Quantum Closures for Quantum Moments

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Motivation

- We have been simulating core-collapse supernovae for ~60 years often with the world's biggest / best computers.
  - We can now do simulations in 3D for post-bounce times beyond 1 s and with ever improving microphysics and spatial fidelity.
- 3D simulations indicate that the explosion is due to a combination of neutrino heating and turbulence.
  - The neutrino heating depends upon the neutrino flavor
- 3D simulations are computationally expensive and still require some mixture of approximations to make them feasible.
  - The approximations are typically in the neutrino transport but simplifications appear elsewhere too e.g. the nucleosynthesis.
- Neutrino flavor transformation is not included in the simulation

- Calculations of SN neutrino flavor transformation usually post-process a ‘classical’ simulation.

see Stapleford et al, PRD, **102,** 081301 (2020) and

Xiong et al PRD **107** 083016 (2023) for two exceptions

- What has been found is that the flavor transformation occurs in several places due to different reasons.

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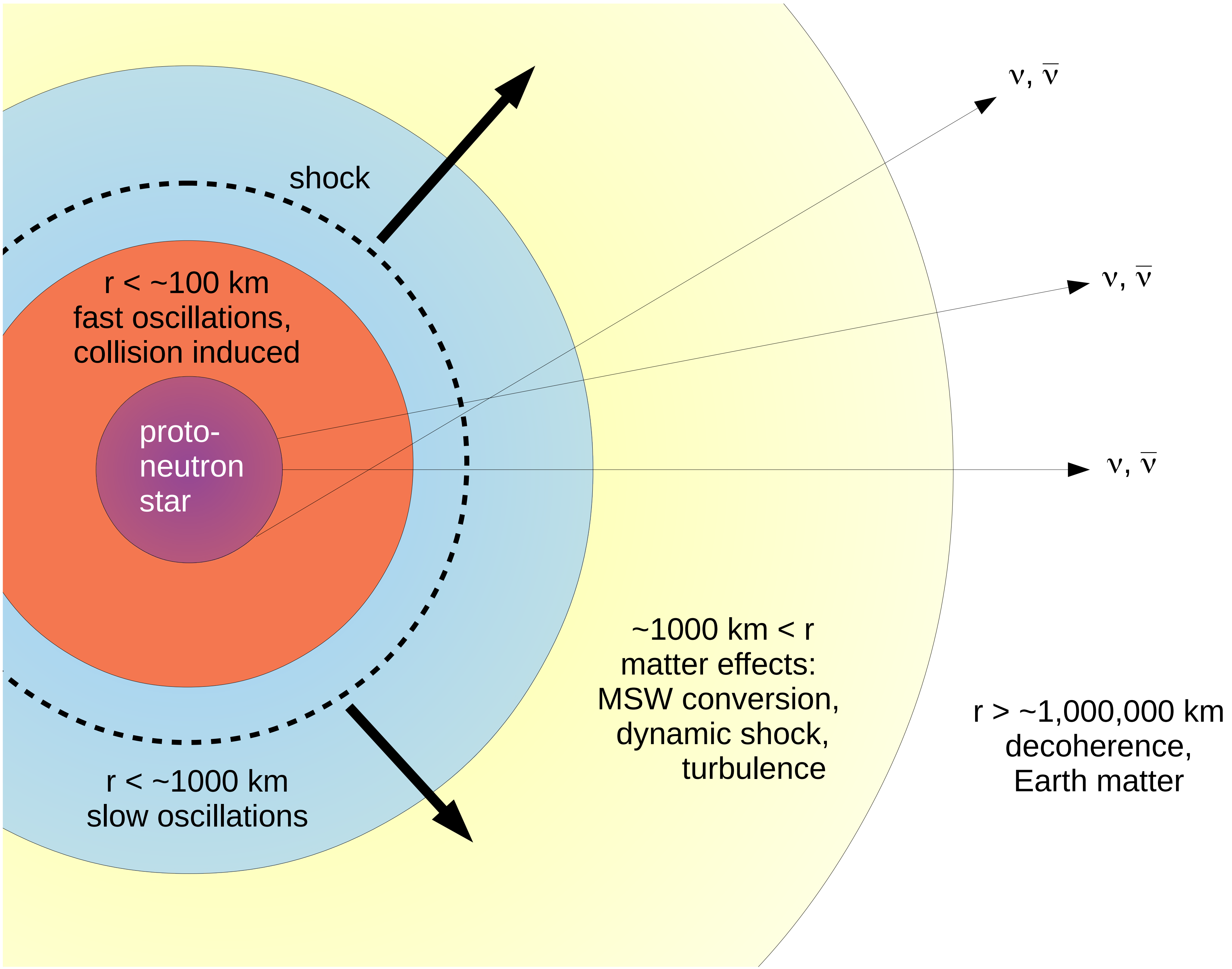

ν, ν̄
ν, ν̄
ν, ν̄
shock
r < ~100 km
fast oscillations,
collision induced
proto-
neutron
star
~1000 km < r
matter effects:
MSW conversion,
dynamic shock,
turbulence
r > ~1,000,000 km
decoherence,
Earth matter
r < ~1000 km
slow oscillations

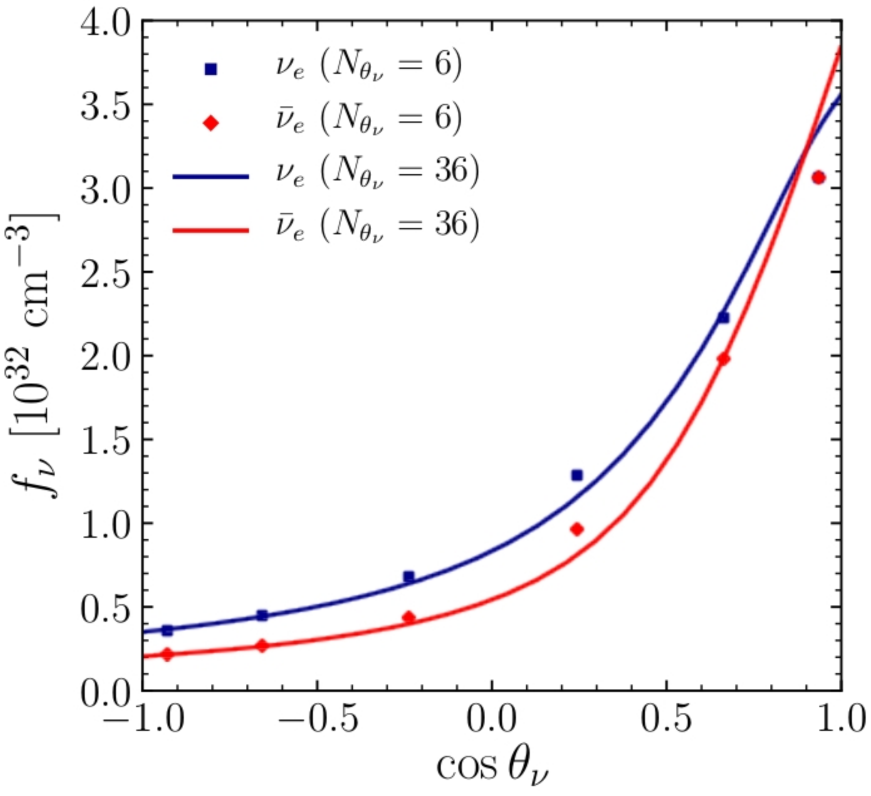
- Fast Oscillations occur due to differences in the angular distribution of the neutrinos versus antineutrinos

Sawyer, PRD **72**, 045003 (2005)

Mirizzi & Serpico, PRL **108**, 231102 (2012)

Izaguirre, Raffelt & Tamborra, PRL **118**, 021101 (2017)

and many many more



Abbar et al, PRD, **100** 043004 (2019)

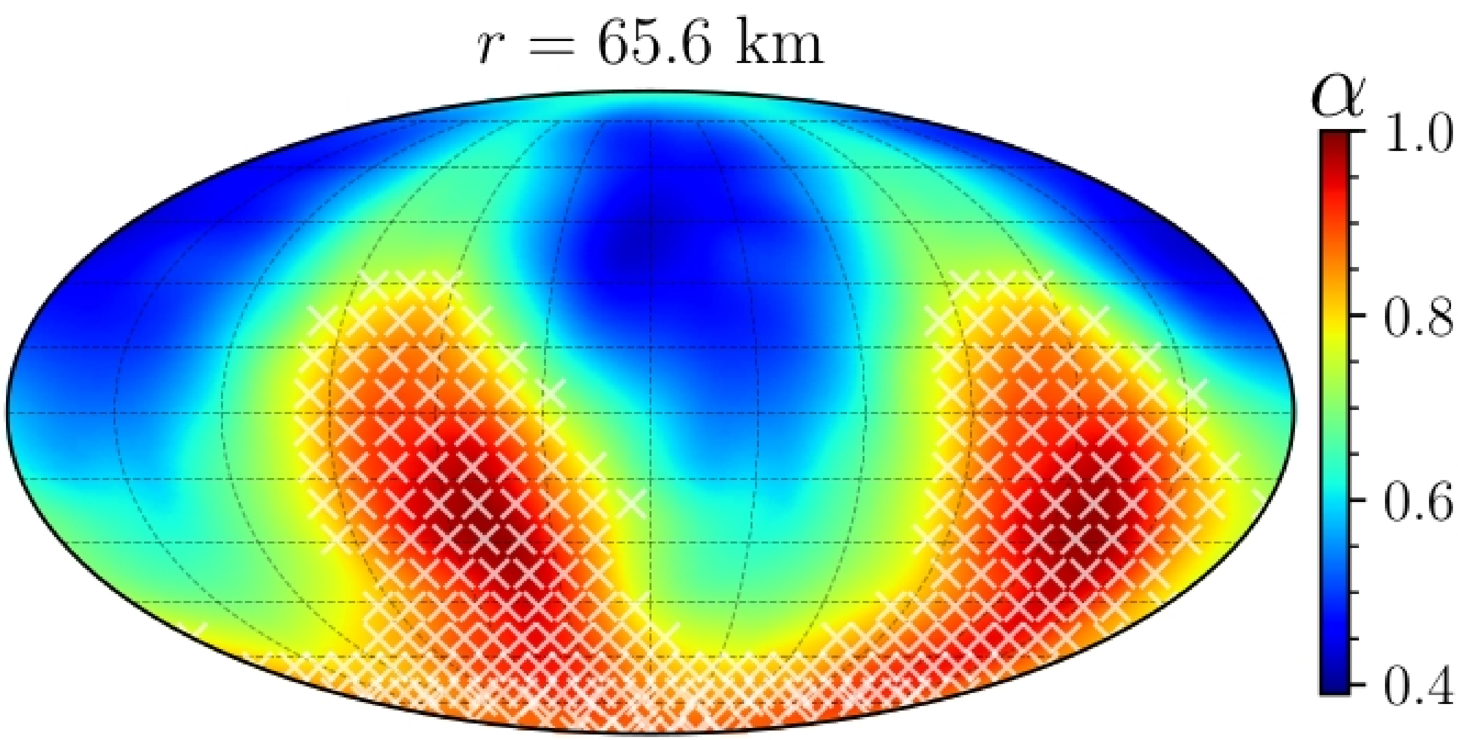
- Tamborra et al did not find angular crossings in their analysis of a 1D simulation.

Tamborra et al, ApJ, **839** 132 (2017)

- Abbar et al examined 2D and 3D simulations and found locations and times where FFO could occur.

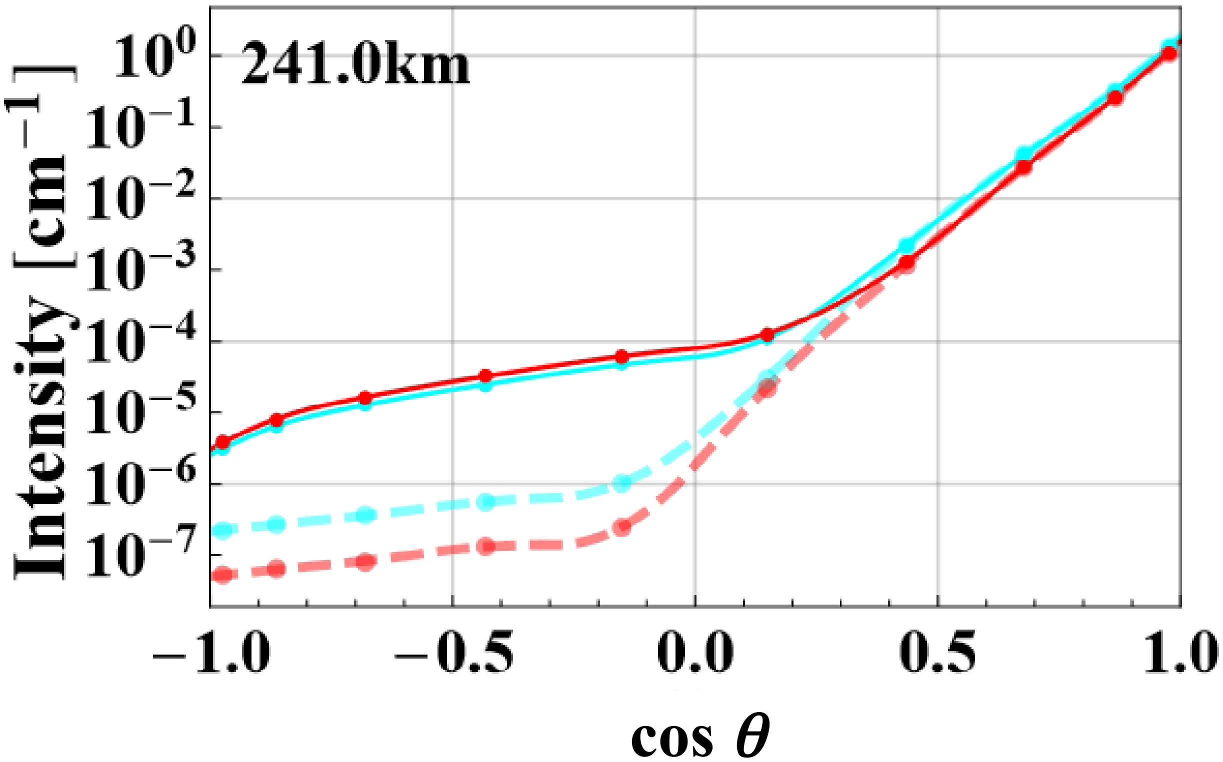
Abbar et al, PRD, **100** 043004 (2019)

see also Nagakura et al, ApJ **886** 139 (2019)



- Angular crossings in 1D were later found above the shock due to greater amount of scattering of the electron antineutrinos.

Morinaga et al PRR **2** 012046 (2020)

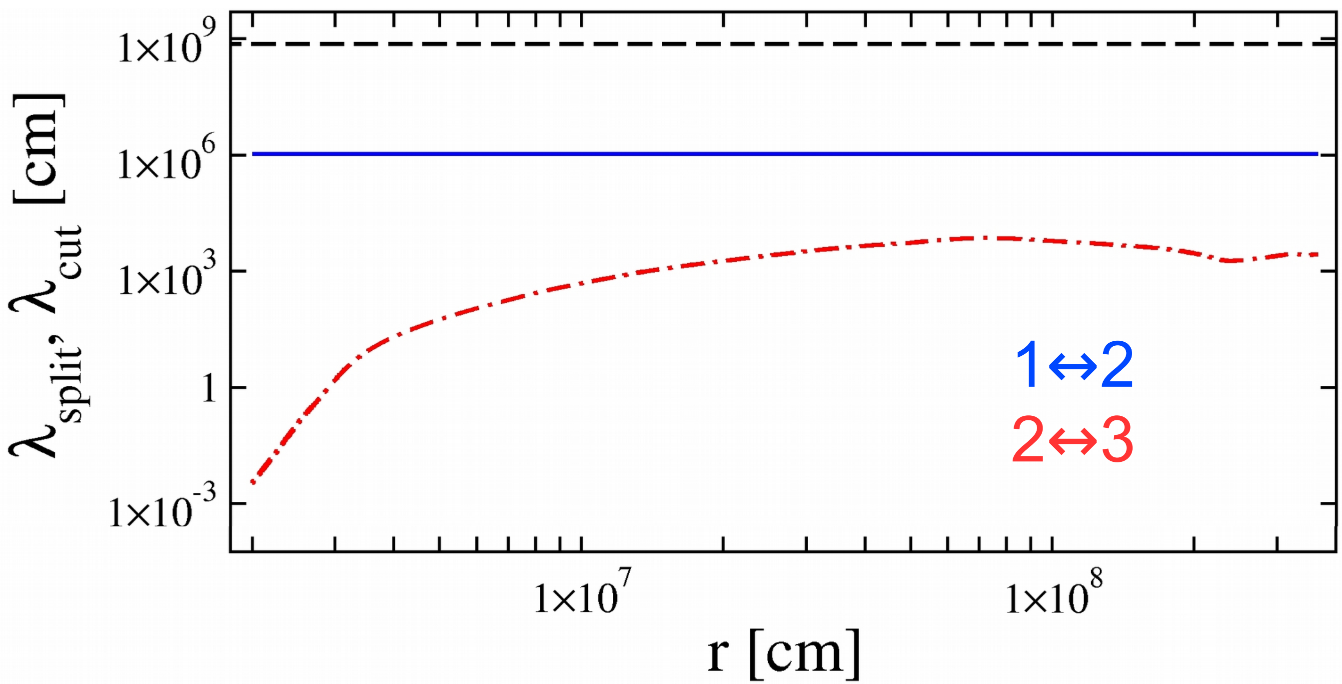


- The current picture of SN, the neutrinos and their flavor transformation is not self-consistent.
  - Any flavor transformation below the shock will change the simulation.

- In order to regain self-consistency, we need to include neutrino flavor transformation as we do the simulations.

How hard can it be?

- The spatial resolution of the simulations will have to increase considerably.
  - The best SN simulations have spatial grid zones ~100 m – 1 km
  - The oscillation lengthscale around the neutrinosphere is ~10 microns



Fischer et al (2010), 10.8 $M_\odot$ model, t = 0.3 s pb, NMO

Kneller & de los Reyes, JPG **44** 084008 (2017)

- As the spatial grid zones become smaller, the time steps taken by the simulation also shrink: μs → ps

- The computational expense will also increase due to the much finer angular resolution required for the neutrino distributions
- State of the art SN simulations typically use ~10 angle groups and assume axial symmetry.

- Multi-angle neutrino flavor oscillation calculations need many hundreds to thousands of angle bins.
- It has been shown the axial symmetry is spontaneously broken so we should really add the other angle dimension.

Raffelt, Sarikas & Seixas PRL **111** 091101 (2013)

- Including neutrino flavor transformation in simulations will increase the runtime of even a 1D simulation by a lot.
  - it takes Agile-Boltztrann ~100 to 1000 core hours to run to ~1 s postbounce.

- To make quantum supernova simulations feasible we will have to get creative:
  - e.g. Nagakura & Zaizen rescaled the neutrino Hamiltonian down by a factor of 10^{-4} then extrapolated their results back to the proper strength.

Nagakura & Zaizen PRL **129** 261101 (2022)

see also Xiong et al PRD **107** 083016 (2023)

Oscillations with moments

- Many classical SN simulations evolve the neutrino field using angular moments.
  - the number of moments evolved is usually just 2 in 1D, 4 in 3D.

- It is possible to do neutrino transformation with moments.

Strack and Burrows, PRD **71** 093004 (2005)

Zhang and Burrows, PRD **88** 105009 (2013)

Myers et al, PRD **105** 123036 (2022)

Grohs et al, arXiv:2207.02214

- We define a quantum angular moment of the distribution *f* as

$$M_n(q) = \int q \cos^n \theta \quad f \, d\Omega_q$$

- where q is the energy of the neutrino, θ the angle relative to the radial direction
- The first few moments have well-known names
  - n = 0 is the (differential) energy density E_q
  - n = 1 is the (differential) radial component of the energy flux F_q
  - n = 2 is the ‘rr’ component of the (differential) pressure tensor P_q

- Assuming spherical symmetry, the moments evolve according to

$$\frac{\partial E_q}{\partial t}+\frac{\partial F_q}{\partial r}+\frac{2F_q}{r}=-i[H_V+H_M+H_E,E_q]+i[H_F,F_q]$$

$$\frac{\partial F_q}{\partial t}+\frac{\partial P_q}{\partial r}+\frac{3P_q-E_q}{r}=-i[H_V+H_M+H_E,F_q]+i[H_F,P_q]$$

$$\vdots$$

  - the H's are contributions to the Hamiltonian,
  - the absorption / emission / collisions have been omitted
- The infinite tower of equations can be truncated at what ever level one desires.
  - Usually one considers two schemes: a one-moment (M0) and a two-moment (M1)
- We need a relationship between the moments to close the equations.
- This relationship is called 'The Closure'

Are moment-based approaches any good?

- We want to compare moment-based approaches against other methods e.g. Discrete Ordinates, Particle-In-Cell, MC
- We compared moments with the multi-angle calculations based on the neutrino Bulb Model.

Duan et al PRL **97** 241101 (2006)

  - The neutrinosphere is a hard surface with spherically symmetric neutrino emission.
  - No collisions or absorption / emission beyond the neutrinosphere.
  - The neutrino field is in steady state.
  - The neutrino field has axial symmetry around the radial direction.
- There is an exact solution for the moments in the classical limit.

- For the M0 moment calculation, we use a scalar closure that is the exact classical solution

$$F_q = \frac{\left(1 + \cos\theta_{max}\right)}{2} E_q$$

- where θ_{max} is the largest angle between the neutrino velocity vectors at some radius r, and the radial direction.

- For the M1 calculation the scalar closure is again taken to be the exact classical solution

$$P_q = \frac{\left(1 + \cos\theta_{max} + \cos^2\theta_{max}\right)}{3} E_q$$

- First consider the MSW problem using 1 MeV neutrinos emitted from a spherical source of radius 10 km.
    - There is an exact solution for a single neutrino.
- The density of the matter was set to 8000 g/cm^3 in order to put the neutrinos on the MSW resonance.
- The angular distribution of the neutrino emission is taken to be

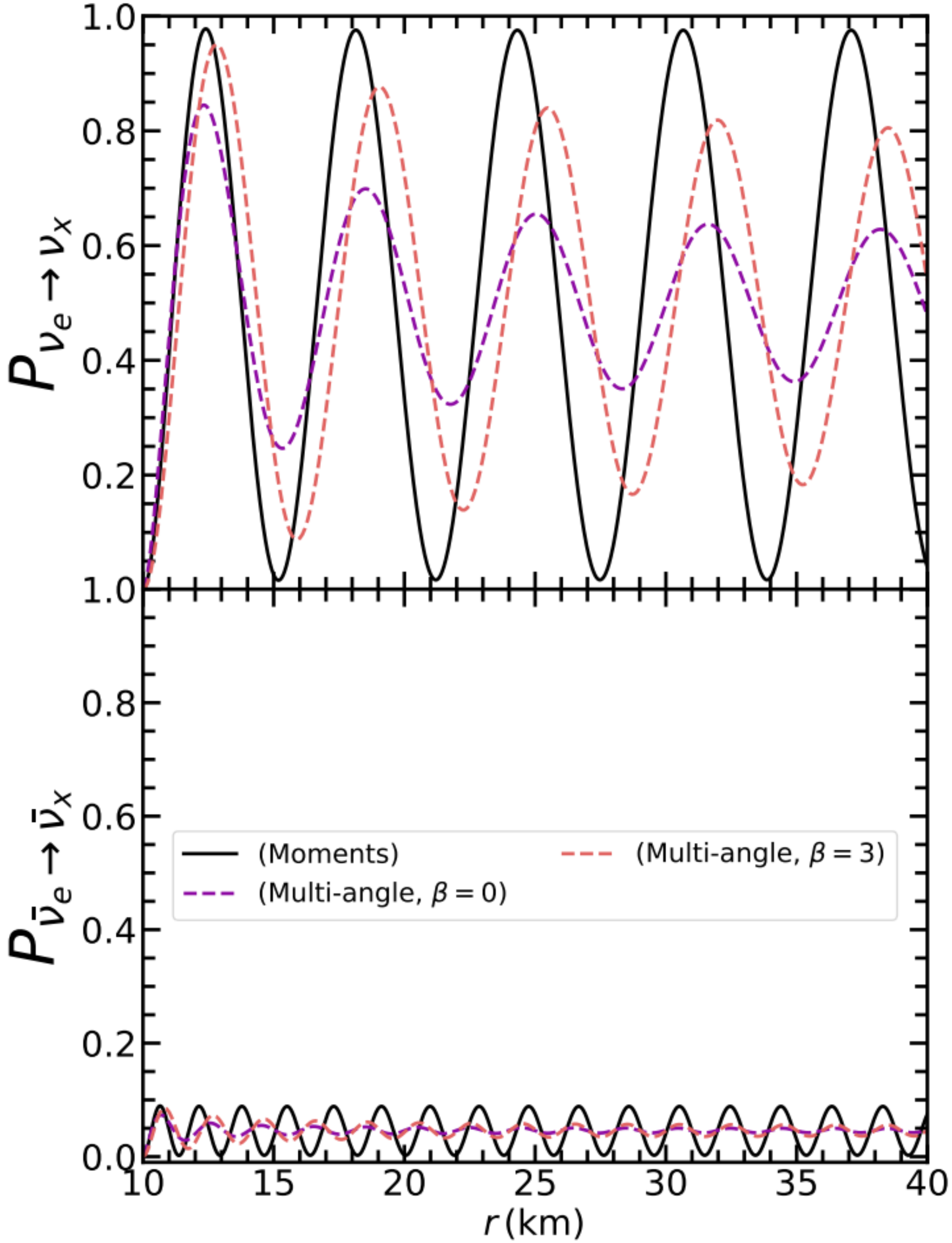
$$F_{aa}(R_\nu, \theta) \propto \Theta_a(\theta)$$

- with

$$\Theta_a \propto \cos^{\beta_a} \theta$$

    - β = 0 is a half-isotropic distribution
- The transition probability is computed from the flux moment.

$$P_{\nu_e \to \nu_x} = \frac{r^2 F_{xx}(r) - R_\nu^2 F_{xx}(R_\nu)}{R_\nu^2 F_{ee}(r) - R_\nu^2 F_{xx}(R_\nu)}$$

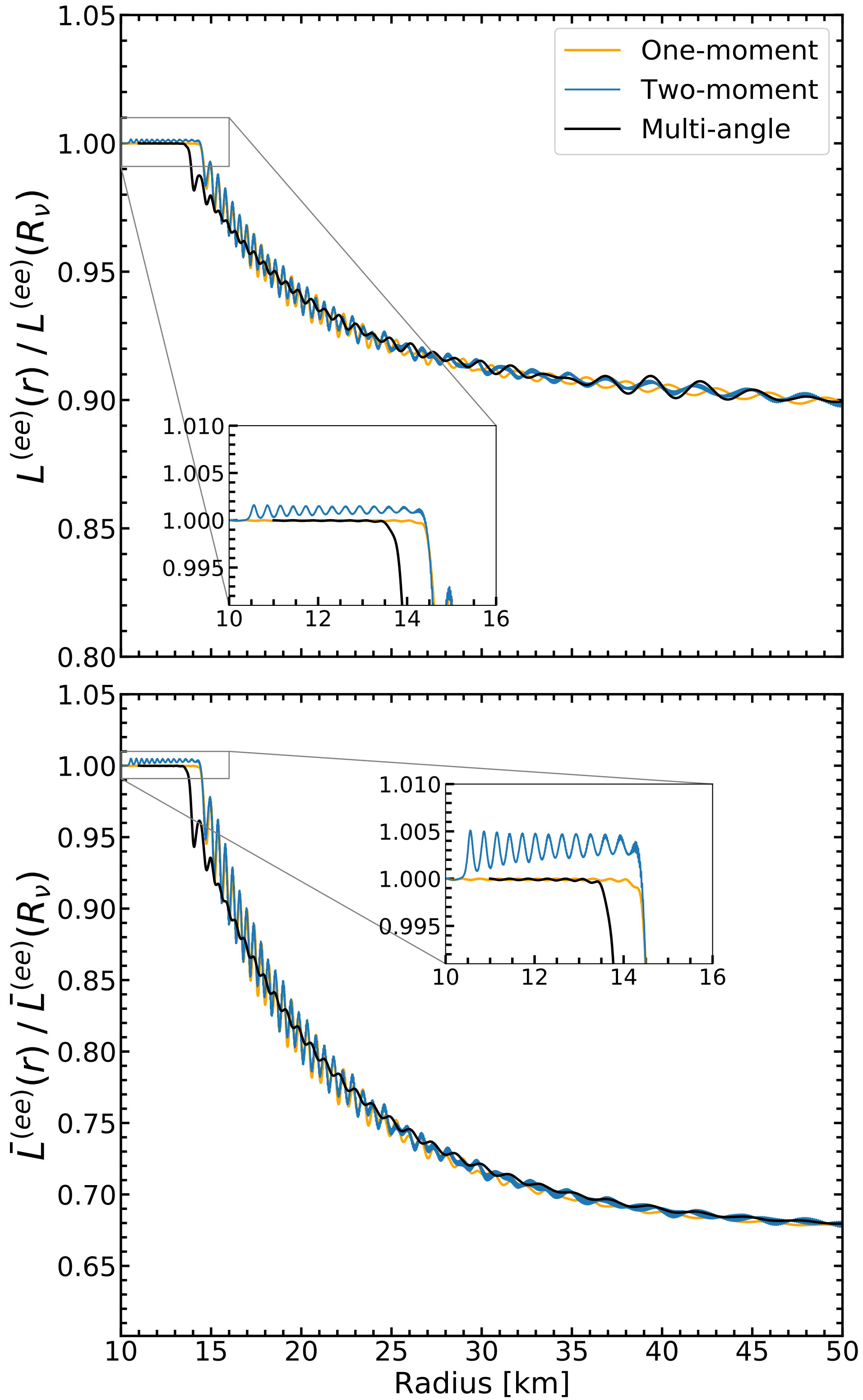


- The difference between the multi-angle and moments grows with r.
  - The amplitude of the moment result is larger than the multi-angle.
- In the multi-angle results we see the effect of neutrinos losing coherence.
- The moments using a scalar closure overestimates the coherence.

- Consider another case now with the self-interaction included.
- We adjusted the luminosities so that the flavor transformation occurs close to the neutrinosphere.

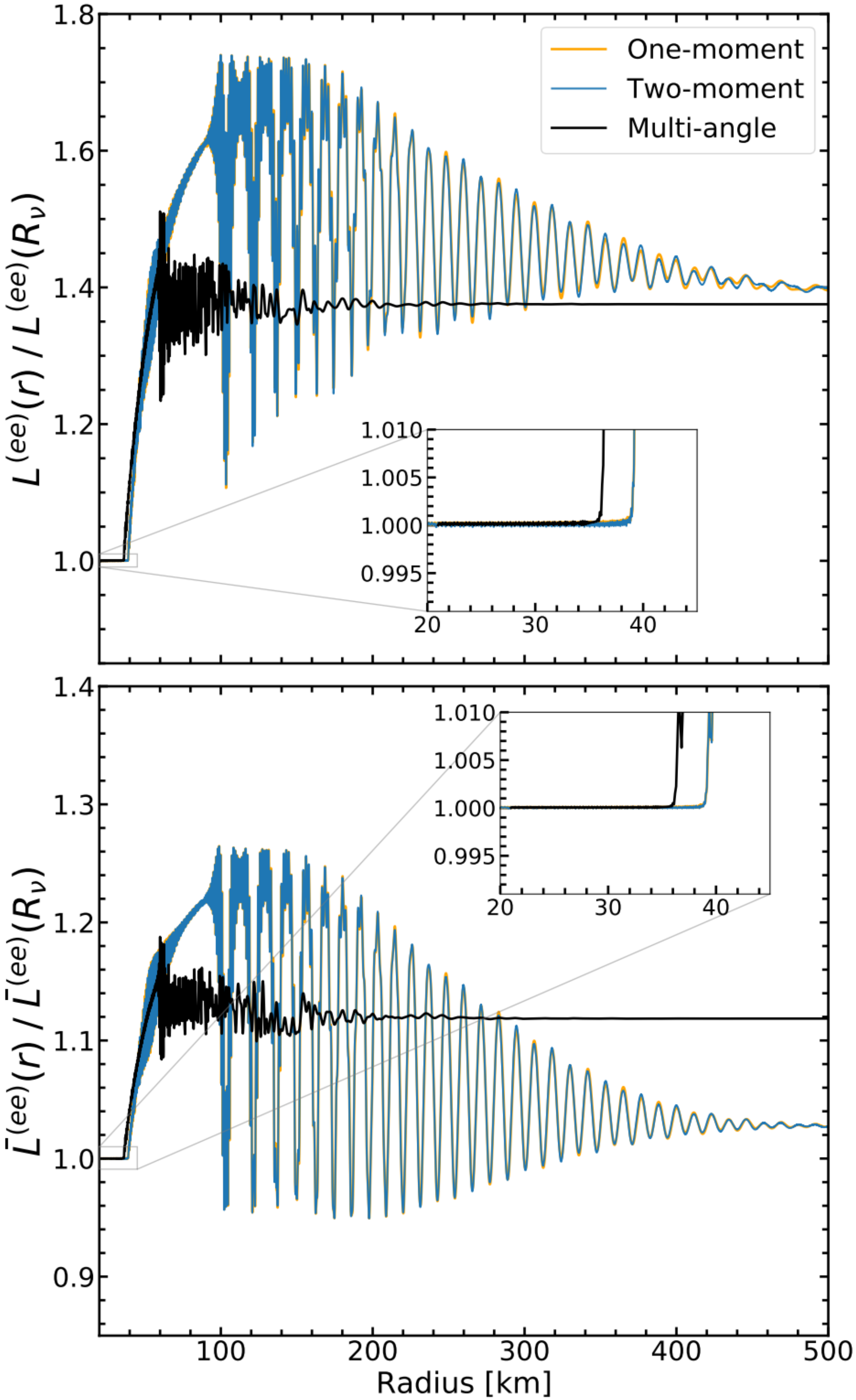
	L [erg/s]	$\langle E \rangle$ [MeV]	T [MeV]	η
ν_e	2.050×10^{49}	9.4	2.1	3.9
$\bar{\nu}_e$	2.550×10^{49}	13	3.5	2.3
ν_x	1.698×10^{49}	15.8	4.4	2.1
$\bar{\nu}_x$	1.698×10^{49}	15.8	4.4	2.1

- The moment calculation does very well.



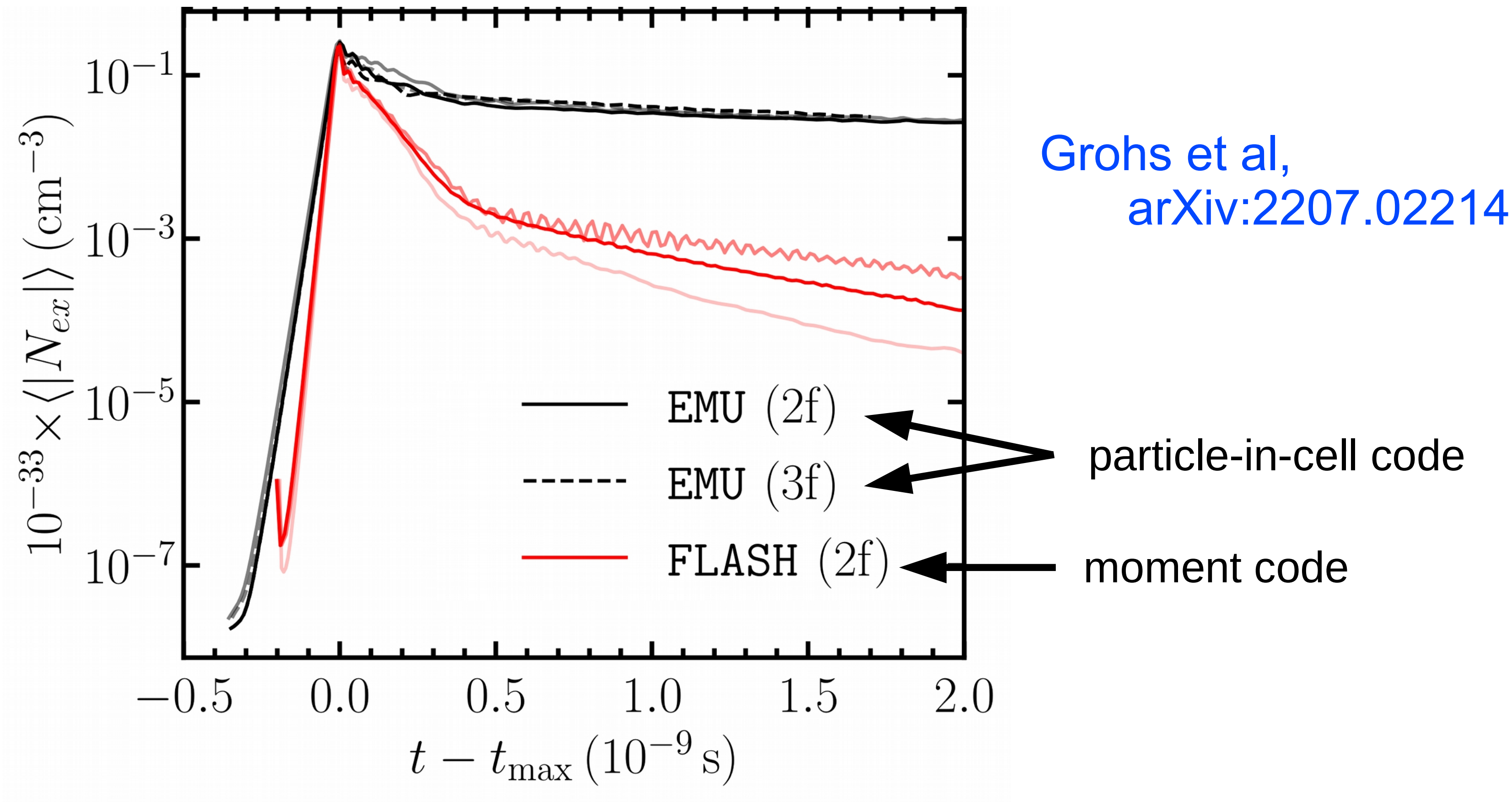
- Consider a third case, again with the self-interaction, where there are equal numbers of electron and x-flavor neutrinos.

	L [erg/s]	$\langle E \rangle$ [MeV]	T [MeV]	η
ν_e	1.8×10^{52}	12	2.1	3.9
$\bar{\nu}_e$	2.2×10^{52}	15	3.5	2.3
ν_x	2.7×10^{52}	18	4.4	2.1
$\bar{\nu}_x$	2.7×10^{52}	18	4.4	2.1



- The multi-angle separates from the moments at ~60 km.

- More recently we looked at how well moments and a scalar closure capture fast-flavor oscillations.
  - this is a demanding test: fast flavor oscillations depend upon angular distributions which is something the moments don't have.



Grohs et al, arXiv:2207.02214

- These results are encouraging but the agreement is inconsistent.
- Is the scalar closure the problem?
- What is a more general, quantum, closure?

Quantum Closures

- Our starting ansatz is that two moments, e.g. E and P, are related by

$$P = L\,E\,R$$

- Since E and P are Hermitian they have eigenvalue matrices

$$E = U_E \Lambda_E U_E^\dagger \qquad P = U_P \Lambda_P U_P^\dagger$$

- Hermiticity means that the closure must be such that

$$P = R^\dagger E\, L^\dagger$$

- The closure must be of the form

$$P = L\,E\left(U_E D\, U_E^\dagger\right) L^\dagger$$

- where D is a diagonal Hermitian matrix.

- For the time being, D = 1 so that

$$P = L\,E\,L^{\dagger}$$

- If we knew E and P, we can find L.

- E and P are positive definite allowing us to write

$$E = \epsilon\,\epsilon^{\dagger} \qquad P = \rho\,\rho^{\dagger}$$

- so L must be

$$L = \rho\,\epsilon^{-1}$$

- Again writing E and P as

$$E = U_E \Lambda_E U_E^\dagger \qquad P = U_P \Lambda_P U_P^\dagger$$

- we see

$$\epsilon = U_E \Lambda_E^{1/2} S_E \qquad \rho = U_P \Lambda_P^{1/2} S_P$$

- where S_E and S_P are arbitrary unitary matrices.
- We can factorize U_E and U_P as

$$U_E = \Upsilon_E \Phi_E \qquad U_P = \Upsilon_P \Phi_P$$

- where Υ_E and Υ_P are Hermitian unitary matrices which can be written in terms of the elements of E and P, and Φ_E and Φ_P are arbitrary diagonal unitary matrices.

- Inserting all these expressions we obtain

$$L = \Upsilon_P \Phi_P \Lambda_P^{1/2} S_P S_E^{\dagger} \Lambda_E^{-1/2} \Phi_E^{\dagger} \Upsilon_E$$

- If we set

$$\Phi_P S_P S_E^{\dagger} \Phi_E^{\dagger} = 1$$

- L reduces to

$$L = \Upsilon_P X^{1/2} \Upsilon_E$$

- where

$$X = \Lambda_P \Lambda_E^{-1}$$

- For two flavors

$$X = \begin{pmatrix} \chi_1 & 0 \\ 0 & \chi_2 \end{pmatrix} = \chi \begin{pmatrix} \dfrac{1+v_P}{1+v_E} & 0 \\ 0 & \dfrac{1-v_P}{1-v_E} \end{pmatrix}$$

- and Y is parameterized in terms of two angles

$$\mathrm{Y} = \begin{pmatrix} \cos(\theta/2) & \sin(\theta/2)e^{-i\phi} \\ \sin(\theta/2)e^{i\phi} & -\cos(\theta/2) \end{pmatrix}$$

- A quantum closure can be understood as a three-step relation:

$$P = \Upsilon_P \left(X^{1/2} \left(\Upsilon_E E \Upsilon_E \right) X^{1/2} \right) \Upsilon_P$$

  - Υ_E rotates E to its eigenvalue matrix
  - $X^{1/2}$ rescales the eigenvalues
  - Υ_P rotates away from a diagonal matrix

- This same relation could be written as

$$P = \left(\Upsilon_P \Upsilon_E \right) E \left(\Upsilon_E X \Upsilon_E \right) \left(\Upsilon_E \Upsilon_P \right)$$

- which is the general form given previously.
- This is the form one would use if you want to relate E and F because F is not positive definite.

Moment alignment

- We can measure the ‘alignment’ between moments using the Frobenius Inner Product and Frobenius norm

$$\cos\xi = \frac{\langle E - \tfrac{1}{2}Tr(E),\, P - \tfrac{1}{2}Tr(E)\rangle_F}{|E - \tfrac{1}{2}Tr(E)|_F\,|P - \tfrac{1}{2}Tr(E)|_F}$$

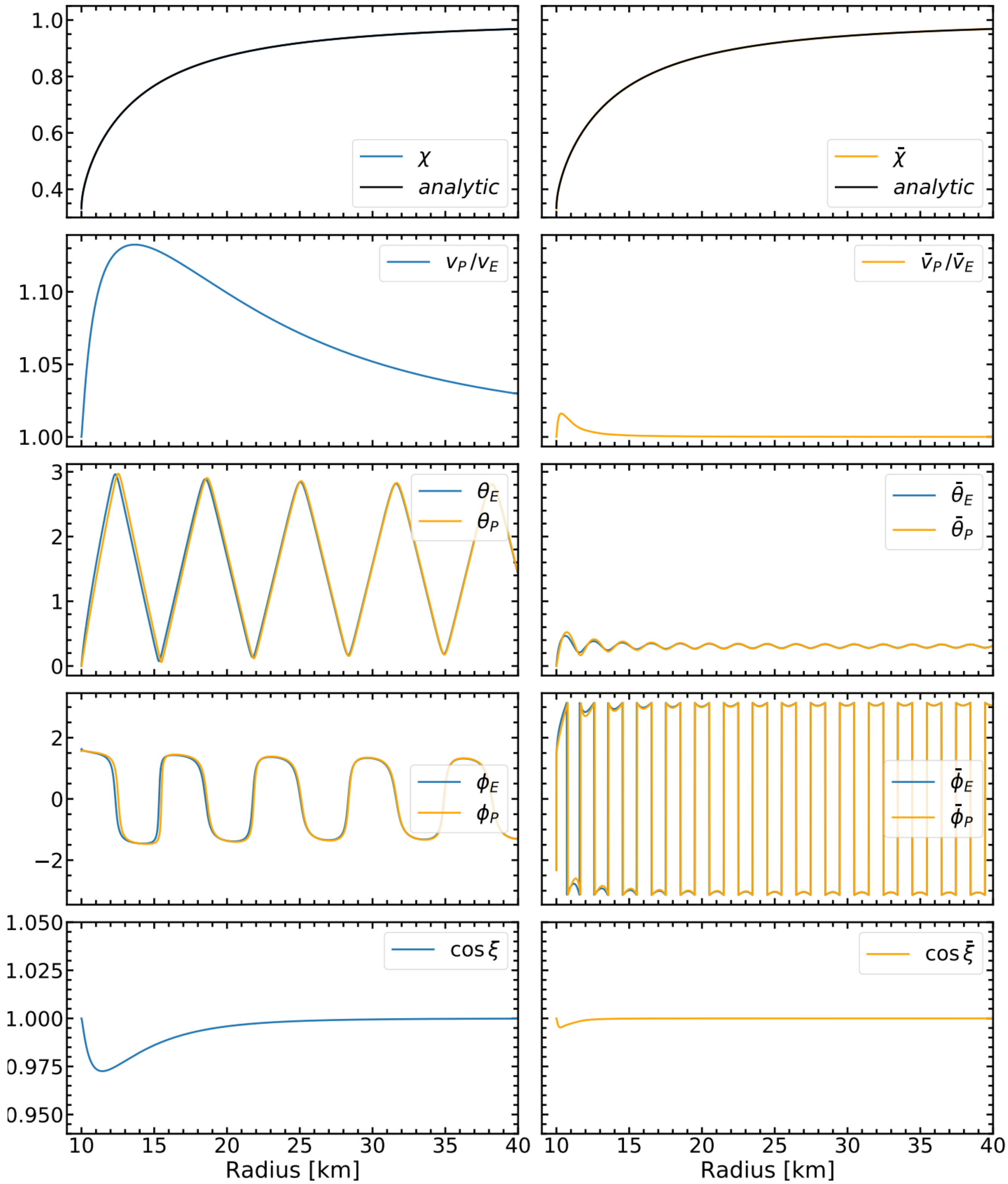
- The Frobenius Inner Product and norm are

$$\langle E, P\rangle_F = Tr\left(E^{\dagger} P\right)$$

$$|E|_F = \left(\langle E, E\rangle_F\right)^{1/2}$$

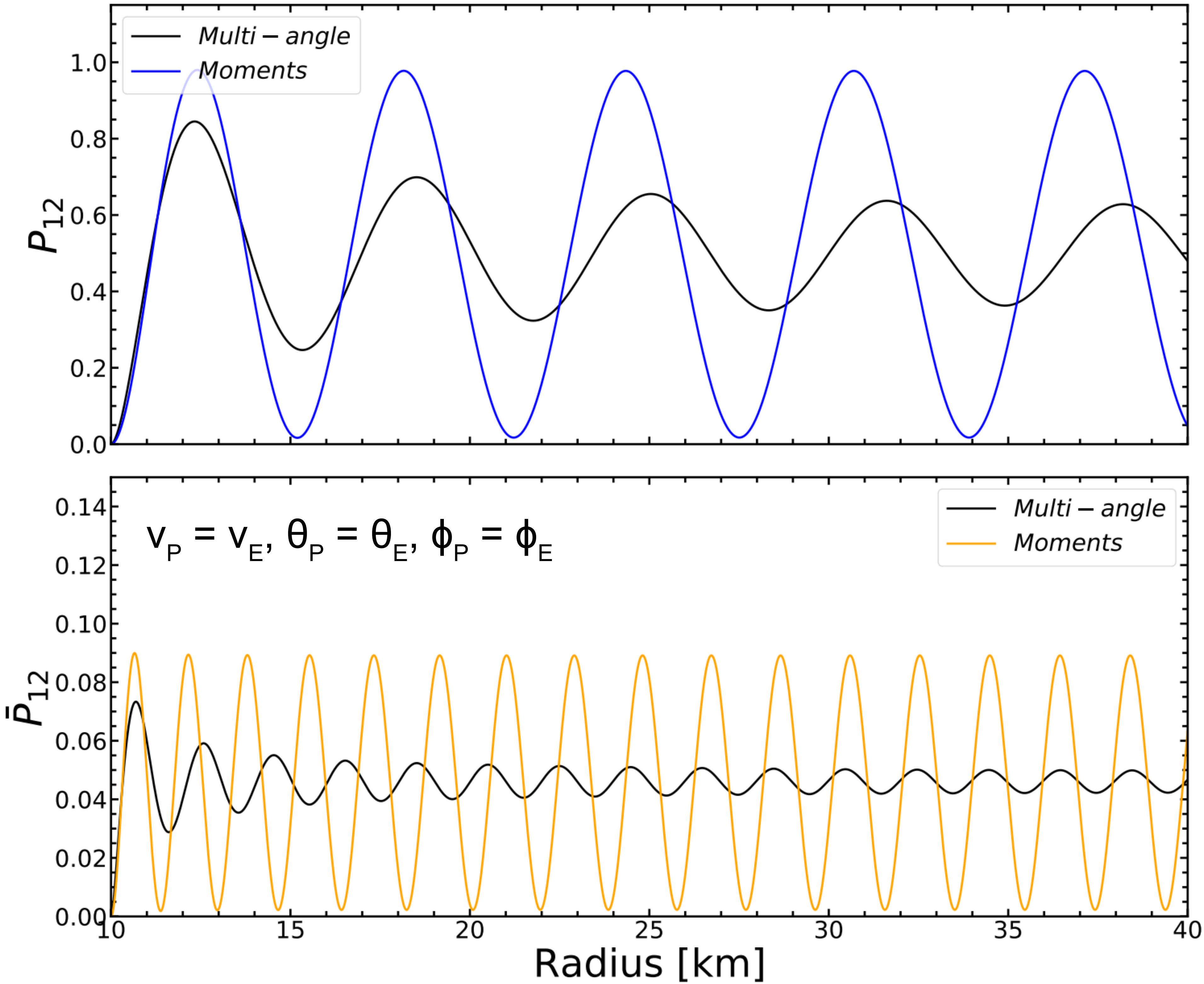
- We can go back to the test problems and determine the closure parameters.

- First the MSW problem

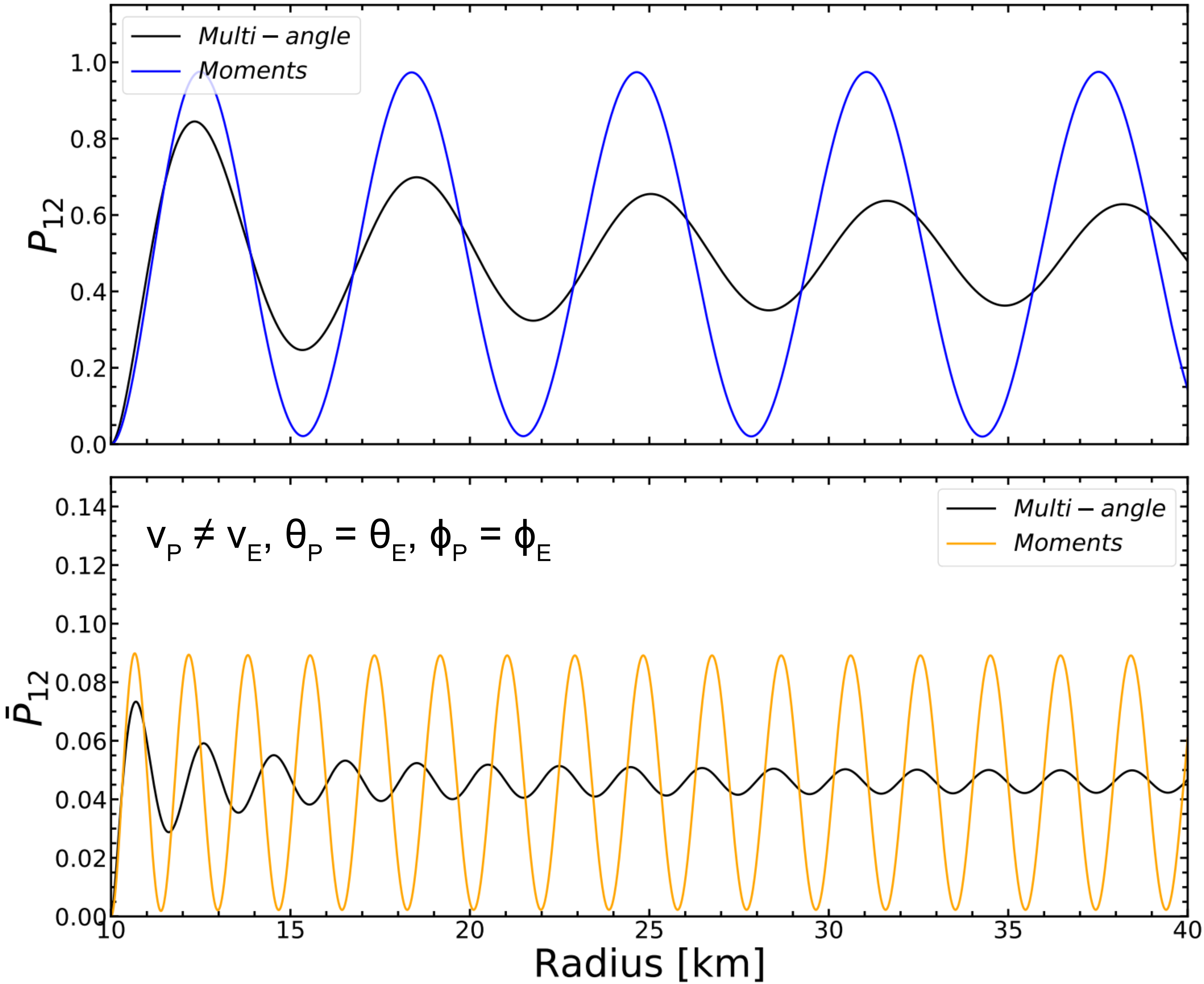


χ
analytic
$\bar{\chi}$
analytic
v_P/v_E
$\bar{v}_P/\bar{v}_E$
θ_E
θ_P
$\bar{\theta}_E$
$\bar{\theta}_P$
ϕ_E
ϕ_P
$\bar{\phi}_E$
$\bar{\phi}_P$
$\cos\xi$
$\cos\bar{\xi}$
Radius [km]
Radius [km]

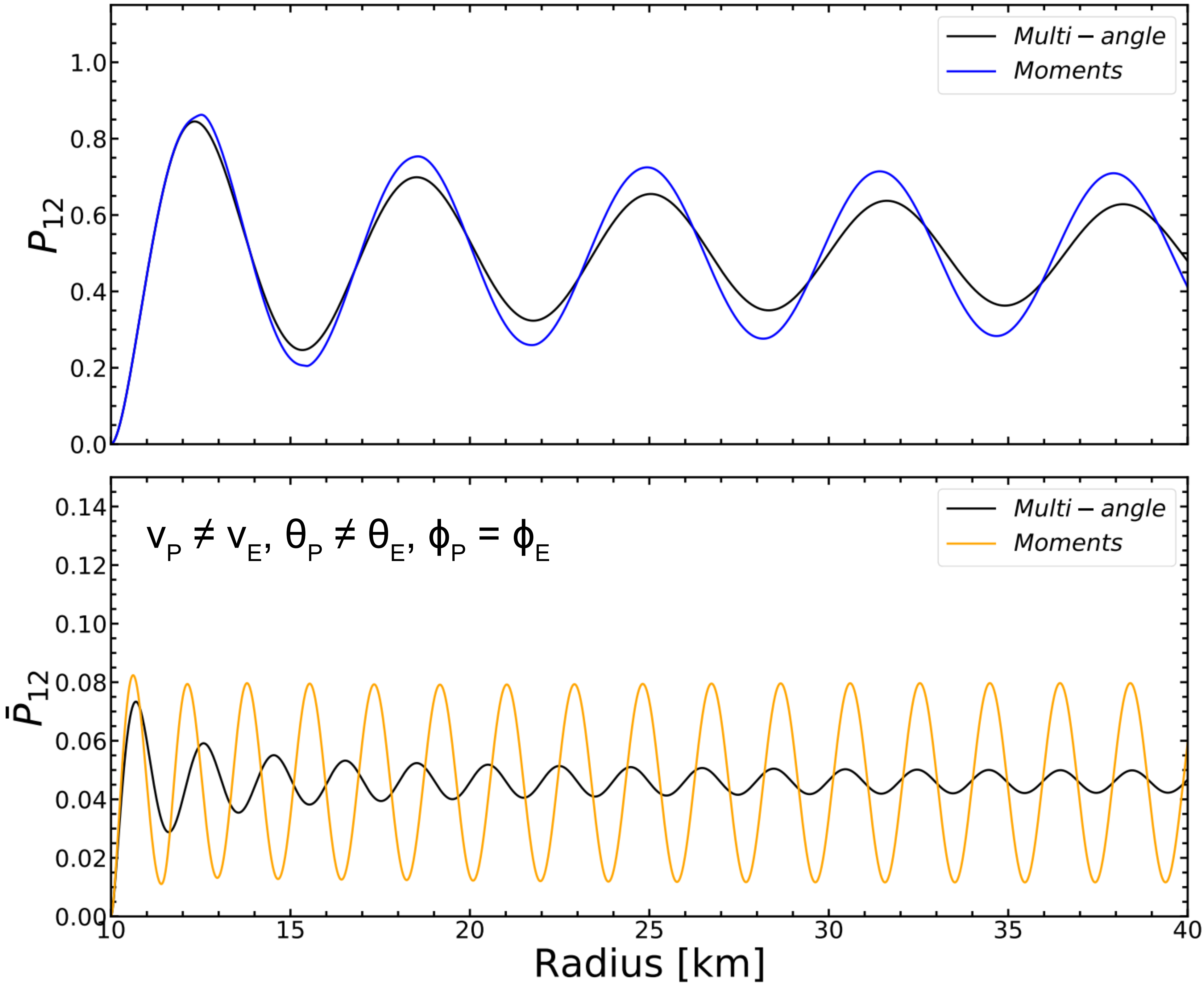
- Now that we know the closure parameters, we can redo the moment calculation with a quantum closure.



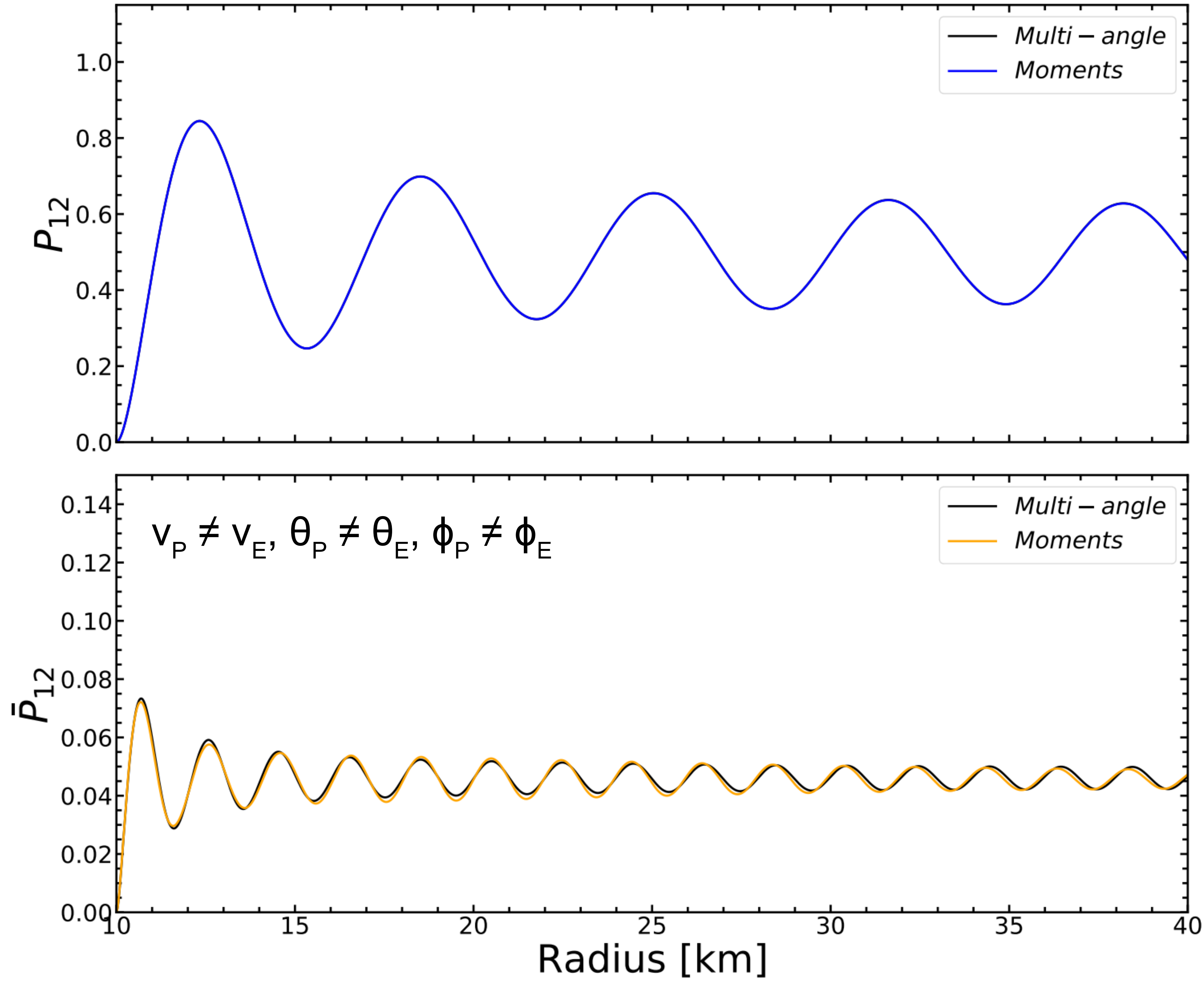
Multi − angle
Moments
1.0
0.8
0.6
0.4
0.2
0.0
P_{12}
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
$\bar{P}_{12}$
$v_P = v_E$, $\theta_P = \theta_E$, $\phi_P = \phi_E$
Multi − angle
Moments
10
15
20
25
30
35
40
Radius [km]



Multi − angle
Moments
P_{12}
1.0
0.8
0.6
0.4
0.2
0.0
$v_P \neq v_E, \theta_P = \theta_E, \phi_P = \phi_E$
Multi − angle
Moments
$\bar{P}_{12}$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
10
15
20
25
30
35
40
Radius [km]

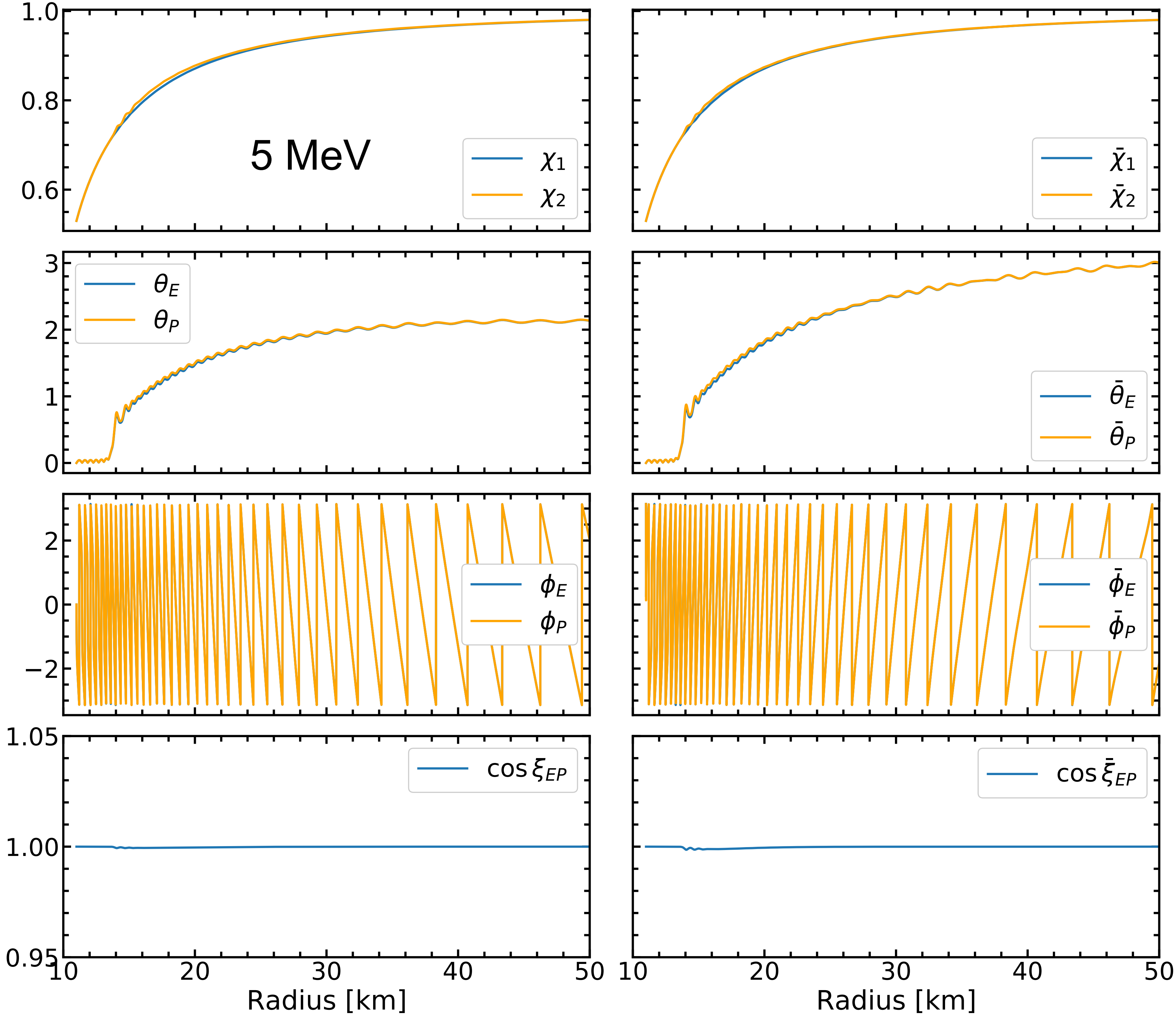


P_{12}
Multi − angle
Moments
1.0
0.8
0.6
0.4
0.2
0.0
$\bar{P}_{12}$
$v_P \neq v_E$, $\theta_P \neq \theta_E$, $\phi_P = \phi_E$
0.14
0.12
0.10
0.08
0.06
0.04
0.02
0.00
10
15
20
25
30
35
40
Radius [km]

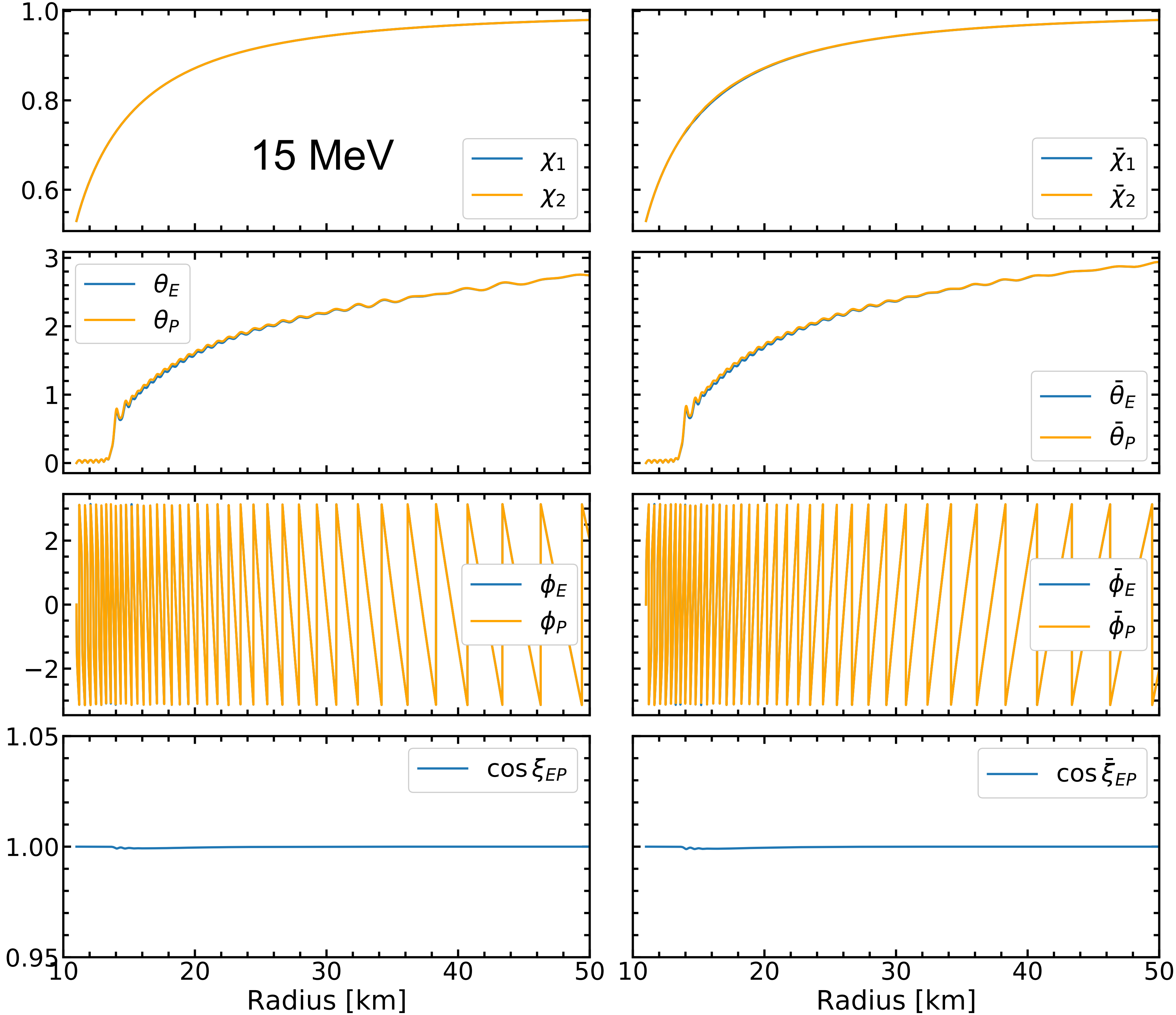


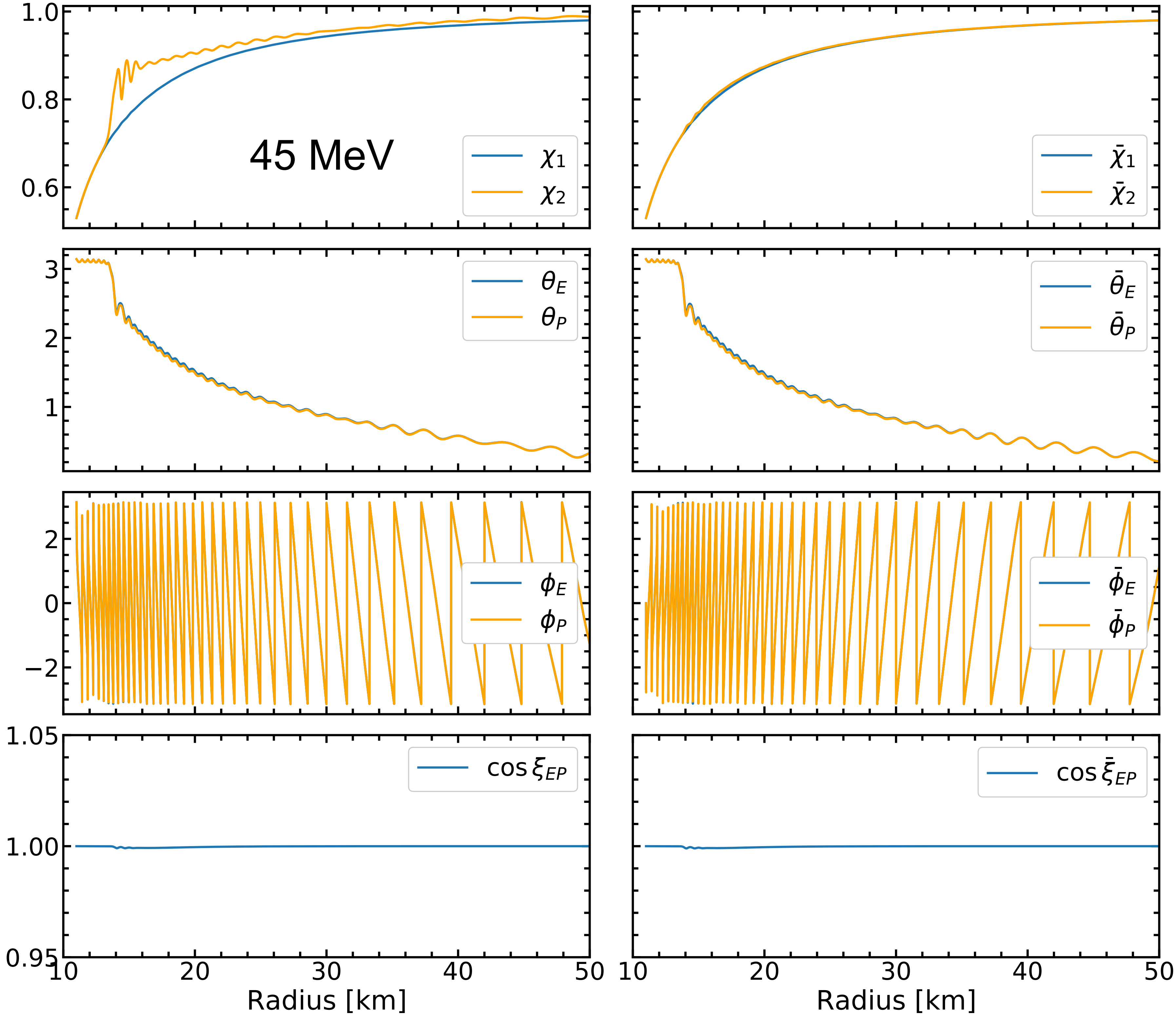
P_{12}
Multi − angle
Moments
$\bar{P}_{12}$
$v_P \neq v_E$, $\theta_P \neq \theta_E$, $\phi_P \neq \phi_E$
Multi − angle
Moments
Radius [km]

- Next, the first self-interaction test case where the scalar closure worked well.



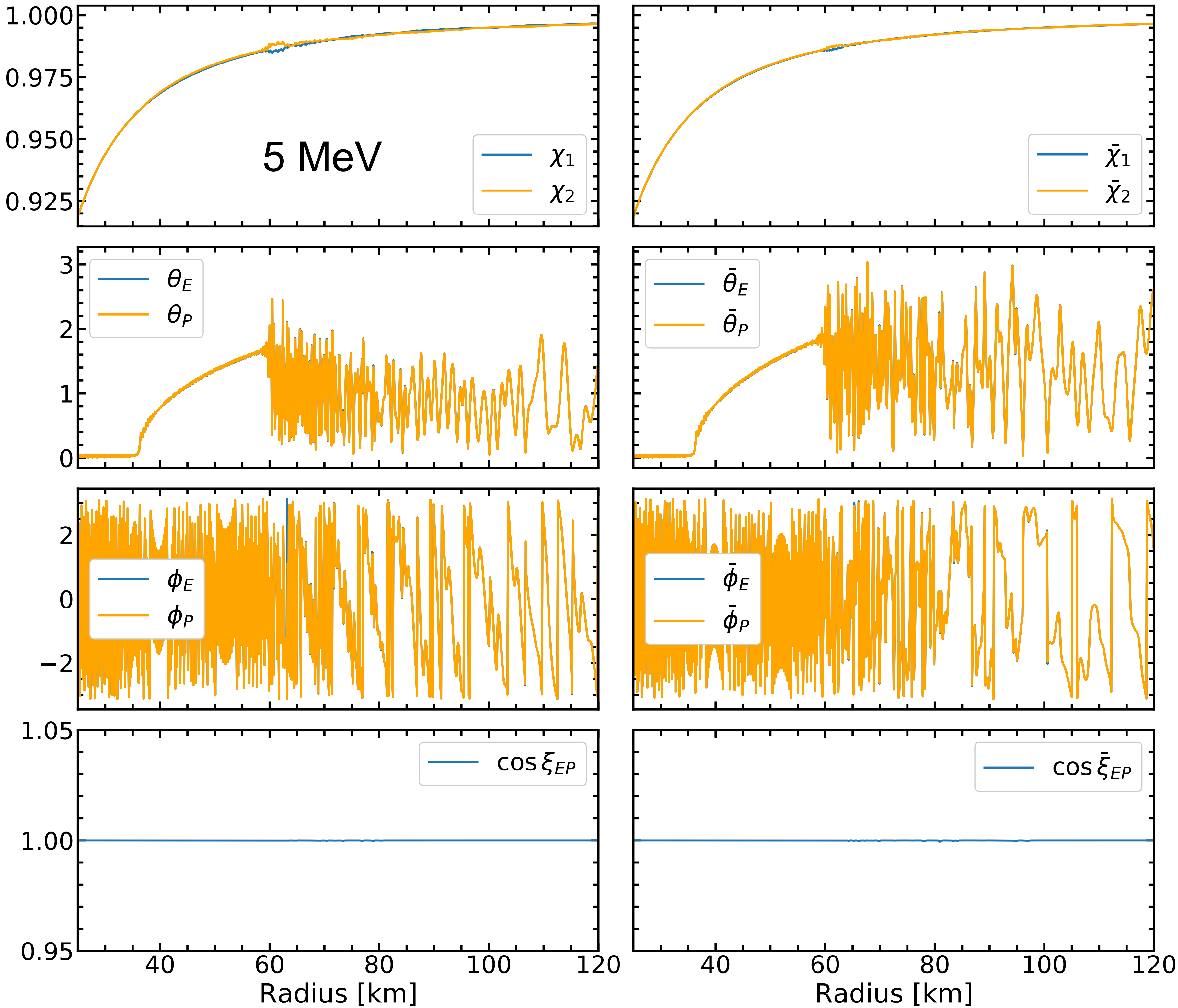
5 MeV
χ_1
χ_2
$\bar{\chi}_1$
$\bar{\chi}_2$
θ_E
θ_P
$\bar{\theta}_E$
$\bar{\theta}_P$
ϕ_E
ϕ_P
$\bar{\phi}_E$
$\bar{\phi}_P$
$\cos \xi_{EP}$
$\cos \bar{\xi}_{EP}$
Radius [km]
Radius [km]


15 MeV
χ_1
χ_2
$\bar{\chi}_1$
$\bar{\chi}_2$
θ_E
θ_P
$\bar{\theta}_E$
$\bar{\theta}_P$
ϕ_E
ϕ_P
$\bar{\phi}_E$
$\bar{\phi}_P$
$\cos \xi_{EP}$
$\cos \bar{\xi}_{EP}$
Radius [km]
Radius [km]

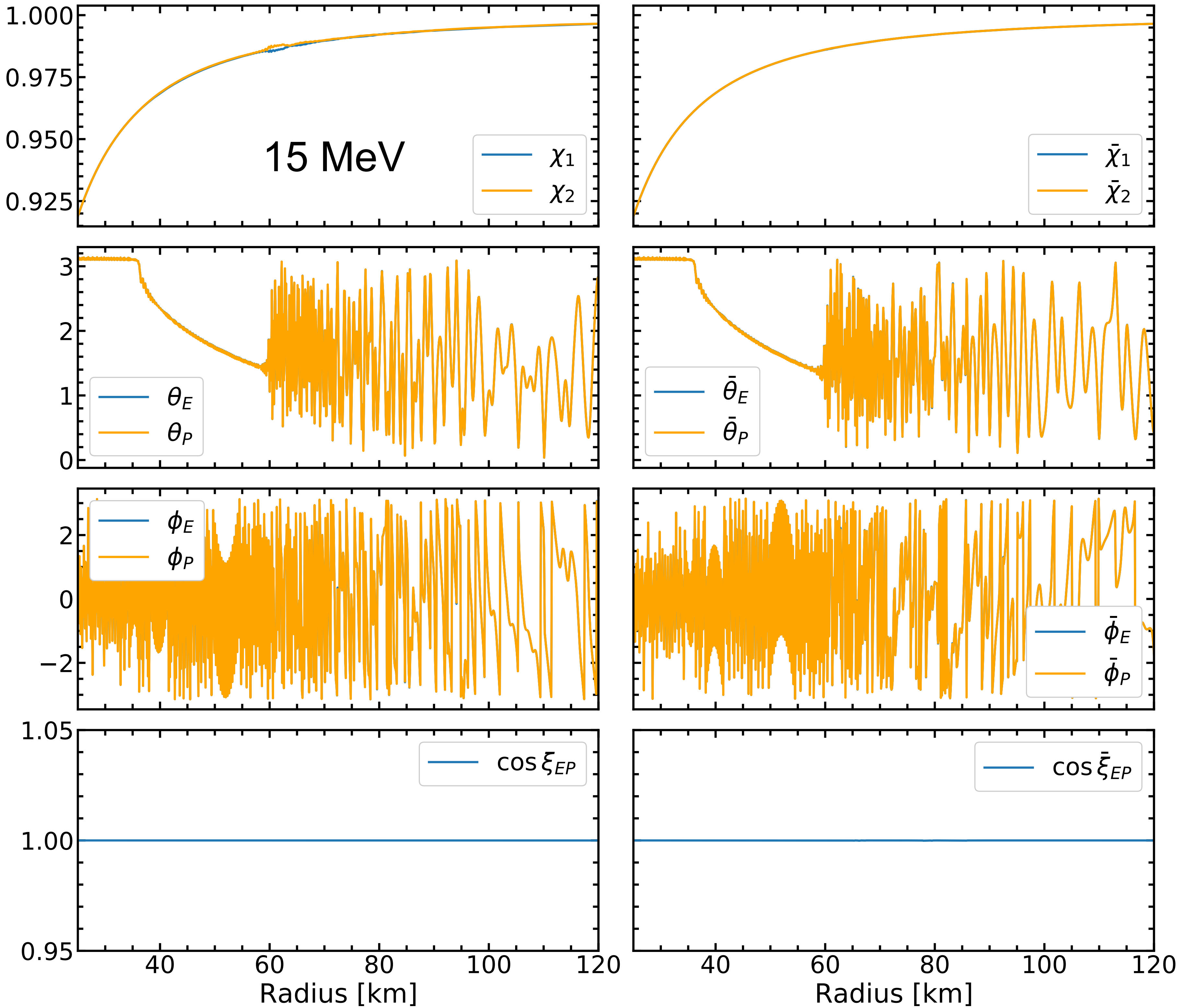

45 MeV
χ_1
χ_2
$\bar{\chi}_1$
$\bar{\chi}_2$
θ_E
θ_P
$\bar{\theta}_E$
$\bar{\theta}_P$
ϕ_E
ϕ_P
$\bar{\phi}_E$
$\bar{\phi}_P$
$\cos \xi_{EP}$
$\cos \bar{\xi}_{EP}$
Radius [km]
Radius [km]

- The closure parameters in this test case are:
  - $\chi_1 = \chi_2$
  - $\theta_E = \theta_P$
  - $\phi_E = \phi_P$
- The moments are very close to perfectly aligned.

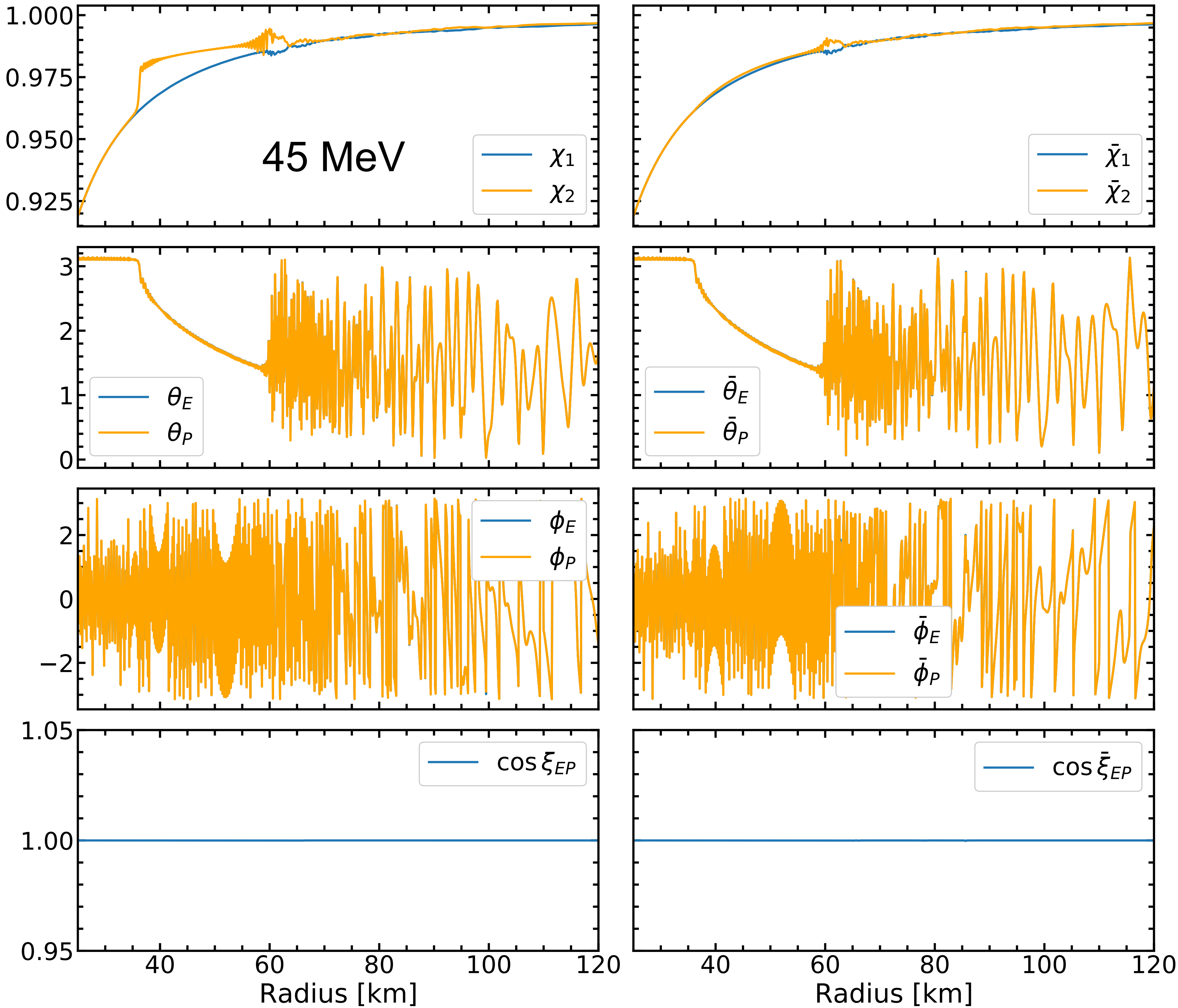
- Finally, the second self-interaction test case where the scalar closure worked well below 60 km but not above.



5 MeV
χ_1
χ_2
$\bar{\chi}_1$
$\bar{\chi}_2$
θ_E
θ_P
$\bar{\theta}_E$
$\bar{\theta}_P$
ϕ_E
ϕ_P
$\bar{\phi}_E$
$\bar{\phi}_P$
$\cos \xi_{EP}$
$\cos \bar{\xi}_{EP}$
Radius [km]
Radius [km]



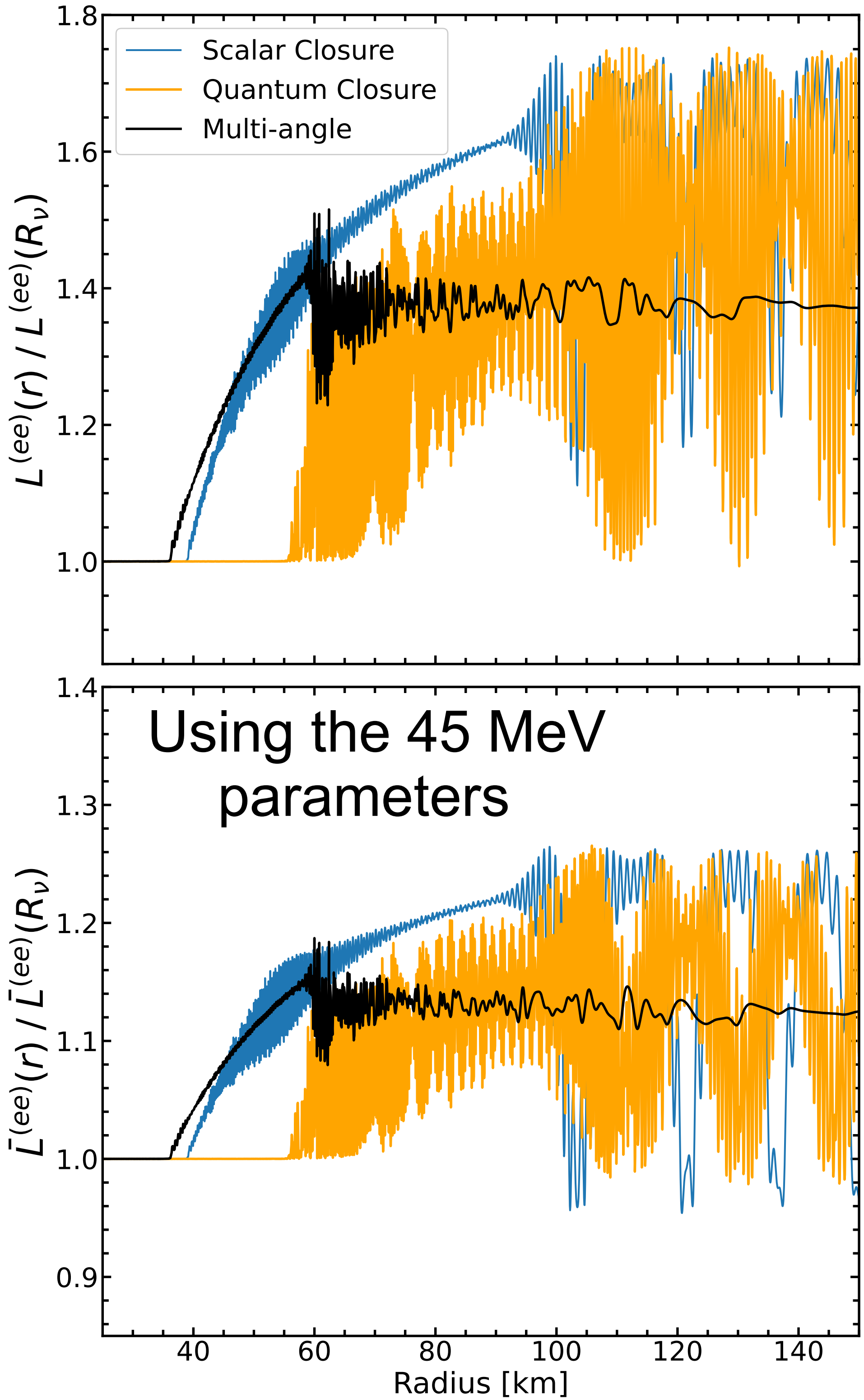
15 MeV
χ_1
χ_2
$\bar{\chi}_1$
$\bar{\chi}_2$
θ_E
θ_P
$\bar{\theta}_E$
$\bar{\theta}_P$
ϕ_E
ϕ_P
$\bar{\phi}_E$
$\bar{\phi}_P$
$\cos\xi_{EP}$
$\cos\bar{\xi}_{EP}$
1.000
0.975
0.950
0.925
3
2
1
0
2
0
−2
1.05
1.00
0.95
40
60
80
100
120
Radius [km]
Radius [km]



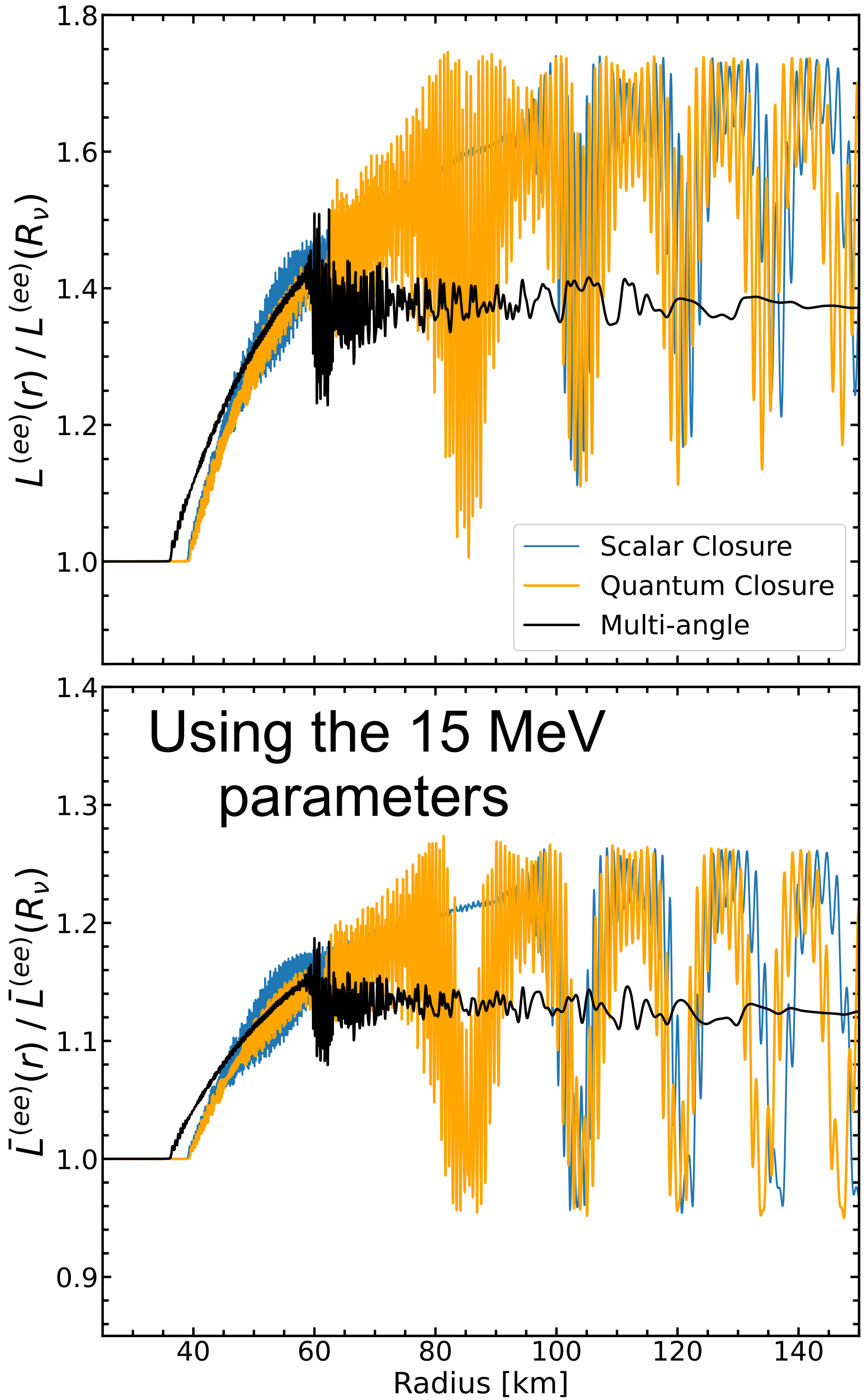
45 MeV
χ_1
χ_2
$\bar{\chi}_1$
$\bar{\chi}_2$
θ_E
θ_P
$\bar{\theta}_E$
$\bar{\theta}_P$
ϕ_E
ϕ_P
$\bar{\phi}_E$
$\bar{\phi}_P$
$\cos\xi_{EP}$
$\cos\bar{\xi}_{EP}$
Radius [km]
Radius [km]

- Surprisingly the closure parameters in this test case evolve very similarly to the previous one:
  - $\chi_1 \approx \chi_2$
  - $\theta_E = \theta_P$
  - $\phi_E = \phi_P$
- and the moments are again very close to perfectly aligned.

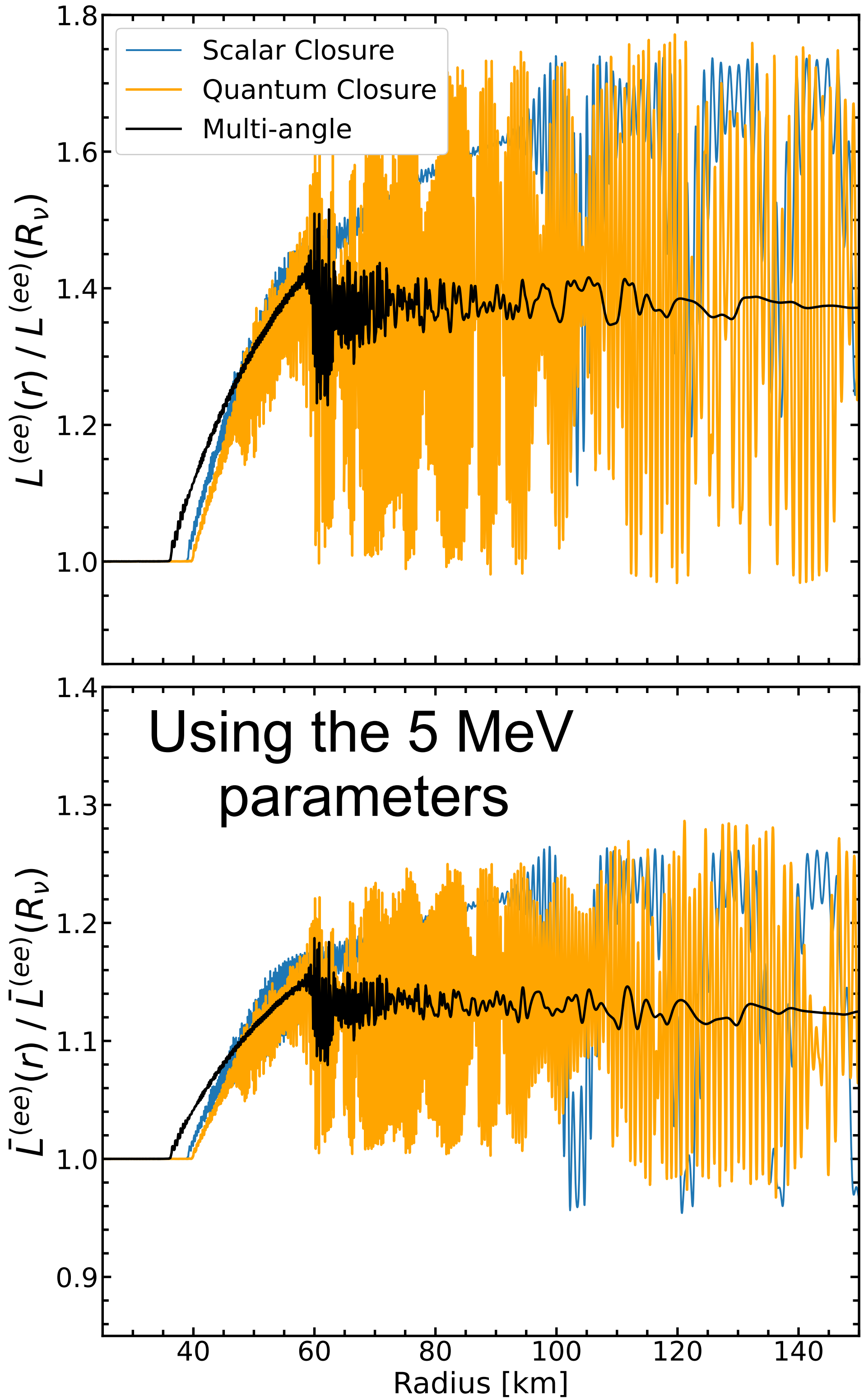
- We take the closure parameters for a single energy and use them in the moment code for all energies.



Scalar Closure
Quantum Closure
Multi-angle
$L^{(ee)}(r) / L^{(ee)}(R_\nu)$
Using the 45 MeV parameters
$\bar{L}^{(ee)}(r) / \bar{L}^{(ee)}(R_\nu)$
Radius [km]



$L^{(ee)}(r) / L^{(ee)}(R_\nu)$
Scalar Closure
Quantum Closure
Multi-angle
Using the 15 MeV parameters
$\bar{L}^{(ee)}(r) / \bar{L}^{(ee)}(R_\nu)$
Radius [km]



Scalar Closure
Quantum Closure
Multi-angle
$L^{(ee)}(r) / L^{(ee)}(R_\nu)$
Using the 5 MeV parameters
$\bar{L}^{(ee)}(r) / \bar{L}^{(ee)}(R_\nu)$
Radius [km]

Summary

- Moments are a more efficient way of doing neutrino flavor transformation calculations.
- If the correct closure is used, the results from moment calculations are exact.

- We have developed a formulation for quantum closures.
- Initial test cases indicate that the closure for self-interaction scenarios is almost classical i.e. a scalar.
- The future goal is to figure out how / why the departure from the scalar closure occurs.