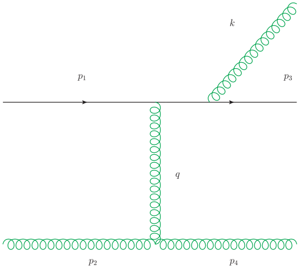
Backup: Guarantee E and p conservation for 2→2+n process

First do 2→2 process, record $q_\perp$ and p_2

$$p_3 = p_1 - q - k,$$
$$p_4 = p_2 + q,$$

Using on shell condition and E and p conservation:

$$(p_1 - q - k)_0^2 = (p_1 - q - k)_x^2 + (p_1 - q - k)_y^2 + (p_1 - q - k)_z^2 + M^2,$$
$$(p_2 + q)_0^2 = (p_2 + q)_x^2 + (p_2 + q)_y^2 + (p_2 + q)_z^2$$



One can get q_0 and q_z from above equation which used E and p conservation.