Finite-size scaling (FSS) of net proton susceptibility: What does it imply? Can it be interpreted as a CP signal?

G. Kovacs, L. McLerran, A. Sorensen

History of FSS

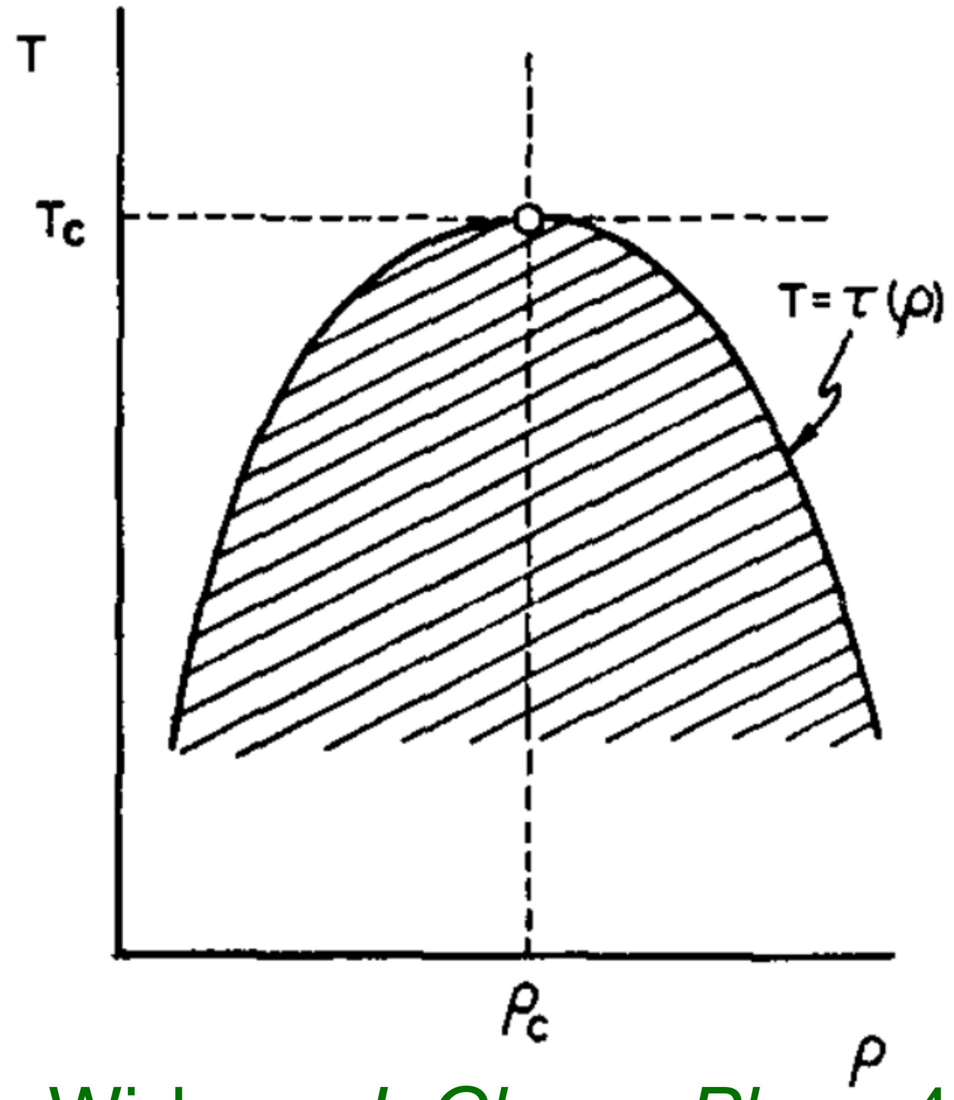
Widom scaling 1965: Widom suggests that near a CP, a singular part of the free energy follows a homogenous function, implying scaling behaviors, hyper-scaling relationships, and explaining previously disparate scaling observations.

Kadanoff in 1966: If a system is scale free, rescaling (grouping/ decimation) leads to the same form for the free energy but with different couplings, giving a microscopic, intuitive picture of Widom scaling. Wilson formalizes this in RG, deriving universality classes and critical exponents from fixed points.

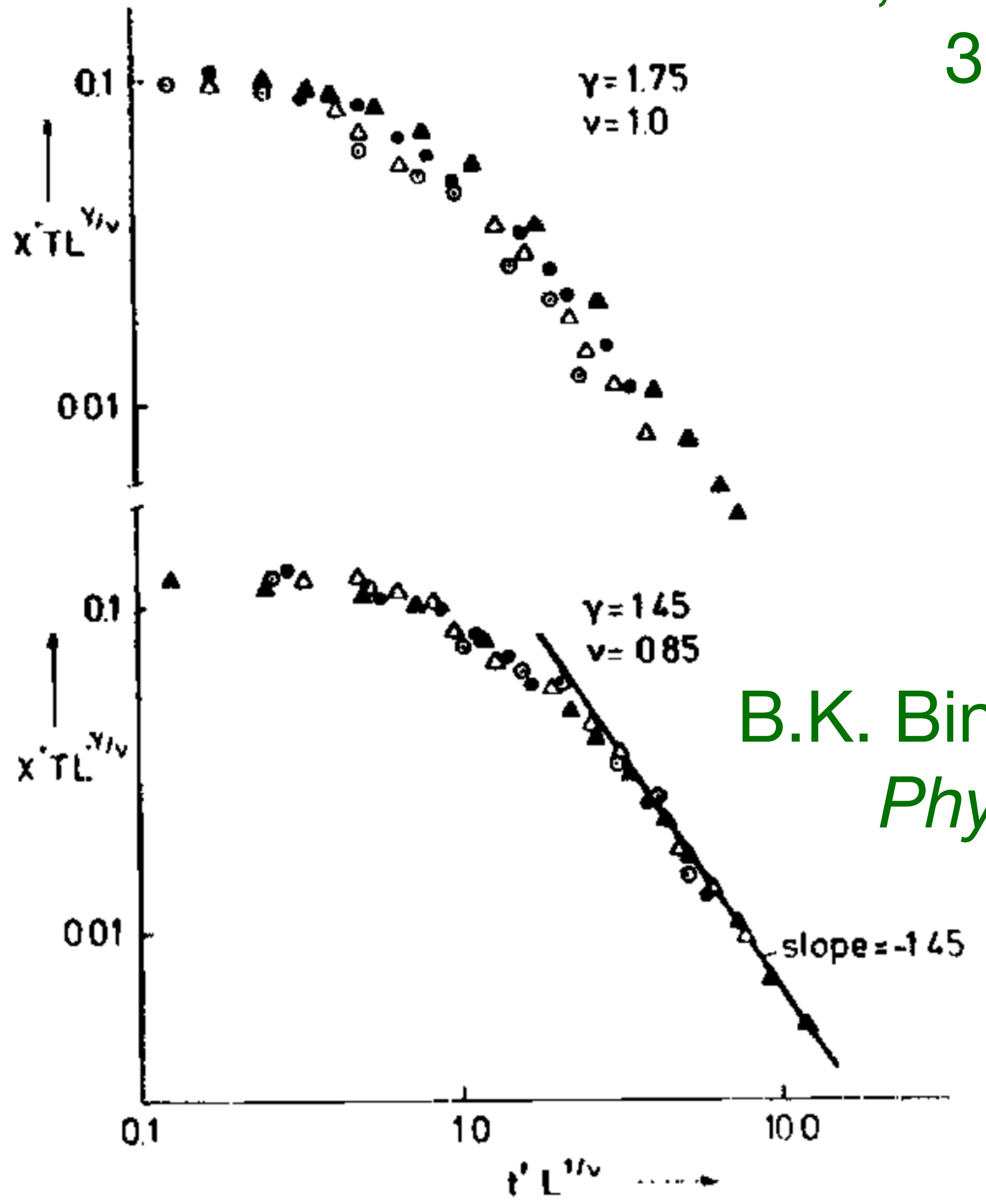
Fischer and Barber 1972: In finite systems, correlations lengths are bounded, peaks are rounded and shifted. Treat size as another scaling variable in a homogeneous function describing singular part of the free energy.

Binder in 1980s: Monte Carlo simulations, sub-volumes, various methods for finding CP, Binder cumulants, etc.

Also in 1980s: FSS began being used in early lattice QCD studies.



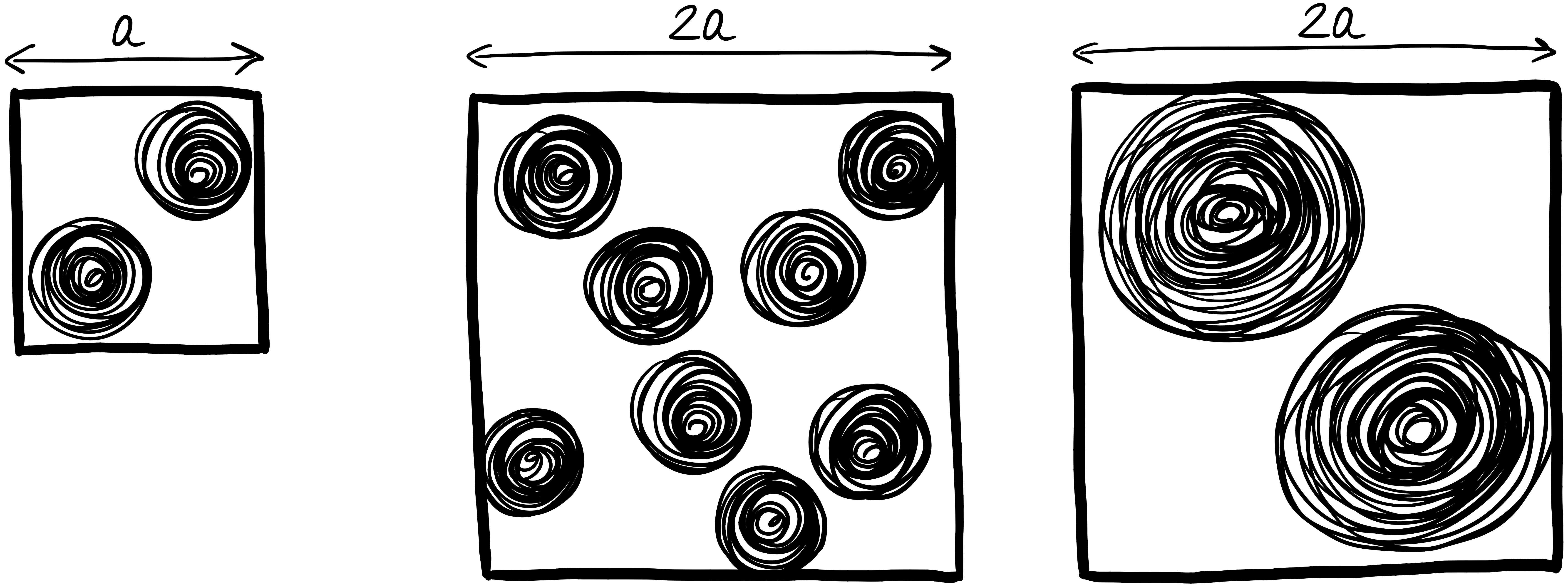
B. Widom, *J. Chem. Phys.* 43, 3898–3905 (1965)



B.K. Binder, D. P. Landau, *Phys. Rev. B* 20,1941 (1980)

Introduction to FSS

The foundation for RG and FSS is invariance under scale transformation:



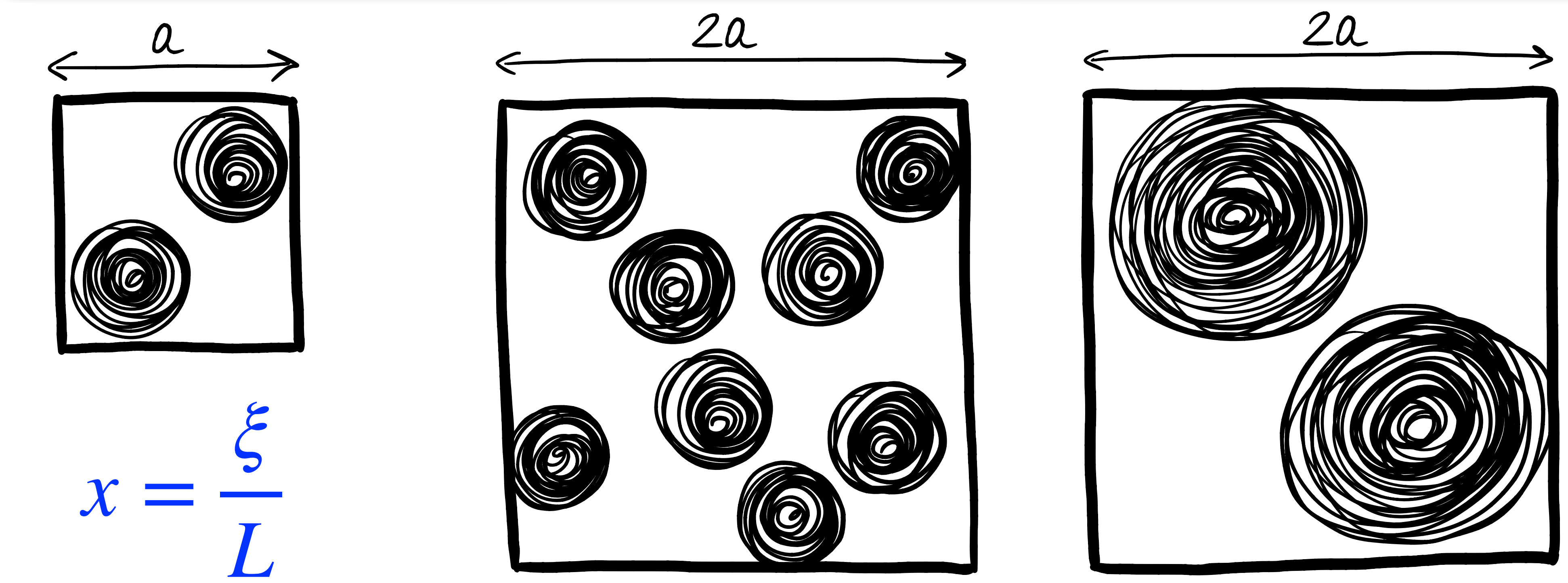
$L = a,\ \xi = \frac{a}{2} \ \Rightarrow\ x = \frac{1}{2}$ $\quad$ $L = 2a,\ \xi = \frac{a}{2} \ \Rightarrow\ x = \frac{1}{4}$ $\quad$ $L = 2a,\ \xi = a \ \Rightarrow\ x = \frac{1}{2}$

What the system "looks like" doesn't depend on the individual scales ξ, L, but on their ratio $x = \frac{\xi}{L}$

Usually: ξ constrained by finite range of the interaction, not thermodynamic conditions.
Near the CP: changing T or μ_B leads to appreciable changes in ξ (diverging when $T \approx T_c$ and $\mu_B \approx \mu_{B,c}$)

As a result: if you change L and want to get a given x, simply adjust T to get the necessary ξ
(more important for us: if you change T (i.e., ξ) and want to get a given x, adjust L)

Introduction to FSS



$x = \frac{\xi}{L}$

$L = a,\ \xi = \frac{a}{2} \ \Rightarrow \ x = \frac{1}{2}$ $\quad$ $L = 2a,\ \xi = \frac{a}{2} \ \Rightarrow \ x = \frac{1}{4}$ $\quad$ $L = 2a,\ \xi = a \ \Rightarrow \ x = \frac{1}{2}$

$$\xi \sim \left|\frac{T - T_c}{T_c}\right|^{-\nu} = |t|^{-\nu}$$

scale which controls the system's behavior: $x = \frac{|t|^{-\nu}}{L}$

convenient to use *the scaling variable* $u = x^{-1/\nu} = |t| L^{1/\nu}$

plotting observables at different T and L against u:
systems which "look the same" will have the same position on the u-axis

Can we also align observables measured at different T and L? Yes!

provided that we use the correct value of u = we know T_c

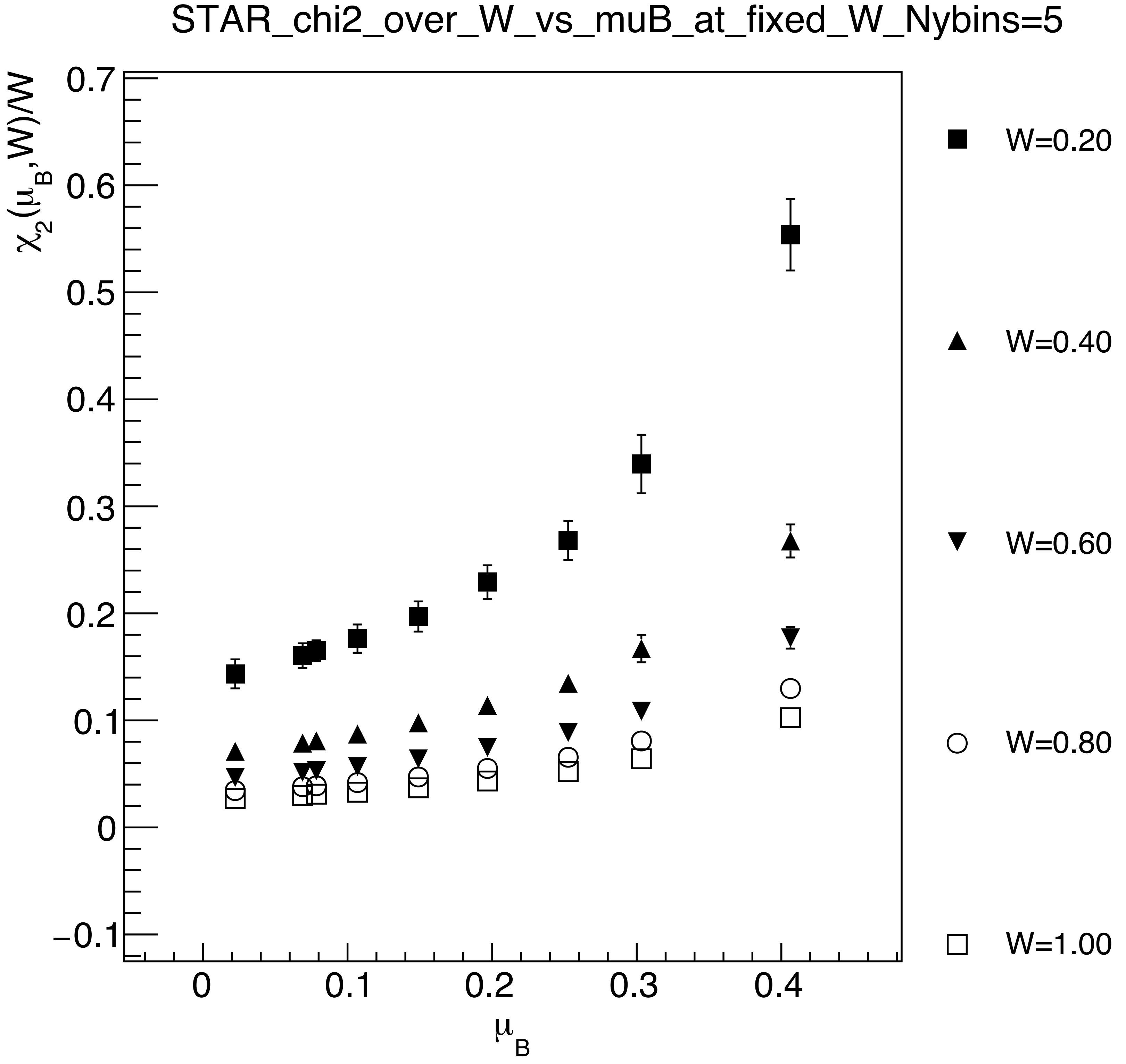
susceptibility: $\chi \sim |t|^{-\gamma} \sim \xi^{\gamma/\nu} = (xL)^{\gamma/\nu} = u^{-\gamma} L^{\gamma/\nu}$

one can find T_c by plotting:
test different T_c until things align!

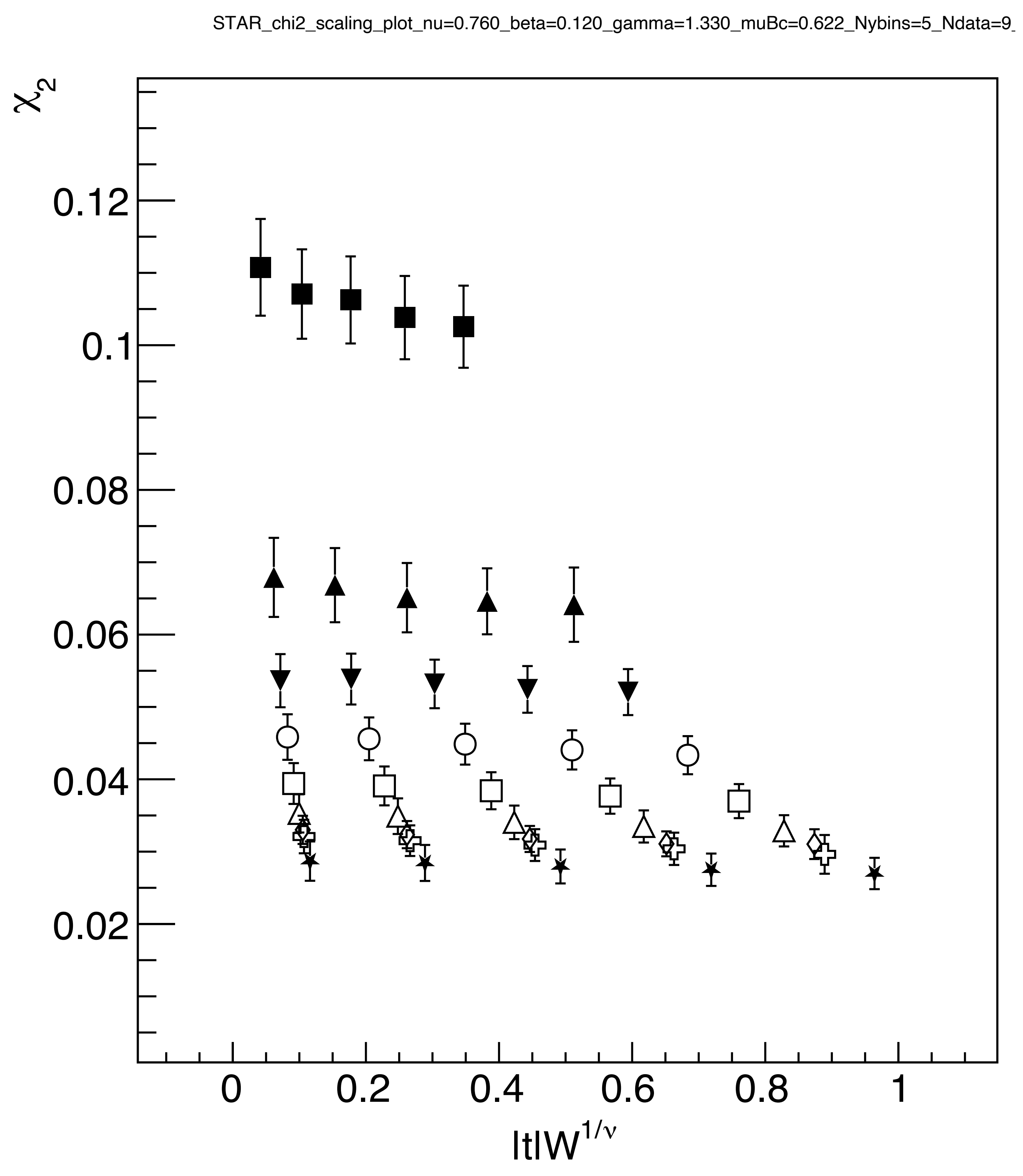
plotting $\tilde{\chi} = \chi L^{-\gamma/\nu}$ at different T and L against u:
systems which "look the same" will have the same position on the u-axis *and* the same value on the $\tilde{\chi}$-axis

Introduction to FSS

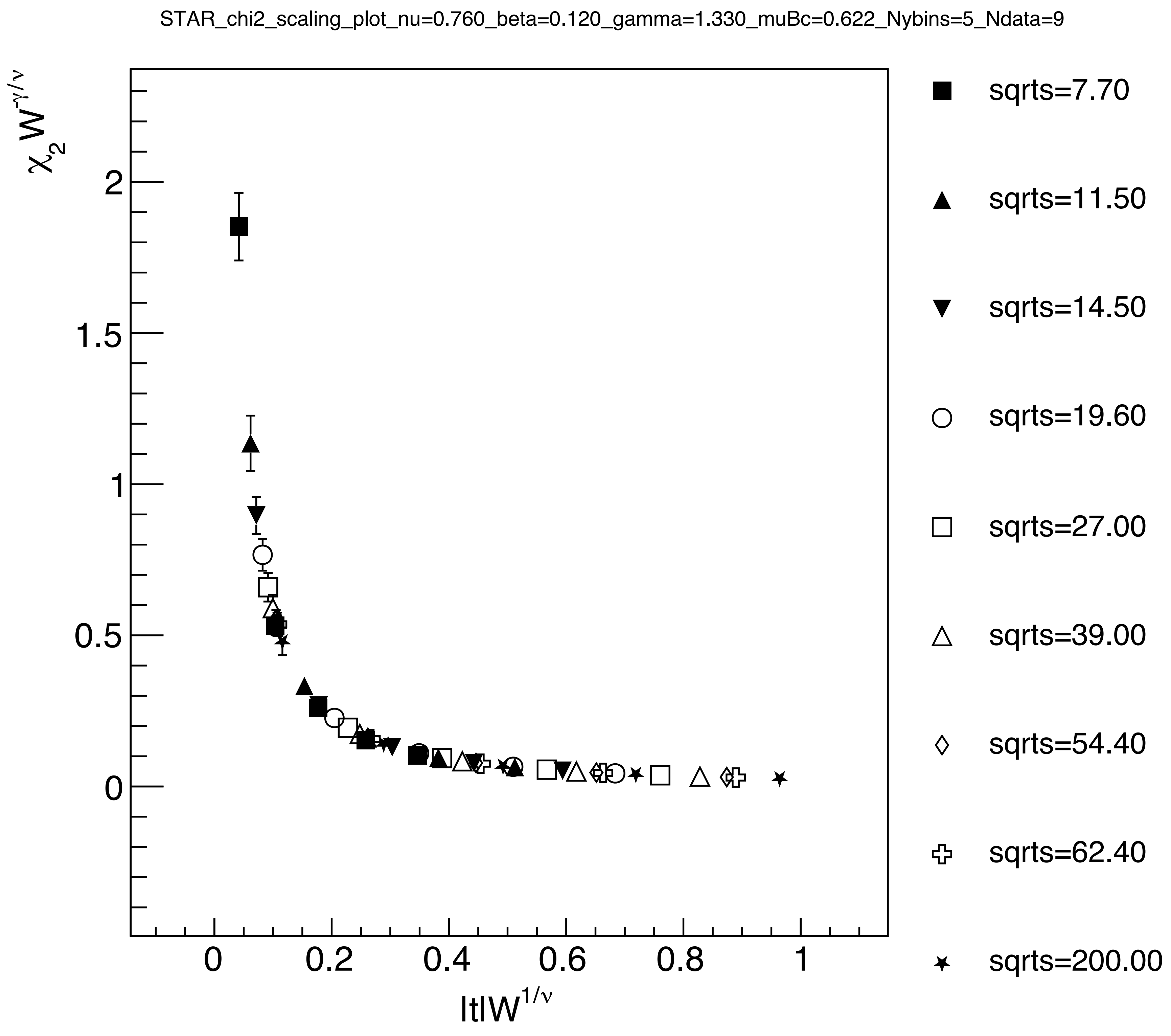
unscaled:



use u on the x-axis:



use u and scale χ_2:



$$t = \frac{\mu_B - \mu_c}{\mu_c}$$

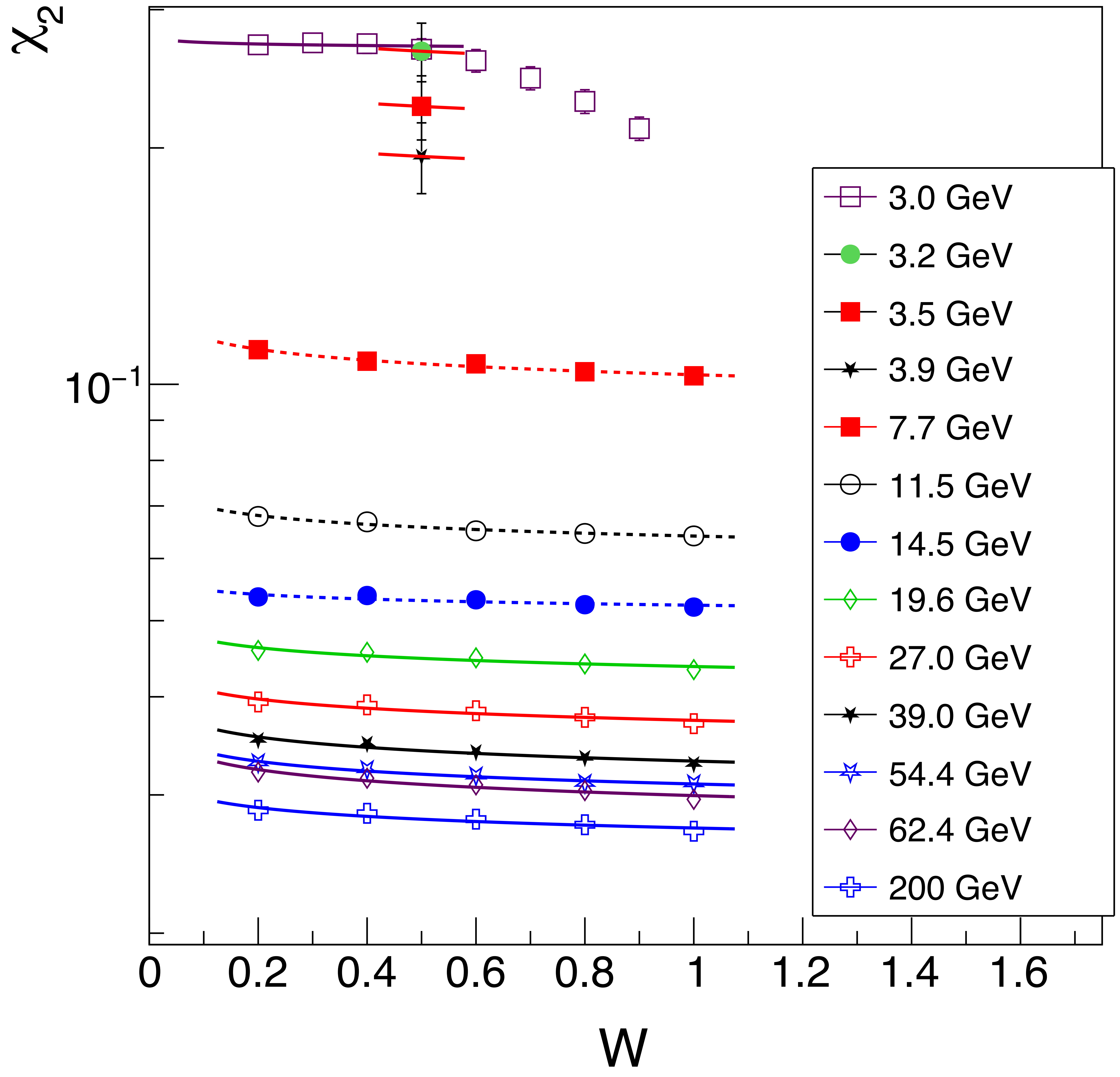
Calculating Susceptibility

Calculate susceptibility χ_2 in sub-volumes from the second moment C_2 of the net-baryon number vs rapidity bin width W.

$$\chi_2(t, W) = \frac{C_2}{VT^3} = \frac{C_2}{WdV/dyT^3}$$

where $t = (\mu - \mu_c)/\mu_c$.

Thermal model fits give dV/dy, μ and T.



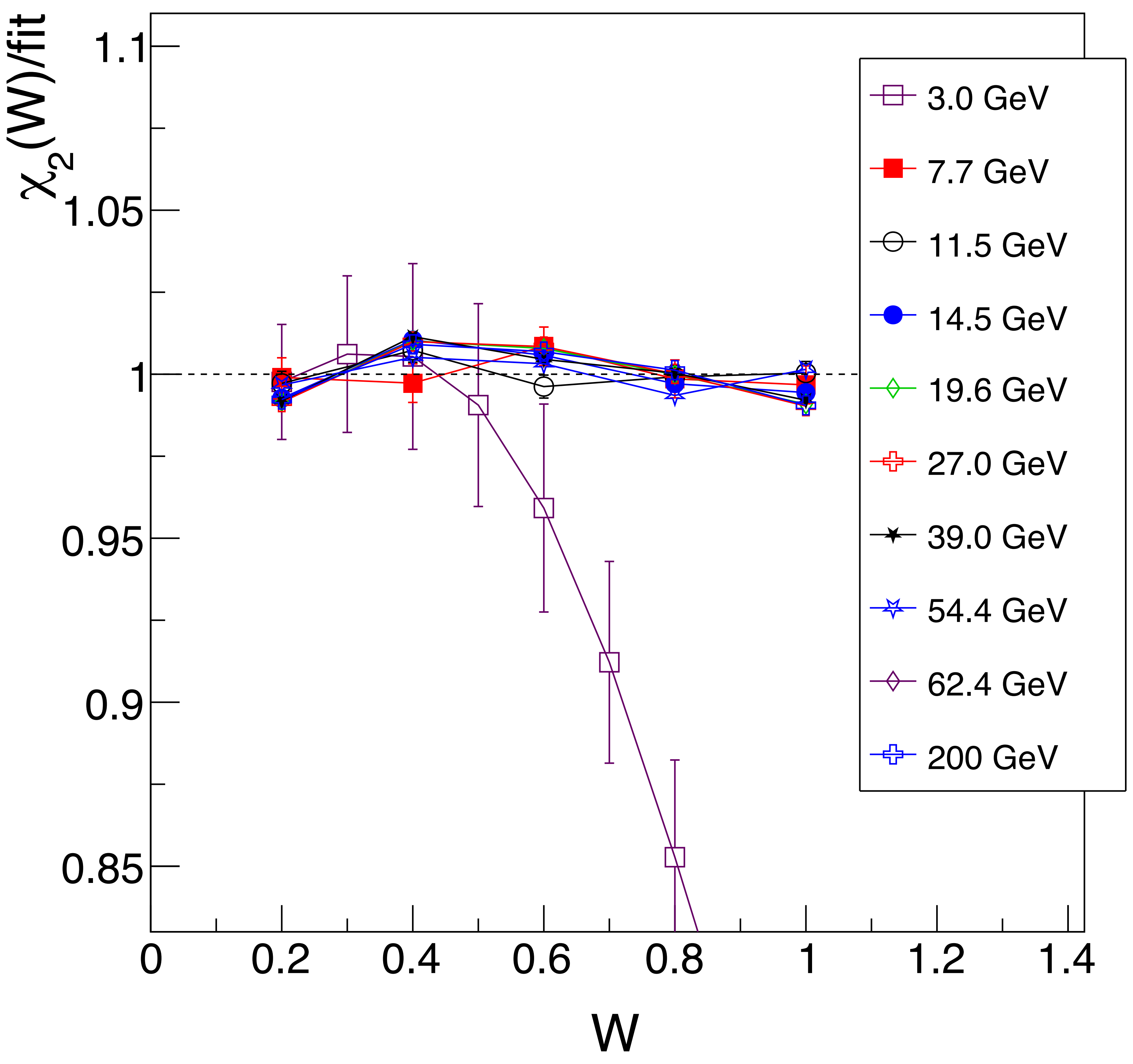
Above 7.7 GeV, χ_2 depends weakly on W:

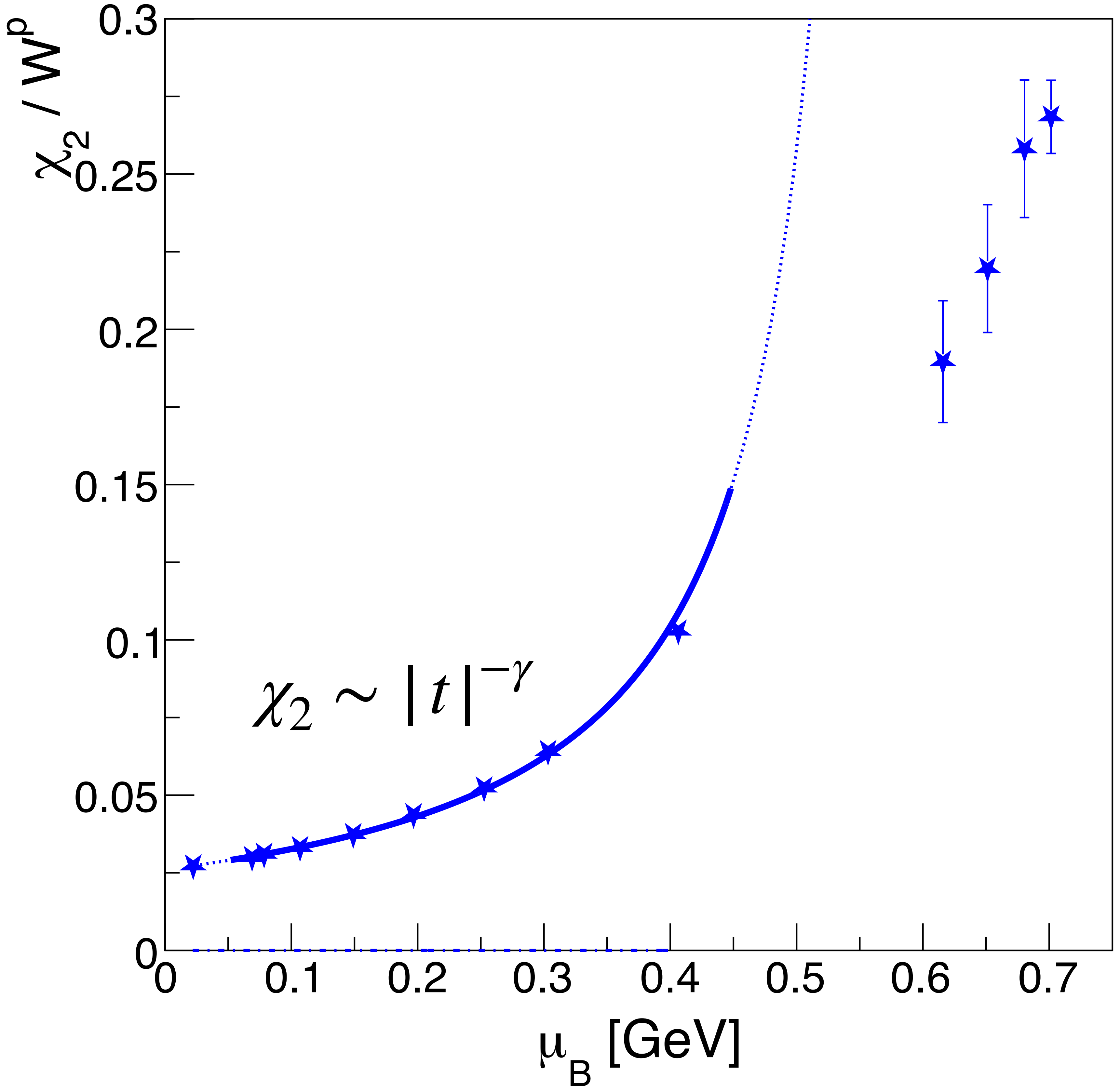
$$\chi_2 \propto W^p \text{ with p=-0.041}$$

Does χ_2 follow a power law as a function of t ?

A Look Under the Hood

We fit χ_2 with a homogeneous function $\chi_2(W, t) = \alpha W^p \,|\, t \,|^{-\gamma}$ where the parameters are p, γ, μ_c, and α. The fit is good above 7.7 GeV.





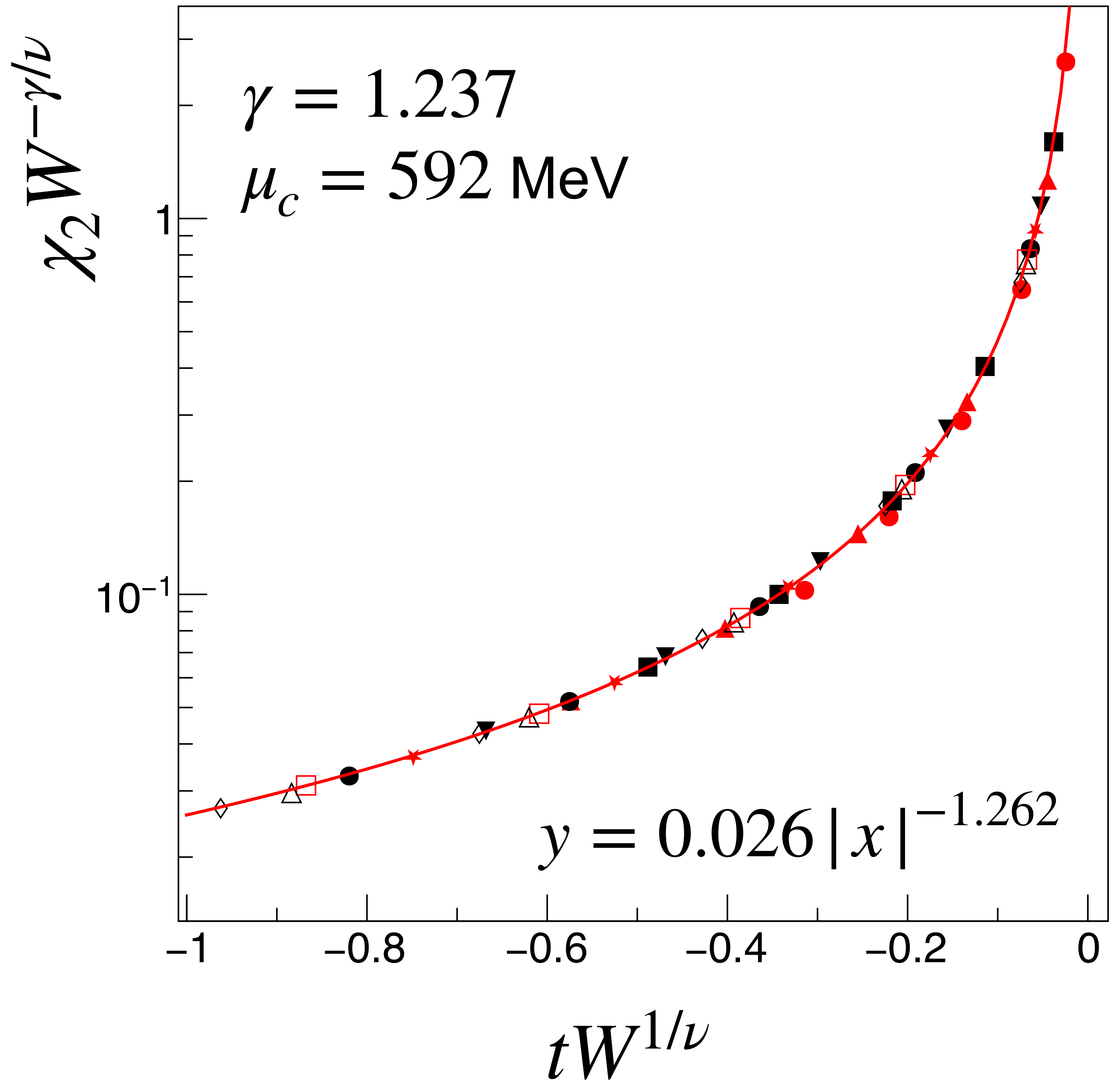
The fit is consistent with $\chi_2 \sim \,|\, t \,|^{-\gamma}$ with a divergence near 590 MeV. Is that the CP?

Observation of FSS Scaling

FSS: a plot of $\chi_2 W^{-\frac{\gamma}{\nu}}$ vs $|t|W^{1/\nu}$

- for a system which exhibits self-similarity with a relevant scale $x = \xi/L$
- for a system where $\xi \sim |t|^{-\nu}$
- for which one uses the correct value for μ_c

should collapse to a single curve



Above 7.7 GeV the data scales very well.

Fixed target data do not.

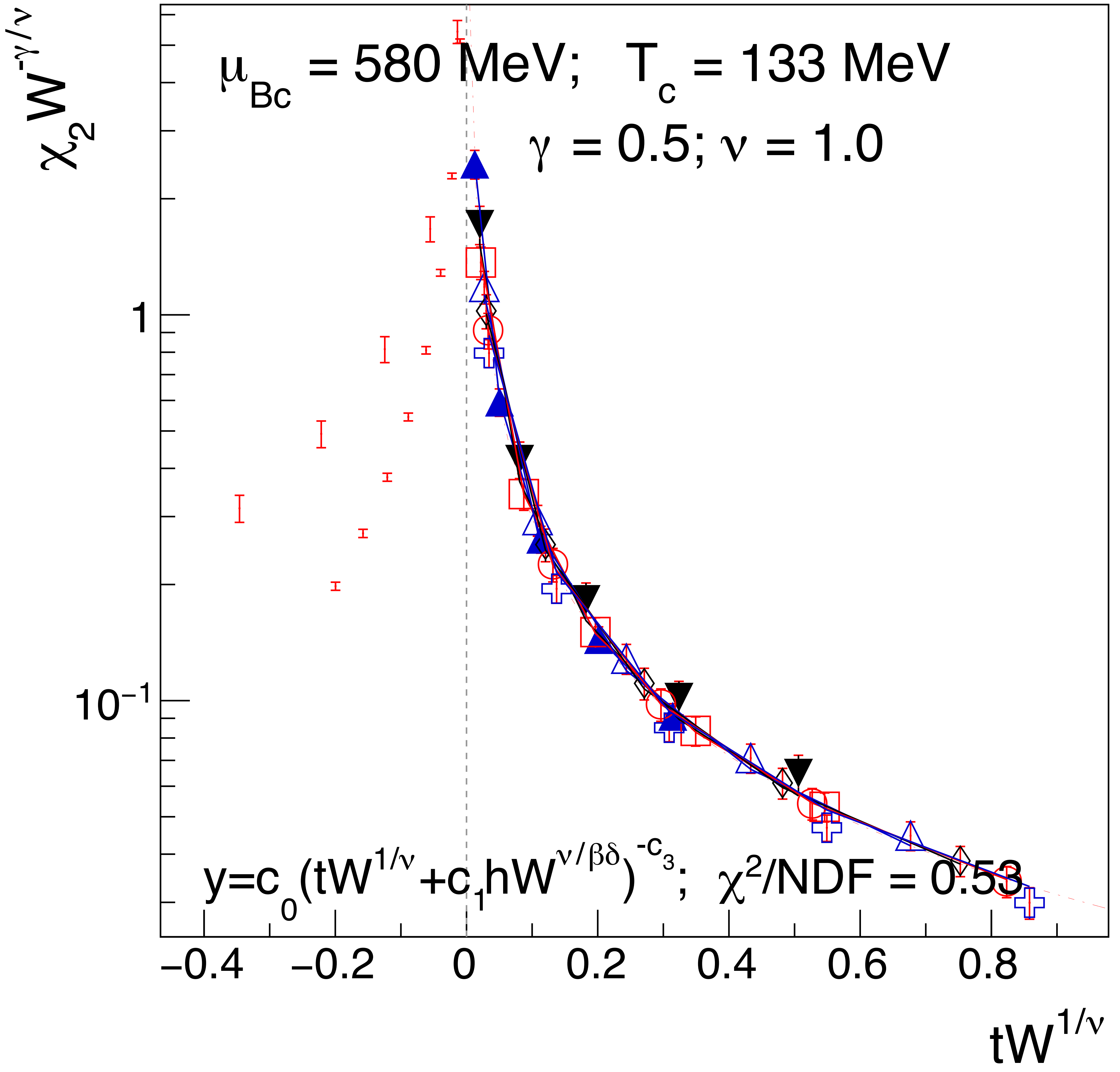
Can we interpret the fit parameters in terms of a CP?

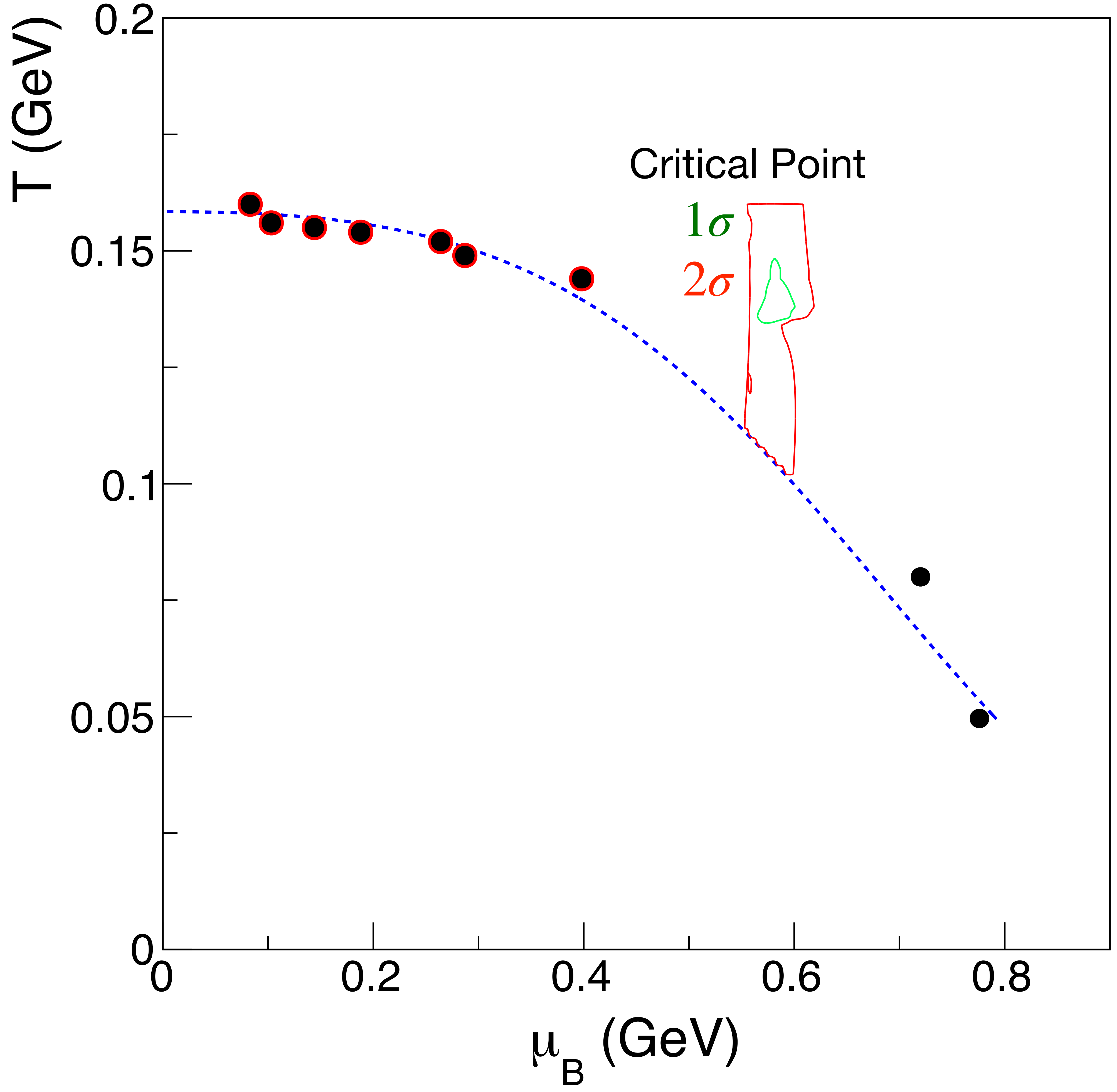
Can we understand the scaling in some other way?

Is the scaling function reasonable, compared to known scaling functions for the universality class?

2-D fit

Also performed a 2D fit:





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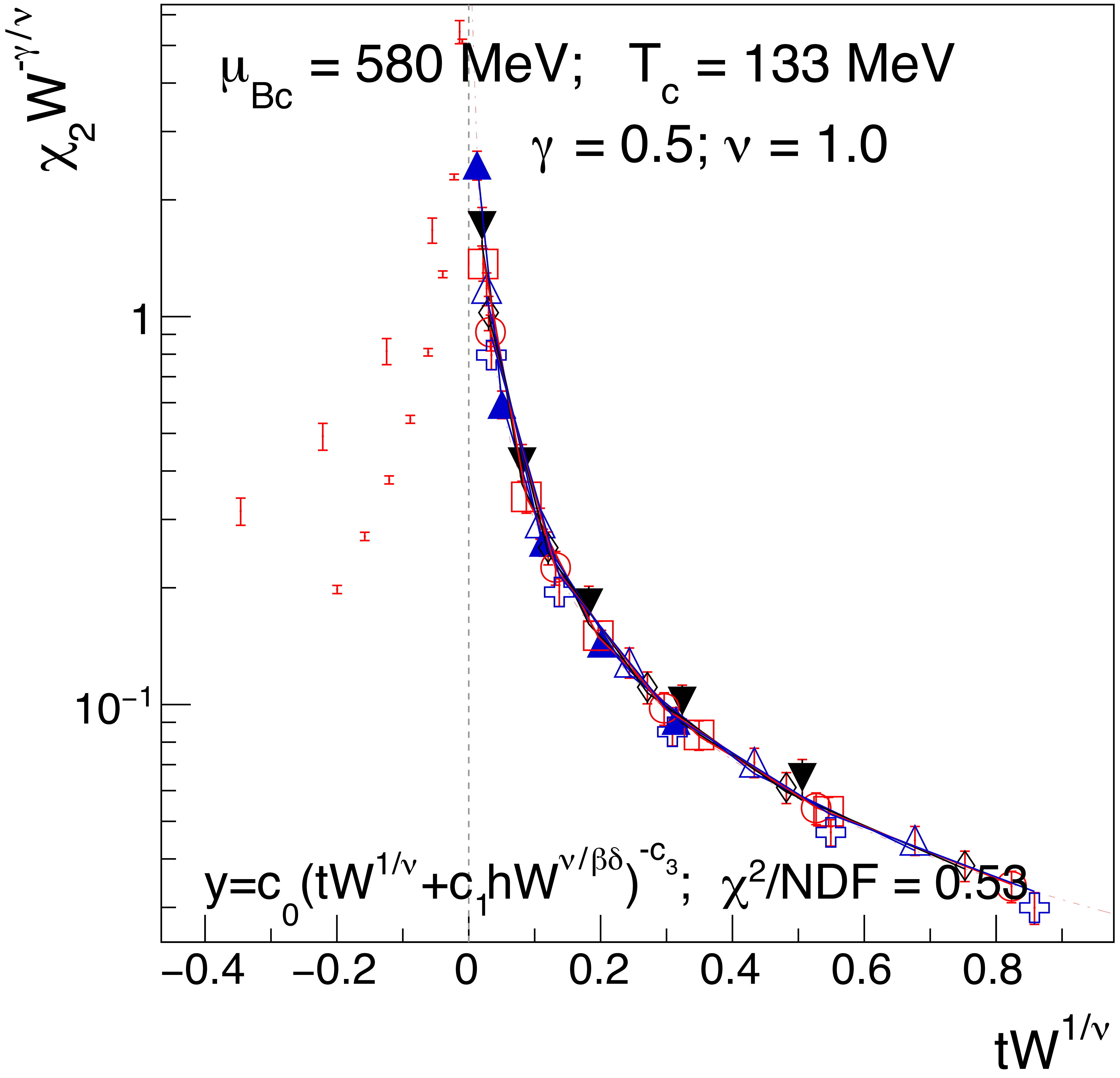
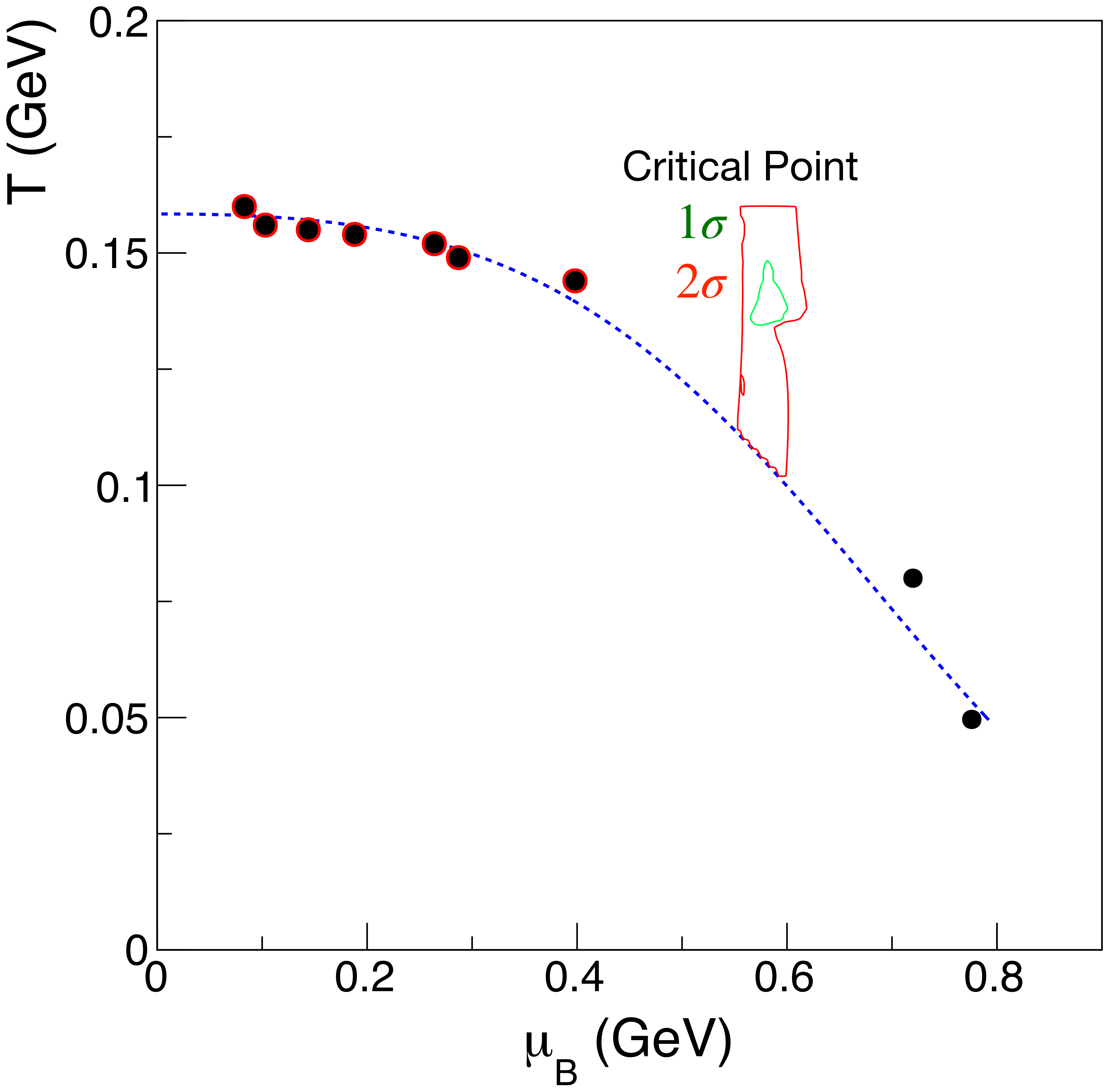




Relationships between exponents

The singular part of the free energy: $f_s(t,h) \sim |t|^{2-\alpha} F\left(\frac{h}{|t|^{\beta\delta}}\right)$

Magnetization: $M \sim \frac{\partial f_s}{\partial h} \sim |t|^{2-\alpha-\beta\delta} F'\left(\frac{h}{|t|^{\beta\delta}}\right)$ since $M \sim |t|^{\beta}$ $\Rightarrow$ $2-\alpha-\beta\delta=\beta$

Susceptibility: $\chi_2 \sim \frac{\partial M}{\partial h} \sim \frac{\partial}{\partial h}\left[|t|^{\beta} F'\left(\frac{h}{|t|^{\beta\delta}}\right)\right] \sim |t|^{-[\beta(\delta-1)]} F''\left(\frac{h}{|t|^{\beta\delta}}\right)$

since $\chi_2 \sim |t|^{-\gamma}$ $\Rightarrow$ $\gamma = \beta(\delta-1) \iff \beta\delta = \gamma+\beta$

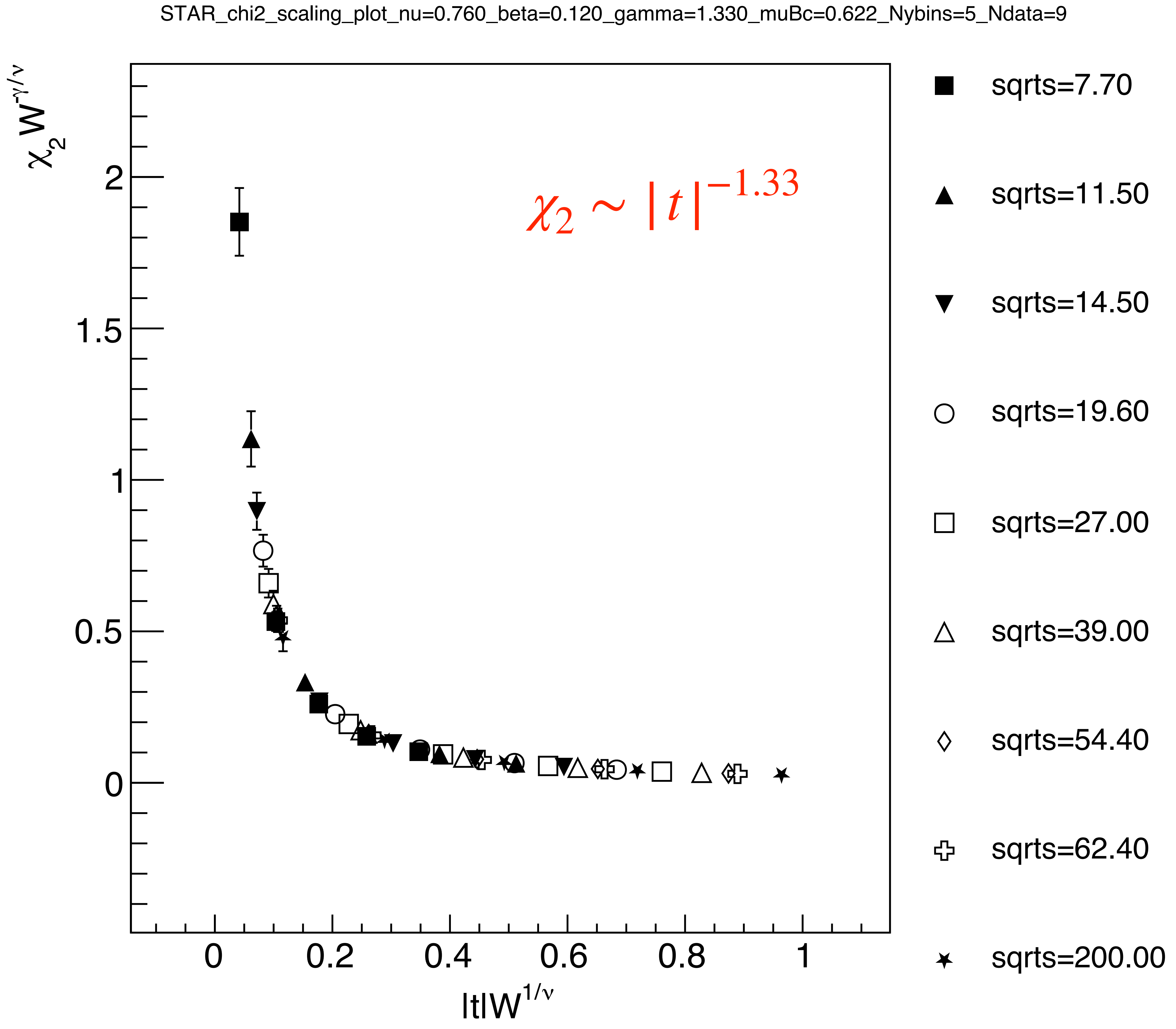
Higher orders: $\chi_n \sim |t|^{-\left[\gamma+(n-2)(\gamma+\beta)\right]} F^{(n)}\left(\frac{h}{|t|^{\beta\delta}}\right)$ in particular $\chi_3 \sim |t|^{-\left[\gamma+(\gamma+\beta)\right]} = \chi_2 |t|^{-(\gamma+\beta)}$

MF: $\nu=0.5$, $\gamma=1$, $\beta=\frac{1}{2}$, $\chi_2 \sim t^{-1}$, $\chi_3 \sim t^{-2.5}$, 3D Ising: $\nu=0.630$, $\gamma=1.237$, $\beta=0.326$, $\chi_2 \sim t^{-1.237}$, $\chi_3 \sim t^{-2.8}$

4D fit ν, β, γ, μ_c for STAR data and SMASH cascade

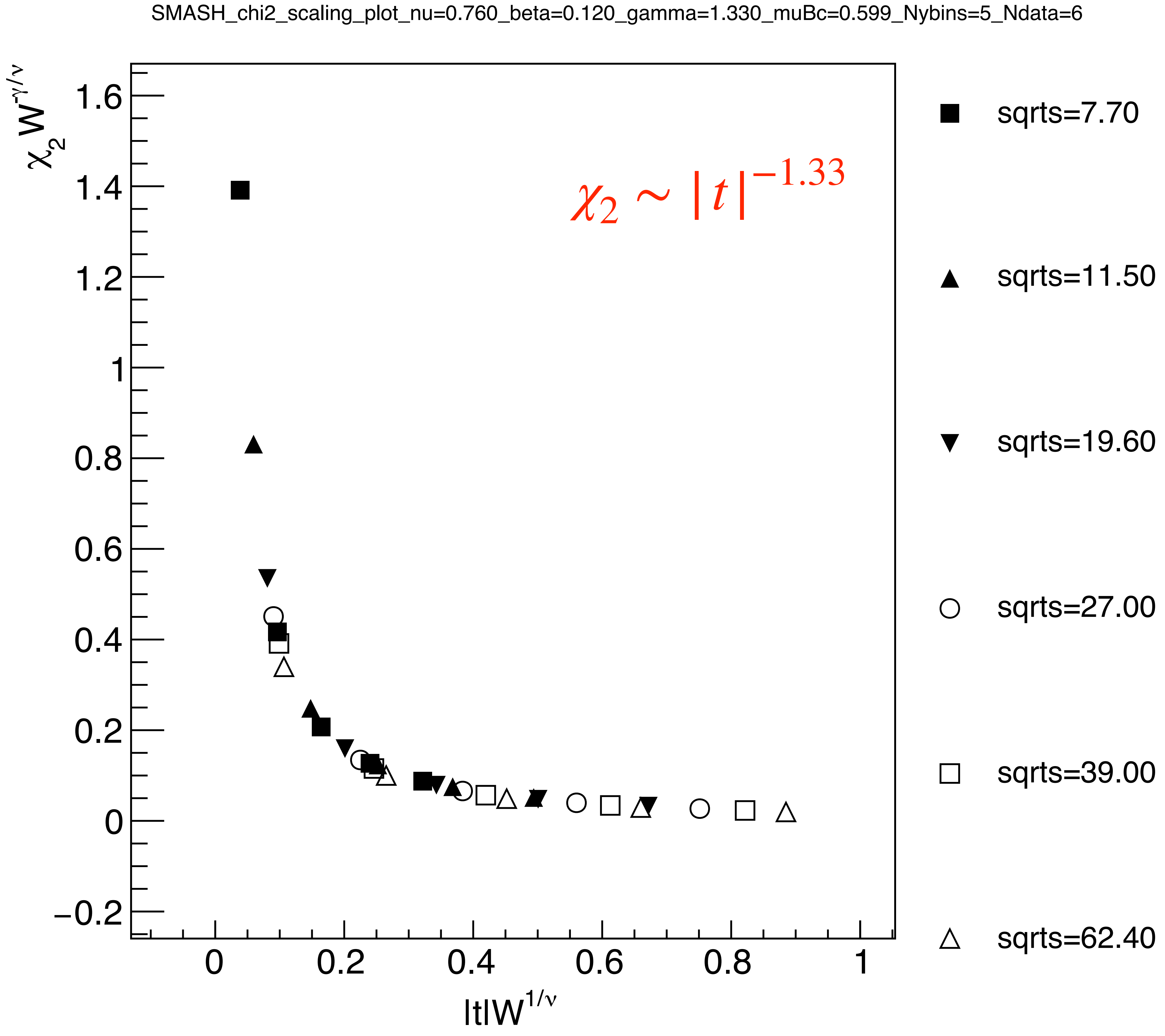
STAR: χ_2

$\nu = 0.76$, $\beta = \mathrm{N/A}$, $\gamma = 1.33$, $\mu_c = 622$ MeV



SMASH: χ_2

$\nu = 0.76$, $\beta = \mathrm{N/A}$, $\gamma = 1.33$, $\mu_c = 599$ MeV



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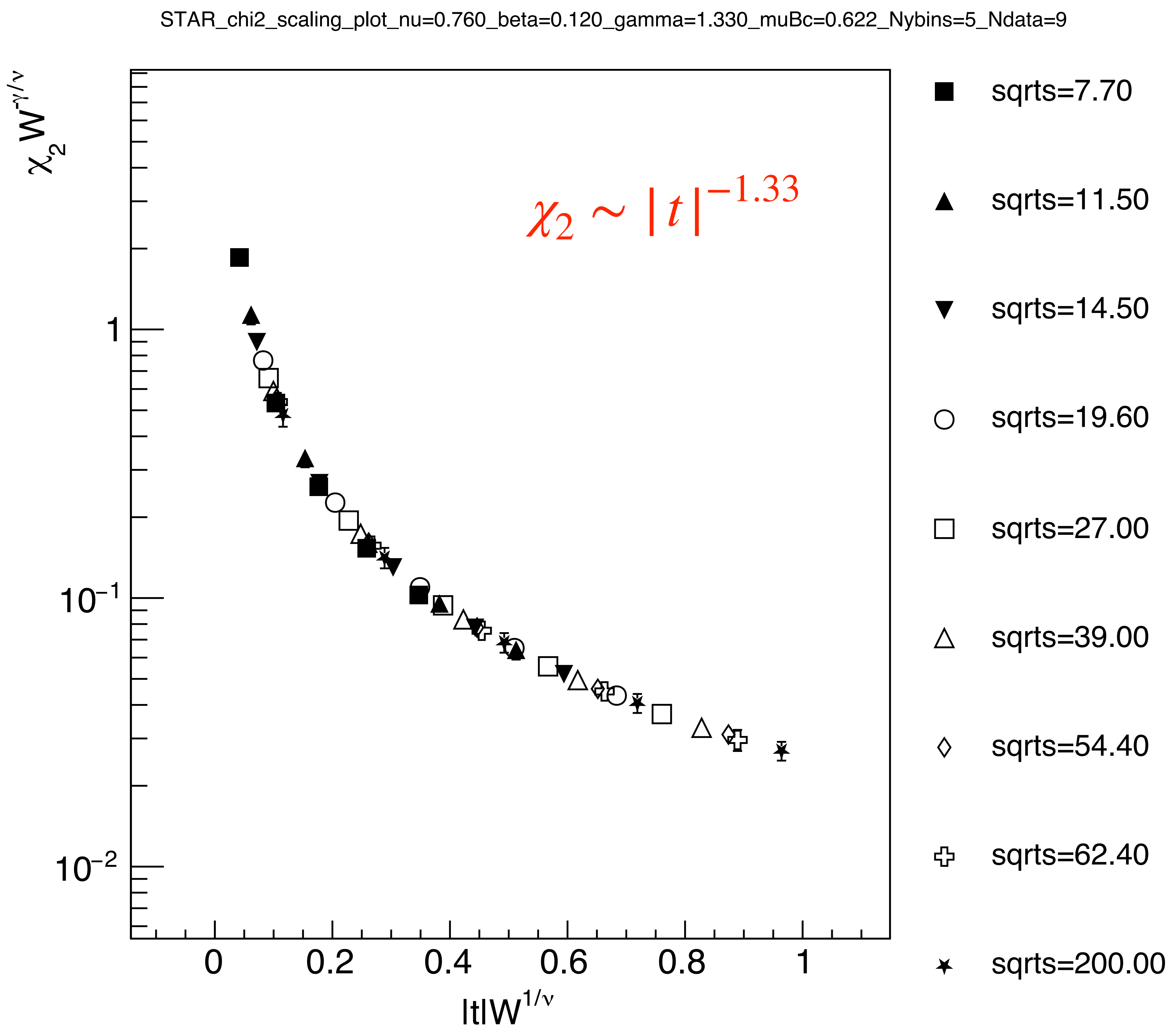

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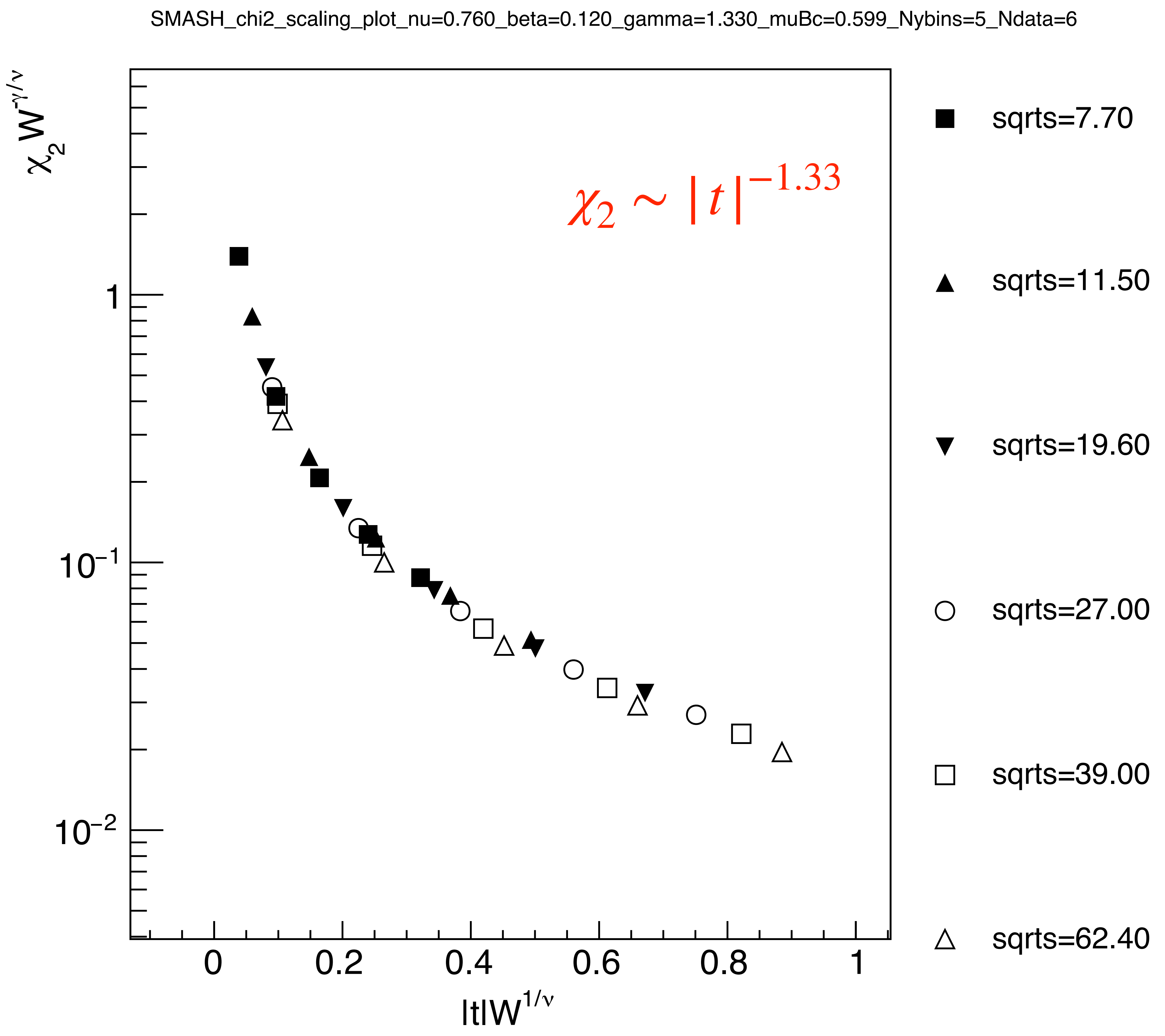

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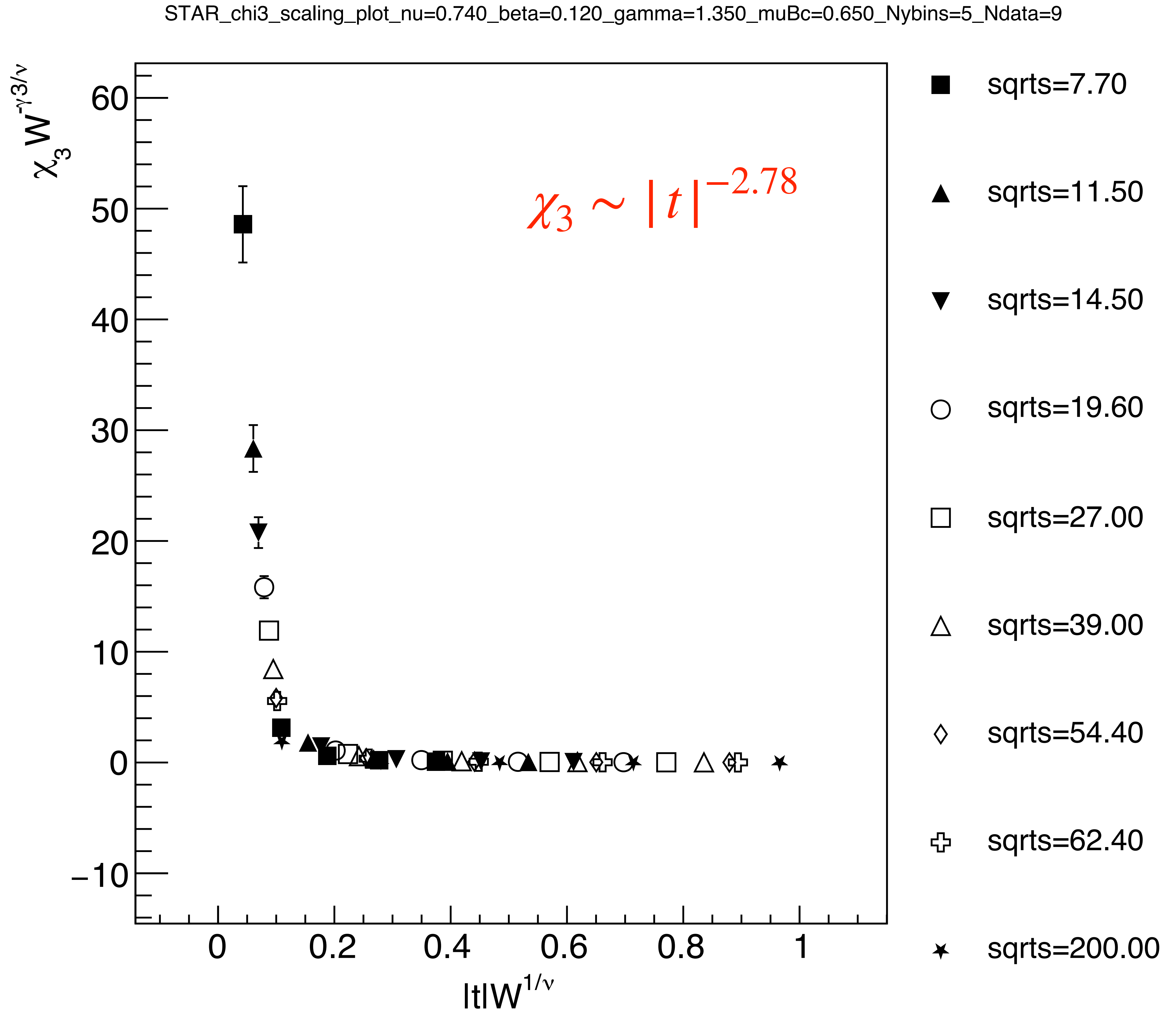
4D fit ν, β, γ, μ_c for STAR data and SMASH cascade

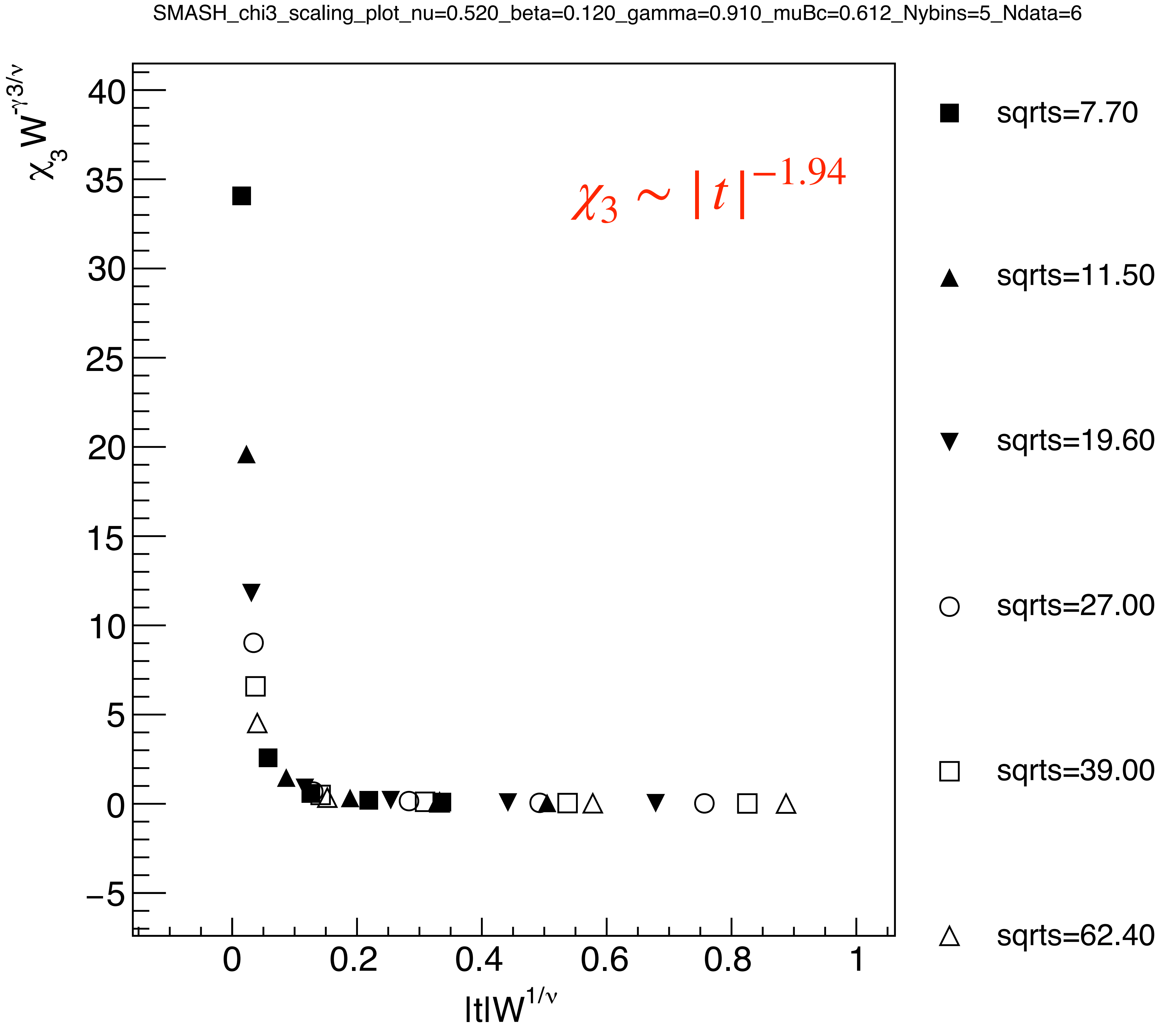
STAR: χ_3

$\nu = 0.74\ , \quad \beta = 0.12\ , \quad \gamma = 1.35\ , \quad \mu_c = 650\ \mathrm{MeV}$



SMASH: χ_3

$\nu = 0.52\ , \quad \beta = 0.12\ , \quad \gamma = 0.91\ , \quad \mu_c = 612\ \mathrm{MeV}$



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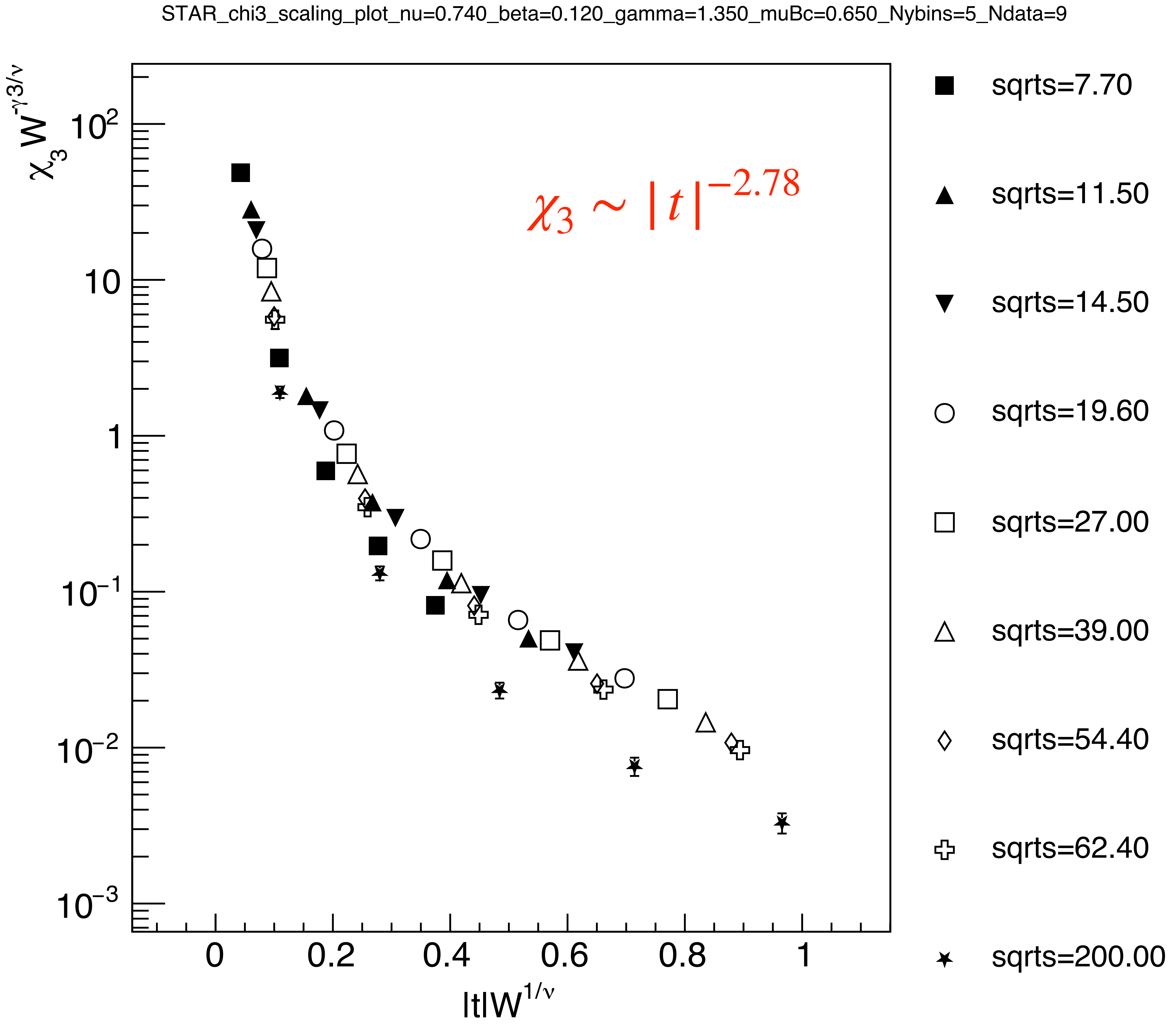

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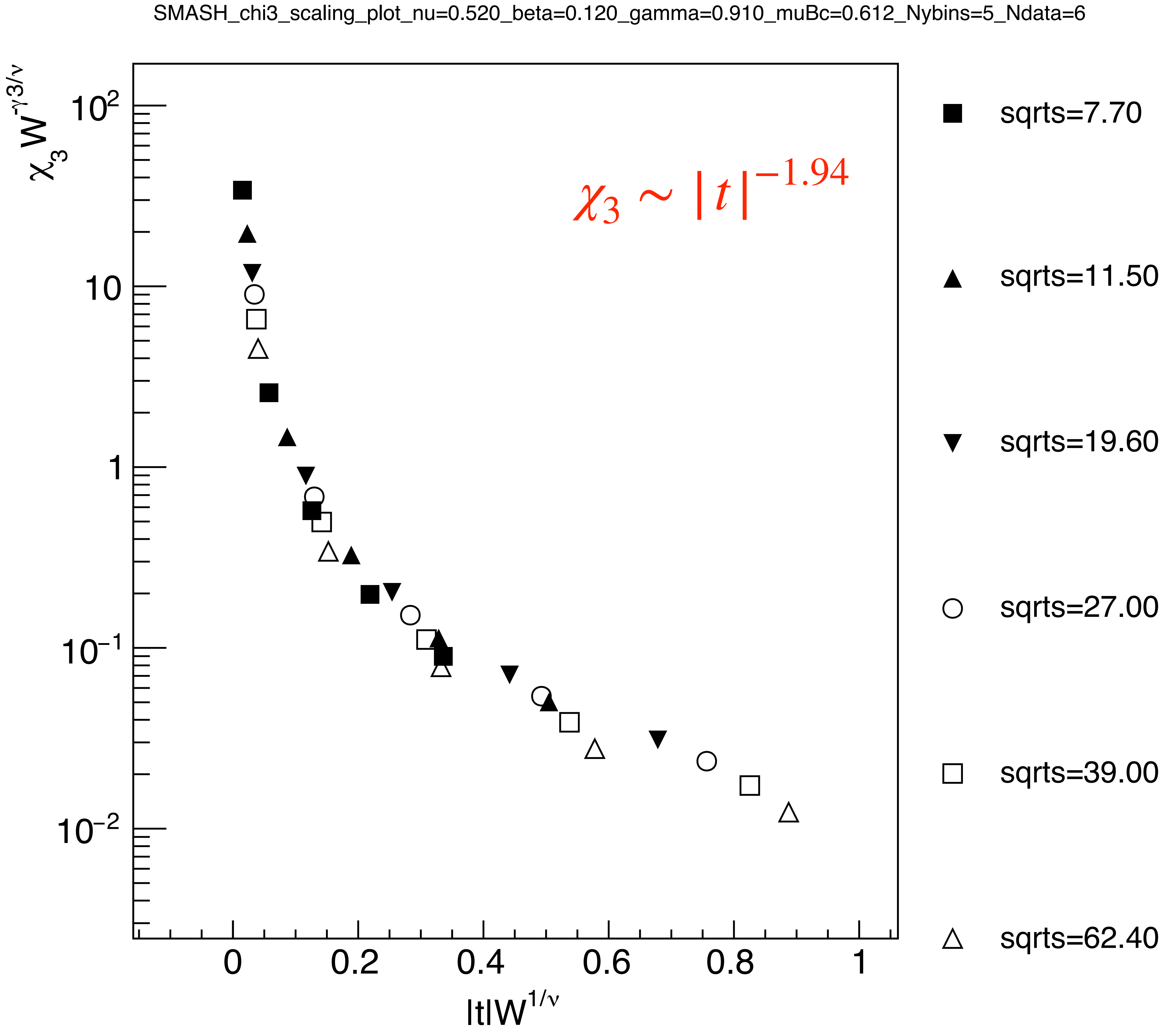

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Binder cumulant

K. Binder, Z. Phys. B 43, 119 (1981).

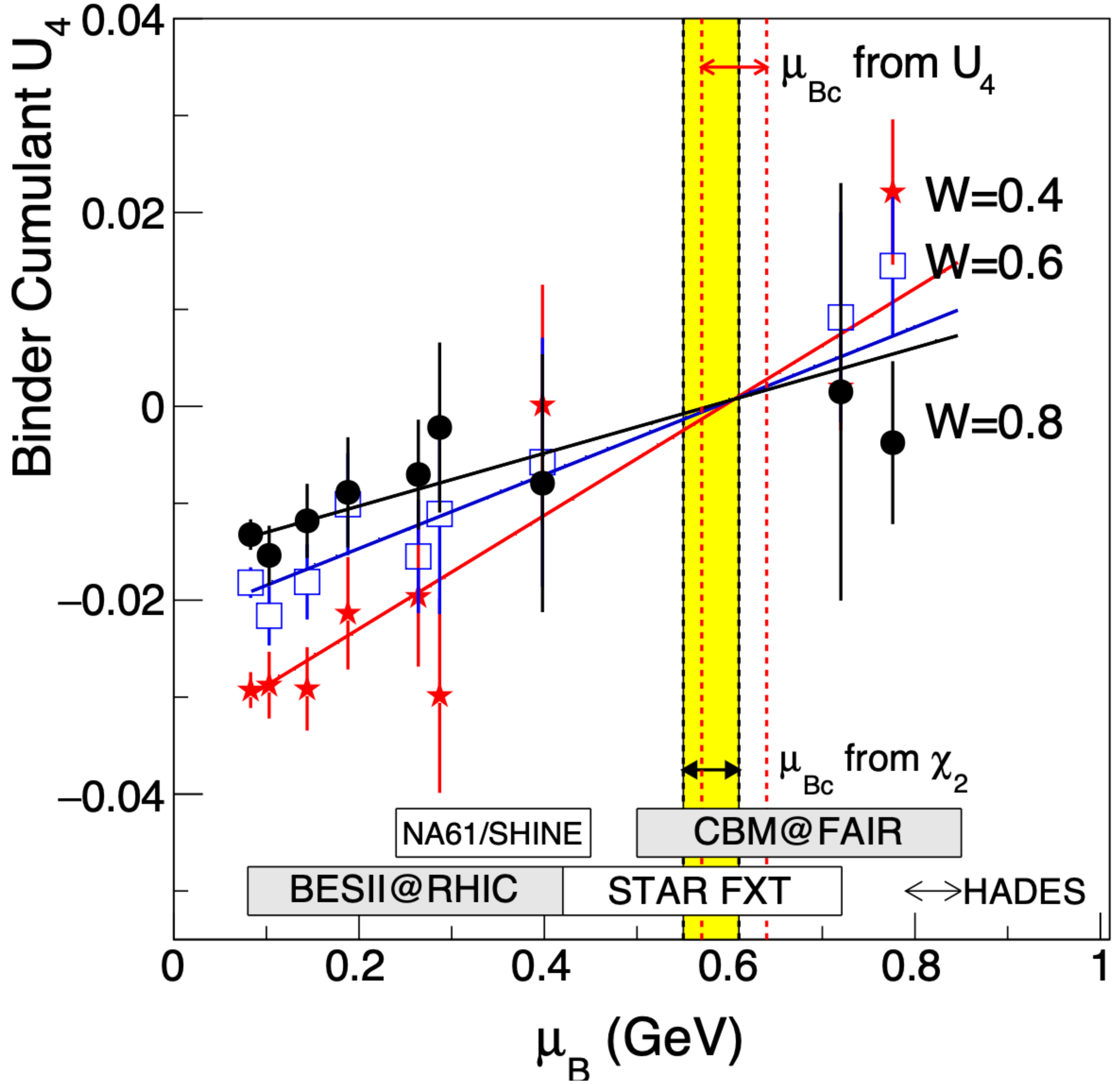
$$U_4 = -\,C_4/(3C_2^2)$$

- Expectation: $U_4 = 0$ (Gaussian), 2/3 (bimodal), crosses at the critical point
$$U_4 \approx c_1 + c_2(\mu - \mu_c)W^{1/\nu}$$

- At low μ_b, U_4 follows Skellam with $U_4(W=0.8)>U_4(0.6)>U_4(0.4)$.

- At $\mu_b>400$, the ordering appears to reverse.

- Data are consistent with a critical point between μ_b of 400 and 800 MeV



Key Points

Reasons to be skeptical:

- The data are outside of the region where we know that $\xi \sim t^{-\nu}$ and the free energy is dominated by a singular part.
- The range over which power law behavior is observed is not that wide.
- The general trends are captured even in a hadronic cascade.
- The rapidity bin width dependence is very weak
- No rounding off of the peak is observed

Reasons to be optimistic

- The conditions in the collider region are ideal: $\langle m_T \rangle$ is flat and there is a well separated mid-rapidity region.
- The fits to χ_2 are extremely good.
- The resultant scaling function is consistent with expectations.
- The critical point position is reasonable.
- χ_3 and the Binder cumulant also seem to provide a consistent picture with the Binder cumulant crossing in the same region.