Effect of long-range interactions on topological ground state degeneracy

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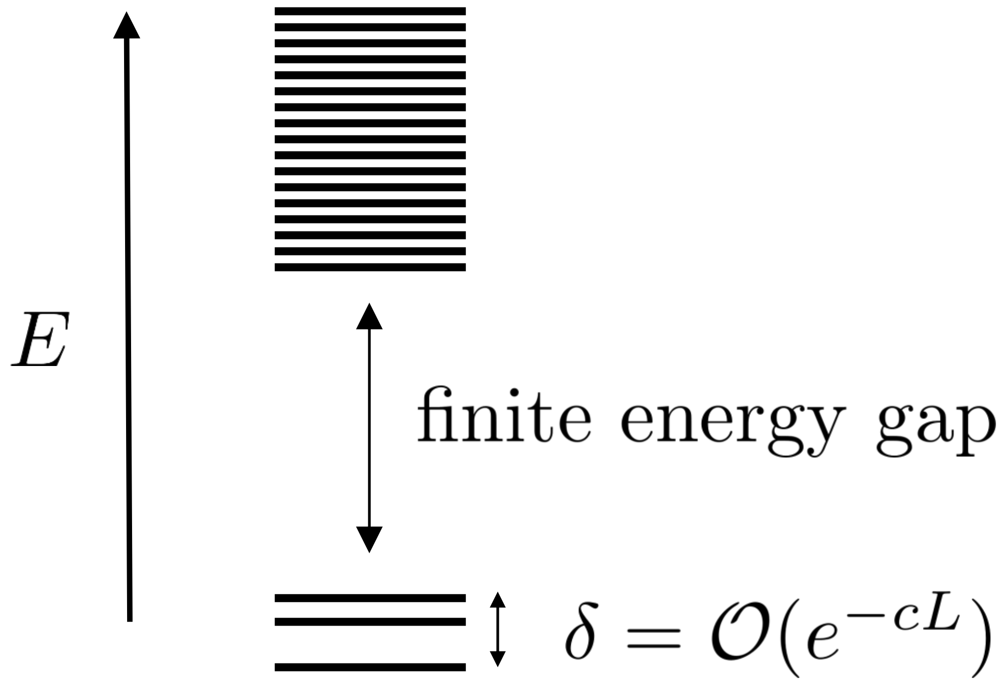
arXiv:2107.11396 (w/ Matt Lapa)

Ongoing work w/ Etienne Granet

Some gapped many-body systems have exponentially small ground state splitting:



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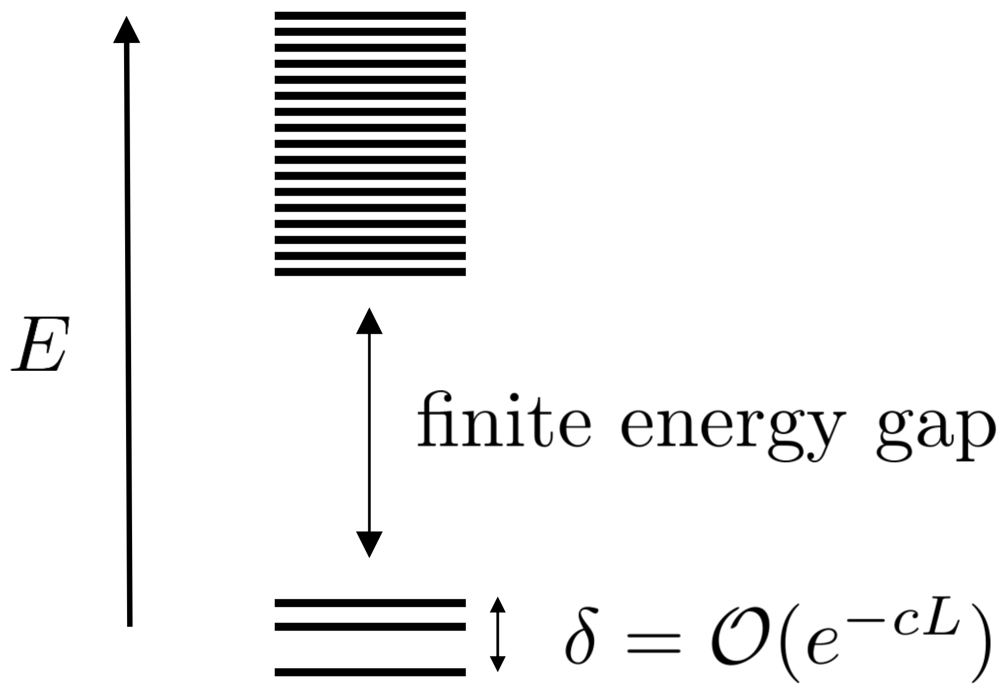

Robust phenomenon: no fine tuning

Examples:

- Systems with (discrete) symmetry breaking
- 1D topological superconductors (with open b.c.)
- 2D fractional quantum Hall systems (in torus geometry)

Main question

Arguments for robust, exponentially small splitting require *short-range* interactions

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What about long-range (power law) interactions?

Why worry about long-range interactions?

- Relevant to many experimental systems
- Conceptual question: how much locality is necessary for topological phenomena?

A simple 1D model

$$H = -\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + \lambda \sum_{j,k=1}^{L} f(j-k) \sigma_j^x \sigma_k^x$$

long-range interaction
e.g. $f(r) \sim \frac{1}{|r|^\alpha}$.

Ising symmetry: $\mathcal{S} = \prod_{j=1}^{L} \sigma_j^x$

$\lambda = 0$: symmetry breaking, exact GSD

What happens when we turn on small $\lambda \neq 0$?

1. Does the gap stay open?
2. How does ground state splitting δ scale with system size?

Physical motivation for model

Map to fermion model via Jordan-Wigner transformation:

$$\sigma_i^x \to 2c_i^\dagger c_i - 1$$

$$\sigma_i^z \sigma_{i+1}^z \to (c_i^\dagger + c_i)(c_{i+1} - c_{i+1}^\dagger)$$

$$H \to -\sum_{i=1}^{L-1} (c_i^\dagger c_{i+1} + c_i c_{i+1} + h.c) + 4\lambda \sum_{jk} f(j-k) \left(n_j - \frac{1}{2}\right) \left(n_k - \frac{1}{2}\right)$$

$\Longrightarrow$ Kitaev model for a 1D topological superconductor with long-range density-density interactions

Why long-range interactions are different

Denote ground states of $H(\lambda = 0)$ by:

$$|+\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle + |\downarrow\downarrow\cdots\downarrow\rangle)$$

$$|-\rangle = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\cdots\uparrow\rangle - |\downarrow\downarrow\cdots\downarrow\rangle)$$

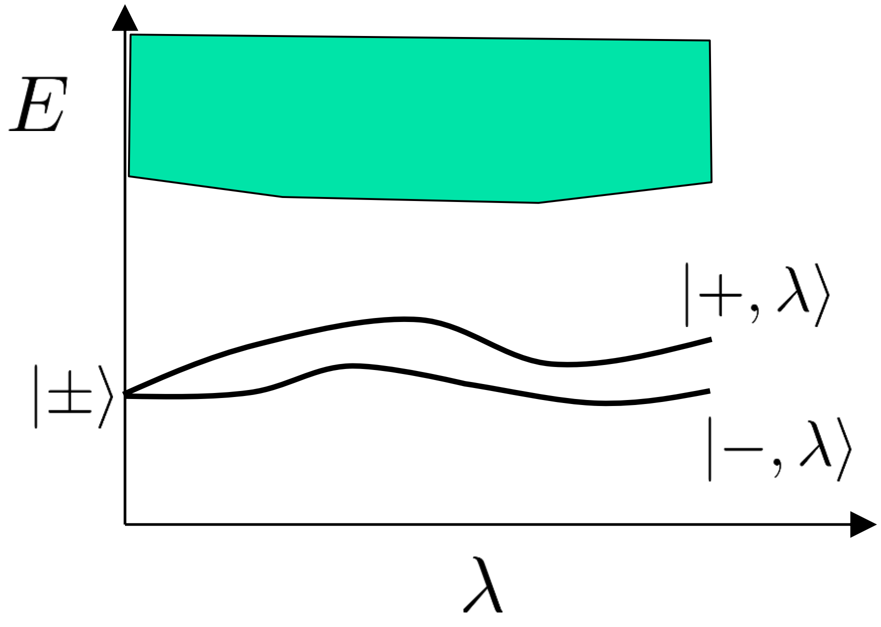
$$|+\rangle \in \mathcal{H}_+$$

$$|-\rangle \in \mathcal{H}_-$$

subspaces with $\mathcal{S} = \pm 1$

(Hastings, Wen, 2005)

Suppose energy gap stays open for small $\lambda \neq 0$:



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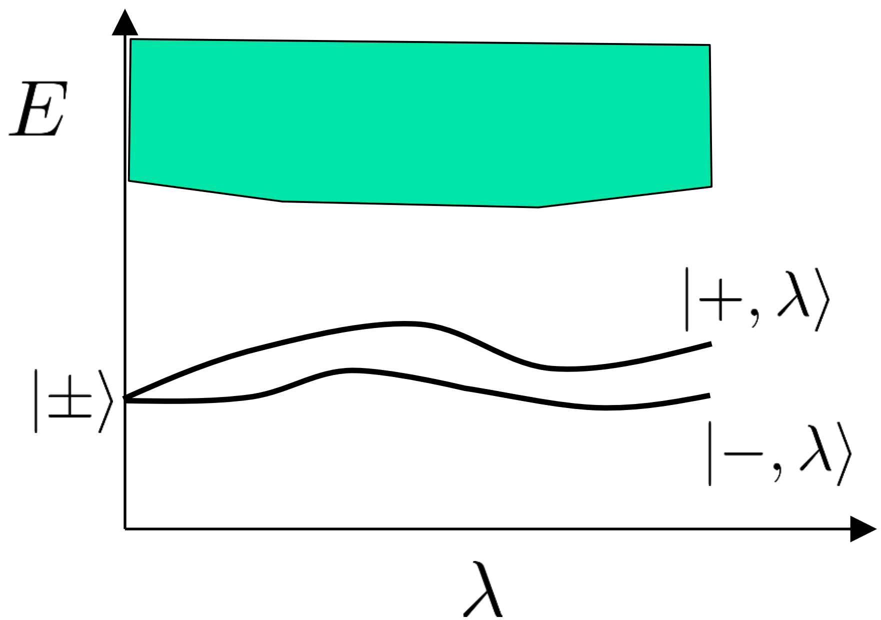

If interactions are *short-range*, can use quasi-adiabatic continuation to construct unitary U_λ with:

$$|\pm,\lambda\rangle = U_\lambda|\pm\rangle$$

(Hastings, Wen, 2005)

In short-range case, U_λ is locality preserving:

$$U_\lambda : \quad O \quad \rightarrow \quad U_\lambda^\dagger O U_\lambda$$

local "superpolynomially local"

(derived from Lieb-Robinson bounds)

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Energy splitting:

$$\begin{aligned}\delta &= \langle + | U_\lambda^\dagger H U_\lambda | + \rangle \; - \; \langle - | U_\lambda^\dagger H U_\lambda | - \rangle \\ &= 2 \,\mathrm{Re}\left(\langle \uparrow\uparrow \cdots \uparrow | U_\lambda^\dagger H U_\lambda | \downarrow\downarrow \cdots \downarrow \rangle \right) \\ &= \mathcal{O}(L^{-\infty})\end{aligned}$$

sum of "superpolynomially local" operators

(Michalakis, Zwolak, 2013)

However, if f is long-range (power law) then:

$$U_\lambda : \quad O \quad \to \quad U_\lambda^\dagger O U_\lambda$$

local "polynomially local"

$\implies$ $U_\lambda^\dagger H U_\lambda$ has power-law tails

$\implies$ can only get *power-law* bound: $\delta = \mathcal{O}(L^{-\beta})$

Should we even expect exponentially small splitting for power law interactions?

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Yes! (for sufficiently high power laws)

(Lapa, ML, arXiv:2107.11396)

Stability Theorem: Suppose that f obeys:

$$\sum_{k=1}^{L} |f(j-k)| \leq C_0$$

"summability condition"

(e.g. $f(r) \sim \frac{1}{|r|^\alpha}$ with $\alpha > 1$)

Then, there exists $\lambda_0 > 0$ such that, if $|\lambda| < \lambda_0$, then:

1. H has a unique ground state and a finite energy gap in $\mathcal{H}_\pm$.

2. The ground state splitting between sectors obeys the bound

$$|E_+(\lambda) - E_-(\lambda)| = \mathcal{O}(e^{-cL})$$

Beyond the stability theorem

What happens if $f(r) = \frac{1}{|r|^{\alpha}}$ with $\alpha \leq 1$?

Beyond the stability theorem

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Focus on the case $\lambda > 0$

($\lambda < 0 \implies$ instability at infinitesimal λ)

Toy model for power-law interactions

$$H = -\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z + \frac{\lambda}{4L^\alpha} \left(\sum_{j=1}^{L} \sigma_j^x \right)^2 \qquad \begin{array}{l} \alpha < 1 \\ \lambda > 0 \end{array}$$

(corresponds to $f(r) = \frac{1}{4L^\alpha}$)

How does ground state splitting scale with L?

(Granet, ML, in prep.)

Exact (asymptotic) result

$$\log \delta = -\sqrt{\frac{4}{\lambda}} L^{(1+\alpha)/2} + o(L^{(1+\alpha)/2})$$

e.g. for $\alpha = 1/2$,

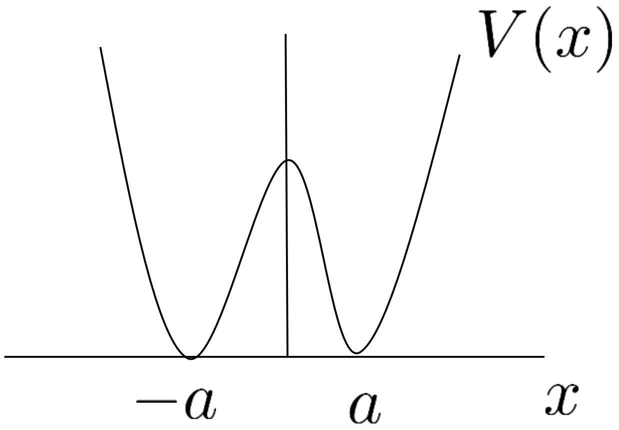
$$\delta \sim e^{-2L^{3/4}/\sqrt{\lambda}}$$

$\Longrightarrow$ *stretched* exponential scaling

(Coleman, 1977)

Warm-up: Particle in a symmetric double well potential

$$H = \frac{P^2}{2m} + V(X)$$



How can we compute GS splitting δ in semi-classical limit ($\hbar \to 0$)?

Let $\mathcal{S}$ = reflection operator (i.e. $\mathcal{S}|x\rangle = |-x\rangle$)

$$\delta \approx \frac{2}{\beta}\frac{\mathrm{Tr}(e^{-\beta H}\mathcal{S})}{\mathrm{Tr}(e^{-\beta H})} \qquad \text{for large } \beta$$

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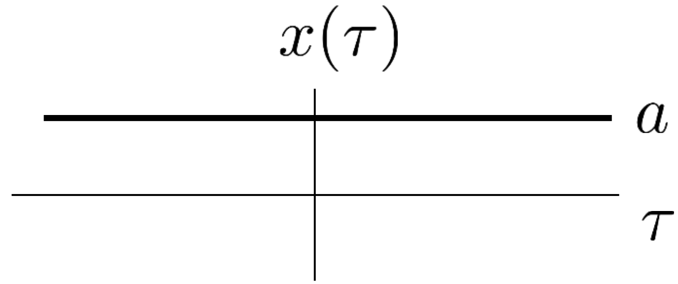
Path integral representation:

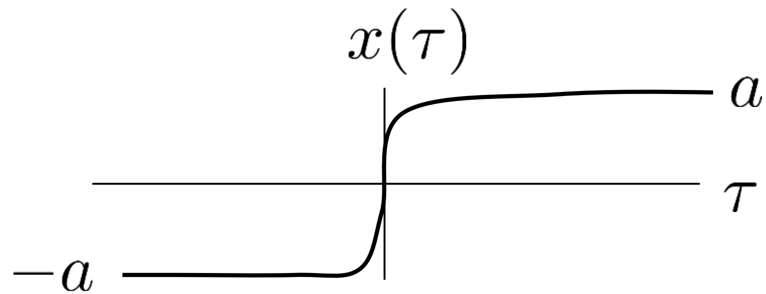
$$\delta = \frac{2}{\beta}\frac{\int_{x(\beta)=-x(0)} \mathcal{D}x \exp\left[-\frac{1}{\hbar}\int_0^\beta d\tau (\frac{m}{2}\dot{x}^2 + V(x)\right]}{\int_{x(\beta)=x(0)} \mathcal{D}x \exp\left[-\frac{1}{\hbar}\int_0^\beta d\tau (\frac{m}{2}\dot{x}^2 + V(x)\right]}$$

Path integrals controlled by saddle points

Saddle point for denominator:

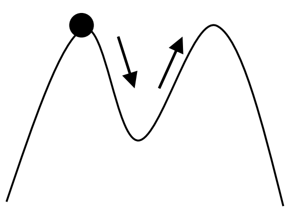


Saddle point for numerator:



"instanton"

(obtain by solving classical eq. of motion
in potential $-V(x)$)



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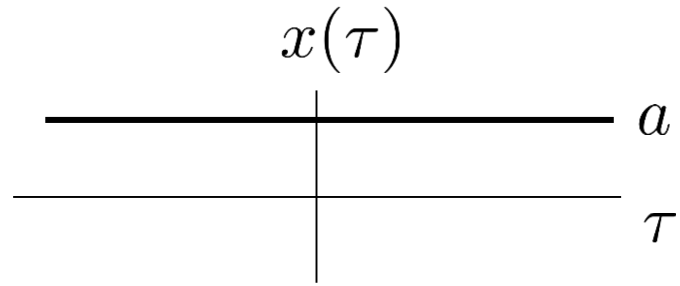

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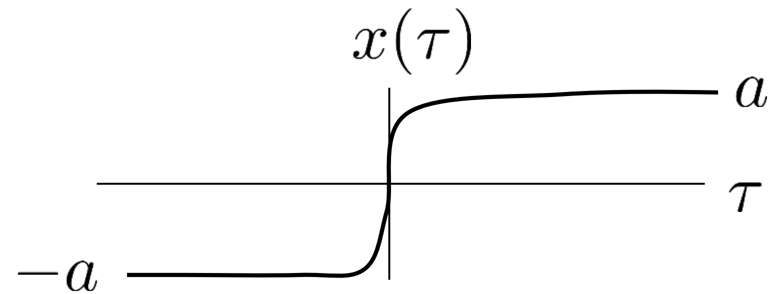

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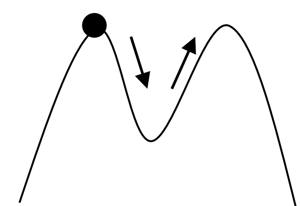

$$\Longrightarrow \quad \delta = 2\frac{e^{-S[\quad]/\hbar}}{e^{-S[\quad]/\hbar}} \cdot \frac{\sqrt{\det(\quad)}}{\sqrt{\det'(\quad)}} \cdot (1 + \mathcal{O}(\sqrt{\hbar}))$$

Back to spin model

$$H = H_0 + \frac{\lambda}{L^\alpha}(M^x)^2$$

$$H_0 = -\sum_{j=1}^{L-1} \sigma_j^z \sigma_{j+1}^z, \qquad M^x = \frac{1}{2}\sum_{j=1}^{L} \sigma_j^x$$

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$$\delta \approx \frac{2}{\beta} \frac{\mathrm{Tr}[e^{-\beta H}\mathcal{S}]}{\mathrm{Tr}[e^{-\beta H}]}$$

$$= \frac{2}{\beta} \frac{\mathrm{Tr}[e^{-\beta(H_0 + \frac{\lambda}{L^\alpha}(M^x)^2)}\mathcal{S}]}{\mathrm{Tr}[e^{-\beta(H_0 + \frac{\lambda}{L^\alpha}(M^x)^2)}]}$$

$$\delta = \frac{2}{\beta} \lim_{M\to\infty} \frac{\mathrm{Tr}\left[\left(e^{-\frac{\beta}{M}H_0} e^{-\frac{\beta}{M}\frac{\lambda}{L^\alpha}(M^x)^2}\right)^M \mathcal{S}\right]}{\mathrm{Tr}\left[\left(e^{-\frac{\beta}{M}H_0} e^{-\frac{\beta}{M}\frac{\lambda}{L^\alpha}(M^x)^2}\right)^M\right]}$$

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Use Hubbard-Stratonovich identity:

$$e^{-\Delta\tau\frac{\lambda}{L^\alpha}(M^x)^2} = \sqrt{\frac{\Delta\tau L^\alpha}{4\pi\lambda}} \int_{-\infty}^{\infty} e^{-iM^x\phi\Delta\tau} e^{-\frac{L^\alpha\phi^2\Delta\tau}{4\lambda}} d\phi$$

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$$\implies \qquad \delta = \frac{2}{\beta} \frac{\int \mathcal{D}\phi \exp\left(-S_a[\phi] - \int_0^\beta d\tau \frac{L^\alpha\phi(\tau)^2}{4\lambda}\right)}{\int \mathcal{D}\phi \exp\left(-S_p[\phi] - \int_0^\beta d\tau \frac{L^\alpha\phi(\tau)^2}{4\lambda}\right)}$$

$$e^{-S_a[\phi]} = \mathrm{Tr}\left[\mathcal{T}\exp\left(-\int_0^\beta d\tau\ [H_0 + iM^x\phi(\tau)]\right)\mathcal{S}\right]$$

$$e^{-S_p[\phi]} = \mathrm{Tr}\left[\mathcal{T}\exp\left(-\int_0^\beta d\tau\ [H_0 + iM^x\phi(\tau)]\right)\right]$$

Make change of variables: $\theta(\tau) = \int_0^\tau \phi(\tau')d\tau'$

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$$\implies \quad \delta = \frac{2}{\beta} \frac{\int \mathcal{D}\theta \exp\left(-S_a[\theta] - \int_0^\beta d\tau \, \frac{L^\alpha \dot{\theta}(\tau)^2}{4\lambda}\right)}{\int \mathcal{D}\theta \exp\left(-S_p[\theta] - \int_0^\beta d\tau \, \frac{L^\alpha \dot{\theta}(\tau)^2}{4\lambda}\right)}$$

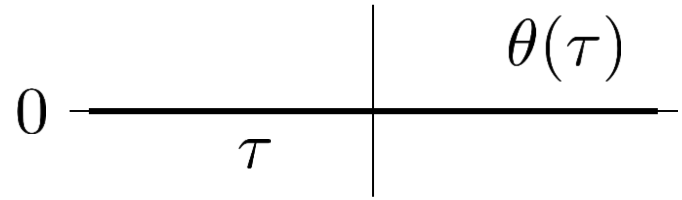
$$e^{-S_a[\theta]} = \mathrm{Tr}\left[\mathcal{T} \exp\left(-\int_0^\beta d\tau [H_0 + iM^x \dot{\theta}(\tau)]\right) \overbrace{e^{i\pi M^x}}^{\mathcal{S}}\right]$$

$$e^{-S_p[\theta]} = \mathrm{Tr}\left[\mathcal{T} \exp\left(-\int_0^\beta d\tau [H_0 + iM^x \dot{\theta}(\tau)]\right)\right]$$

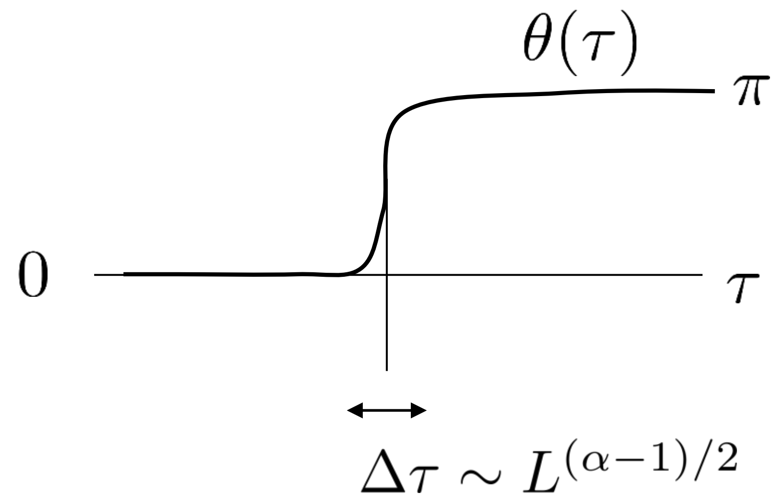
Expect $S_a[\theta], S_p[\theta] \propto L$

$\implies$ can use saddle-point approximation in limit $L \to \infty$

Saddle point for denominator:



Saddle point for numerator:

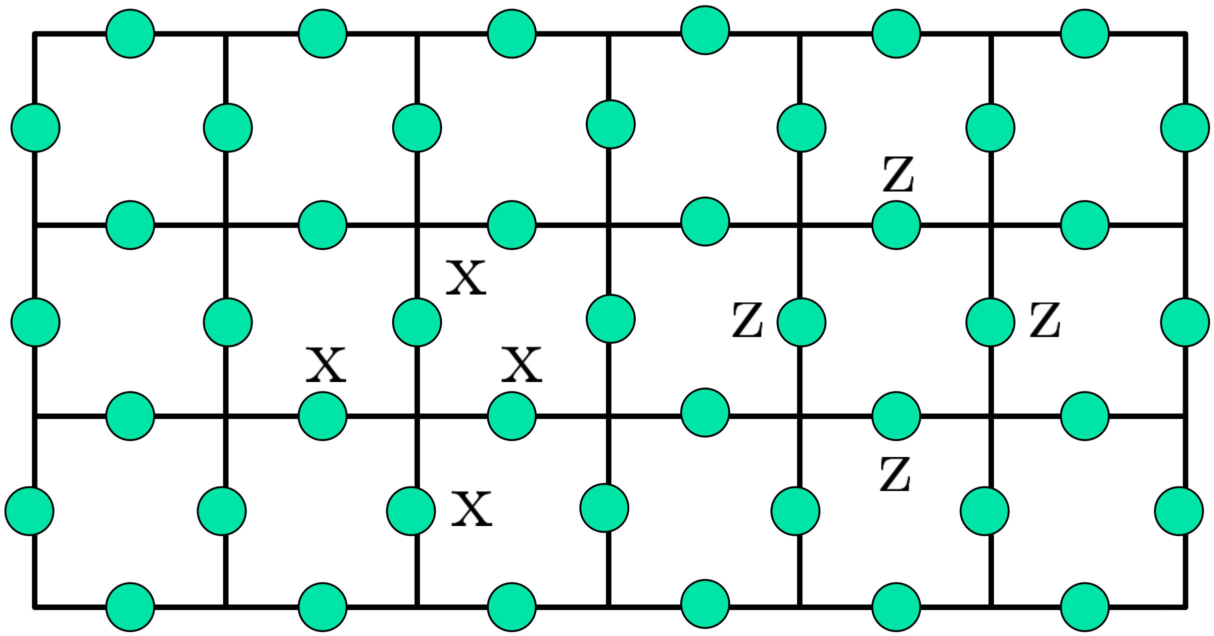


$$\Longrightarrow \quad \delta \propto \frac{e^{-S[\quad]}}{e^{-S[\quad]}} = e^{-\sqrt{\frac{4}{\lambda}}L^{(1+\alpha)/2}}$$

2D model

$$H = H_{tc} + \lambda \sum_{rr'} f(r-r')\sigma^x_r \sigma^x_{r'}$$

$$H_{tc} = -\sum_{+} \sigma^x_1\sigma^x_2\sigma^x_3\sigma^x_4 - \sum_{\square} \sigma^z_1\sigma^z_2\sigma^z_3\sigma^z_4$$



$\lambda = 0$: exact topological GSD on torus

Turn on small $\lambda > 0 \implies$ degeneracy will split

How does splitting scale with L?

(Levin, Burnell, Koch-Janusz, Stern, 2011)

Motivation for model

Exact mapping onto (another) 2D model with:

- $U(1)$ charge conservation
- Fractionally charged excitations
- Long-range density-density interactions

$\Longrightarrow$ captures many features of FQH systems

Results for 2D model

Case 1: Any $f(r)$ such that $\sum_{r'} |f(r - r')| \le C_0$
(e.g. $f(r) \sim \frac{1}{r^\alpha}$ with $\alpha > 2$)

$$\delta = \mathcal{O}(e^{-cL}) \qquad \text{(generalized stability theorem)}$$

Case 2: $f(r) \sim \frac{1}{L^\alpha}$ with $\alpha < 1$:

$$\delta \sim e^{-cL^{(1+\alpha)/2}/\sqrt{\lambda}} \qquad \text{(instanton calculation)}$$

Summary

1D Ising model:

$f(r) = 1/\|r\|^\alpha, \quad \alpha > 1$	$\delta \leq e^{-L}$ (theorem)
$f(r) = 1/\|r\|^\alpha, \quad \alpha < 1$	$\delta \sim e^{-L^{(1+\alpha)/2}}$ (conjecture)
$f(r) = 1/\|r\|^\alpha, \quad \alpha = 1$	??

2D toric code model:

$f(r) = 1/\|r\|^\alpha, \quad \alpha > 2$	$\delta \leq e^{-L}$ (theorem)
$f(r) = 1/\|r\|^\alpha, \quad \alpha < 1$	$\delta \sim e^{-L^{(1+\alpha)/2}}$ (conjecture)
$f(r) = 1/\|r\|^\alpha, \quad 1 \leq \alpha \leq 2$	??