Comparing low-order fluctuation measurements to theory: What have we learned? Which observables are the most informative?

Volker Koch, Anar Rustamov, Győző Kovacs, Volodymyr Vovchenko

INT-25-3a Program "The QCD Critical Point: Are We There Yet?"

WG Lower-order fluctuations: Theory

October 29, 2025

QCD phase diagram

What we hope to know

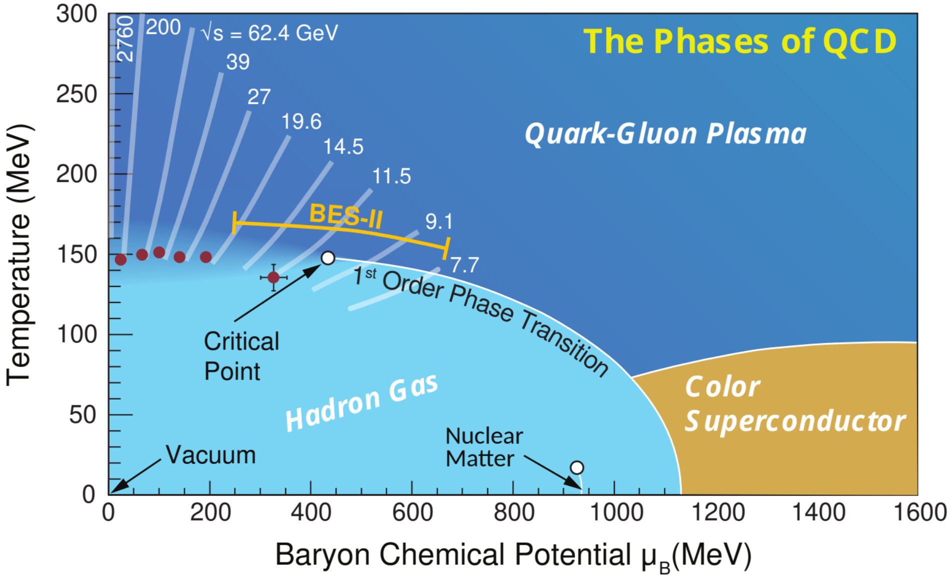
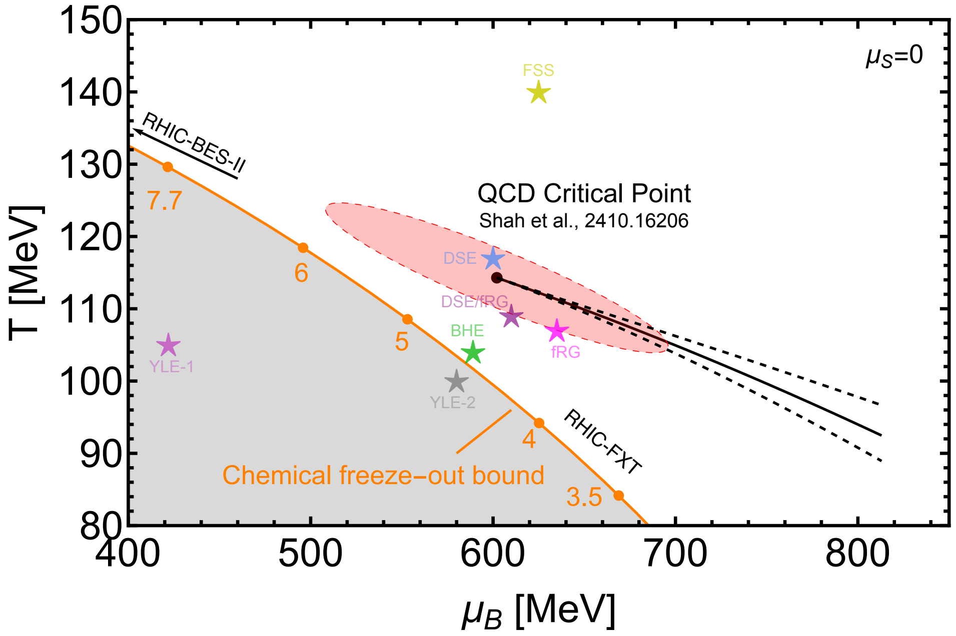


Figure from Bzdak et al., Phys. Rept. '20 & 2015 US Nuclear LRP

Recent CP estimates



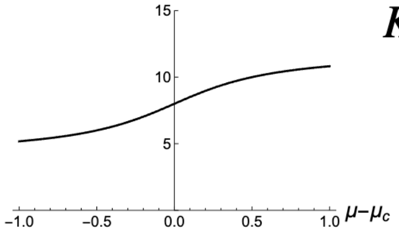
- Lattice QCD excludes CP at $\mu_B < 450$ MeV on (one-sided) 2σ level

  Borsanyi et al., arXiv:2502.10267

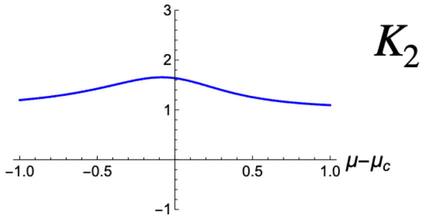
- Multiple approaches predict CP at $\mu_B \sim 600$ MeV, potentially accessible at lower energies

- RHIC measurements of cumulants at $\mu_B < 450$ MeV (collider) and $\mu_B < 650$ MeV (fixed target)
  - Are they consistent or indicative of a CP?

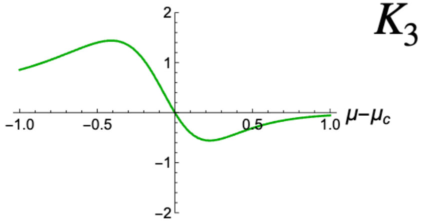
Why cumulants



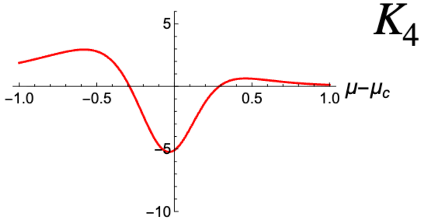
$$K_1 = \langle N \rangle$$



$$K_2 = \frac{\partial}{\partial(\mu/T)} K_1$$



$$K_3 = \frac{\partial}{\partial(\mu/T)} K_2$$



$$K_4 = \frac{\partial}{\partial(\mu/T)} K_3$$

Statistical mechanics:

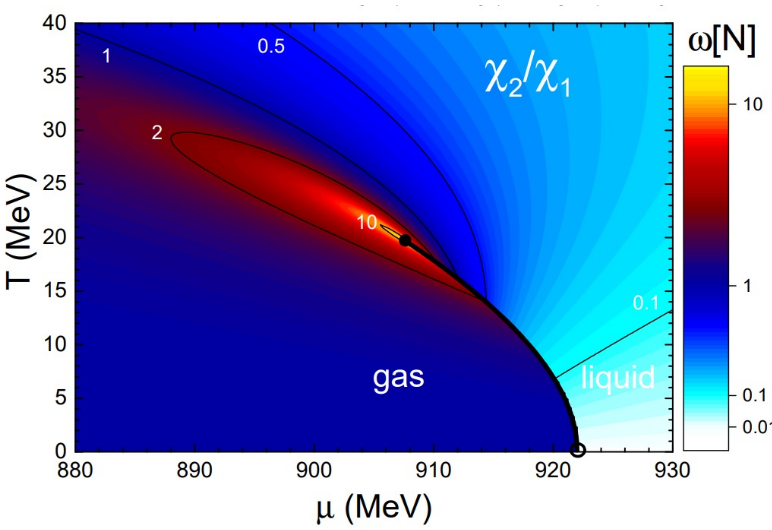
GCE: $$\ln Z^{\text{gce}}(T, V, \mu) = \ln \left[\sum_N e^{\mu N} Z^{\text{ce}}(T, V, N) \right],$$

$$\boxed{\kappa_n \propto \frac{\partial^n (\ln Z^{\text{gce}})}{\partial(\mu_N)^n}}$$

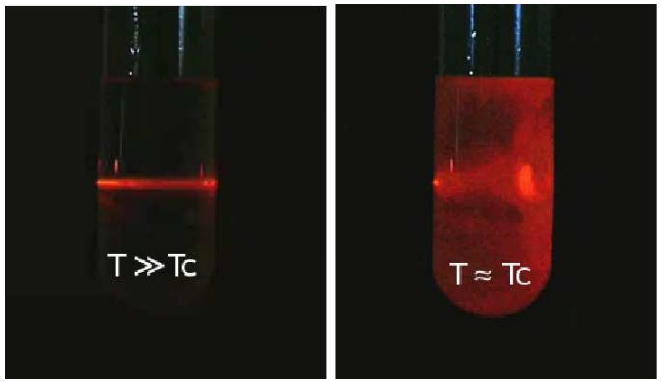
Cumulants measure μ derivatives of the (QCD) EoS

Critical point: large correlation length, equilibrium fluctuations diverge

van der Waals model



Critical opalescence



$$\langle N^2 \rangle - \langle N \rangle^2 \sim \langle N \rangle \sim 10^{23}$$

in equilibrium

Why lower-order cumulants

1. Lower-order cumulants are easier to measure (both statistical and systematic errors)

2. Lower-order cumulants are easier to model

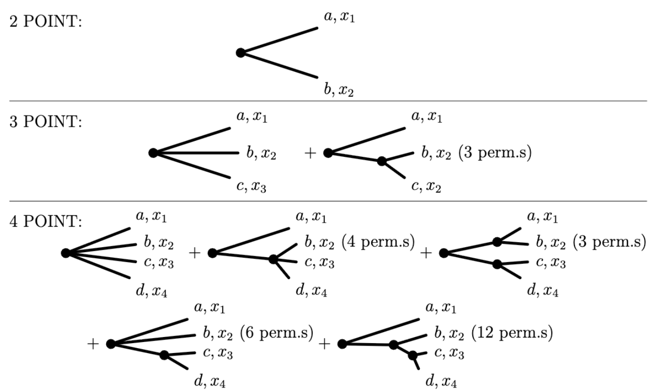
2-point correlator

$$
\begin{aligned}
\mathcal{C}^{(\mathrm{tot})}_{ab}(\boldsymbol{r}_1,\boldsymbol{r}_2,t) &= \langle\delta\rho_a(\boldsymbol{r}_1,t)\delta\rho_b(\boldsymbol{r}_2,t)\rangle\\
&= \chi^{(2)}_{ab}(\boldsymbol{r}_{12},t)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)+C^{(1;1)}_{a;b}(\boldsymbol{r}_1,\boldsymbol{r}_2,t),\\
\boldsymbol{r}_{ij} &\equiv (\boldsymbol{r}_i+\boldsymbol{r}_j)/2.
\end{aligned}
$$

3-point correlator

$$
\begin{aligned}
\mathcal{C}^{(\mathrm{tot})}_{abc}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t) &= \langle\delta\rho_a(\boldsymbol{r}_1,t)\delta\rho_b(\boldsymbol{r}_2,t)\delta\rho(\boldsymbol{r}_3,t)\rangle\\
&= C^{(1;1;1)}_{a;b;c}(\boldsymbol{r}_1,\boldsymbol{r}_2,\boldsymbol{r}_3,t)+C^{(2;1)}_{ab;c}(\boldsymbol{r}_{12},\boldsymbol{r}_{\mathbf{3}},t)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2)\\
&\quad+C^{(2;1)}_{ac;b}(\boldsymbol{r}_{13},\boldsymbol{r}_{\mathbf{2}})\delta(\boldsymbol{r}_1-\boldsymbol{r}_3,t)+C^{(2;1)}_{bc;a}(\boldsymbol{r}_{23},\boldsymbol{r}_{\mathbf{1}})\delta(\boldsymbol{r}_2-\boldsymbol{r}_3,t)\\
&\quad+\chi^{(3)}_{abc}(\boldsymbol{r}_{123},t)\delta(\boldsymbol{r}_{12}-\boldsymbol{r}_3)\delta(\boldsymbol{r}_1-\boldsymbol{r}_2).
\end{aligned}
$$

S. Pratt, PRC 101, 014914 (2020)



3. Lower-order cumulants should be well controlled before high-orders can be reliably analyzed

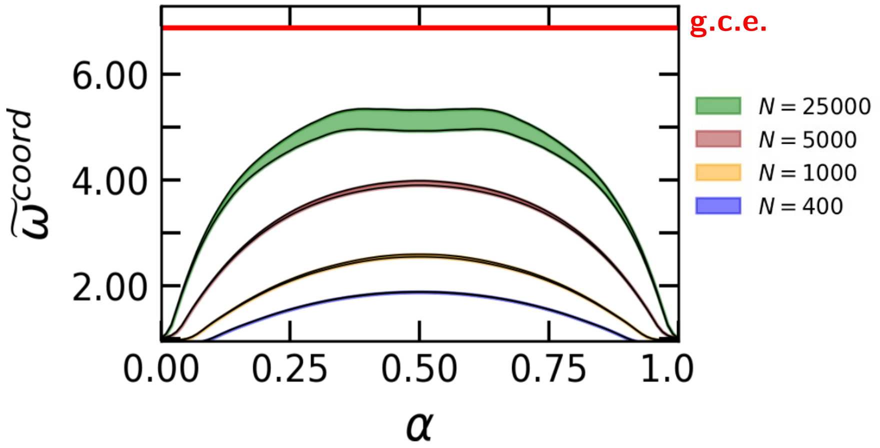
Example: Critical fluctuations in microscopic simulation

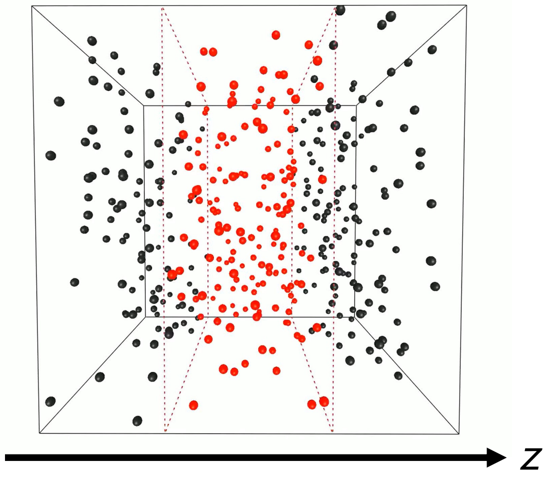
V. Kuznietsov et al., Phys. Rev. C 105, 044903 (2022)

Classical molecular dynamics simulations of the **Lennard-Jones fluid** near 3D-Ising critical point ($T \approx 1.06 T_c$, $n \approx n_c$) of the liquid-gas transition

Scaled variance in coordinate space acceptance $|z| < z^{max}$

$$\boxed{\tilde{\omega}^{\mathrm{coord}} = \frac{1}{1-\alpha} \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle}}$$

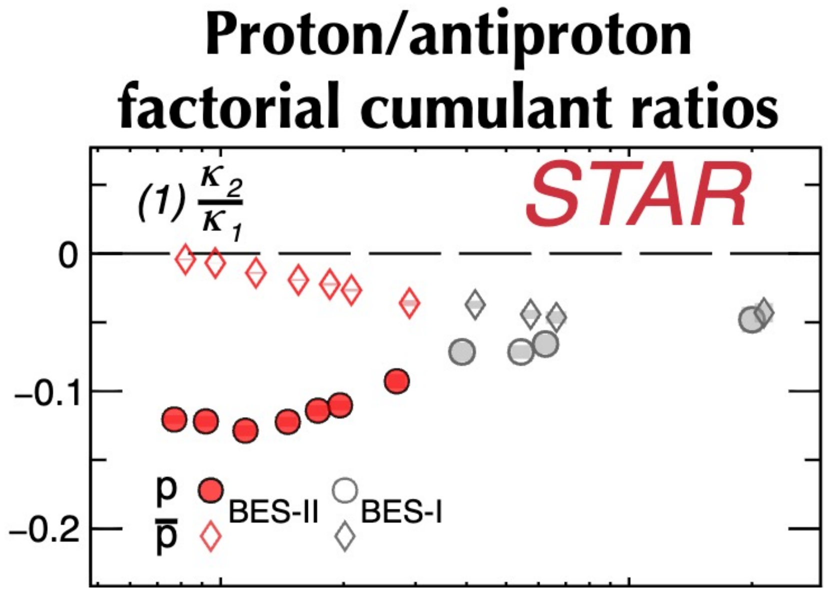


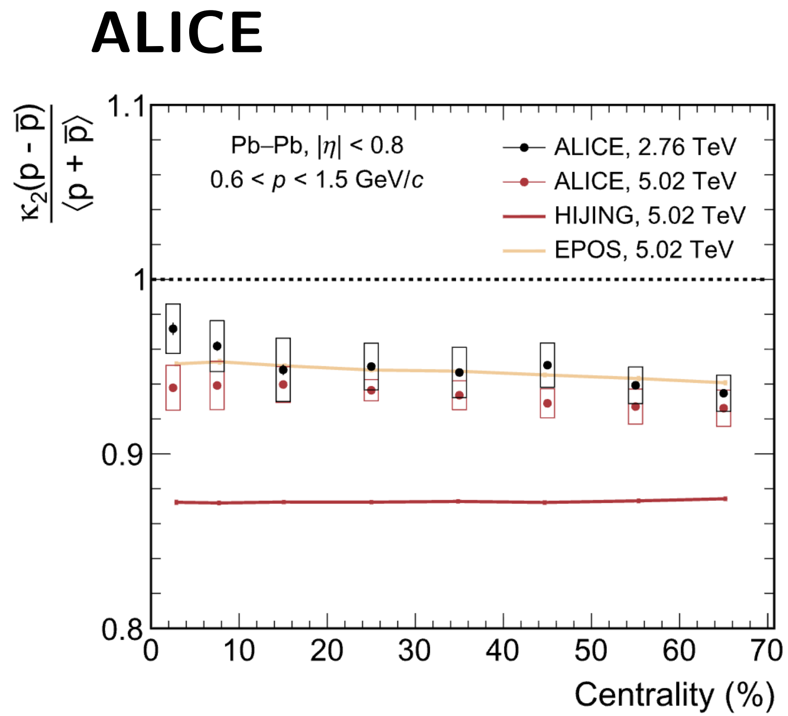


- Large fluctuations survive despite strong finite-size effects

Critical point leads to enhancement of fluctuations relative to something. **Relative to what?**

Experimental measurements



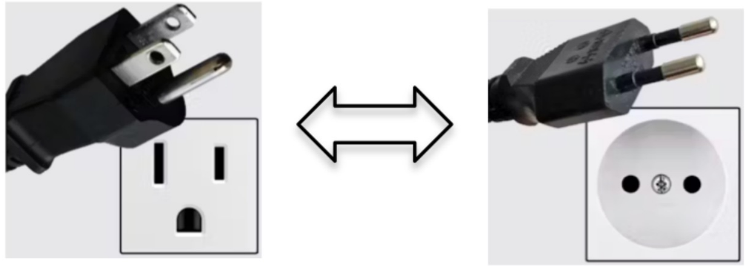


- Measurements at collider energies indicate suppression of fluctuations, not enhancement
- If there is a critical point signal, it can only be visible by subtracting a baseline

A note on notation

STAR
Cumulants (C)
Factorial cumulants (κ)

Others
Cumulants (κ)
Factorial cumulants (C)



M. Arslandok, QM 2025

Cumulants (κ)
Factorial cumulants (FC)

Cumulants (κ)
Factorial cumulants ($F^{(n,0)}$)

Cumulants (κ)
Factorial cumulants ($\hat{C}$)

Non-critical baseline

Ingredients

- baryon conservation (total net baryon number does not fluctuate)
- non-critical EoS effects in χ_2 (ideal gas or purely repulsive)

Density-density correlator

$$\mathcal{C}_2(\mathbf{r}_1,\mathbf{r}_2) = \chi_2\delta(\mathbf{r}_1-\mathbf{r}_2) - \frac{\chi_2}{V}$$

local fluctuations (equation of state)

balancing contribution (e.g. baryon conservation)

HRG CE: ideal gas in the canonical ensemble $\chi_2 \sim \langle N_B + N_{\bar{B}} \rangle$ leads to

$$\frac{\kappa_2[p-\bar{p}]}{\langle p+\bar{p}\rangle} \approx 1-\alpha, \qquad \alpha = \frac{\langle N_p^{\mathrm{acc}} + N_{\bar{p}}^{\mathrm{acc}}\rangle}{\langle N_B^{4\pi} + N_{\bar{B}}^{4\pi}\rangle}$$

$\sqrt{s_{NN}} \searrow \quad \Rightarrow \quad \alpha \nearrow$

Braun-Munzinger et al., NPA 1008, 122141 (2021)

Hydro EV: additional repulsion via excluded volume (fitted to lattice data)

- Implemented at Cooper-Frye stage of MUSIC

VV, V. Koch, C. Shen, Phys. Rev. C 105, 014904 (2022)

UrQMD: no interactions, baryon number conserved

- Non-equilibrium evolution and volume fluctuations

Measurements vs baselines

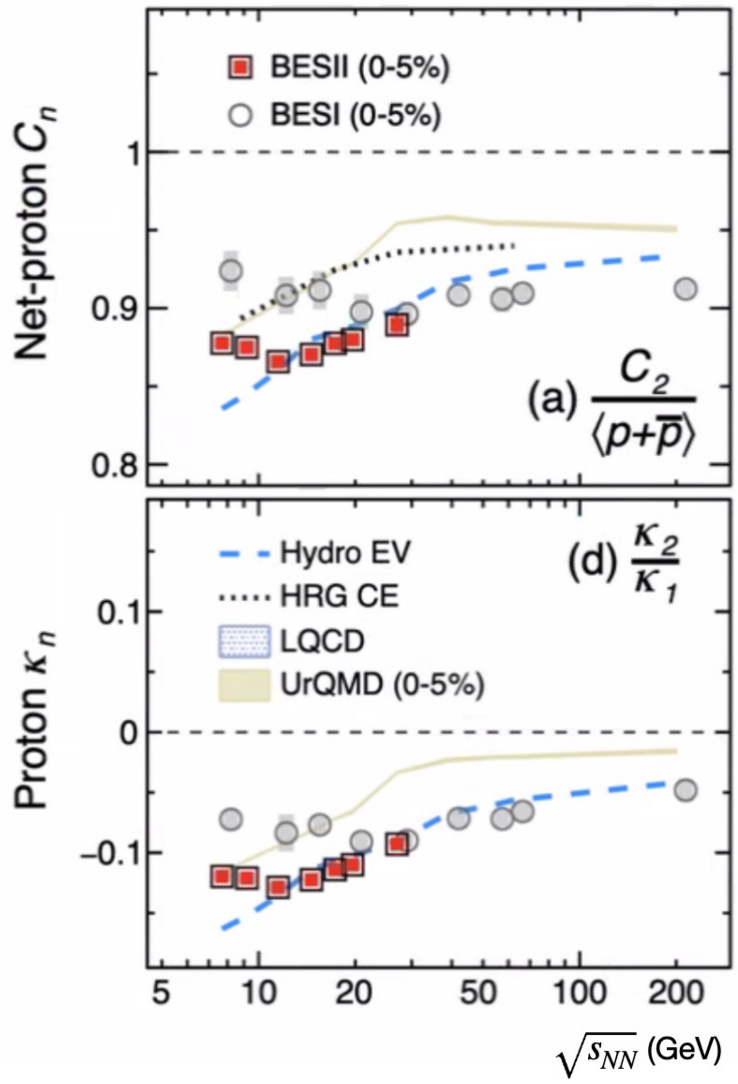
- The suppression relative to Poisson at collider energies appears to be driven by baryon conservation

$$\frac{\kappa_2[p-\bar{p}]}{\langle p+\bar{p}\rangle} \approx 1-\alpha, \qquad \alpha = \frac{\langle N_p^{\mathrm{acc}} + N_{\bar{p}}^{\mathrm{acc}}\rangle}{\langle N_B^{4\pi} + N_{\bar{B}}^{4\pi}\rangle} \qquad \sqrt{s_{NN}} \searrow \Longrightarrow \alpha \nearrow$$

- Additional repulsion (excluded volume) improves the agreement
    - Same result in an alternative implementation of repulsion

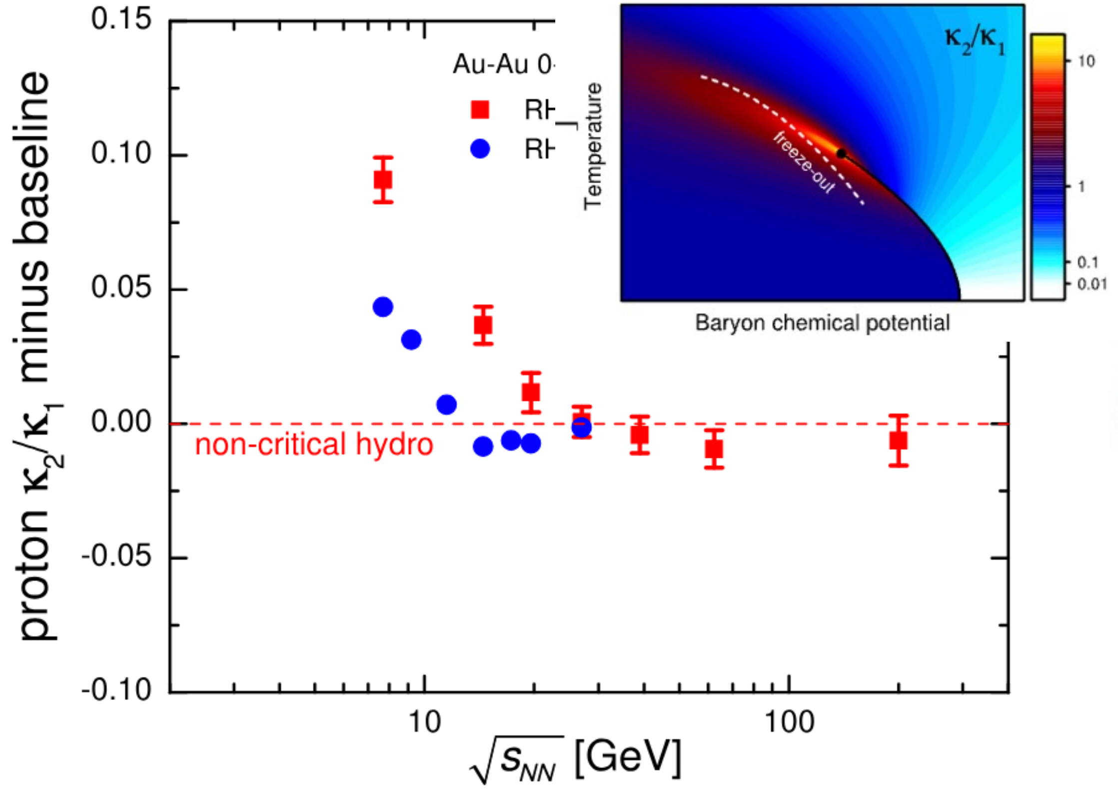
Friman, Redlich, Rustamov, arXiv:2508.18879

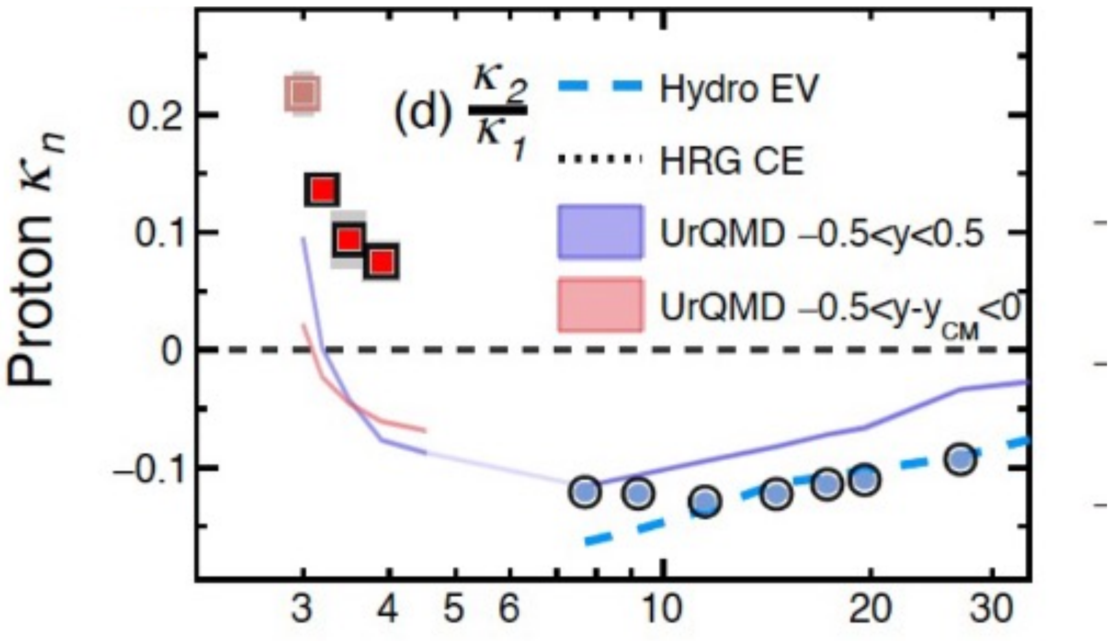
- Change of trend emerges at $\sqrt{s_{NN}}$~10 GeV



Subtracting the baseline

If $\kappa_2^{tot} \approx \kappa_2^{crit} + \kappa_2^{reg}$ try to isolate the critical part* by subtracting the baseline (here hydro EV)





Enhancement relative to the baseline at lower $\sqrt{s_{NN}}$ which continues at fixed target energies

*May be a useful quantity for finite-size scaling analysis compared to the bare κ_2/κ_1

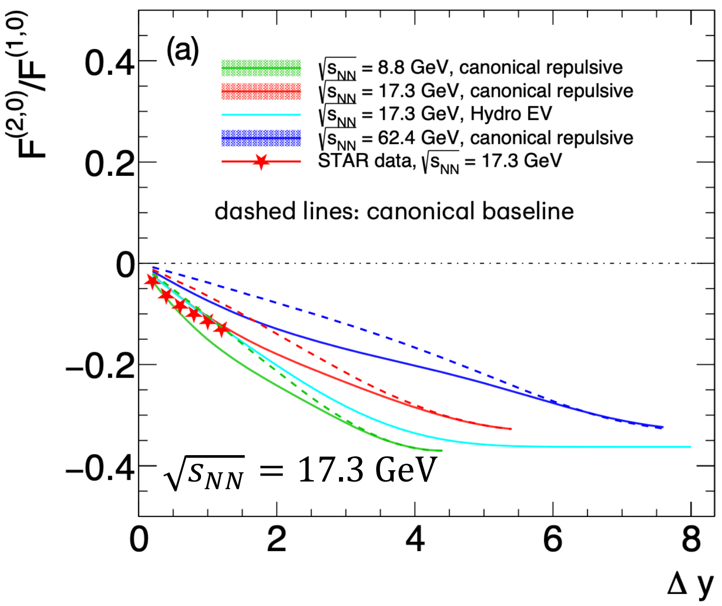
Attraction vs repulsion

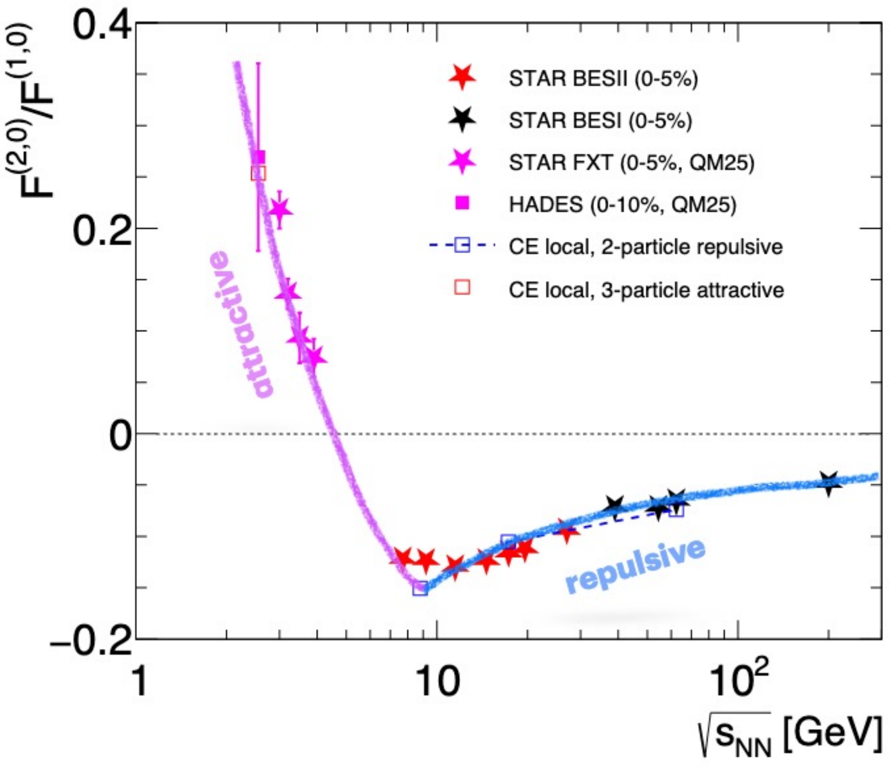
Friman, Redlich, Rustamov, arXiv:2508.18879

Recently, attractive and repulsive interactions implemented through a potential in rapidity

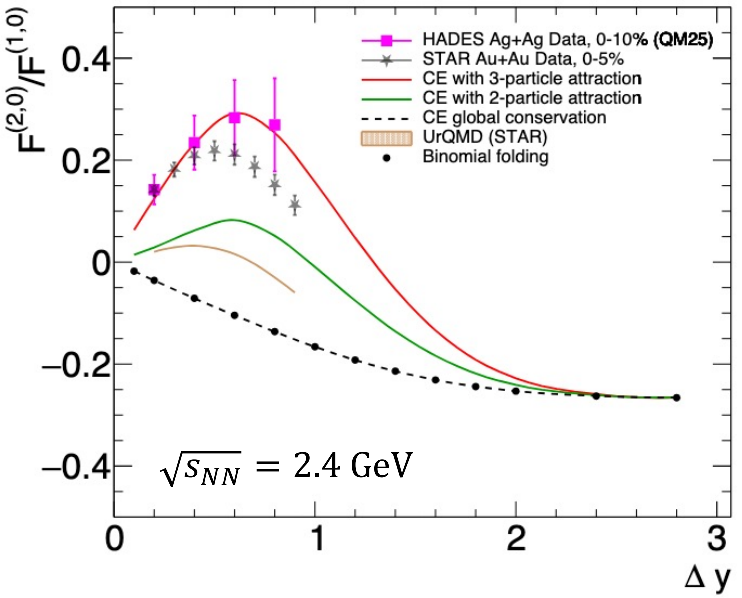
$$E_r(y_1, y_2) = \alpha_r e^{-|y_1 - y_2|/\rho_r} \qquad P(y_1, y_1) = \frac{e^{-E(y_1,y_2)}}{Z} \qquad E_a(y_1, y_2) = \alpha_a |y_1 - y_2|^{\beta_a}$$

repulsion





attraction



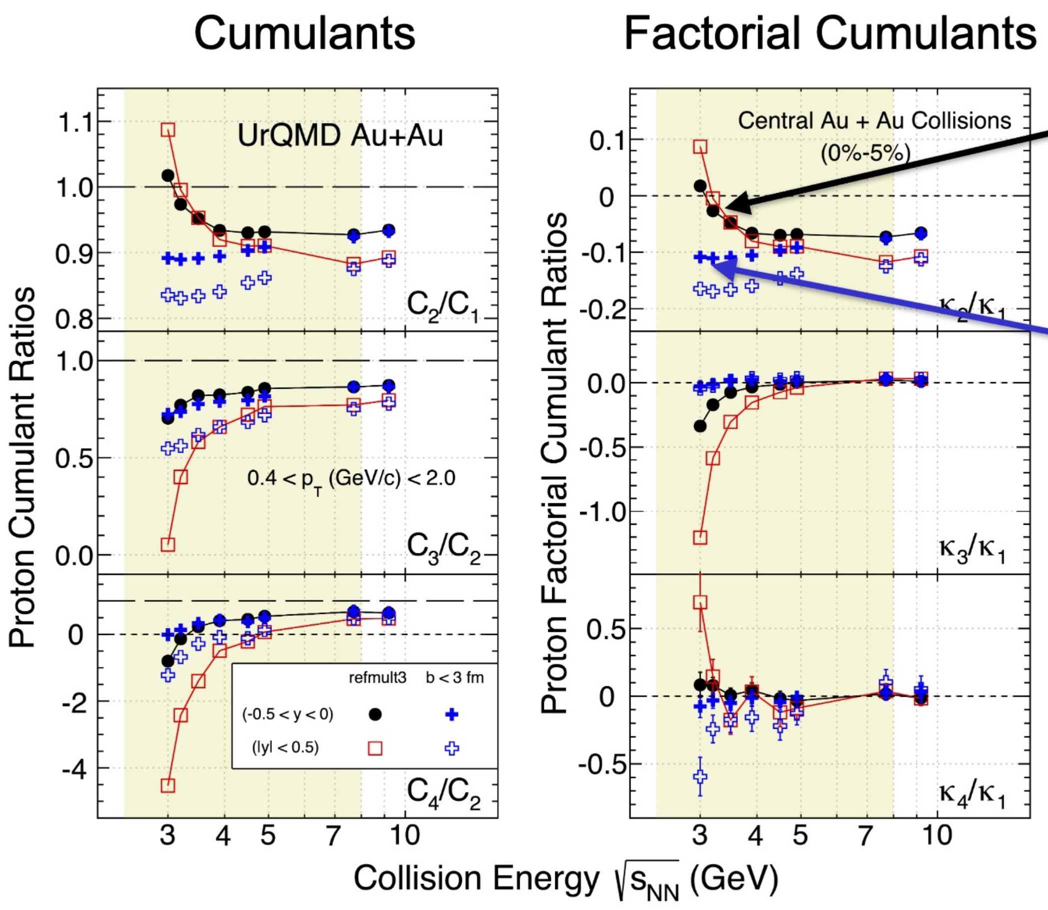
Interplay of repulsive (high $\sqrt{s_{NN}}$) and attractive (low $\sqrt{s_{NN}}$) interactions?

Time to celebrate!

Adapted from V. Koch, ERICE2025

The (possible) culprit

X. Zhang, Y. Zhang, X. Luo, N. Xu, arXiv: 2506.18832



Fluctuating impact parameter
STAR centrality selection

Fixed impact parameter $(b < 3 \text{ fm})$
minimal volume fluctuations.

N.B.: Centrality Bin Width Corrections applied to both

Possible culprit:
volume fluctuations/centrality selection

Test for baseline: acceptance dependence of couplings

Bzdak et al. introduced reduced correlation functions – "couplings" [Bzdak, Koch, Strodthoff, PRC 95, 054906 (2017)]

$$\boxed{\hat{c}_k = \frac{\hat{C}_k}{\langle N \rangle^k}}$$

$$c_k = \frac{\int \rho_1(y_1)\cdots\rho_1(y_k)c_k(y_1,\ldots,y_k)dy_1\cdots dy_k}{\int \rho_1(y_1)\cdots\rho_1(y_k)dy_1\cdots dy_k}$$

integrated correlation function in rapidity

Long-range correlations in rapidity lead to acceptance-independent couplings, for example

- Global (not local) baryon conservation
  [Bzdak, Koch, Skokov, EPJC 77, 288 (2017); Bzdak, Koch, PRC 96, 054905 (2017)]
- $+$ volume fluctuations
  [Holzmann, Koch, Rustamov, Stroth, arXiv:2403.03598]

$$c_2 = -\frac{1}{B}, \qquad c_3 = \frac{2}{B^2}, \qquad c_4 = -\frac{6}{B^3}.$$

$$\tilde{\hat{c}}_{i,j} = \hat{c}_{i,j} + \frac{\kappa_2[V]}{\langle V \rangle^2}, \quad \text{for} \quad i+j=2.$$

all lead to

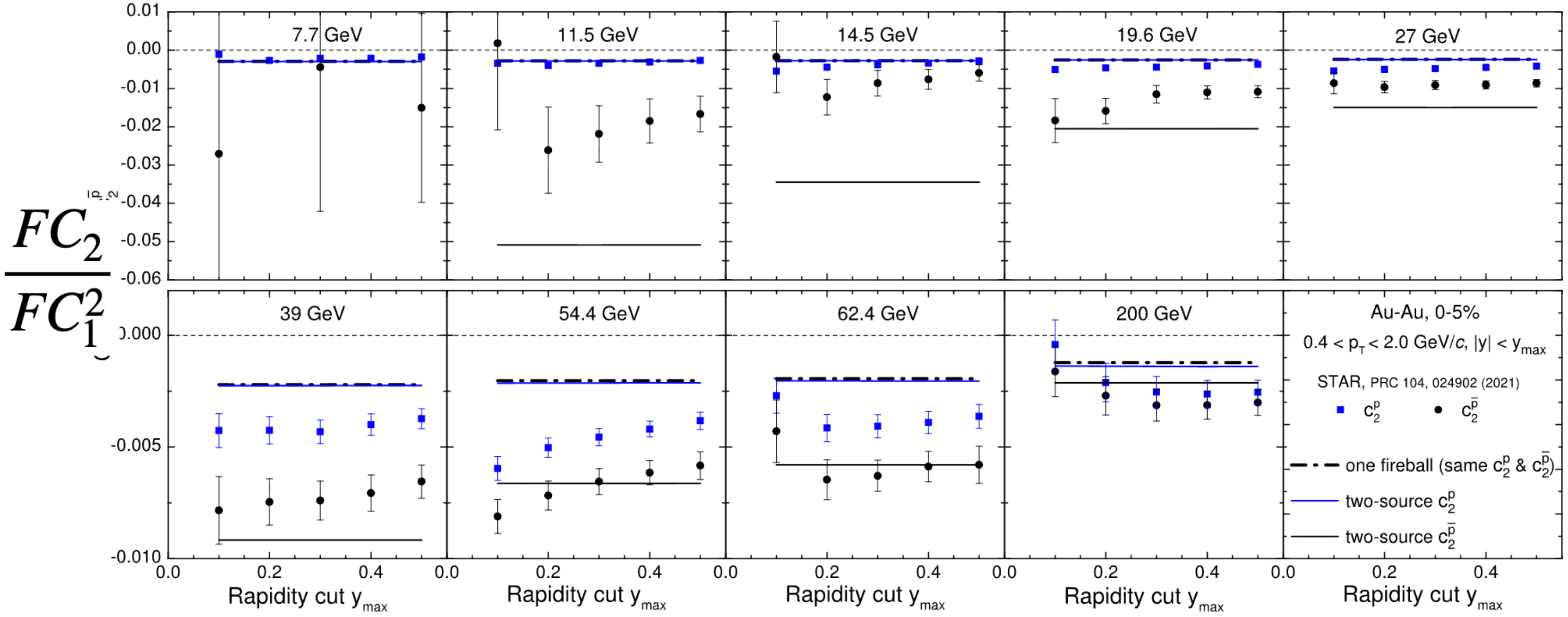
$$\boxed{\frac{\hat{C}_k}{\langle N \rangle^k} = const.}$$ at a given $\sqrt{s_{NN}}$ and $$\boxed{\frac{\hat{c}_2^{p}}{\langle N_p \rangle^2} \approx \frac{\hat{c}_2^{\bar{p}}}{\langle N_{\bar{p}} \rangle^2} = -\frac{1}{\langle N_B + N_{\bar{B}} \rangle_{4\pi}}}$$ at a given $\sqrt{s_{NN}}$

Can be tested *without* CBWC/volume fluctuations correction Bzdak, Koch, Vovchenko, arXiv:2503.16405

Test for baseline: BES-I data

- $\dfrac{FC_2}{FC_1^2}$ more or less constant

- $\dfrac{FC_2[p]}{FC_1^2[p]} \neq \dfrac{FC_2[\bar{p}]}{FC_1^2[\bar{p}]}$



- baseline OK for protons

- No good for anti-protons ??

- How will it look with BES II data?

Two component model

2 sources: stopped and produced particles

- All anti-protons are produced
- protons come from produced and stopped sources

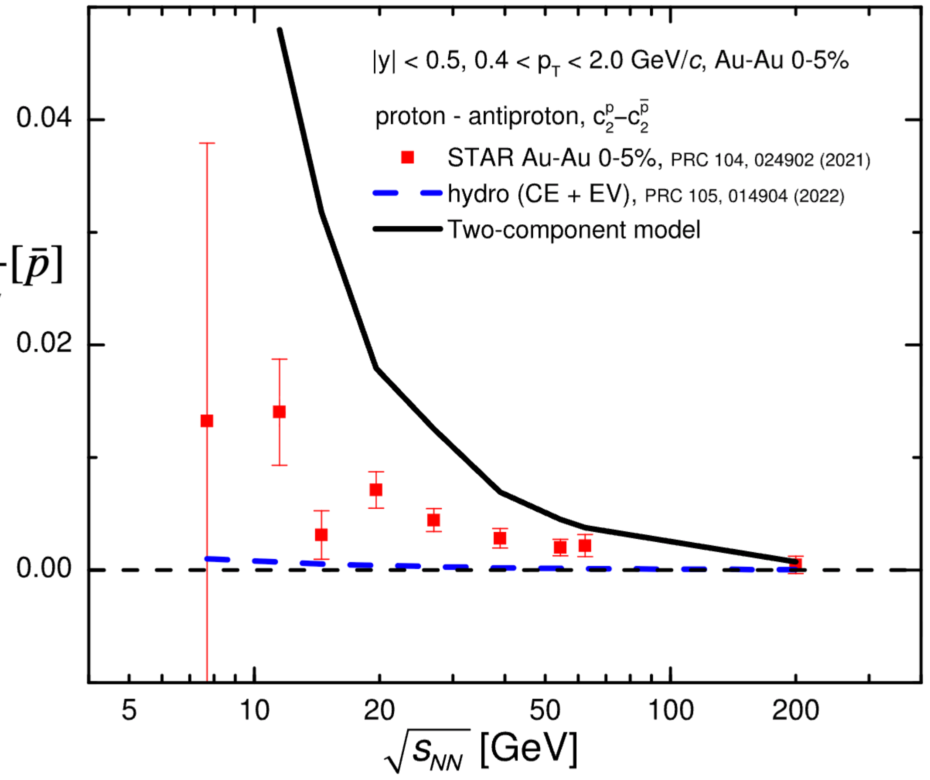
$$N_p(\text{produced}) = \mathrm{N}_{\bar{\mathrm{p}}}$$

$$N_p = N_p(\text{stopped}) + \mathrm{N}_{\bar{\mathrm{p}}}$$

- Produced source: Thermal with zero net baryon number $\langle B - \bar{B} \rangle = 0$

- Stopped source: Follows binomial distribution

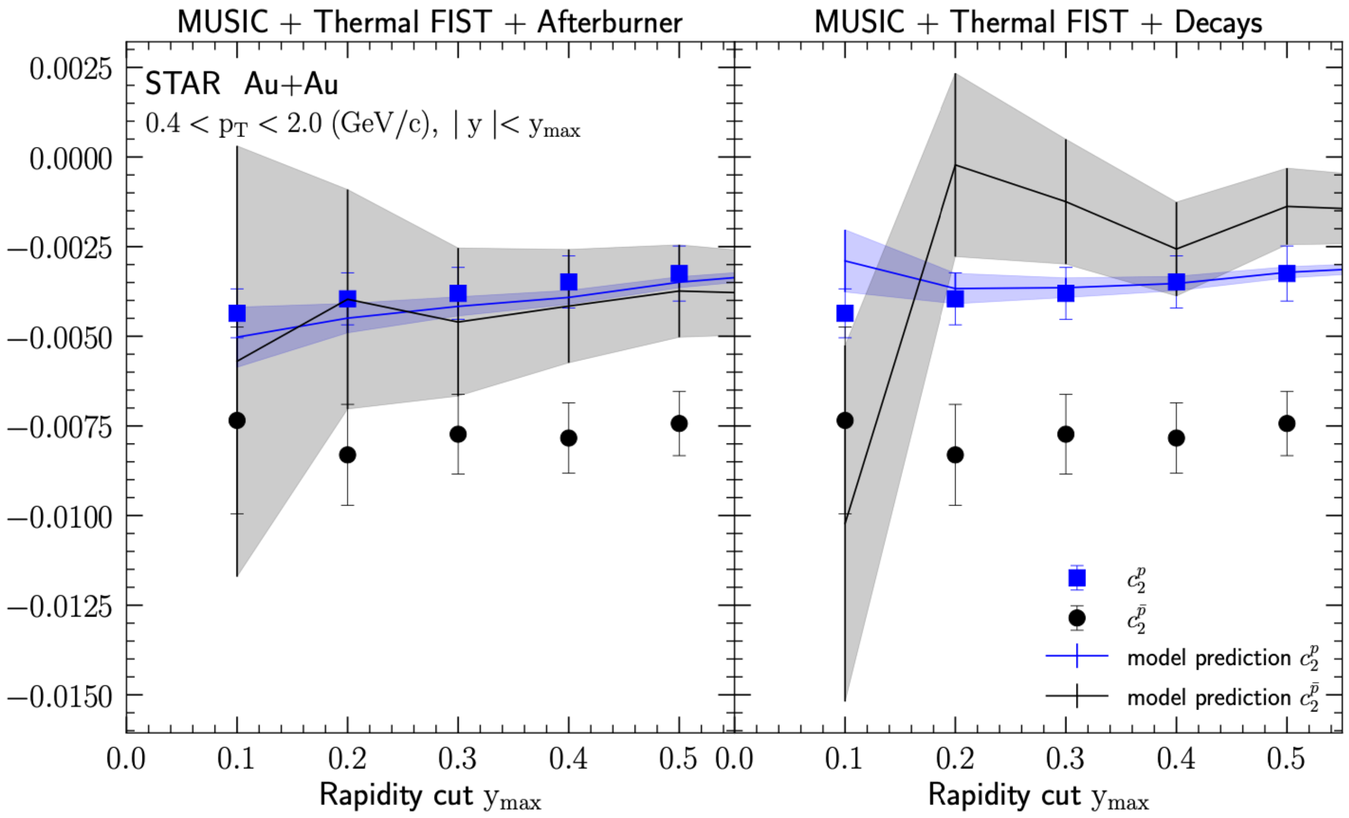
$$\frac{FC_2}{FC_1^2}[p] - \frac{FC_2}{FC_1^2}[\bar{p}]$$



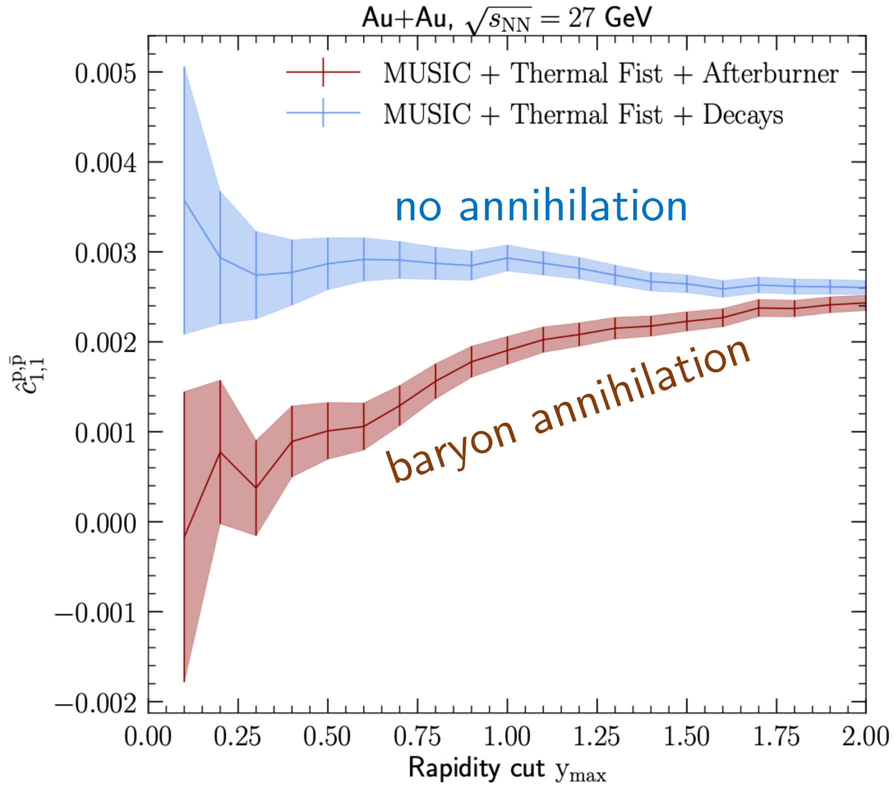
Scaled factorial cumulants and baryon annihilation

Extending Hydro EV to incorporate hadronic phase (UrQMD)

Au-Au, $\sqrt{s_{NN}} = 27$ GeV G. Pihan, VV, in progress



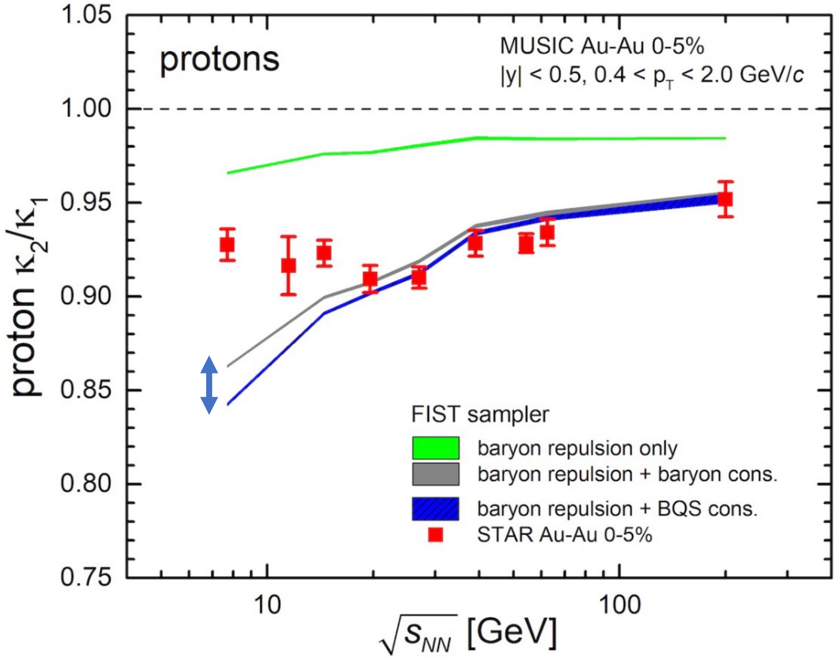
Covariance $c_{11}^{p\bar{p}}$



- Hadronic phase appears unlikely to resolve the antiproton puzzle (more statistics needed)
- Acceptance dependence of proton-antiproton covariance shows clear effect of hadronic phase

(Some of) Missing pieces at lower energies

Charge conservation



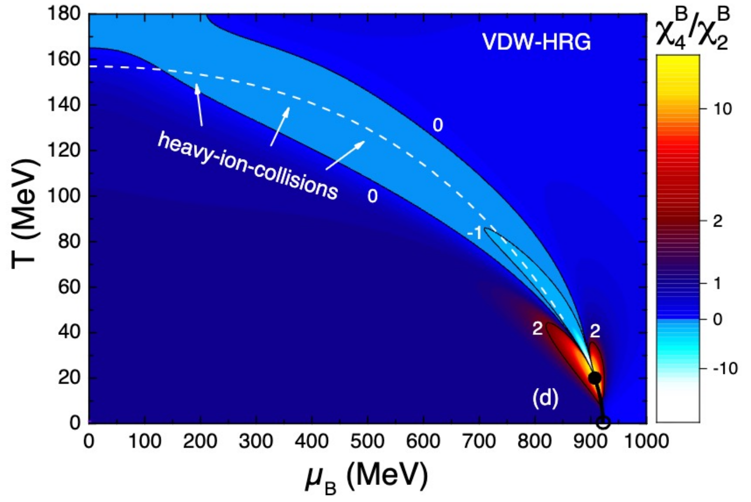
Light nuclei

particle	multiplicity	uncertainty	Ref.
p	77.6	± 2.4	[15]
$p + n \to {}^2H$	28.7	± 0.8	[15]
$p + 2n \to {}^3H$	8.7	± 1.1	[15]
$p + p + n \to {}^3He$	4.6	± 0.3	[15]
p (bound)	46.5	± 1.5	[15]
π^+	9.3	± 0.6	[18]
π^-	17.1	± 1.1	[18]
K^+	$5.98\,10^{-2}$	$\pm 6.79\,10^{-3}$	[16]
K^-	$5.6\,10^{-4}$	$\pm 5.96\,10^{-5}$	[16]
Λ	$8.22\,10^{-2}$	${}^{+5.2}_{-9.2}\,10^{-3}$	[17]

TABLE I. Preliminary particle yields measured by the HADES collaboration at SIS18 accelerator, $\sqrt{s_{NN}} = 2.4$ GeV, for 10% most central Au-Au collisions. Protons bound in nuclei can be accounted all as free under the assumption that the nuclei are formed after kinetic freeze-out. The data compilation is extracted from [1].

HADES preliminary

Nuclear liquid-gas



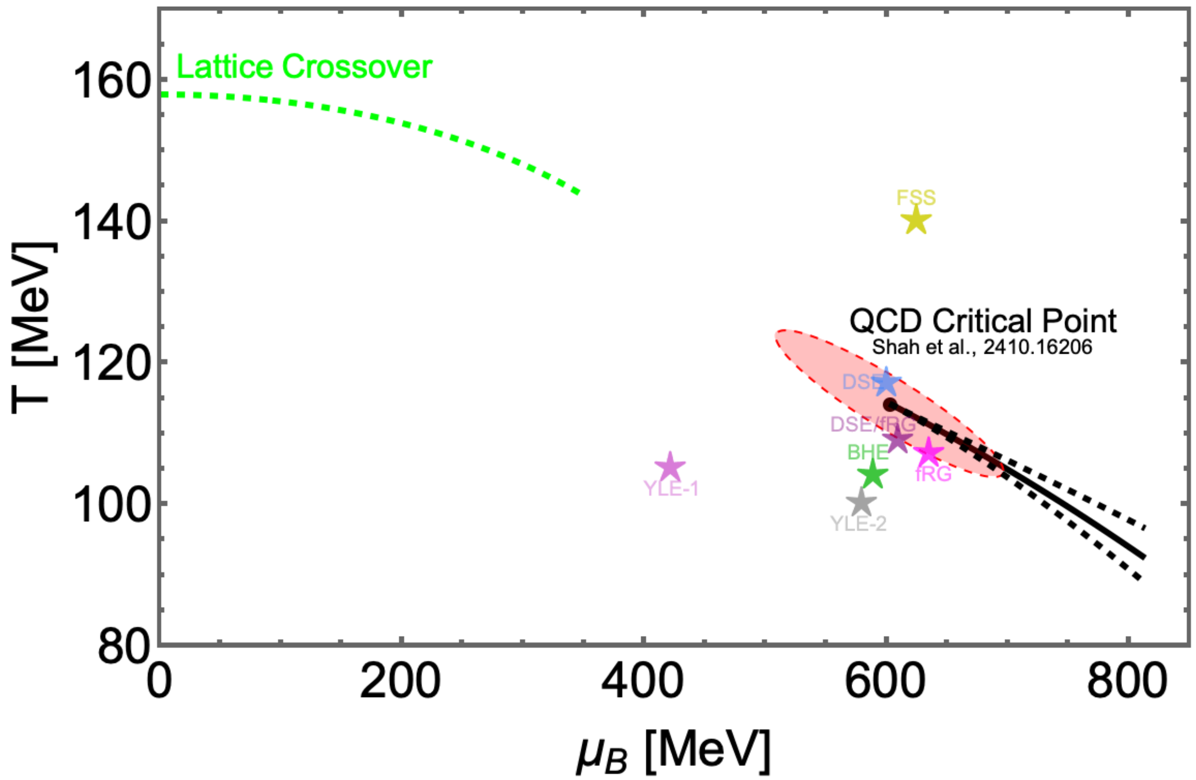
Summary

- Non-critical baseline with baryon number conservation and repulsive interaction tuned to LQCD agrees with data down to $\sqrt{s_{NN}} \sim 10$ GeV

- Data below $\sqrt{s_{NN}} \sim 10$ GeV seem to require some kind of attraction
  - However, UrQMD gets the trend in the energy dependence right. Volume fluctuations a low energy?
  - Other missing effects (charge conservation, light nuclei, liquid-gas)?

- Possible test of a baseline involving baryon number conservation and volume fluctuations via **acceptance dependence** ratio of factorial cumulants $\frac{\hat{C}_2^p}{\langle N_p \rangle^2}$
- Anti protons from BES I are NOT understood (the baseline fails). BESII comparison needed. Two source model?
- Verify other observables (differential measures e.g. balance functions, other species)

Critical point estimates



Critical point estimate at $O(\mu_B^2)$:

$T_c = 114 \pm 7$ MeV, $\mu_B = 602 \pm 62$ MeV

Estimates from recent literature:

YLE-1: D.A. Clarke et al. (Bielefeld-Parma), arXiv:2405.10196

YLE-2: G. Basar, PRC 110, 015203 (2024)

BHE: M. Hippert et al., arXiv:2309.00579

fRG: W-J. Fu et al., PRD 101, 054032 (2020)

DSE/fRG: Gao, Pawlowski., PLB 820, 136584 (2021)

DSE: P.J. Gunkel et al., PRD 104, 052022 (2021)

FSS: A. Sorensen et al., arXiv:2405.10278

Optimist's view: Different estimates converge onto the same region because QCD CP is likely there

Pessimist's view: Different estimates converge onto the same region because it's the closest not yet ruled out by LQCD

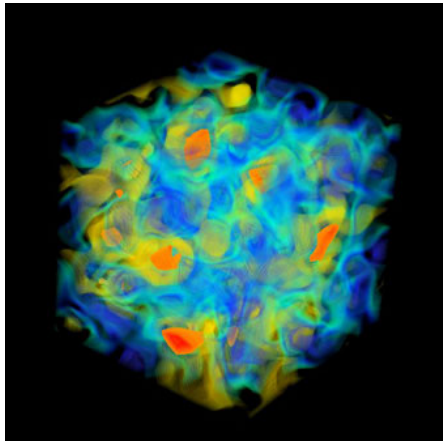
"…experimental measurements are essential to determine whether a QCD critical point exists."

2023 Long Range Plan for Nuclear Science

Theory vs experiment: Challenges for fluctuations

UNIVERSITY OF HOUSTON

Theory



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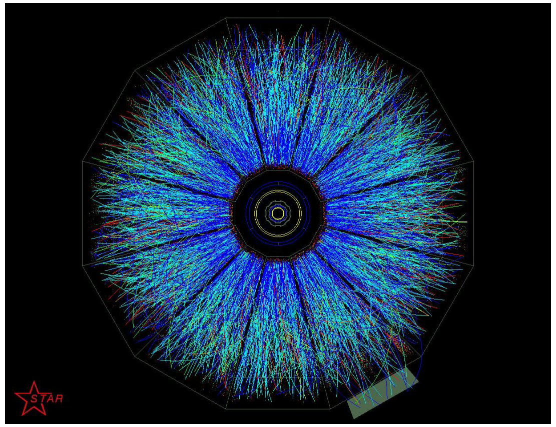
- Coordinate space
- In contact with the heat bath
- Conserved charges
- Uniform
- Fixed volume

Experiment



STAR event display

- Momentum space
- Expanding in vacuum
- Non-conserved particle numbers
- Inhomogeneous
- Fluctuating volume

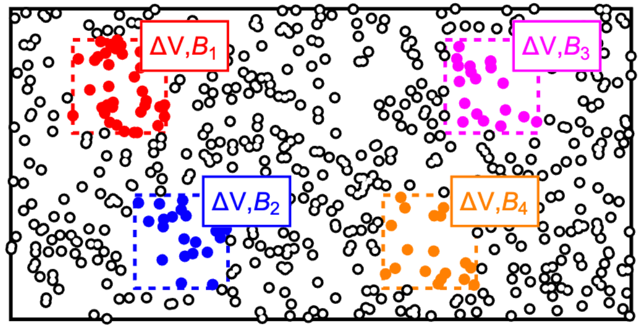
Comparing theory and experiment should be done very carefully

Additional slides

Exact charge conservation

VV, Savchuk, Poberezhnyuk, Gorenstein, Koch, PLB 811, 135868 (2020); VV, arXiv:2409.01397

Utilizing the canonical partition function in thermodynamic limit compute **n-point density correlators**



$$\mathcal{C}_1(\mathbf{r}_1) = \rho(\mathbf{r}_1)$$

$$\mathcal{C}_2(\mathbf{r}_1,\mathbf{r}_2) = \chi_2\delta(\mathbf{r}_1-\mathbf{r}_2) - \frac{\chi_2}{V}$$

local correlation — balancing contribution (e.g. baryon conservation)

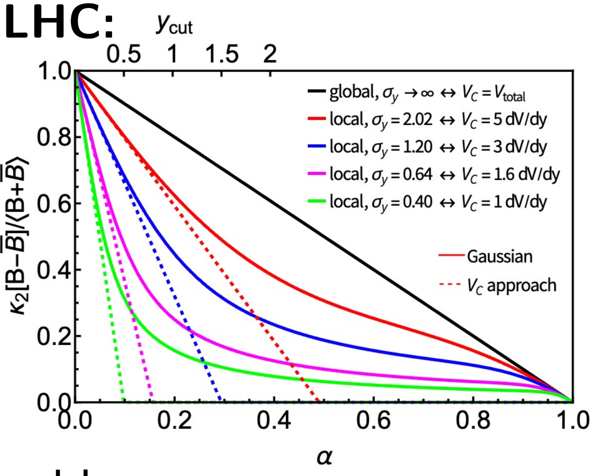
$$\mathcal{C}_3(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3) = \chi_3\delta_{1,2,3} - \frac{\chi_3}{V}[\delta_{1,2}+\delta_{1,3}+\delta_{2,3}] + 2\frac{\chi_3}{V^2} \qquad \delta_{1,\dots,n} = \prod_{i=2}^{n}\delta(\mathbf{r}_1-\mathbf{r}_i)$$

local correlation — balancing contributions

$$\mathcal{C}_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = \chi_4\delta_{1,2,3,4} - \frac{\chi_4}{V}[\delta_{1,2,3}+\delta_{1,2,4}+\delta_{1,3,4}+\delta_{2,3,4}] - \frac{(\chi_3)^2}{\chi_2 V}[\delta_{1,2}\delta_{3,4}+\delta_{1,3}\delta_{2,4}+\delta_{1,4}\delta_{2,3}]$$
$$+ \frac{1}{V^2}\left[\chi_4+\frac{(\chi_3)^2}{\chi_2}\right][\delta_{1,2}+\delta_{1,3}+\delta_{1,4}+\delta_{2,3}+\delta_{2,4}+\delta_{3,4}] - \frac{3}{V^3}\left[\chi_4+\frac{(\chi_3)^2}{\chi_2}\right].$$

local correlation — balancing contributions

LHC:



Integrating the correlator yields cumulant inside a subsystem of the canonical ensemble

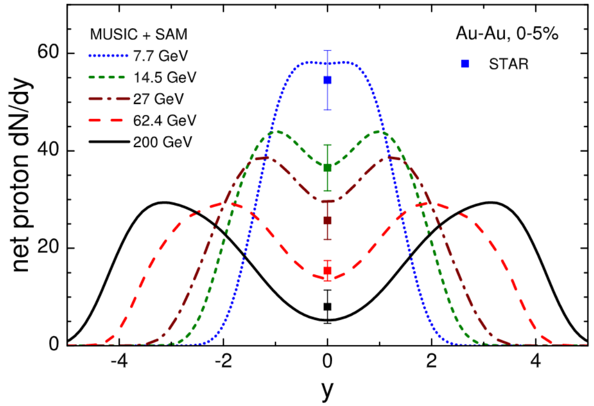
$$\kappa_n[B_{V_s}] = \int_{\mathbf{r}_1\in V_s} d\mathbf{r}_1 \dots \int_{\mathbf{r}_n\in V_s} d\mathbf{r}_n\, \mathcal{C}_n(\{\mathbf{r}_i\})$$

Momentum space: Fold with Maxwell-Boltzmann in LR frame and integrate out the coordinates

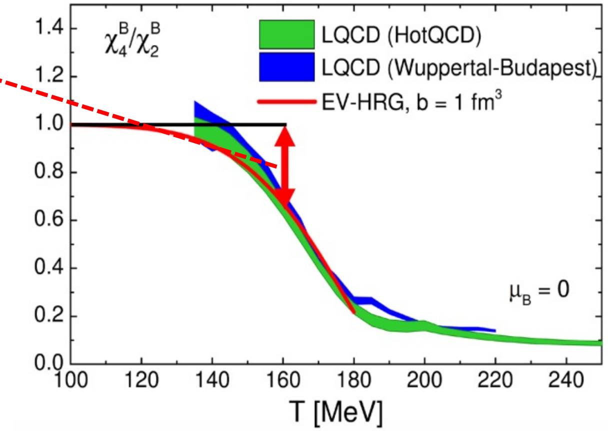
Hydro EV: Non-critical hydro baseline at RHIC-BES

VV, V. Koch, C. Shen, Phys. Rev. C 105, 014904 (2022)

- (3+1)-D viscous hydrodynamics evolution (MUSIC-3.0)
  - Collision geometry-based 3D initial state [Shen, Alzhrani, PRC 102, 014909 (2020)]
  - Crossover equation of state based on lattice QCD [Monnai, Schenke, Shen, Phys. Rev. C 100, 024907 (2019)]



- Non-critical contributions computed at particlization ($\epsilon_{sw} = 0.26$ GeV/fm^3)
  - QCD-like baryon number distribution (χ_n^B) via **excluded volume** b = 1 fm^3 [VV, V. Koch, Phys. Rev. C 103, 044903 (2021)]
  - **Exact global baryon conservation*** (and other charges)
    - Subensemble acceptance method 2.0 (analytic) [VV, Phys. Rev. C 105, 014903 (2022)]
    - or FIST sampler (Monte Carlo) [VV, Phys. Rev. C 106, 064906 (2022)] https://github.com/vlvovch/fist-sampler



- Included: baryon conservation, repulsion, kinematical cuts
- Absent: critical point, local conservation, initial-state/volume fluctuations, hadronic phase

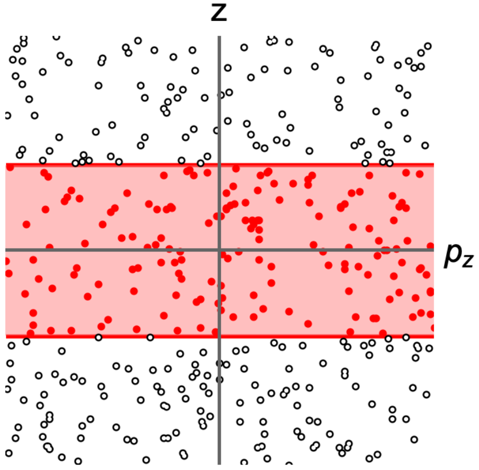
*If baryon conservation is the only effect (no other correlations), non-critical baseline can be computed without hydro
Braun-Munzinger, Friman, Redlich, Rustamov, Stachel, NPA 1008, 122141 (2021)

Coordinate vs Momentum space

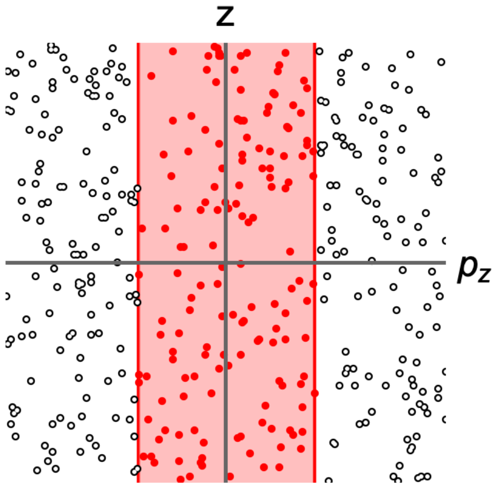
Box setup: Coordinates and momenta are uncorrelated

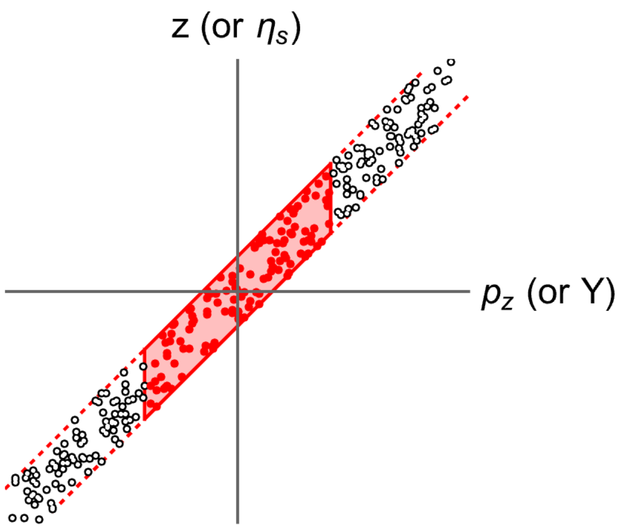
HICs: Flow (e.g. Bjorken)

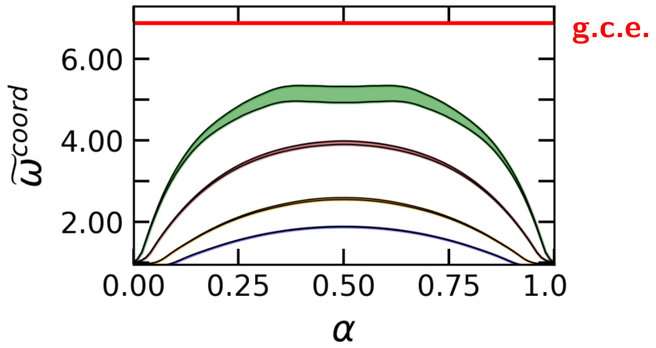
Coordinate space cut



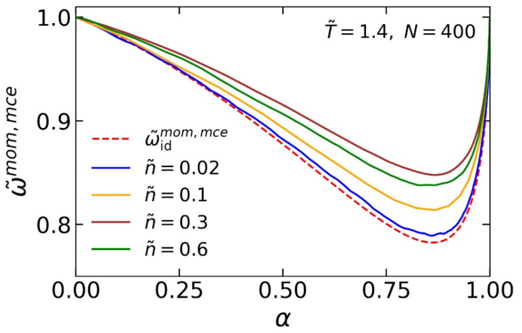
Momentum space cut



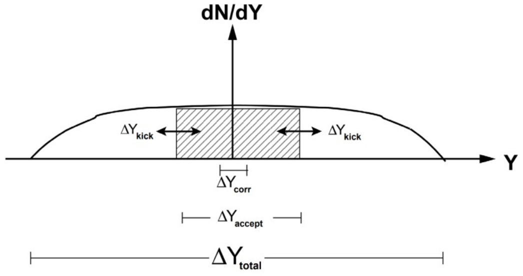




Large correlations



Nothing left

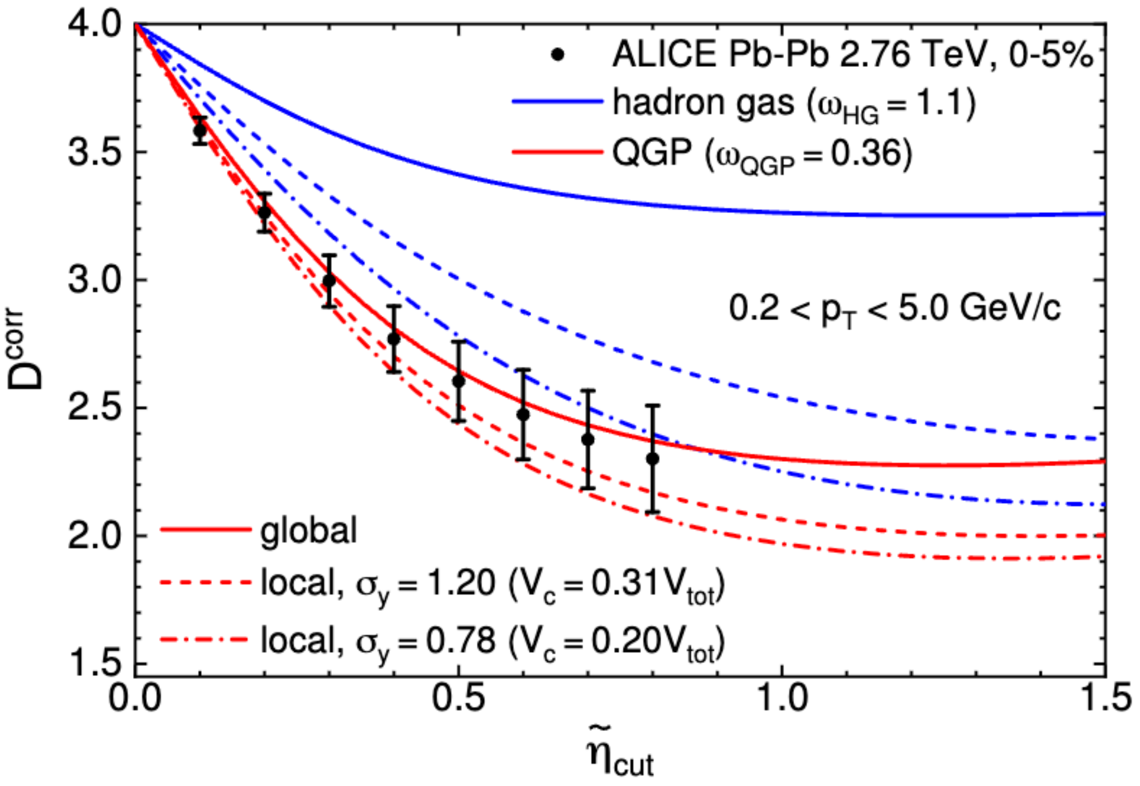


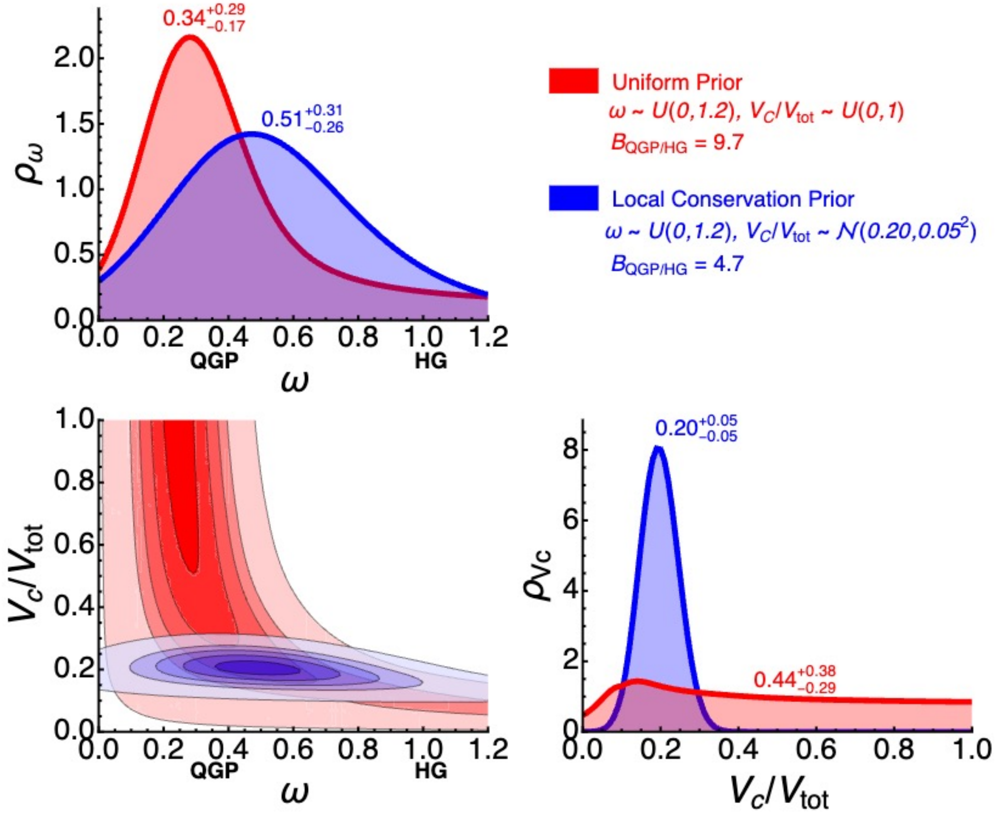
V. Koch, arXiv:0810.2520

momentum cut $\sim$ coordinate cut $+$ smearing

D-measure of charge fluctuations

$$D = 4\frac{\kappa_2[N_+ - N_-]}{\langle N_{\rm ch}\rangle} = 4\frac{\kappa_2[Q]}{\langle Q^+ + Q^-\rangle} = 4\left\{1 - \left(1 - \frac{\omega}{\gamma_Q}\right)\frac{\langle p^2(\eta)\rangle}{\langle p(\eta)\rangle} - \frac{\omega}{\gamma_Q}\frac{\langle p(\eta_1)p(\eta_2)\rangle_{\varkappa}}{\langle p(\eta)\rangle}\right\}$$



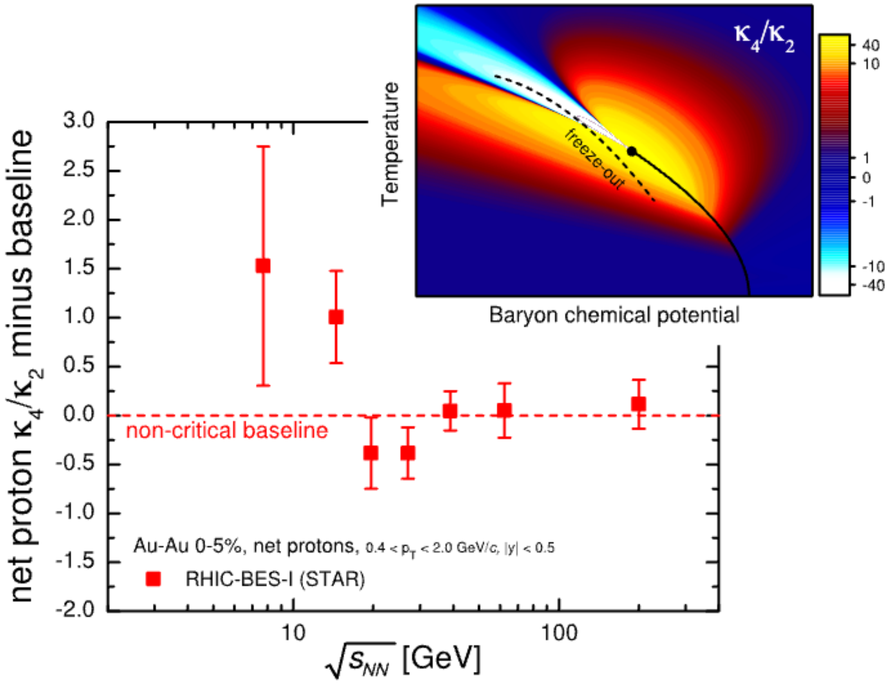


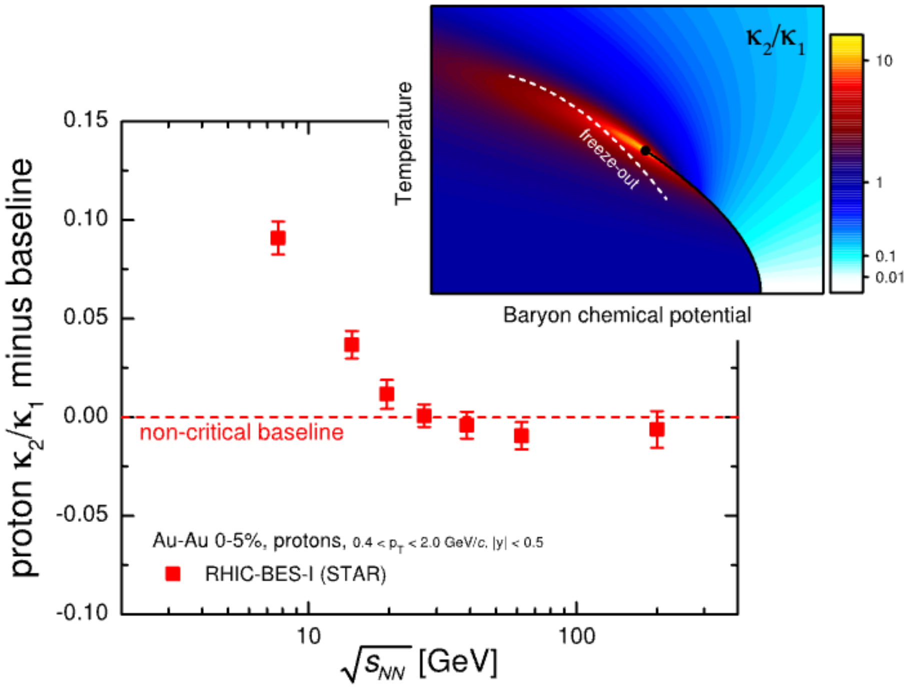
J. Parra, R. Poberezhniuk, V. Koch, C. Ratti, VV, arXiv:2504.02085

Hints from RHIC-BES-I

VV, V. Koch, C. Shen, Phys. Rev. C 105, 014904 (2022)

Subtracting the hydrodynamic non-critical baseline

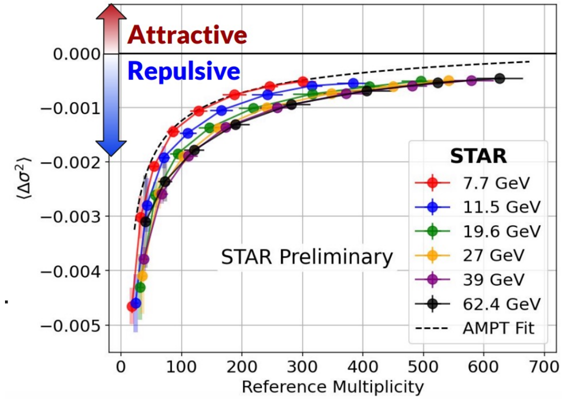




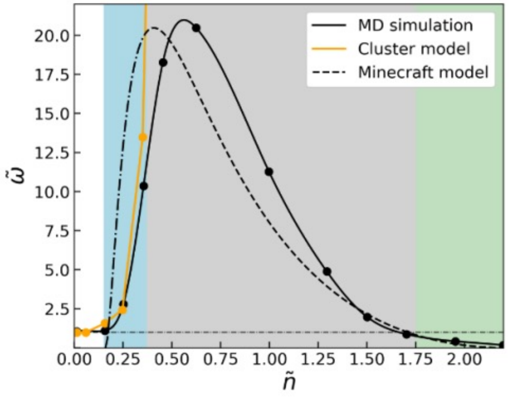
⚠️ **Notation:** Here we use κ_n for cumulants and $\hat{C}_n$ for factorial cumulants, STAR Collaboration uses the opposite

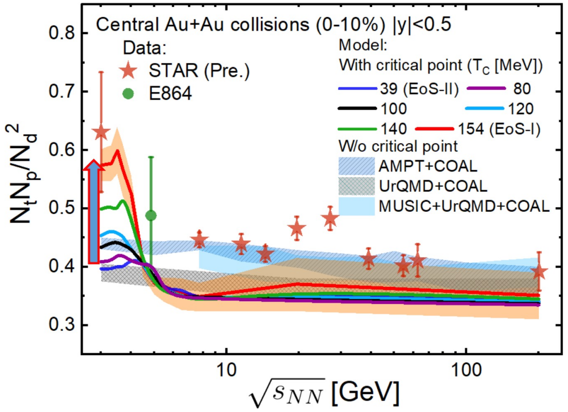
Other observables

- Azimuthal correlations of protons
  - points to repulsion at RHIC-BES



- Light nuclei
  - Spinodal/critical point enhancement of density fluctuations and light nuclei production





- Proton intermittency
  - No structure indicating power-law seen by NA61/SHINE

- Directed flow, speed of sound

Consistency in understanding all the observables is required

Dependence on the switching energy density