Kinetic energy spectrum in compressible turbulence

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Outline

- ☞ Astrophysical motivation
- ☞ Basic questions:
  - Is there a Kolmogorov-like inertial range in compressible turbulence?
  - What is the right statistical proxy to describe energy transfer?
- ☞ Introduction:
  - Energy transfer proxy in K41
  - Equations and ideal invariants for isothermal fluids
- ☞ Two-point energy correlation/structure functions for homogeneous turbulence
- ☞ Illustrative examples from DNS of supersonic turbulence
- ☞ Summary

JWST: Cosmic cliffs in 3D

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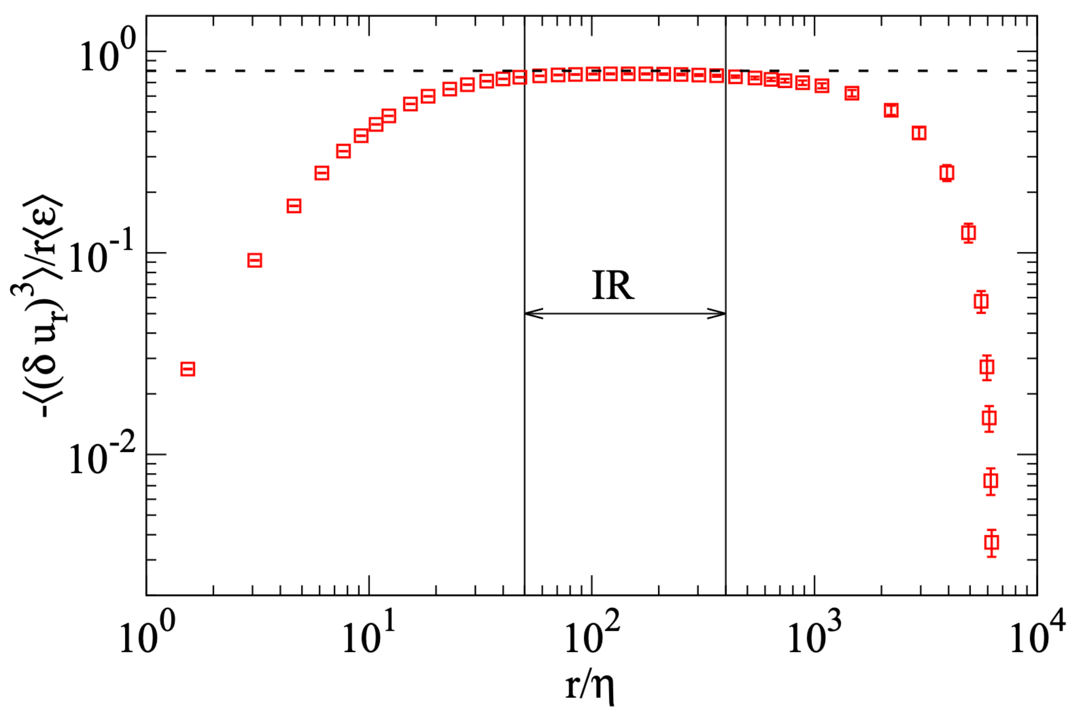
Compressible turbulence: $Re \sim 10^8$, $M_t \sim 4$, and $\frac{\langle \varepsilon_d \rangle}{\langle \varepsilon_s \rangle} < 1$

The 4/5 law of K41

$$S_3^{\|}(r) \equiv \langle(\delta \boldsymbol{u}_{\|}(r))^3\rangle = -\tfrac{4}{5}\varepsilon r \text{ for } \eta \ll r \ll L$$



- Normalized 3rd-order longitudinal velocity structure function [Iyer et al. (2017)]
- 8192^3 DNS of isotropic turbulence at $R_\lambda = 1300$
- A primitive version of the 4/5 law: $\langle\boldsymbol{\delta} u_{\|}(\delta \boldsymbol{u})^2\rangle = -\frac{4}{3}\varepsilon r$ [Antonia et al. (1997)]

Governing equations

Compressible Navier–Stokes system

$$\partial_t \rho + \nabla\cdot(\rho \boldsymbol{u}) = 0, \tag{1}$$

$$\partial_t(\rho \boldsymbol{u}) + \nabla\cdot(\rho \boldsymbol{u}\otimes\boldsymbol{u}) = -\nabla p + \boldsymbol{d} + \boldsymbol{f}, \tag{2}$$

- Viscous terms based on Stokes formula $\boldsymbol{d} \equiv \mu\nabla^2\boldsymbol{u} + (\mu/3)\nabla\theta$, where $\theta \equiv \nabla\cdot\boldsymbol{u}$
- Random forcing $\boldsymbol{f}(\boldsymbol{x},t) = \rho\boldsymbol{a}(\lambda_f,\varepsilon)$
- Large-scale solenoidal or mixed external acceleration with finite correlation time
- Energy injection rate ε; injection scale λ_f and wave number $k_f = 2\pi/\lambda_f$
- Isothermal equation of state: $p(\rho) = c_\mathrm{s}^2\rho$, sound speed $c_\mathrm{s} = \mathrm{const}$
- Total energy is an inviscid invariant of the isothermal system

$$E = \langle \rho\boldsymbol{u}^2/2 + \rho e\rangle \tag{3}$$

- In the isothermal case, $e = c_\mathrm{s}^2\ln(\rho/\rho_0)$ is the Gibbs free energy per unit mass
- Energy equation: $\partial_t E = \langle\boldsymbol{u}\cdot\boldsymbol{f}\rangle + \langle\boldsymbol{u}\cdot\boldsymbol{d}\rangle$

Kinetic energy correlator: Available options

☞ **Incompressible fluids**

Energy invariant is quadratic: $E = \langle u^2 \rangle / 2$

$\Rightarrow$ unique point-split version: $E(\boldsymbol{r}) = \langle \boldsymbol{u} \cdot \boldsymbol{u}' \rangle / 2$, where $\boldsymbol{u} \equiv \boldsymbol{u}(\boldsymbol{x}, t)$, $\boldsymbol{u}' \equiv \boldsymbol{u}(\boldsymbol{x} + \boldsymbol{r}, t)$, and homogeneity is assumed

☞ **Compressible fluids**

Kinetic energy is not quadratic: $K = \langle \rho \boldsymbol{u} \cdot \boldsymbol{u} \rangle / 2$

$\Rightarrow$ no unique point-split version

Split options available in the literature:

1. $K_1(\boldsymbol{r}) = \langle \boldsymbol{w} \cdot \boldsymbol{w}' \rangle / 2$, where $\boldsymbol{w} \equiv \rho^{1/2} \boldsymbol{u}$ [Kida & Orszag (1990)]
2. $K_2(\boldsymbol{r}) = \langle \boldsymbol{j} \cdot \boldsymbol{u}' + \boldsymbol{j}' \cdot \boldsymbol{u} \rangle / 4$, where $\boldsymbol{j} \equiv \rho \boldsymbol{u}$ [Graham et al. (2010)]
3. $K_3(\boldsymbol{r}) = \langle \boldsymbol{j} \cdot \boldsymbol{u}' + \boldsymbol{j}' \cdot \boldsymbol{u} \rangle / 4 - \langle \rho u'^2 + \rho' u^2 \rangle / 8$ [Ferrand et al. (2020)]

Other split options:

4. $K_4(\boldsymbol{r}) = \langle \boldsymbol{j} \cdot \boldsymbol{j}' V' + \boldsymbol{j}' \cdot \boldsymbol{j} V \rangle / 4 - \langle V j'^2 + V' j^2 \rangle / 8$, where $V \equiv 1/\rho$
5. $K_5(\boldsymbol{r}) = a K_2 + b K_3 + c K_4$, where $a + b + c = 1$

Gibbs free energy is not quadratic: $U = \langle \rho e \rangle$ where $e \equiv c_s^2 \ln(\rho / \rho_0)$,

but the point-split version is still unique (up to a const. factor): $U(\boldsymbol{r}) = \langle \rho e' + \rho' e \rangle / 2$

Point-splitting a product of three quantities

☞ Take the product $\langle abc \rangle$

$w_1 \langle a' bc \rangle + w_2 \langle ab' c \rangle + w_3 \langle abc' \rangle + \ldots$, where w_i are weights and symmetric counterparts (e.g., $\langle ab' c' \rangle$, etc.) are dropped for brevity

To avoid discrimination, the weights, generally, should be equal (in absolute value), but perhaps can have different signs.

☞ Example 1. For kinetic energy substitute $a = \rho$, $\boldsymbol{b} = \boldsymbol{c} = \boldsymbol{u}$

Kinetic energy $4K_3(r) = w_1 \langle \rho' \boldsymbol{u} \cdot \boldsymbol{u} \rangle + w_2 \langle \rho \boldsymbol{u}' \cdot \boldsymbol{u} \rangle + w_3 \langle \rho \boldsymbol{u} \cdot \boldsymbol{u}' \rangle$

With $w_1 = -1,\ w_2 = w_3 = 1$ we get (experimentally $\langle \rho' \boldsymbol{u} \cdot \boldsymbol{u} \rangle < 0$):

$4K_3(r) = -\langle \rho' u^2 \rangle + 2 \langle \boldsymbol{j} \cdot \boldsymbol{u}' \rangle$, cf. [Ferrand et al. (2020)]

☞ Example 2. For energy injection substitute $a = \rho$, $\boldsymbol{b} = \boldsymbol{u}$, $\boldsymbol{c} = \boldsymbol{a}$, where $\boldsymbol{a}$ is large-scale external acceleration

Kinetic energy injection $4F_3(r) = w_1 \langle \rho' \boldsymbol{u} \cdot \boldsymbol{a} \rangle + w_2 \langle \rho \boldsymbol{u}' \cdot \boldsymbol{a} \rangle + w_3 \langle \rho \boldsymbol{u} \cdot \boldsymbol{a}' \rangle$

With the same weights $w_1 = -1,\ w_2 = w_3 = 1$ we get:

$4F_3(r) = -\langle \frac{\rho'}{\rho} \boldsymbol{u} \cdot \boldsymbol{f} \rangle + \langle \boldsymbol{f} \cdot \boldsymbol{u}' \rangle + \langle \frac{\rho}{\rho'} \boldsymbol{u} \cdot \boldsymbol{f}' \rangle$, where $\boldsymbol{f} = \rho \boldsymbol{a}$, cf. [Ferrand et al. (2020)]

The "magic" of nondiscrimination

☞ $4F_3(r) = -\langle \frac{\rho'}{\rho} \boldsymbol{u} \cdot \boldsymbol{f} \rangle + \langle \boldsymbol{f} \cdot \boldsymbol{u}' \rangle + \langle \frac{\rho}{\rho'} \boldsymbol{u} \cdot \boldsymbol{f}' \rangle + \ldots = -\langle \rho' \boldsymbol{u} \cdot \boldsymbol{a} \rangle + \langle \rho \boldsymbol{a} \cdot \boldsymbol{u}' \rangle + \langle \boldsymbol{j} \cdot \boldsymbol{a}' \rangle + \ldots$

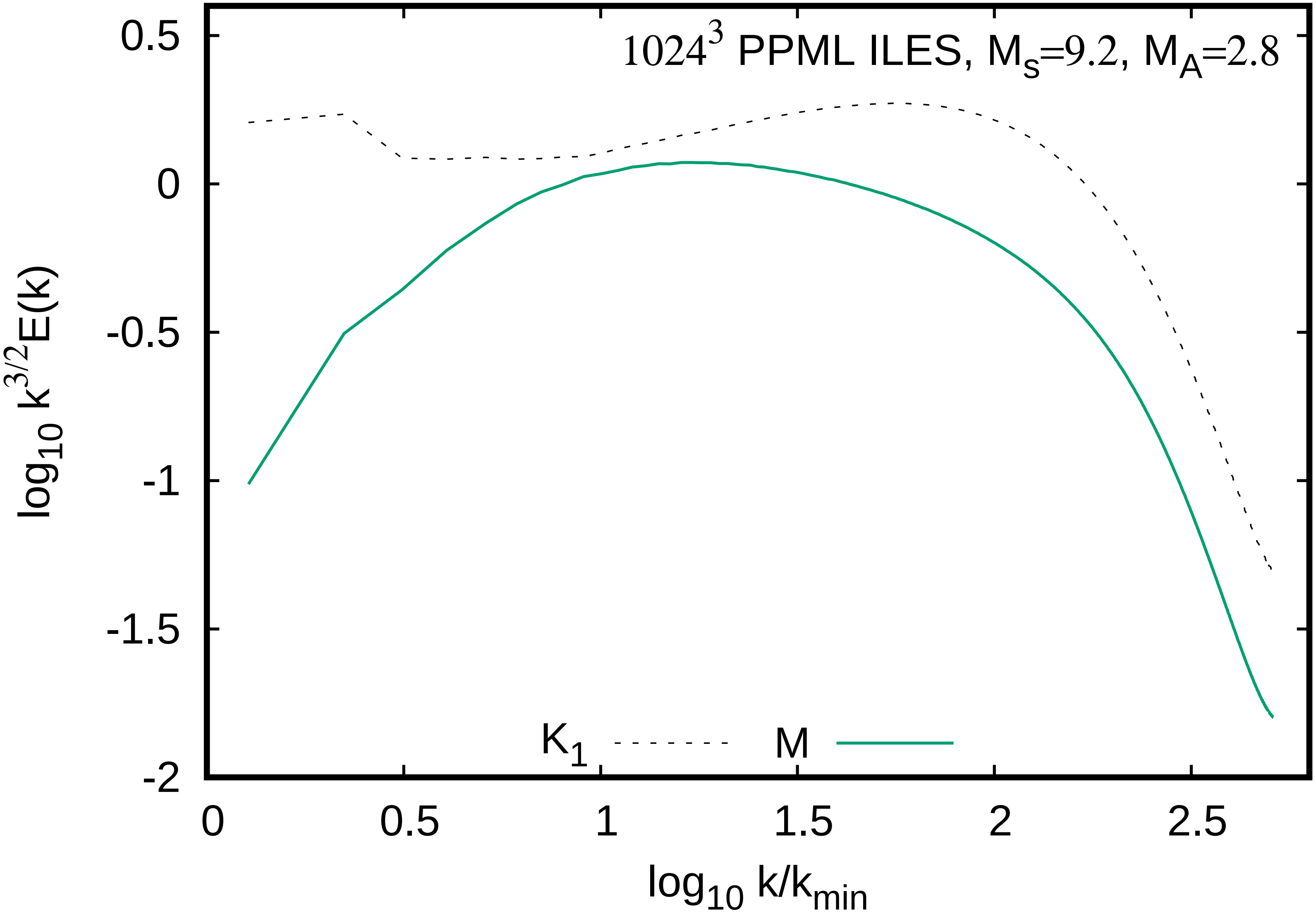
☞ The last term on the r.h.s. $\langle \boldsymbol{j} \cdot \boldsymbol{a}' \rangle \to 0$ at $r \ll r_f$

☞ The first two terms cancel each other exactly at $r \ll r_f$

☞ In numerical experiments, $F_3 \propto r^3$ outside the forcing interval at $r \ll r_f$

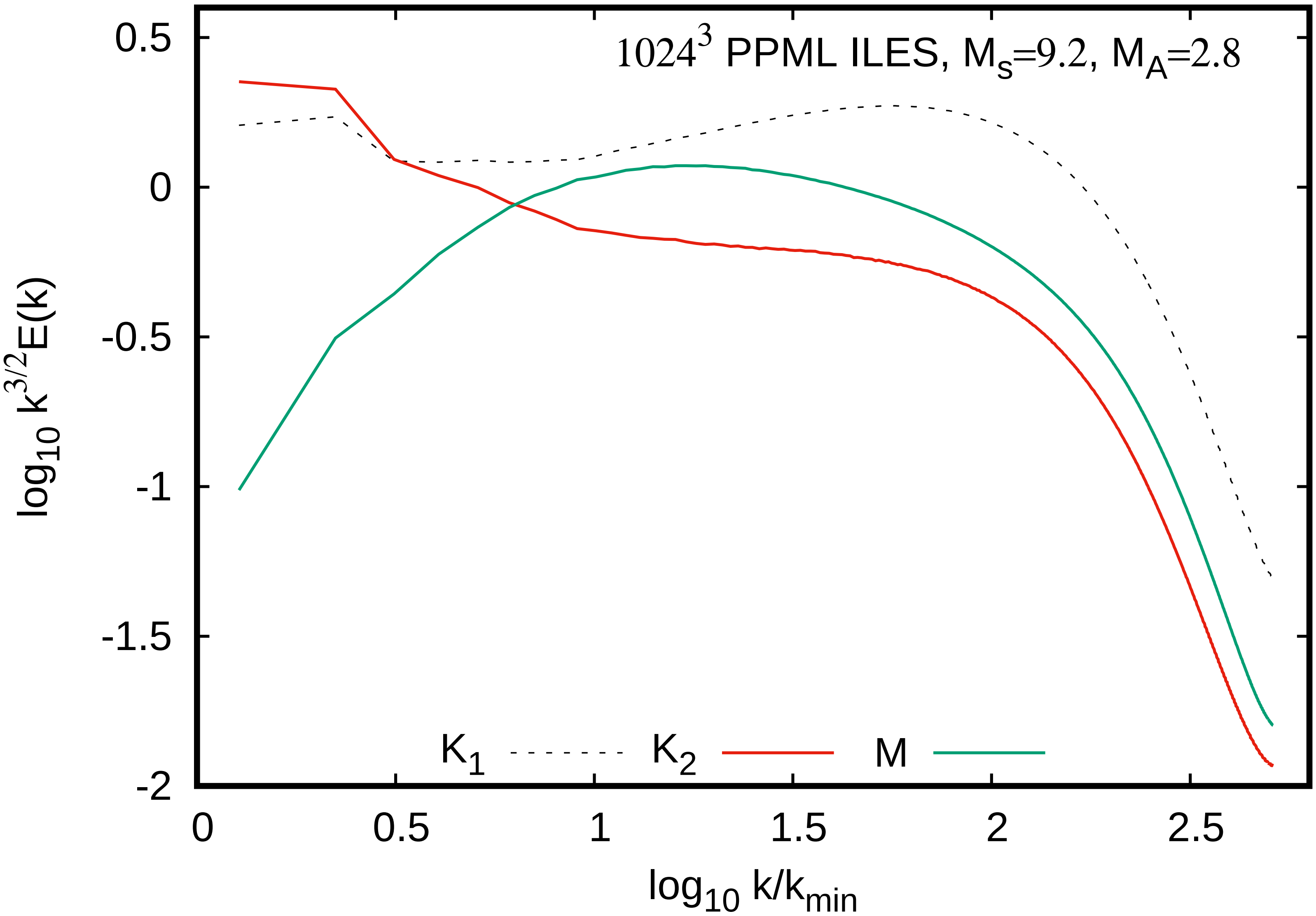
☞ In contrast $F_2 \propto r$ because one of the canceling terms is missing

$K_1(k)$



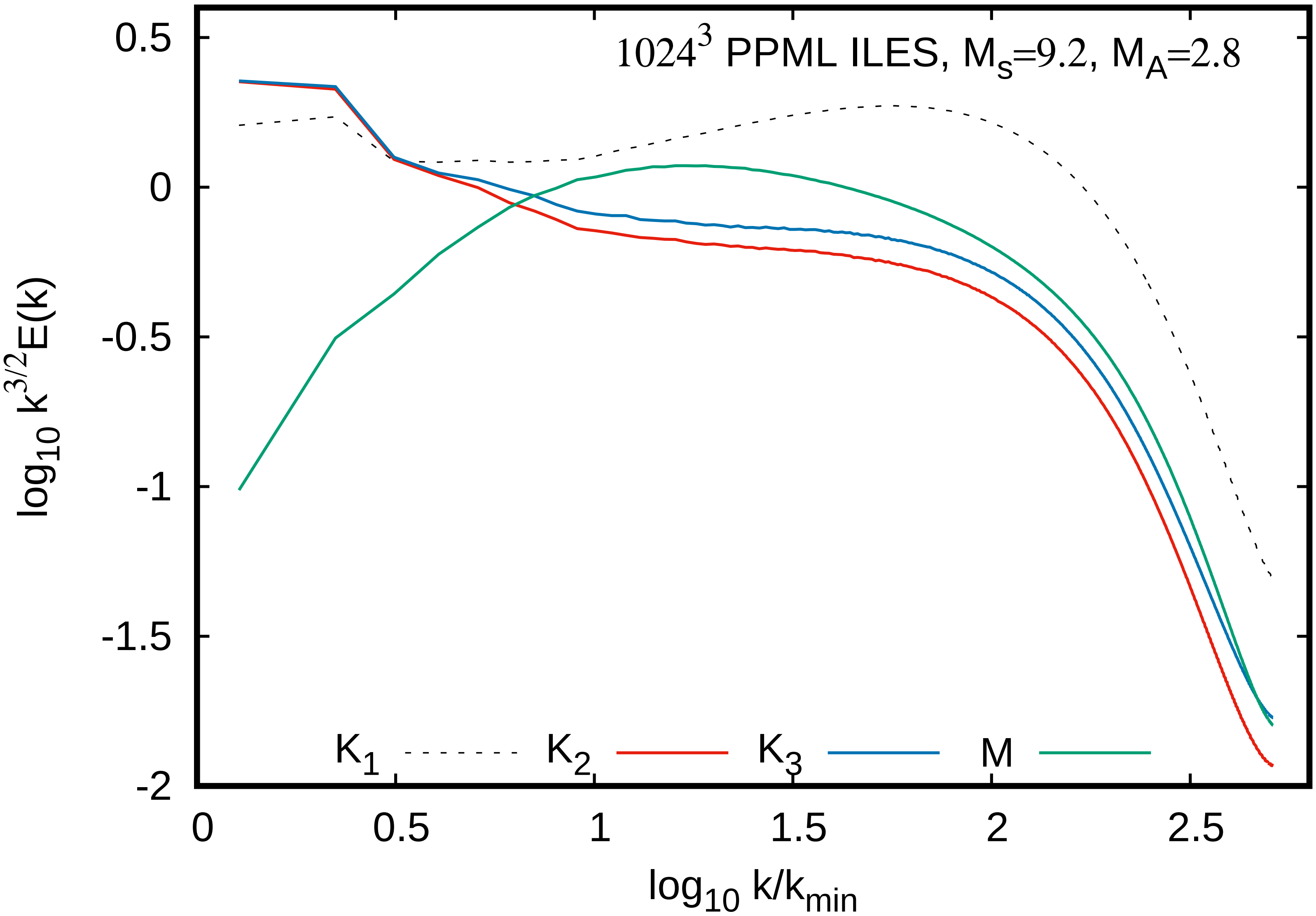
Compensated kinetic and magnetic energy spectra, $M_{\mathrm{s}} = 9.2$, $M_{\mathrm{A}} = 2.8$.

$K_1(k)$ vs. $K_2(k)$



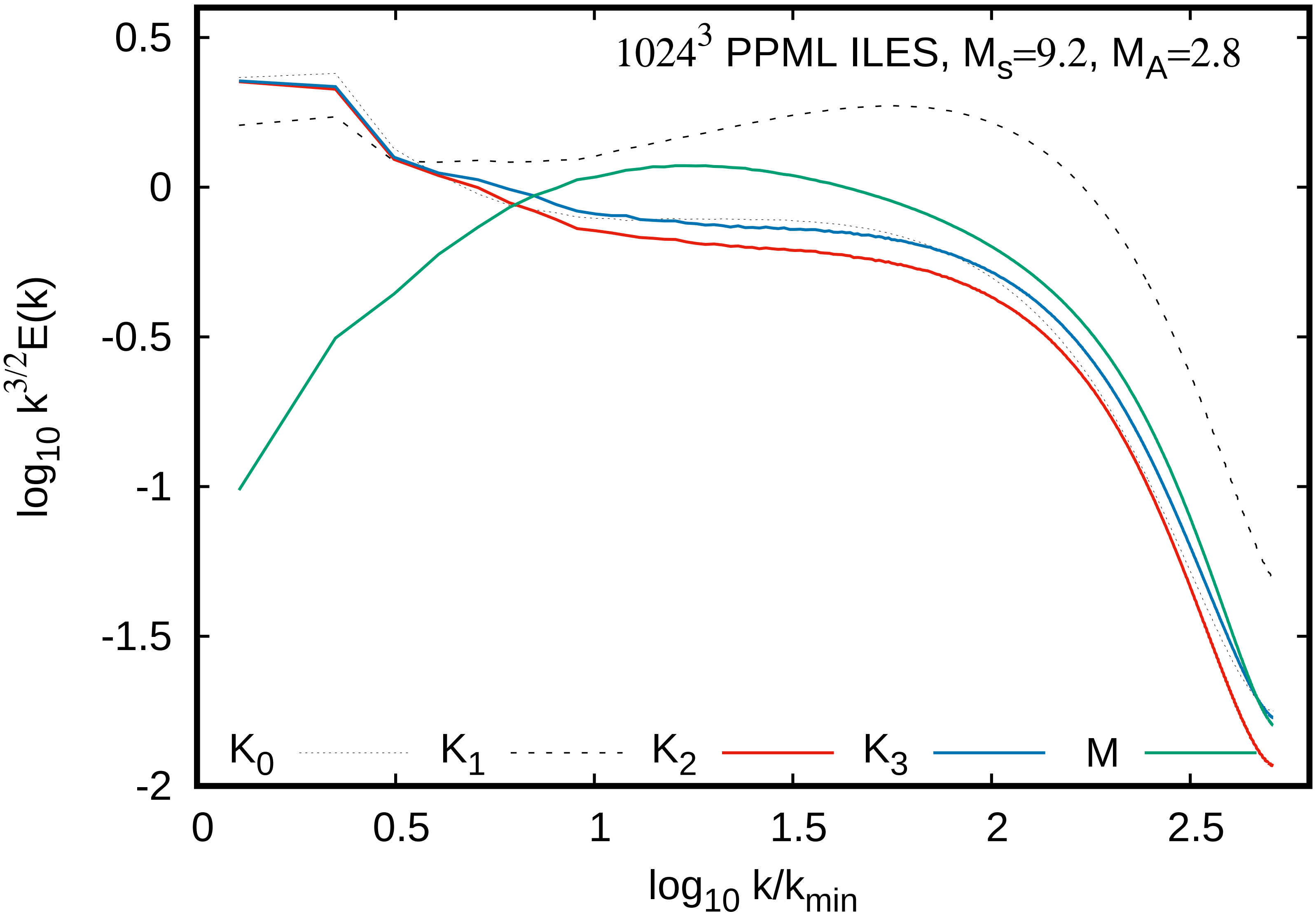
Compensated kinetic and magnetic energy spectra, $M_s = 9.2$, $M_A = 2.8$.

$K_1(k)$ vs. $K_2(k)$ vs. $K_3(k)$



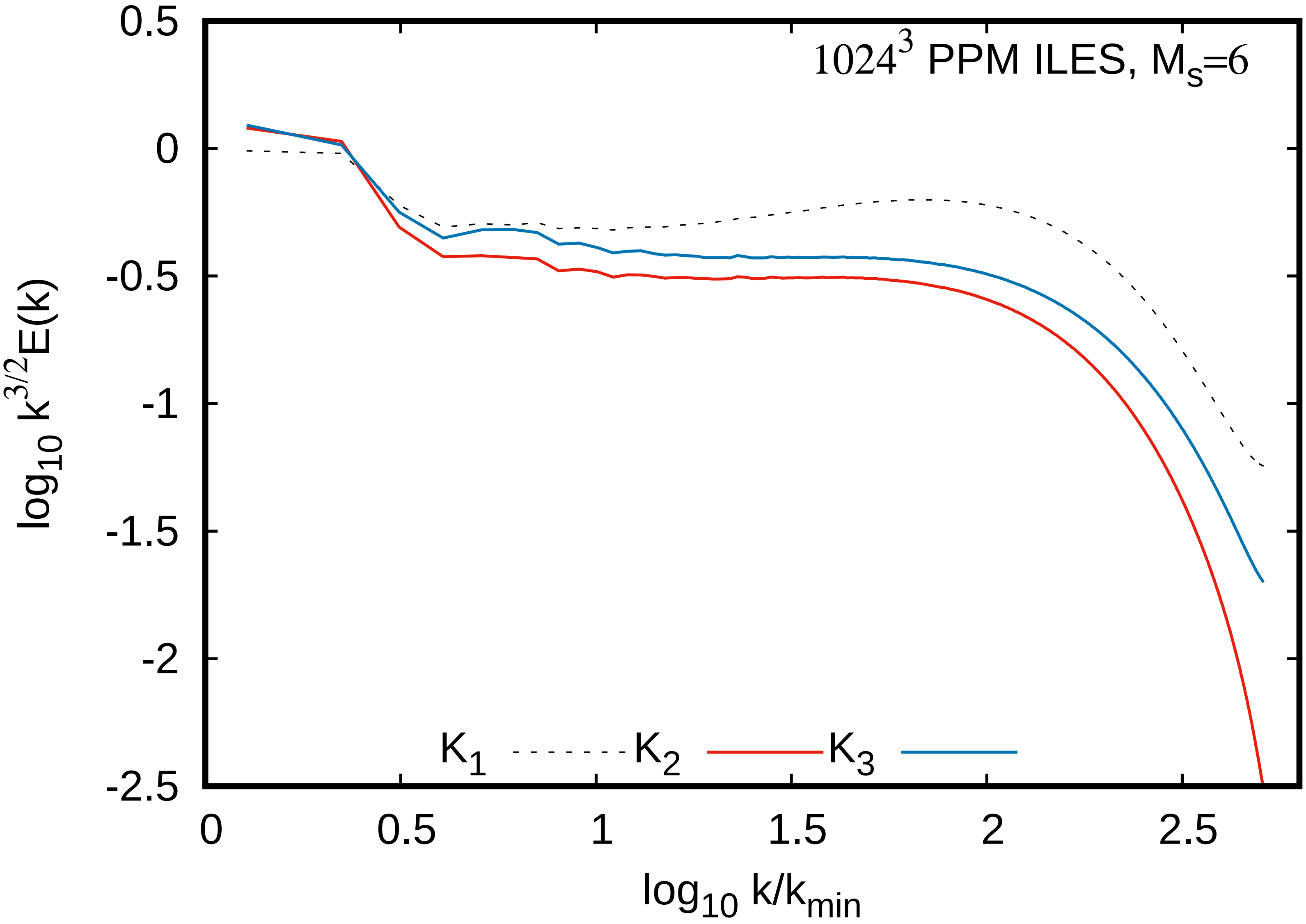
Compensated kinetic and magnetic energy spectra, $M_s = 9.2$, $M_A = 2.8$.

$K_0(k)$ vs. $K_1(k)$ vs. $K_2(k)$ vs. $K_3(k)$



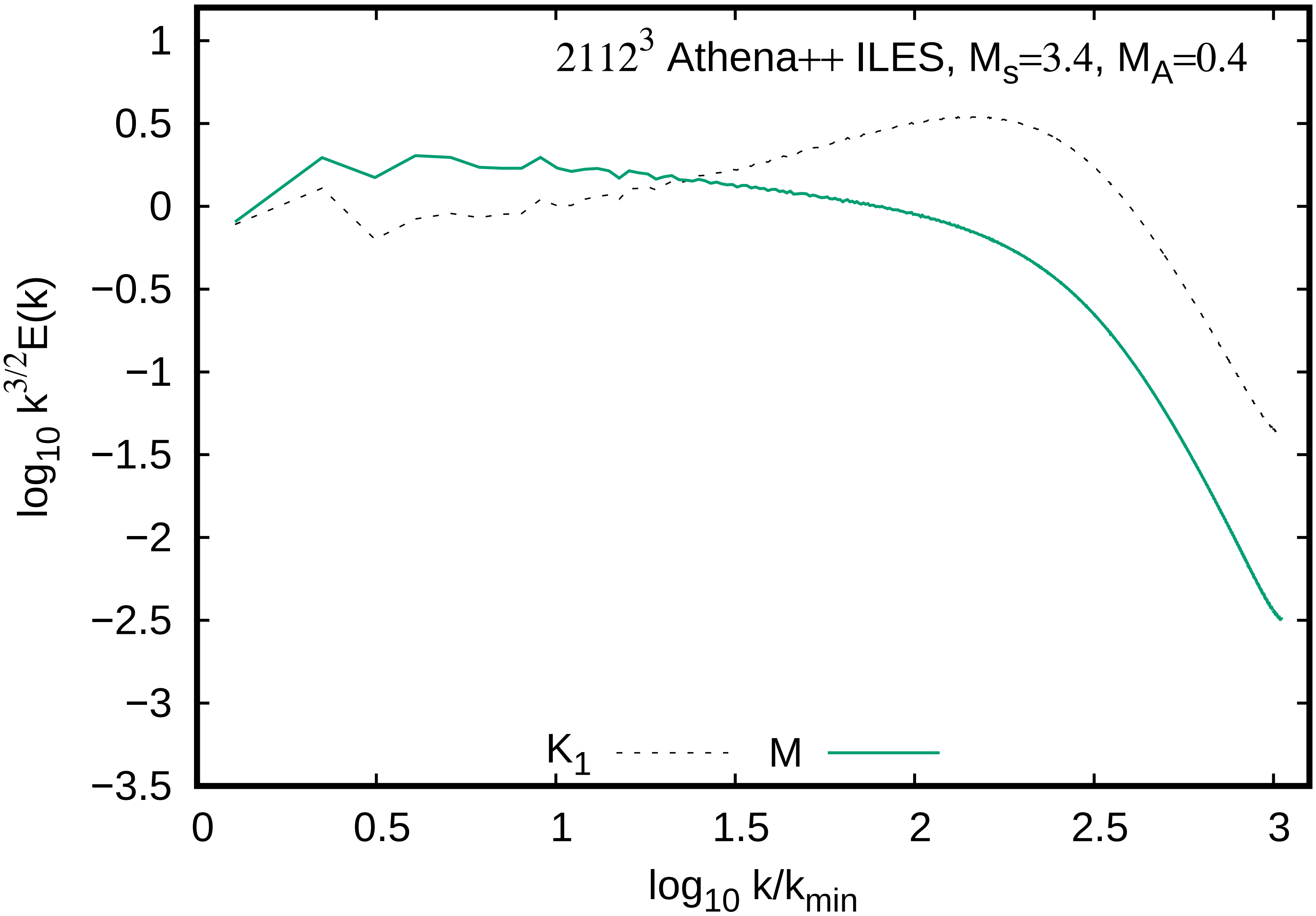
Compensated kinetic and magnetic energy spectra, $M_\mathrm{s} = 9.2$, $M_\mathrm{A} = 2.8$.

$K_1(k)$ vs. $K_2(k)$ vs. $K_3(k)$



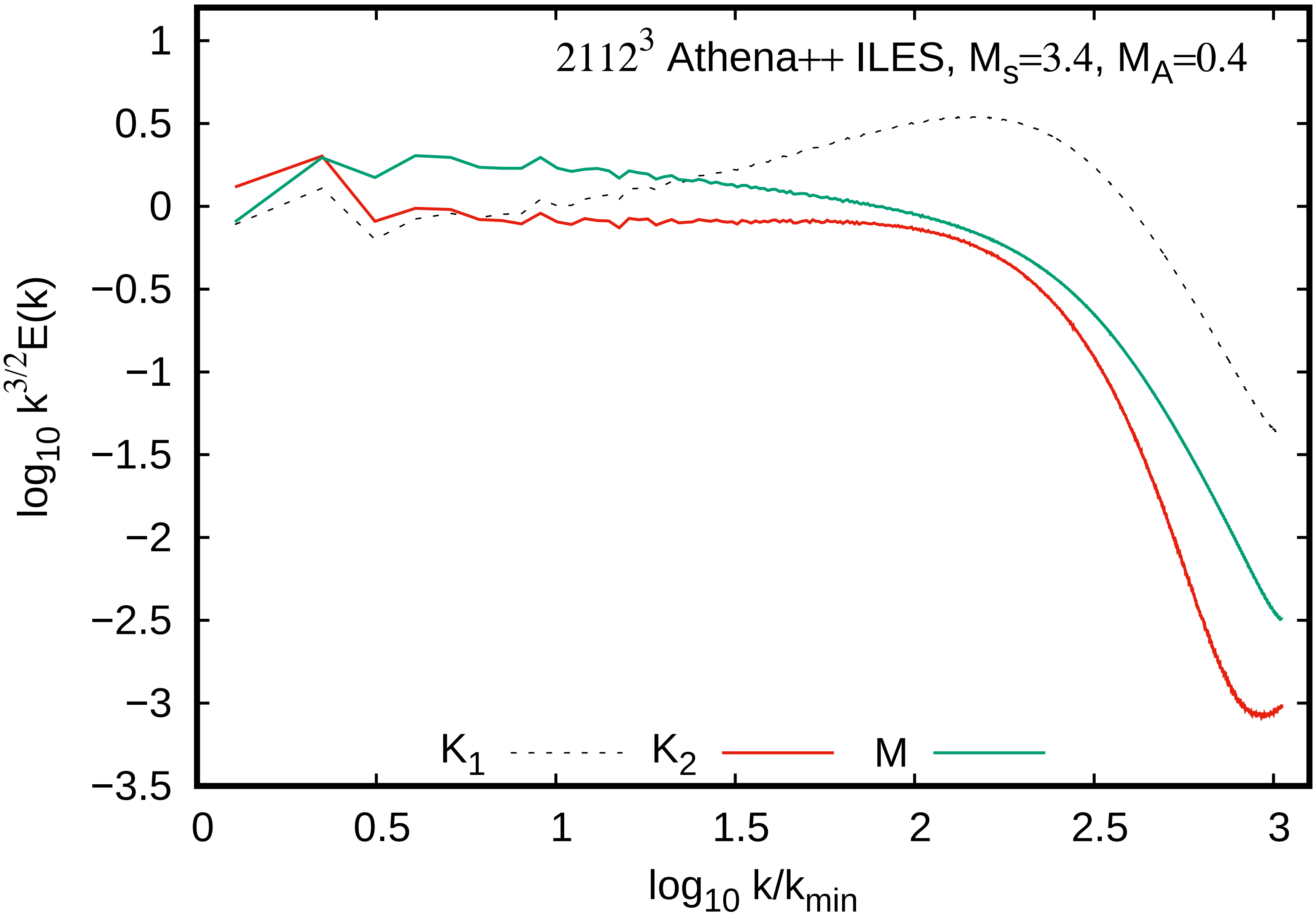
Compensated kinetic energy spectra, $M_s = 6$.

$K_1(k)$



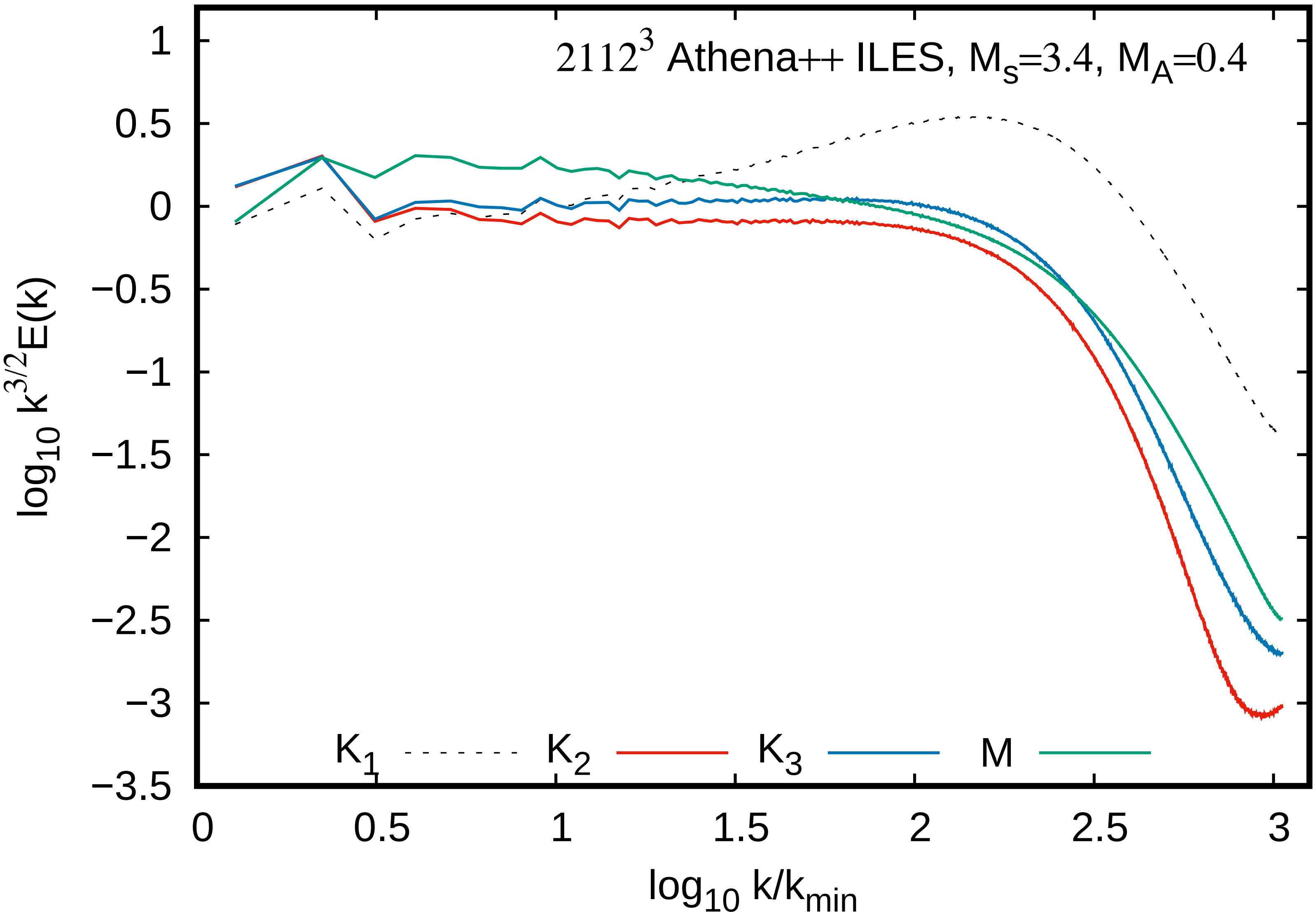
Compensated kinetic and magnetic energy spectra; MP case w. $M_\mathrm{s} = 3.4$, $M_\mathrm{A} = 0.4$.

$K_1(k)$ vs. $K_2(k)$



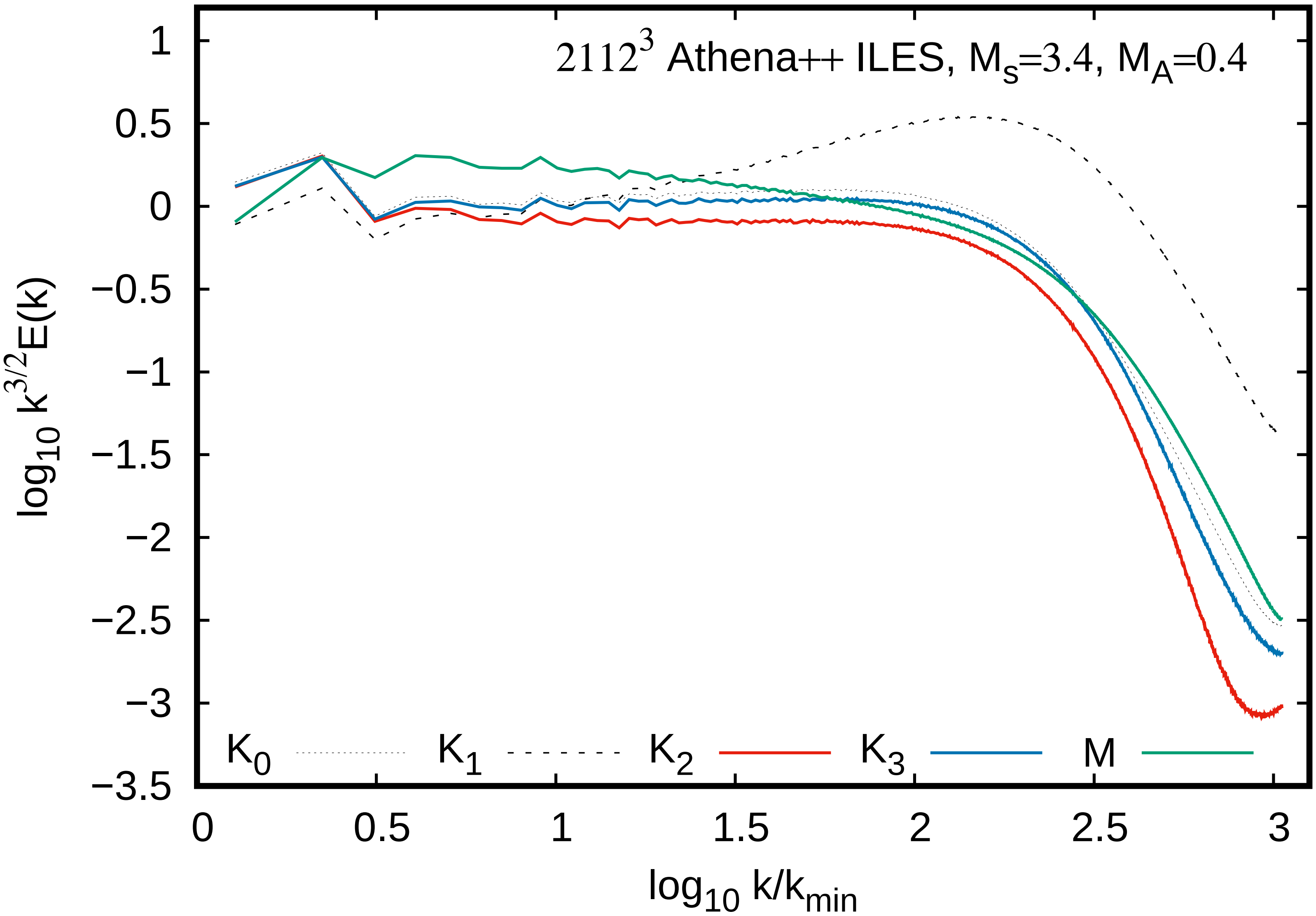
Compensated kinetic and magnetic energy spectra; MP case w. $M_\mathrm{s} = 3.4$, $M_\mathrm{A} = 0.4$.

$K_1(k)$ vs. $K_2(k)$ vs. $K_3(k)$



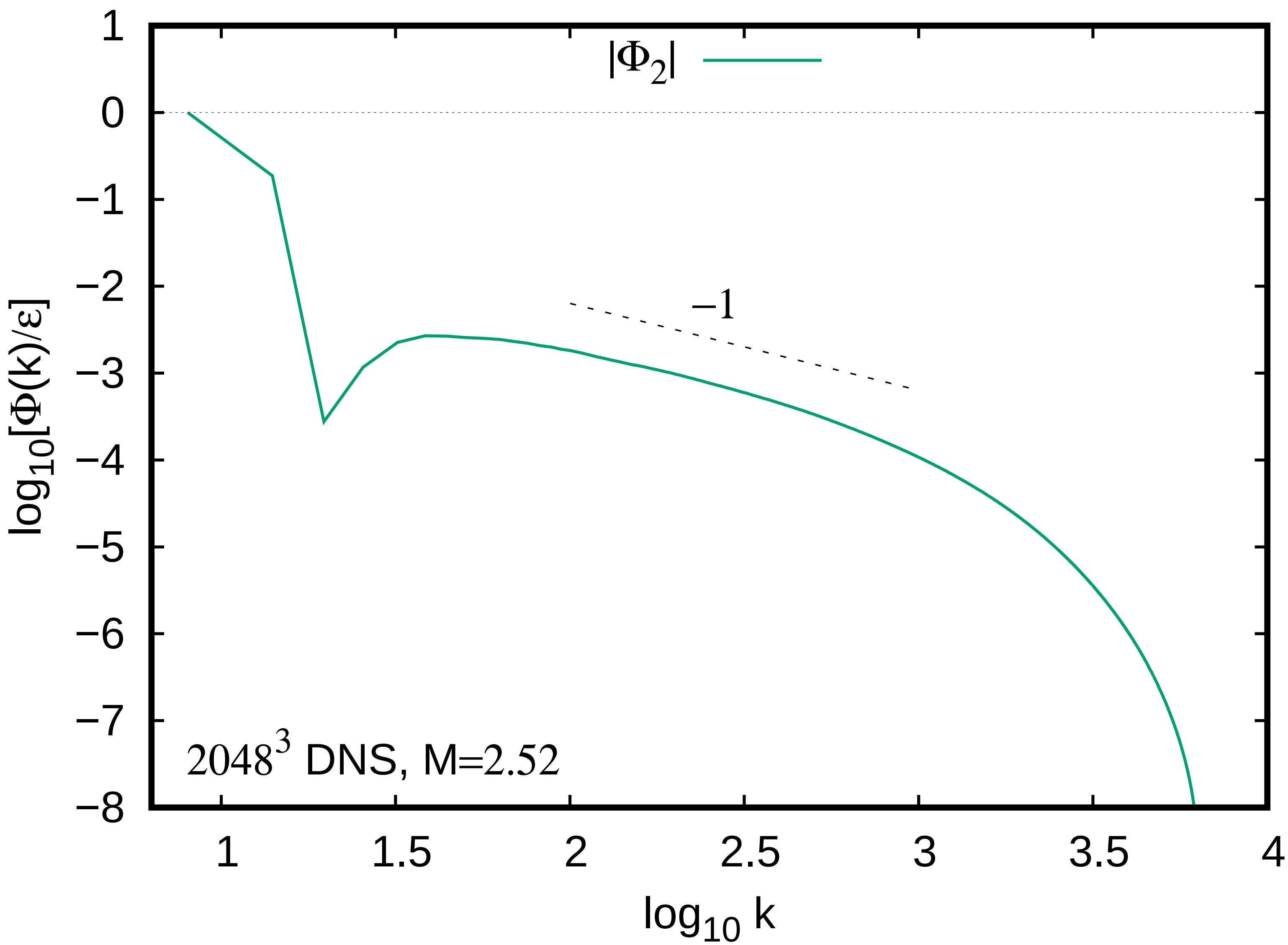
Compensated kinetic and magnetic energy spectra; MP case w. $M_{\rm s} = 3.4$, $M_{\rm A} = 0.4$.

$K_0(k)$ vs. $K_1(k)$ vs. $K_2(k)$ vs. $K_3(k)$



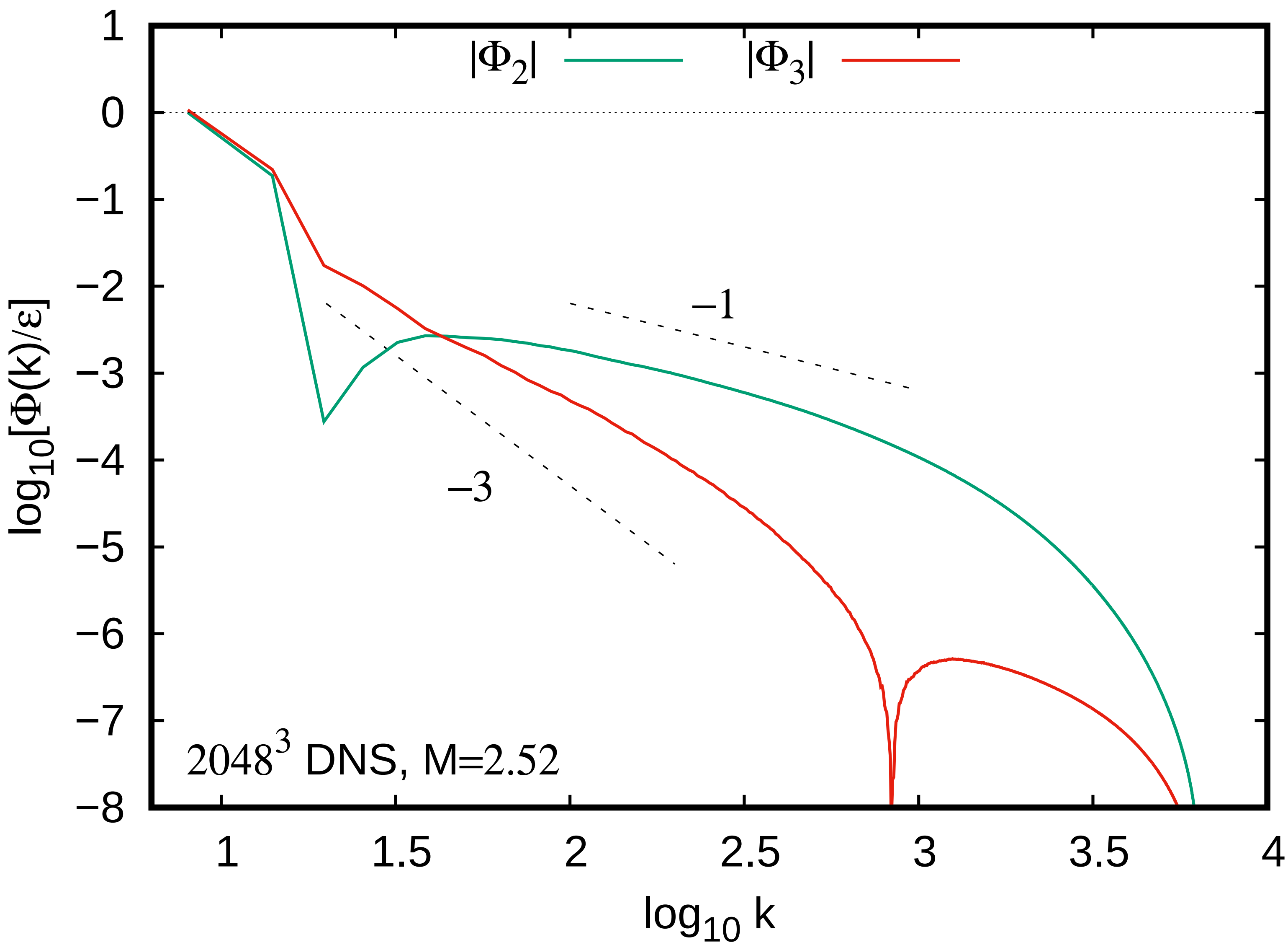
Compensated kinetic and magnetic energy spectra; MP case w. $M_\mathrm{s} = 3.4$, $M_\mathrm{A} = 0.4$.

Energy injection by stochastic forcing



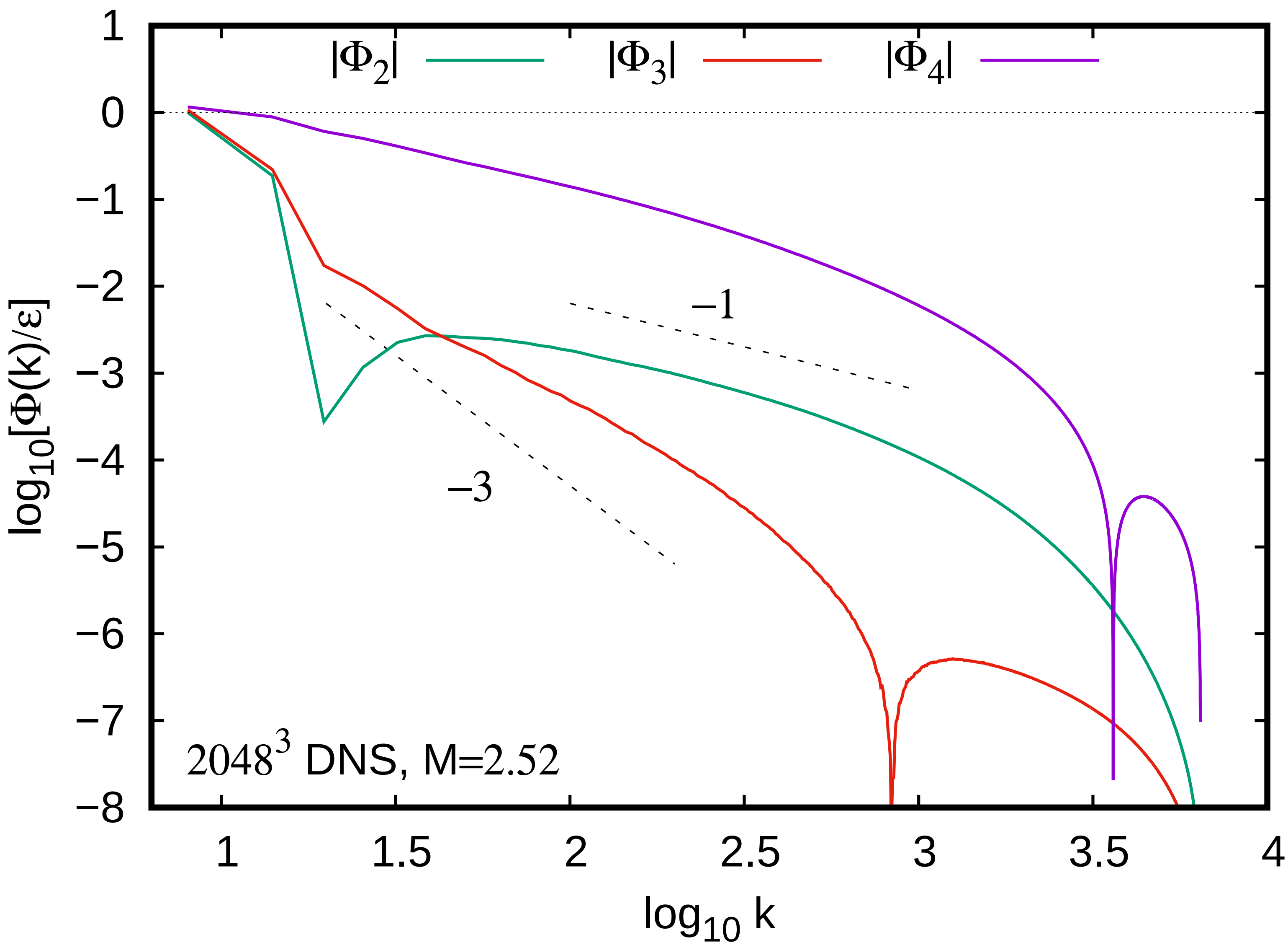
The K_2 formulation yields linear contamination of the inertial range.

Energy injection by stochastic forcing



Contamination in the K_3 formulation is benign as it decays $\propto k^{-3}$ in the inertial range.

Energy injection by stochastic forcing



Clearly, the K_2 and K_4 formulations must be rejected due to contamination of the inertial range.

Energy corrrelator: Evolving trends

☞ Galtier & Banerjee (2011)

$$E_{\mathrm{GB11}}(\boldsymbol{r},t) = \frac{1}{4}\langle \boldsymbol{j}\cdot\boldsymbol{u}' + \boldsymbol{j}'\cdot\boldsymbol{u}\rangle + \frac{1}{2}\langle \rho e' + \rho' e\rangle, \qquad (4)$$

☞ Banerjee & Kritsuk (2017)

$$E_{\mathrm{BK17}}(\boldsymbol{r},t) = \frac{1}{4}\langle \boldsymbol{j}\cdot\boldsymbol{u}' + \boldsymbol{j}'\cdot\boldsymbol{u}\rangle + \frac{1}{4}\langle \rho e' + \rho' e\rangle + \frac{1}{2}\langle \rho e\rangle, \qquad (5)$$

☞ Ferrand et al. (2020)

$$E_{\mathrm{F20}}(\boldsymbol{r},t) = \frac{1}{4}\langle \boldsymbol{j}\cdot\boldsymbol{u}' + \boldsymbol{j}'\cdot\boldsymbol{u}\rangle - \frac{1}{8}\langle \rho \boldsymbol{u}'^2 + \rho' \boldsymbol{u}^2\rangle + \frac{1}{4}\langle \boldsymbol{j}\cdot\boldsymbol{u}\rangle + \frac{1}{4}\langle \rho e' + \rho' e\rangle + \frac{1}{2}\langle \rho e\rangle. \qquad (6)$$

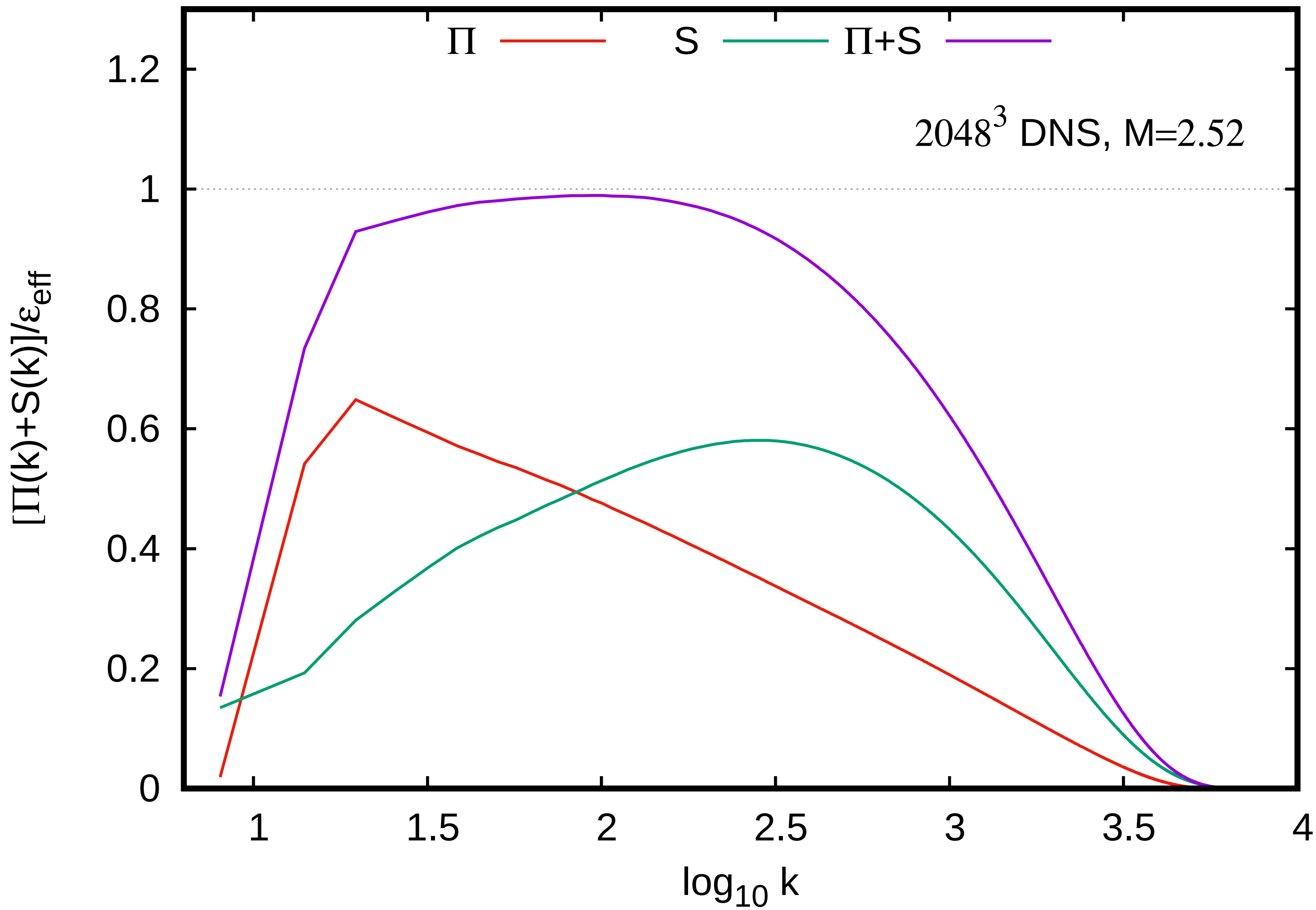
☞ Scale-by-scale balance equations for the Ferrand et al. (2020) formulation

$$\partial_t E(\boldsymbol{r},t) = T_K(\boldsymbol{r},t) + F(\boldsymbol{r},t) + D(\boldsymbol{r},t), \qquad (7)$$

$$\partial_t K(\boldsymbol{r},t) = T_K(\boldsymbol{r},t) - X_{K\to U}(\boldsymbol{r},t) + F(\boldsymbol{r},t) + D(\boldsymbol{r},t), \qquad (8)$$

$$\partial_t U(\boldsymbol{r},t) = X_{K\to U}(\boldsymbol{r},t). \qquad (9)$$

Scale-by-scale balance in Fourier space



Compressibility correction in K_3 significantly affects kinetic energy transfer across scales.

Fluctuating kinetic energy structure functions

☞ **Incompressible fluids**

Energy invariant is quadratic: $E = \langle u^2 \rangle / 2$

$\Rightarrow$ unique 2nd-order structure function: $S_E(\boldsymbol{r}) = \langle (\delta \boldsymbol{u})^2 \rangle / 2$

☞ **Compressible fluids**

Kinetic energy is not quadratic: $K = \langle \rho \boldsymbol{u} \cdot \boldsymbol{u} \rangle / 2$

$\Rightarrow$ no unique structure function version

Available options for fluctuations:

1. $S_{K,1}(\boldsymbol{r}) = \langle (\delta \boldsymbol{w}^2) \rangle / 2$, where $\boldsymbol{w} \equiv \rho^{1/2} \boldsymbol{u}$ [Kida & Orszag (1990)]
2. $S_{K,2}(\boldsymbol{r}) = \langle \delta \boldsymbol{j} \cdot \delta \boldsymbol{u} \rangle / 2$, where $\boldsymbol{j} \equiv \rho \boldsymbol{u}$ cf. [Graham et al. (2010)]
3. $S_{K,3}(\boldsymbol{r}) = \langle \overline{\delta} \rho (\delta \boldsymbol{u})^2 \rangle / 2$, where $\overline{\delta} \rho \equiv \frac{1}{2}(\rho' + \rho)$ [Ferrand et al. (2020)]
4. $S_{K,4}(\boldsymbol{r}) = \langle \delta \rho (\delta \boldsymbol{u})^2 \rangle / 2 \Leftarrow$ does not comply with incompressible limit

☞ A von Kármán-Howarth-Monin equation [Ferrand et al. (2020)]

$$\nabla_{\boldsymbol{r}} \cdot \left\langle \bar{\delta} \rho (\delta \boldsymbol{u})^2 \delta \boldsymbol{u} \right\rangle - \frac{1}{2} \left\langle (\rho \theta' + \rho' \theta)(\delta \boldsymbol{u})^2 \right\rangle = -4\varepsilon$$

Summary

☞ Only E_{F20} definition of the energy correlation function meets the following criteria:

- ✔ Incompressible limit
- ✔ Acoustic limit
- ✘ Forcing contamination in the inertial range
- ✘ Viscous dissipation in the inertial range at $Re \gg 1$
- ✘ Gibbs free energy cascade

☞ Problem solved?

Acknowledgments

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