Defects, Renormalization Group Flows, Magnets, and Wilson Lines

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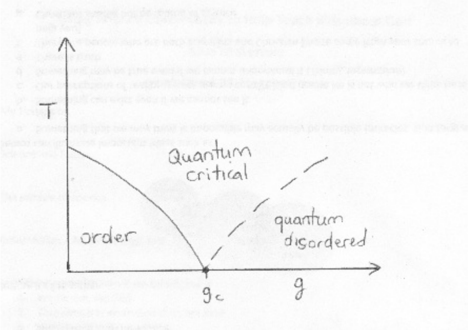
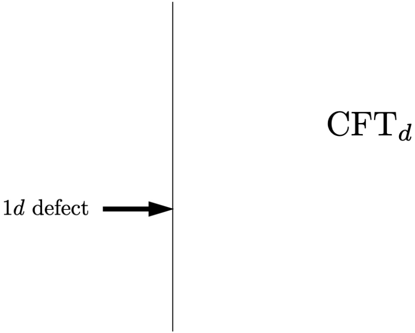
Consider a quantum critical point in d space-time dimensions.



A lot is known about the space of *local* operators. We will assume the critical point is a CFT_d, then the local operators come in representations of the conformal group and there are many well known constraints on correlation functions.

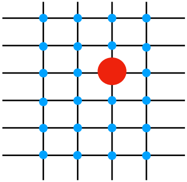
Let us now consider a point-like impurity in space. In space-time, this is a line operator.



In the infrared, the impurity is either completely screened or becomes a nontrivial conformal impurity. A line defect is said to be screened if it is the unit line operator. In other words, it is completely transparent to all bulk correlation functions.

The subject of line defects has been historically extremely productive. The Kondo line defect in 2d has led to the **renormalization group** [Wilson...], to substantial progress on **integrability** [Andrei, Tsvelick-Wiegmann...], and of course to the development of **conformal symmetry** at the end points of the RG flow. The topic of this talk is to explore **line defects in higher dimensions**.

$$H = H_{bulk} + H_{imp}$$



$$H = J_0 \vec{T} \cdot \vec{S} + H_{bulk}$$

We are already familiar with many constructions of line defects in $d > 2$:

- Wilson/'t Hooft loops.
- Twist (symmetry) defects in 2+1 dimensions
- SPT defects
- Worldlines of anyons in 2+1 dimensions
- Pinning Field Defects
- $\cdots$

We will touch briefly upon several subjects:

- RG flows on line defects
- Magnetic field defects
- Spin impurities
- Wilson lines

Consider a straight line in a d-dimensional CFT. It can be conformal or non-conformal. A conformal line preserves

$$SL(2,\mathbb{R}) \times SO(d-1)$$

(we assume the line has no transverse spin). A non-conformal line preserves

$$\mathbb{R} \times SO(d-1) \ .$$

It describes a point-like impurity in space at zero temperature, with a critical bulk. At long distances, the impurity becomes critical (and may or may not be non-trivial).

In this setup there are bulk operators, which are the usual ones, and defect operators, which are local operators acting on the line defect. At the defect fixed point (DCFT), operators are classified by their $SL(2,\mathbb{R}) \times SO(d-1)$ quantum numbers. In general, the space of defect operators has nothing to do with the bulk operators.

There is a bulk-defect OPE, where we expand bulk operators in terms of defect operators

$$O(x_\perp, t) \sim \sum a_k x_\perp^{\Delta_{\hat{O}_k} - \Delta_O} \hat{O}_k(t) \ .$$

This expansion is useful at short distances from the defect.

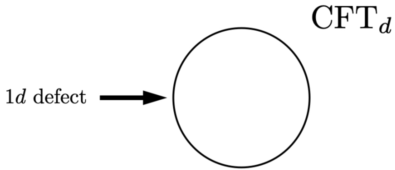
A particularly important observable comes from the unit operator on the r.h.s:

$$\langle O(x_\perp, t) \rangle = \frac{a}{x_\perp^{\Delta_O}} \ .$$

An interesting observable for such a line is its “defect entropy.” We make the line into a Euclidean circle and compute the expectation value of the circle.

$$s = \left(1 - R\frac{\partial}{\partial R}\right)\log\langle L\rangle \equiv \log g \ .$$

The differential operator $\left(1 - R\frac{\partial}{\partial R}\right)$ cancels a scheme dependent linear in R term in $\log\langle L\rangle$ (mass renormalization of the impurity).



Therefore s is a scheme-independent intrinsic observable. At the fixed point of the line defect the value of s is also called $\log g$.

It is hard to directly measure $\log g$.

For line defects in a 2+1 dimensional topological theory, g is called the "quantum dimension." Unlike line defects in topological theories, it is not necessarily true that $g \geq 1$ for general conformal defects, as we will see.

In the event that relevant defect operators exist ($\Delta_{\hat{O}} < 1$), we can deform by $M_0^{1-\Delta_{\hat{O}}} \int dt \hat{O}(t)$. M_0 becomes the physical scale of the flow.

The defect entropy $s = \left(1 - R\frac{\partial}{\partial R}\right) \log\langle L \rangle$ becomes a nontrivial function

$$s = s(M_0 R)$$

We have

$$s(M_0 R) \rightarrow \begin{cases} \log g_{UV} \text{ as } R \rightarrow 0 \\ \log g_{IR} \text{ as } R \rightarrow \infty \end{cases}$$

The renormalization group flow is implemented by changing the radius of the circle of the defect worldline.

One basic result is the following identity (with $\hat{T}_D$ the energy localized at the defect)

$$R\frac{\partial s}{\partial R} = -R^2 \int d\phi_1 d\phi_2 \langle \hat{T}_D(\phi_1)\hat{T}_D(\phi_2)\rangle_c \,(1-\cos(\phi_1-\phi_2)) \ .$$

Since $\langle \hat{T}_D(\phi_1)\hat{T}_D(\phi_2)\rangle_c \geq 0$ at separated points and since $(1-\cos(\phi_1-\phi_2)) \geq 0$ we have that

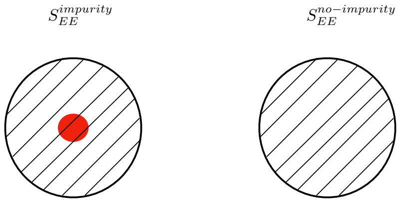
$$R\frac{\partial s}{\partial R} \leq 0 \ ,$$

and therefore also $g_{UV} \geq g_{IR}$. This provides a general non-perturbative constraint on RG flows on point impurities.

This generalizes the familiar results of [Affleck-Ludwig, Friedan-Konechny] to line defects/impurities in higher dimensions.

Note that it follows that g is independent of exactly marginal defect couplings. We will soon see an example of that.

This higher-dimensional g theorem should be connected somehow to entanglement entropy. Our g function is not the same as the additional entanglement entropy of the vacuum with the impurity. See (16') [Casini – Salazar-Landea – Torroba] for the $d = 2$ case, where these two quantities coincide. For $d > 2$ see the recent (22') [Casini – Salazar-Landea – Torroba]



Three ways to construct line defects:

- Start from the trivial line defect ($g_{UV} = 1$). If there is a bulk operator with $\Delta < 1$ then we can integrate it on the line:

$$S = S_{bulk} + M_0^{1-\Delta_O} \int dt O(t)$$

This is called a "pinning field" defect or an external field defect. Physically this is an impurity created by applying external fields in a manner localized in space, independent of time. Example: applying a magnetic field in a critical magnet, but only at a few lattice sites.

- Start from a QM model on the line with d states. Couple some operators acting on these states to the bulk operators:

$$S = S_{bulk} + M_0^{1-\Delta_O-\Delta_T} \int dt T_{QM}(t) O(t)$$

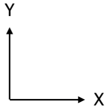
Example: a qubit coupled to some bulk CFT.

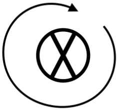
- Start from a QM model on the line with d states in a representation of G and consider a bulk CFT with G gauge symmetry. Couple the two systems by gauging the symmetry G in QM.

$$S_{bulk} + \int dt J^a_{QM}(t) A^a(t)$$

In this case there is no free defect coupling constant since the coefficient of $J^a_{QM}(t)A^a(t)$ is fixed. Charged impurities (Wilson lines) are constructed in this manner.

- A fourth construction: In any 2+1 dimensional quantum system with a $U(1)$ global symmetry we can construct a monodromy (pseudo) point defect with flux $h \in [0, 1)$. The infrared limit of this point impurity is an interesting function of h.





$$\Phi \to e^{2\pi i h}\Phi$$

Dualities may interchange the various descriptions above.

For the remaining part of this talk we will discuss some examples and quote results about their properties.

The Pinning Field in $O(N)$ Models

Consider the $O(N)$ model in $2 \leq d \leq 4$ with an external localized magnetic field:

$$S = S_{O(N)} + h \int dt \phi_1(t)$$

where $S_{O(N)}$ stands for the critical bulk $O(N)$ model in d space-time dimensions and ϕ_1 is the first component of $\vec{\phi}$.

This is a relevant perturbation in $2 \leq d \leq 4$. By the g theorem, this must flow to a nontrivial ($g < 1$) infrared DCFT in any $2 \leq d < 4$. Hence, the external magnetic field cannot be "screened" and cannot disappear.

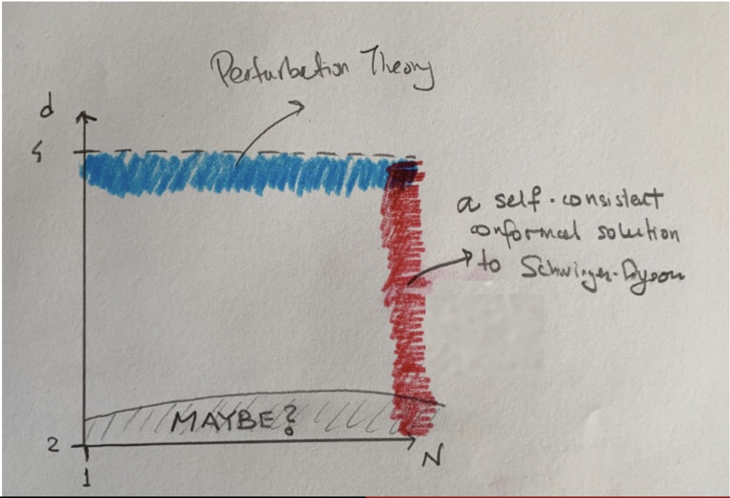
The Pinning Field in $O(N)$ Models

This is physically realizable as a localized magnetic field at zero temperature at a bulk quantum critical point and it can be tested in quantum critical points and also in Monte Carlo [....Assaad, Herbut; Parisen Toldin, Assaad, Wessel....]

This infrared DCFT will have no nontrivial relevant operators whatsoever.

The Pinning Field in $O(N)$ Models

In principle, understanding the infrared is a strongly coupled problem.



The Pinning Field in $O(N)$ Models

Many concrete predictions can be made in the epsilon expansion. For instance,

$$\log g_{IR} = -\frac{N+8}{16}\epsilon + \cdots \ .$$

$$\Delta(\hat{\phi}_1) = 1 + \epsilon - \epsilon^2 \frac{3N^2 + 49N + 194}{2(N+8)^2} + \cdots \ ,$$

Note: the infrared value of h is NOT small in the ϵ expansion. However, the ϵ expansion makes sense since the bulk is weakly coupled.

The Pinning Field in $O(N)$ Models

The line operator

$$e^{-h \int dt \phi_1(t)}$$

has a smooth large N limit which is manifested if we define a 't Hooft coupling $\lambda = h/\sqrt{N}$. The claim is that in the large N limit the coupling λ flows to some $\lambda_* \sim \mathcal{O}(1)$ in the infrared.

There is a saddle point that determines the DCFT observables, e.g. the g function:

$$g = e^{-NS_{classical}}$$

For more analytic work from recently see [Rodriguez Gomez, Popov, Wang, Grau, Lauria, Liendo]

The Pinning Field in $O(N)$ Models

It should be in principle possible to solve for the whole RG flow of the 't Hooft coupling λ in the large N limit.
Here is a sample of results at large N and $d = 3$:

$$\log g = -0.1536N + \mathcal{O}(N^0)$$

$$\Delta(\hat{\phi}_1) = 1.542 + \mathcal{O}(N^{-1})$$

Note: g is exponentially small at large N.

The Pinning Field in $O(N)$ Models

Combining all the data we amassed suggests that in $d = 3$ one should expect $\Delta(\hat{\phi}_1) \sim 1.5$ with rather weak N dependence. This is the first nontrivial $O(N-1)$ singlet operator. It is roughly consistent with Monte Carlo simulations and this along with several other predictions should be testable.

Spin Impurities

Another important line defect especially for the $O(3)$ model comes about as follows: We begin with QM with a spin s representation of $SO(3)$, so just a QM system with Hilbert space of dimension $2s+1$. We then couple the $SO(3)$ generators S^a to the interacting bulk:

$$S = S_{O(3)} - \gamma \int dt S^a(t)\phi^a(t) \ .$$

This is the line operator

$$Tr_s P e^{\gamma \int dt S^a \phi^a} \ .$$

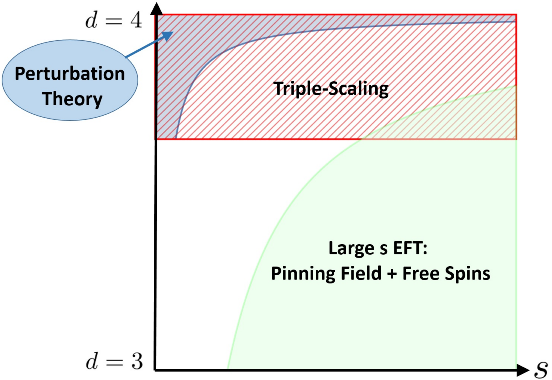
It is similar to Wilson lines but it is just a line defect in a magnet.

Spin Impurities

Physically this is realizable by putting an external atom of spin s in a quantum anti-ferromagnet at the critical point. While there is a lot to say about this problem here I will mention one general result.

Spin Impurities

At $s \to \infty$ the spin impurity breaks up into two almost-decoupled DCFTs, one being the pinning field DCFT we studied above and the other being just the theory of a free spin s. There is a systematic $1/s$ expansion. This statement leads to many predictions that can be checked in the future.



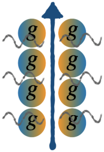
Spin Impurities

For additional recent work on this subject see [Beccaria-Giombi-Tseytlin, Rodriguez Gomez-Russo, Nahum, Weber-Vojta, Grau]

Now we will consider gauge theories in 3 or 4 space-time dimensions. An important line operator is the Wilson line:

$$W_R = Tr_R \left[P \ exp \left(i \int_\gamma dx^\mu A_\mu^a T_R^a \right) \right]$$

This describes the insertion of a probe particle moving on the worldline γ.



In confining theories Wilson lines serve as order parameters for confinement. Wilson lines are order parameters only if there are no dynamical fields with the same quantum numbers.

But today we are interested in de-confined theories where Wilson lines in various representations may settle into conformal line operators or be screened.

Take massless scalar QED_4

$$S = \int d^4x \left[-\frac{1}{4e^2} F^2 + |D\phi|^2 - \lambda |\phi|^4 \right] + q \int dt A_t \ .$$

$D\phi = \partial\phi - iA\phi$.

We have a classical saddle point

$$A_t = \frac{e^2 q}{4\pi r} \ , \quad \phi = 0 \ .$$

An important computation to do is the measure the dimension of the defect operator $\Delta(\hat{\phi}^\dagger\hat{\phi}) =$? For small q it has to be close to the bulk dimension so we expect

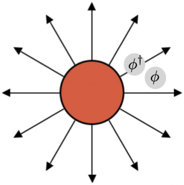
$$\Delta(\hat{\phi}^\dagger\hat{\phi}) = 2 + \# e^4 q^2 + \#(e^4 q^2)^2 + \cdots$$

The answer can be found exactly by studying the Green's function around the saddle point. One finds:

$$\Delta(\hat{\phi}^\dagger\hat{\phi}) = 1 + \sqrt{1 - \frac{e^4 q^2}{4\pi^2}}\ .$$

Clearly for $\frac{e^2|q|}{2\pi} > 1$ there is some sickness.

Intuitively, for $\frac{e^2|q|}{2\pi} > 1$ the electric field is so strong that it leads to pair creation and the saddle point is destablized.

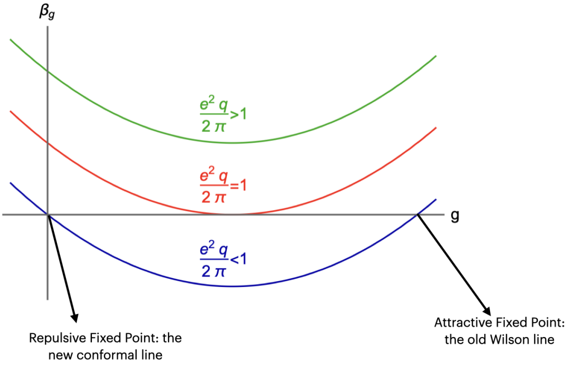


$$A_0 = \frac{e^2 q}{4\pi r}$$

As we approach $\frac{e^2|q|}{2\pi} = 1$ from below, the defect operator $\hat{\phi}^\dagger \hat{\phi}$ becomes closer and closer to being marginal. So we must consider the more general Wilson line

$$W_q^g = P\ exp\left(i \int_\gamma dt\ (q \frac{dx^\mu}{dt} A_\mu - g\hat{\phi}^\dagger \hat{\phi})\right)\ .$$

q cannot be renormalized, being an integer. But g can be renormalized.



β_g
$\frac{e^2 q}{2\pi}>1$
$\frac{e^2 q}{2\pi}=1$
g
$\frac{e^2 q}{2\pi}<1$
Repulsive Fixed Point: the new conformal line
Attractive Fixed Point: the old Wilson line

It is not too hard to guess what happens when $g \to -\infty$; this drives the scalar field to condense on the defect which subsequently triggers a condensate in the bulk.

The annihilation of the two fixed points we have seen above is reminiscent of how QCD exits the conformal phase and also of the BKT transition. It leads to Miransky scaling [Kaplan-Lee-Son-Stephanov].

It is then not surprising that the scalar cloud that forms around the Wilson loop is actually exponentially large in units of the cutoff (e.g. the impurity radius). There is therefore dimensional transmutation!

Also we will see that the cloud completely screens the Wilson line and the infrared DCFT is entirely trivial.

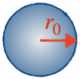
The naive guess for the extent of the cloud should be

$$\Lambda = \mu_0 e^{\frac{-2\pi}{\sqrt{\frac{e^4 q^2}{4\pi^2}-1}}}$$

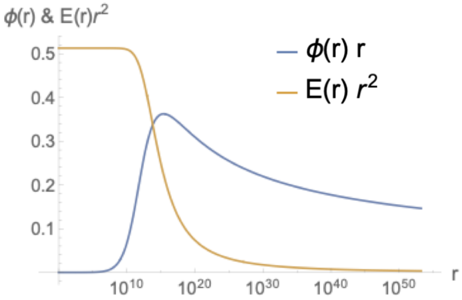
This is the spread of the wave functions of the bosons which are tachyonic around the original saddle point.

In reality we find a larger extent due to the non-linearities of the condensate.

$$F_{0r}|_{r=r_0} = \frac{e^2 q}{4\pi r^2}$$



$$|\phi|^2|_{r=r_0} = 0$$



$$\frac{e^2 q}{2\pi} = 1.02$$

In essence, very similar physics happens also in QED_3 with N_f Dirac fermions (and many other deconfined 2+1 dimensional critical points). We find that Wilson lines are conformally invariant in the infrared for $q \lessapprox N_f/2$ and screened otherwise. This might be observable, too.

Thank You!