# Lattice QCD and the muon $g-2$

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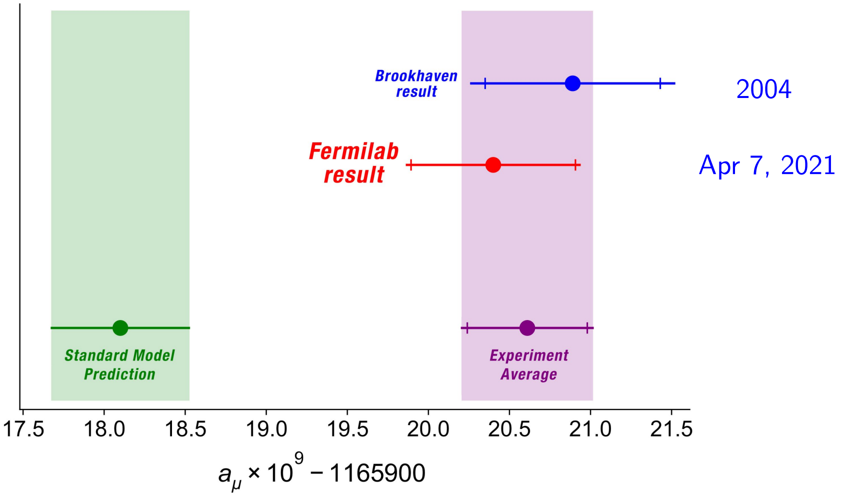
May 11, 2023

New physics searches at the precision frontier

INT Program INT-23-1b

Institute for Nuclear Theory

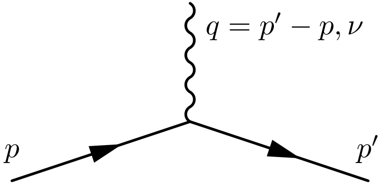
# Outline




- "So far we have analyzed less than 6% of the data that the experiment will eventually collect. Although these first results are telling us that there is an intriguing difference with the Standard Model, we will learn much more in the next couple of years." – Chris Polly, Fermilab scientist, co-spokesperson for the Fermilab muon $g-2$ experiment.

# Anomalous magnetic moments in Euclidean space-time

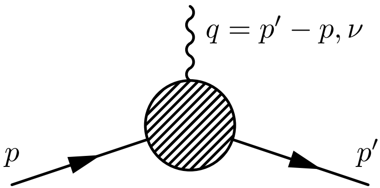





Dirac equation implies:

$$\bar{u}(p')\gamma_\nu u(p)$$

$$g = 2$$




$$\bar{u}(p')\left(F_1(q^2)\gamma_\nu + i\frac{F_2(q^2)[\gamma_\nu, \gamma_\rho]q_\rho}{4m}\right)u(p)$$

(Euclidean space time)

$$a = F_2(q^2 = 0) = \frac{g-2}{2}$$

- The quantity $a$ is called the anomalous magnetic moments.
- Its value comes from quantum correction.

# Muon $g-2$ from the Standard Model

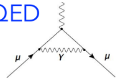


Muon $g-2$ Theory Initiative White paper posted 10 June 2020.

132 authors from worldwide theory + experiment community. [Phys. Rept. 887 (2020) 1-166]

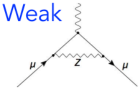
$$\boxed{a_\mu = a_\mu(\text{QED}) + a_\mu(\text{Weak}) + a_\mu(\text{Hadronic})}$$

QED




+…  $116\,584\,718.9\,(1) \times 10^{-11}$  0.001 ppm

Weak

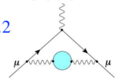



+…  $153.6\,(1.0) \times 10^{-11}$  0.01 ppm

Hadronic…

…Vacuum Polarization (HVP)

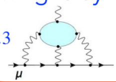
$\alpha^2$




+…  $6845\,(40) \times 10^{-11}$ [0.6%]  0.34 ppm

…Light-by-Light (HLbL)

$\alpha^3$




+…  $92\,(18) \times 10^{-11}$ [20%]  0.15 ppm

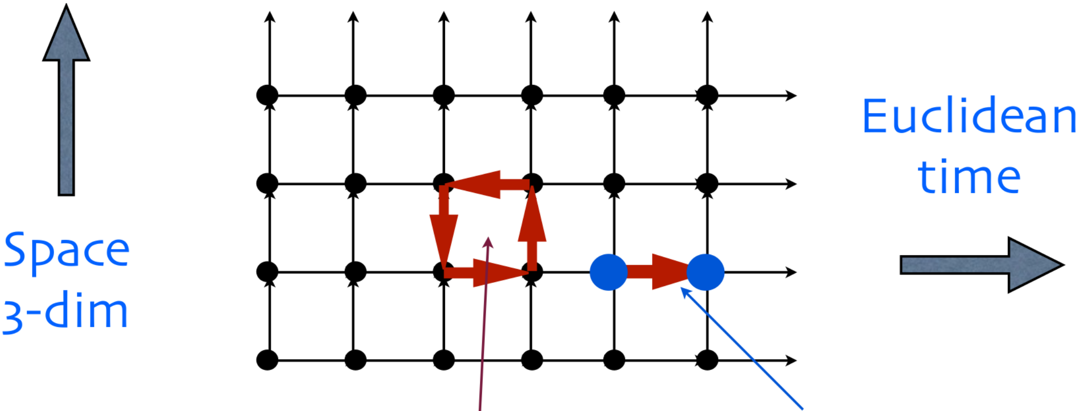
- Two methods: dispersive + data $\leftrightarrow$ lattice QCD

From Aida El-Khadra's theory talk during the Fermilab $g-2$ result announcement.

# Outline

# Lattice QCD: Action






$$S_E^{\mathrm{latt}} = -\sum_{\square} \frac{6}{g^2} \,\mathrm{Re}\; \mathrm{tr}_N (U_{\square,\mu\nu}) - \sum_q \bar{q}(D_\mu^{\mathrm{lat}} \gamma_\mu + a m_q) q$$

Wilson gauge action Lattice fermion action

Figure credit: Stephen R. Sharpe.

$$\langle \mathcal{O}(U,q,\bar{q})\rangle = \frac{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}} \mathcal{O}(U,q,\bar{q})}{\int [\mathcal{D}U] \prod_q [\mathcal{D}q_q][\mathcal{D}\bar{q}_q] e^{-S_E^{\text{latt}}}}$$

$$= \frac{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right) \tilde{\mathcal{O}}(U)}{\int [\mathcal{D}U] e^{-S_{\text{gauge}}^{\text{latt}}} \prod_q \det\left(D_\mu^{\text{latt}}\gamma_\mu + am_q\right)}$$

Monte Carlo:

- The integration is performed for all the link variables: $U$. Dimension is $L^3 \times T \times 4 \times 8$.
- Sample points the following distribution:

$$e^{-S_{\text{gauge}}^{\text{latt}}(U)} \prod_q \det\left(D_\mu^{\text{latt}}(U)\gamma_\mu + am_q\right)$$

- Therefore:

$$\langle \mathcal{O}(U,q,\bar{q})\rangle = \frac{1}{N_{\text{conf}}} \sum_{k=1}^{N_{\text{conf}}} \tilde{\mathcal{O}}(U^{(k)})$$

- Parameters in lattice QCD calculations (e.g. isospin symmetric ($m_u = m_d = m_l$) and three flavor $u, d, s$ theory):

$$g \qquad am_l \qquad am_s$$

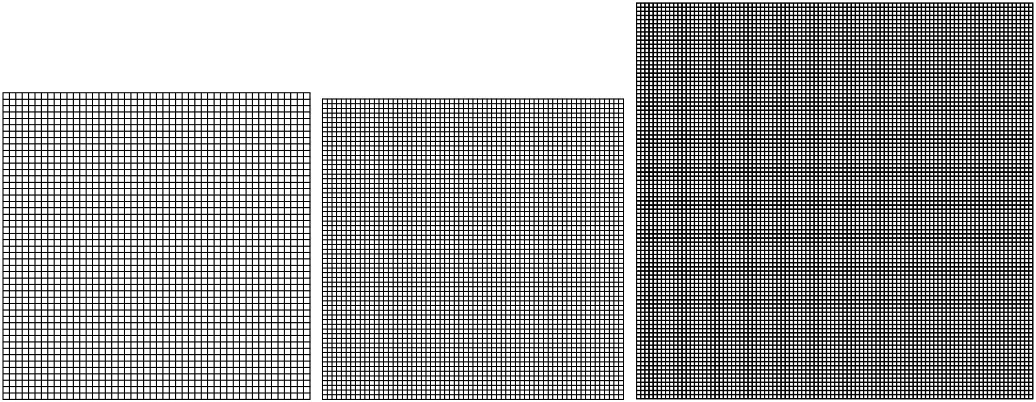
  Note that lattice spacing $a$ is determined by $g$ via the renormalization group equation.

- The experimental inputs needed to determine these parameters can be: $m_\pi/m_\Omega$, $m_K/m_\Omega$.

# Lattice QCD: Lattice size

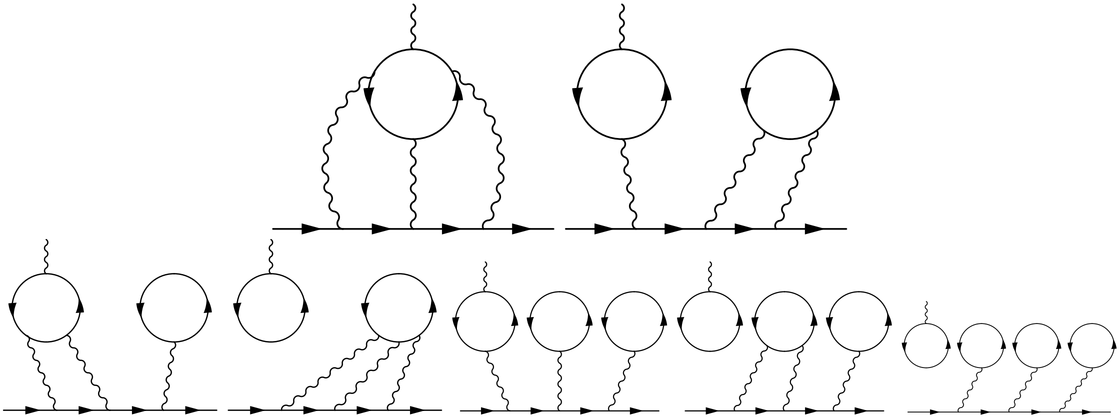


- RBC-UKQCD Domain wall fermion action and Iwasaki gauge action ensembles.
- At physical pion mass (almost).
- 48I, 64I, 96I with $a^{-1} = 1.73,\ 2.36,\ 2.68$ GeV, $L = 5.47,\ 5.36,\ 7.06$ fm.



# Outline

# Muon $g-2$ HLbL: diagrams



- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- There are additional different permutations of photons not shown.
- The second row diagrams are suppressed by flavor SU(3) symmetry (and small charge factors, $1/N_c$, etc). The contributions are numerically very small.

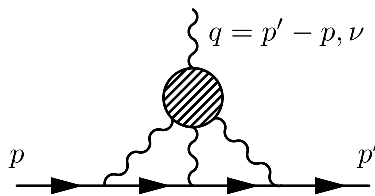
# Muon $g-2$ HLbL: Analytical approach (WP2020)



| Contribution | PdRV(09) [471] | N/JN(09) [472, 573] | J(17) [27] | Our estimate |
|---|---|---|---|---|
| $\pi^0, \eta, \eta'$-poles | 114(13) | 99(16) | 95.45(12.40) | 93.8(4.0) |
| $\pi, K$-loops/boxes | −19(19) | −19(13) | −20(5) | −16.4(2) |
| $S$-wave $\pi\pi$ rescattering | −7(7) | −7(2) | −5.98(1.20) | −8(1) |
| subtotal | 88(24) | 73(21) | 69.5(13.4) | 69.4(4.1) |
| scalars | – | – | – | } − 1(3) |
| tensors | – | – | 1.1(1) | } |
| axial vectors | 15(10) | 22(5) | 7.55(2.71) | 6(6) |
| $u, d, s$-loops / short-distance | – | 21(3) | 20(4) | 15(10) |
| $c$-loop | 2.3 | – | 2.3(2) | 3(1) |
| total | 105(26) | 116(39) | 100.4(28.2) | 92(19) |

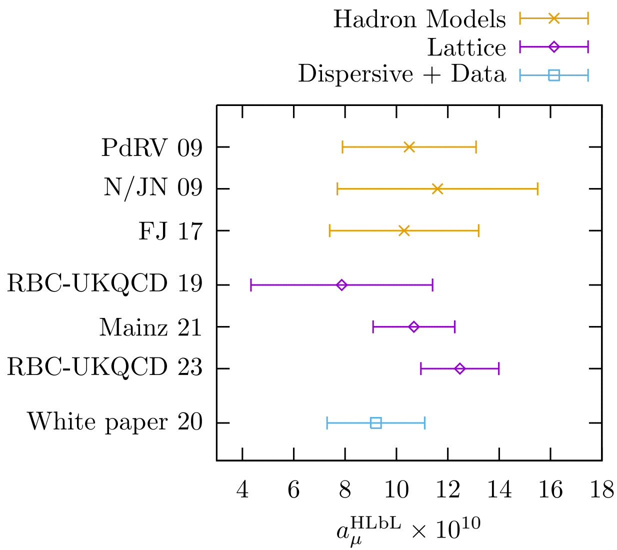
Table 15: Comparison of two frequently used compilations for HLbL in units of $10^{-11}$ from 2009 and a recent update with our estimate. Legend: PdRV = Prades, de Rafael, Vainshtein ("Glasgow consensus"); N/JN = Nyffeler / Jegerlehner, Nyffeler; J = Jegerlehner.

- Values in the table is in unit of $10^{-11}$.
- Uncertainty of the analytically approach mostly come from the short distance part.



- RBC-UKQCD 19:
  Physical $m_\pi$ and QED$_L$.
  Extrapolate to infinite volume via $1/L^2$.

- Mainz 21:
  Heavy $m_\pi$ and QED$_\infty$.
  Calculated all the sub-leading disconnected diagrams.
  Fit the contribution in the long distance region to improve statistics.
  Extrapolate to the physical pion mass and infinite volume via ($e^{-m_\pi L/2}$).



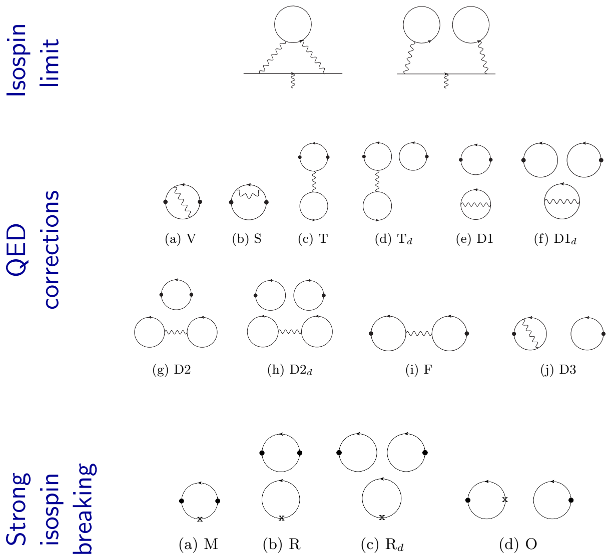

- RBC-UKQCD 23: Physical $m_\pi$ and QED$_\infty$.
  Mainly based on the "48I" ensemble. Several new error reduction techniques are developed to reduce the statistical noise in the long distance region.

- Required precision for HLbL: 10% to match Fermilab's final results. (Very close now.)

# Outline

# Muon $g-2$ HVP: diagrams






- Gluons and sea quark loops (not directly connected to photons) are included automatically to all orders!
- Need to calculate and cross check all the contributions.

T. Blum 2003; D. Bernecker, H. Meyer 2011.

$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j^{em}(\vec{x},t)J_j^{em}(0)\rangle_{\text{QCD}}$$

$$a_\mu^{\text{HVP LO}} = \left(\frac{\alpha}{\pi}\right)^2\int_0^\infty dK^2 f(K^2)\hat{\Pi}(K^2) = \sum_{t=0}^{+\infty} w(t)C(t)$$

QED

and

strong isospin

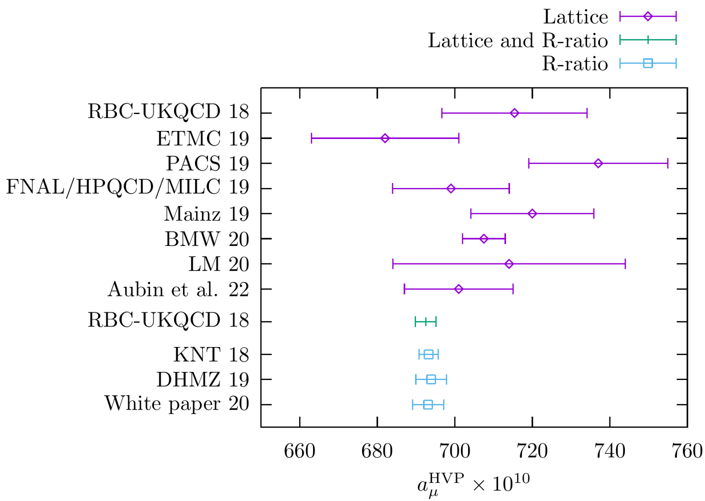
breaking

| $a_\mu^{\text{HVP, LO}}(ud)$ | $a_\mu^{\text{HVP, LO}}(s)$ | $a_\mu^{\text{HVP, LO}}(c)$ | $a_{\mu,\text{disc}}^{\text{HVP, LO}}$ | $\delta a_\mu^{\text{HVP}}$ |
|---|---|---|---|---|
| 650.2(11.6) | 53.2(0.3) | 14.6(0.1) | −13.7(2.9) | 7.2(3.4) |

- From muon $g-2$ theory initiative white paper (2020). Value in unit of $10^{-10}$
- Light quark connected diagram has the largest contribution and largest uncertainty.

# Muon $g-2$ HVP: overview 

- Dispersive method via R-ratio (red points) is mature and reproducible.
- Lattice (blue points) errors are limited by statistics.
  Except for BMW, which beats down the statistical error, result is limited by systematic error:
  BMW 20: $707.5(2.3)_{\mathrm{stat}}(5.0)_{\mathrm{sys}}$
- Lattice-QCD calculations of comparable precision needed.
- Consistency is needed to claim new physics.



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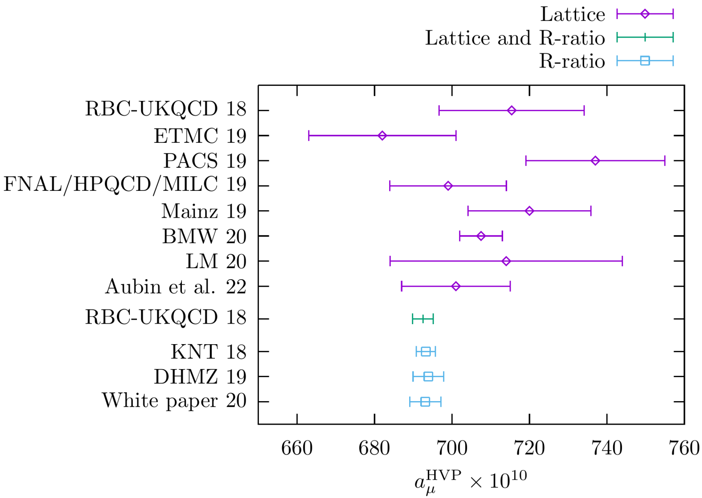

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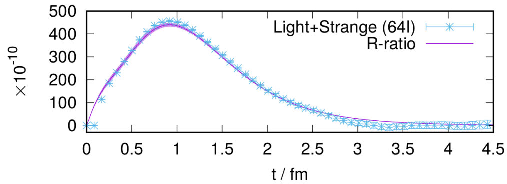
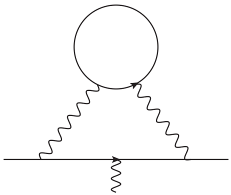



- Statistical error is mostly from:
  Light quark connected diagram at $t \gtrsim 1.5$ fm
  - More configurations (BMW 20 used $\sim 20,000$).
  - Use low modes averaging to gain full volume average. ✓
  - Bounding method on the long distance tail. ✓
  - Study the $\pi\pi$ system spectrum to calculate $C(t)$ large $t$.
    - Not used in any published work yet!
    - On-going efforts with promising initial results.
- Systematic error is mostly from the **continuum extrapolation**.



- Main idea is that: one does not have to calculate the long distance part of the correlation function directly.

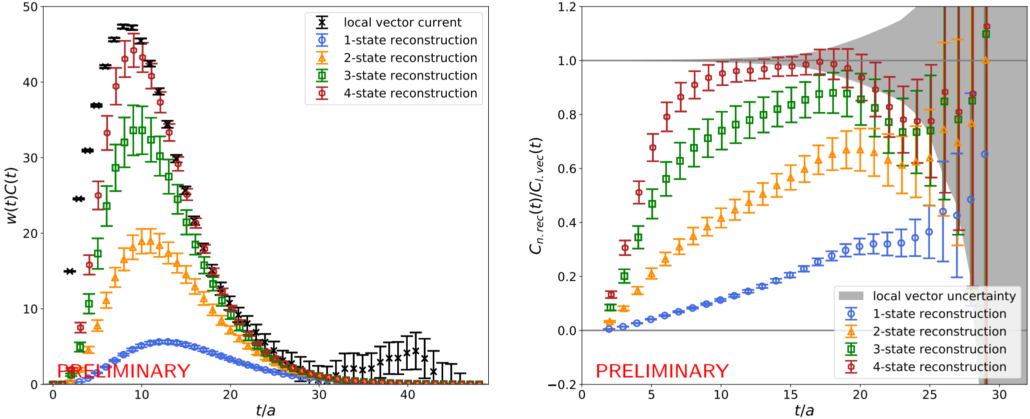
$$C(t) = \frac{1}{3}\sum_{\vec{x}}\sum_{j=0,1,2}\langle J_j(\vec{x},t)J_j(0)\rangle$$
$$= \sum_{n}\frac{V}{3}\sum_{j=0,1,2}\langle 0|J_j(0)|n\rangle\langle n|J_j(0)|0\rangle e^{-E_n t}$$

- The summation over n is limited to zero momentum states and states are normalized to "1".

- At large $t$, only lowest few states contribute. We only need the matrix elements $\langle n|J_j(0)|0\rangle$ and the corresponding energy $E_n$.

- Need to study the spectrum of the $\pi\pi$ system!

- Can reduce the statistical error beyond the gauge noise limit!




- GEVP results to reconstruct long-distance behavior of local vector correlation function needed to compute connected HVP
- Explicit reconstruction good estimate of correlation function at long-distance, missing excited states at short-distance
- More states $\Longrightarrow$ better reconstruction, can replace $C(t)$ at shorter distances

RBC-UKQCD by Aaron Meyer and Christoph Lehner

Preliminary

RBC-UKQCD PRL 121, 022003 (2018)

Window contribution allows a high precision study of the continuum extrapolation.

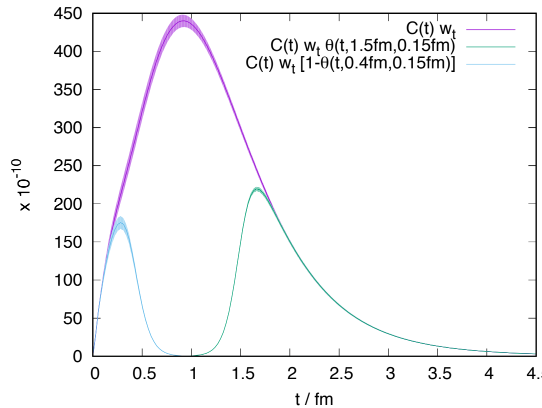
$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t) C(t)$$

$$w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

- Splitting sum into three parts allows crosschecks:
  - short distance $\Leftarrow$ discretization effects
  - long distance $\Leftarrow$ noisy $\pi\pi$ tail
  - intermediate (Window): sweet spot
- Can form windows from $R(e^+e^-)$ dispersive analysis too.

Combine "window" from lattice with dispersive analysis.

- Compare "window" among lattice-QCD calculations



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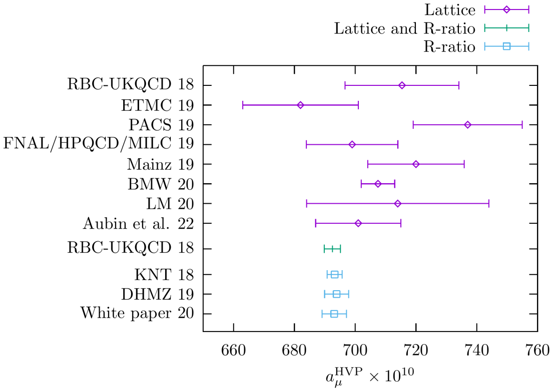

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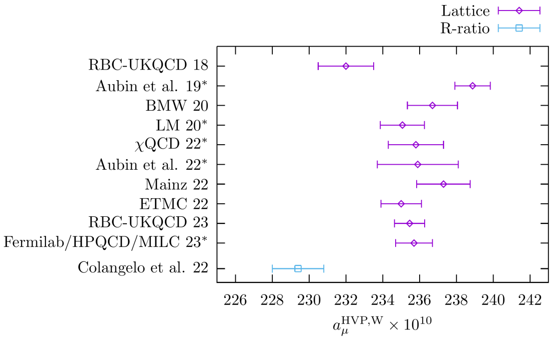


- Lattice QCD community has already reached consensus on the window contribution!
- The consensus has a noticeable tension with the dispersive results.

RBC-UKQCD PRL 121, 022003 (2018)

Window contribution allows a high precision study of the continuum extrapolation.

$$a_\mu^{\text{HVP LO}} = \sum_{t=0}^{+\infty} w(t)C(t)$$

$$w(t) = w^{\text{SD}}(t) + w^{\text{W}}(t) + w^{\text{LD}}(t)$$

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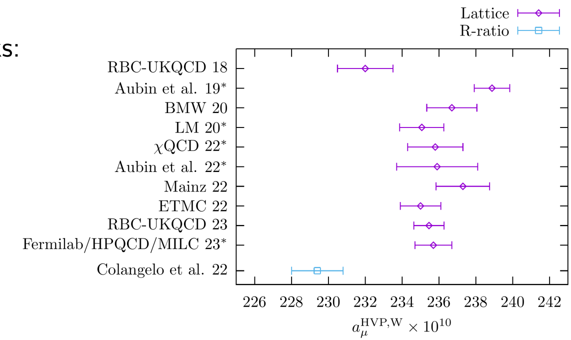
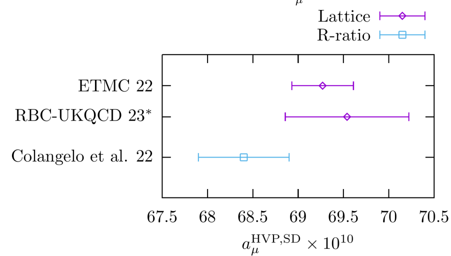


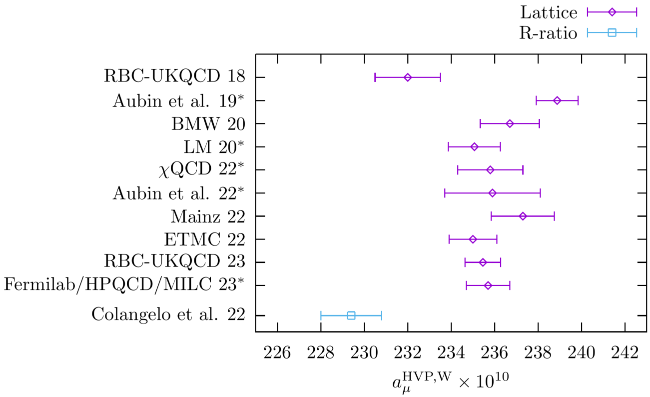
# Muon $g - 2$ HVP: Some comments 

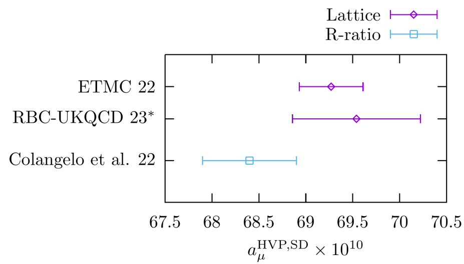
- The current white paper result for the HVP is a community-vetted method average for the data-driven approach. It accounts for spreads in sub-contributions between individual results (KNT/DHMZ) that may not be visible in the agreement of looking at the final results for the HVP. Its error estimate accounts for the tension between BaBar and KLOE experimental inputs. New CMD3 results have larger tension.

- We are now in the fortunate situation that we have a first lattice result with sub-percent precision (BMW). It is clear that to safely assess systematic uncertainties, most notably the one related to the choice of the lattice regulator, calculations by other lattice groups with a similar precision will be essential. The importance of having more than one lattice calculations of the same quantity and obtained with different lattice discretizations is well understood inside the lattice community.

- On the way to a lattice QCD average for HVP, it is prudent to also look at individual sub-contributions and their agreement, similar to what was done for the data driven approach. For example, individual QED corrections should be cross checked (currently there are some tensions).

- The previous tension in the standard iso-symmetric window results has already been resolved among lattice QCD calculations!




$*$: Use iso-symmetric, quark connected, light quark contribution from this work and remaining contributions from RBC-UKQCD 18 (W) or ETMC 22 (SD).

# Outline

# Summary



- The errors of lattice QCD calculations comes from:

  1. finite statistics $\rightarrow$ statistical error
  2. non-zero lattice spacing $\rightarrow$ discretization error
     - smaller lattice spacing $a \lesssim 0.06$ fm
     - compare different lattice actions: Staggered, Wilson, Domain Wall, etc
  3. finite lattice size $\rightarrow$ finite volume error
  4. non-physical pion mass $\rightarrow$ Chiral extrapolation
     Many lattice calculations are now performed with physical pion mass, eliminating this source of the systematic errors.

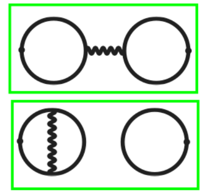
- Lattice QCD calculation is now playing important role in determining the hadronic contribution to muon $g - 2$ and many other physical observables.

- More accurate lattice results are expected when Fermilab releases the final experimental result.

# Thank You!

# HVP: Compare BMW 20 and RBC-UKQCD 18

| | |
|---|---|
| $a_\mu^{\text{ud, conn, isospin}}$ | $649.7(14.2)_S(2.8)_C(3.7)_V(1.5)_A(0.4)_Z(0.1)_{E48}(0.1)_{E64}$ |
| $a_\mu^{\text{s, conn, isospin}}$ | $53.2(0.4)_S(0.0)_C(0.3)_A(0.0)_Z$ |
| $a_\mu^{\text{c, conn, isospin}}$ | $14.3(0.0)_S(0.7)_C(0.1)_Z(0.0)_M$ |
| $a_\mu^{\text{uds, disc, isospin}}$ | $-11.2(3.3)_S(0.4)_V(2.3)_L$ |
| $a_\mu^{\text{QED, conn}}$ | $5.9(5.7)_S(0.3)_C(1.2)_V(0.0)_A(0.0)_Z(1.1)_E$ |
| $a_\mu^{\text{QED, disc}}$ | $-6.9(2.1)_S(0.4)_C(1.4)_V(0.0)_A(0.0)_Z(1.3)_E$ |
| $a_\mu^{\text{SIB}}$ | $10.6(4.3)_S(0.6)_C(6.6)_V(0.1)_A(0.0)_Z(1.3)_{E48}$ |
| $a_\mu^{\text{udsc, isospin}}$ | $705.9(14.6)_S(2.9)_C(3.7)_V(1.8)_A(0.4)_Z(2.3)_L(0.1)_{E48}$ $(0.1)_{E64}(0.0)_M$ |
| $a_\mu^{\text{QED, SIB}}$ | $9.5(7.4)_S(0.7)_C(6.9)_V(0.1)_A(0.0)_Z(1.7)_E(1.3)_{E48}$ |
| $a_\mu^{\text{R-ratio}}$ | |
| $a_\mu$ | $715.4(16.3)_S(3.0)_C(7.8)_V(1.9)_A(0.4)_Z(1.7)_E(2.3)_L$ $(1.5)_{E48}(0.1)_{E64}(0.3)_b(0.2)_c(1.1)_{\overline{S}}(0.3)_{\overline{Q}}(0.0)_M$ |

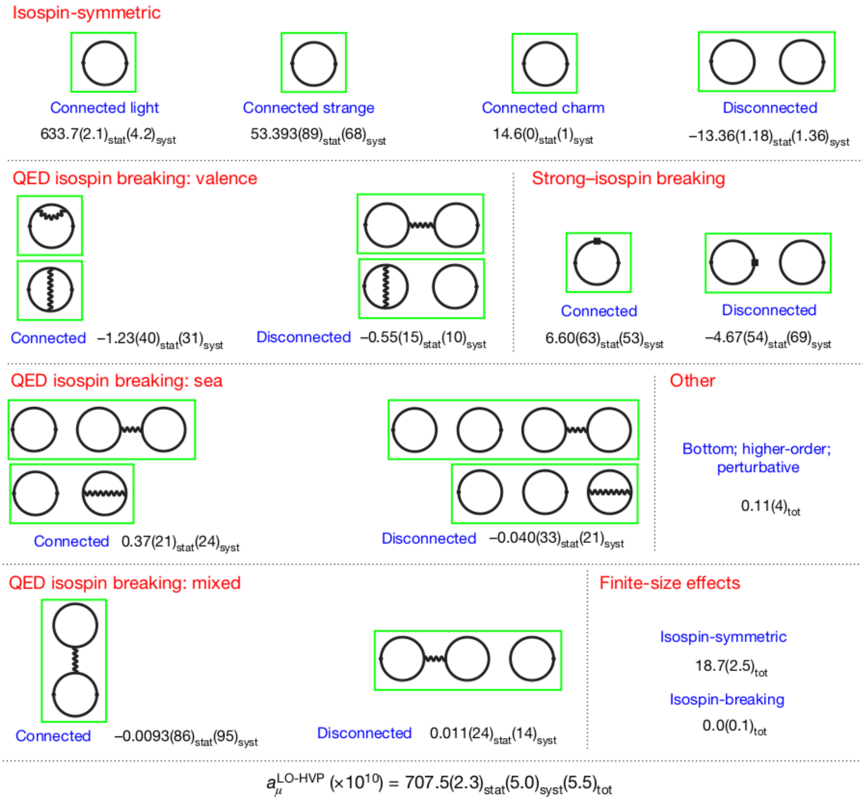


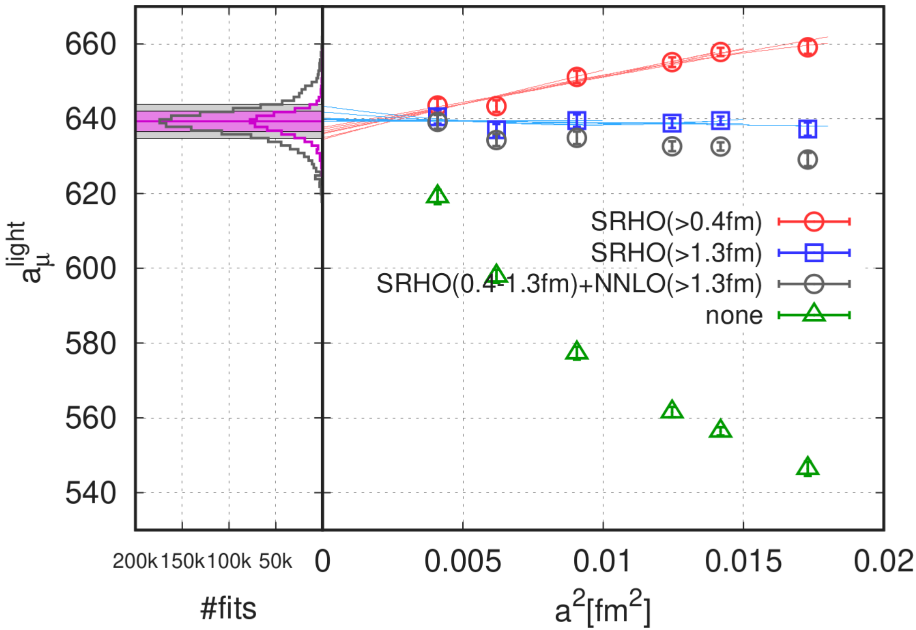
Disconnected $-0.55(15)_{\text{stat}}(10)_{\text{syst}}$

- The left table shows result from RBC-UKQCD 18. The right figure shows the result from BMW 20.
- This discrepancy needs further study and more cross checks.

# HVP: BMW 2021 - all channels





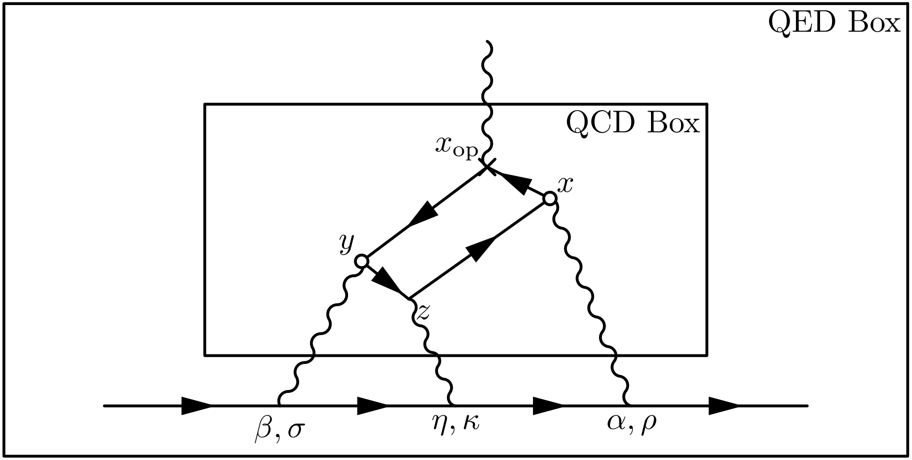



- Staggered fermion has a special lattice artifacts: taste breaking effects.
- Curves show different treatments of correcting the taste breaking effects.

# HLbL: QCD box inside QED box

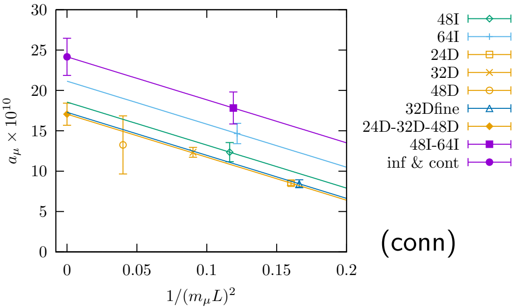


- The basic idea is to right the Feynman diagram in coordinate space, and then it is possible and natural to use infinite volume for QED and finite (by necessity) for QCD.

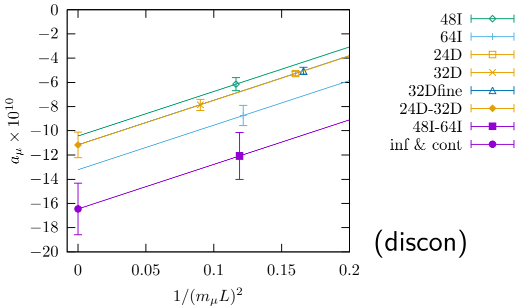



- In a way, this is the HLbL version of the Bernecker-Meyer formula: coordinate space (sum over $t$), infinite-volume QED kernel, finite volume (by necessity) calculation of the correlator.

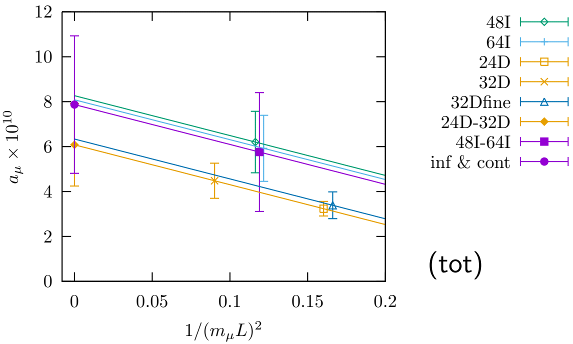
$$a_\mu(L, a^{\mathrm{I}}, a^{\mathrm{D}}) = a_\mu\left(1 - \frac{b_2}{(m_\mu L)^2} - c_1^{\mathrm{I}}(a^{\mathrm{I}}\ \mathrm{GeV})^2 - c_1^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^2 + c_2^{\mathrm{D}}(a^{\mathrm{D}}\ \mathrm{GeV})^4\right)$$




(conn)




(discon)

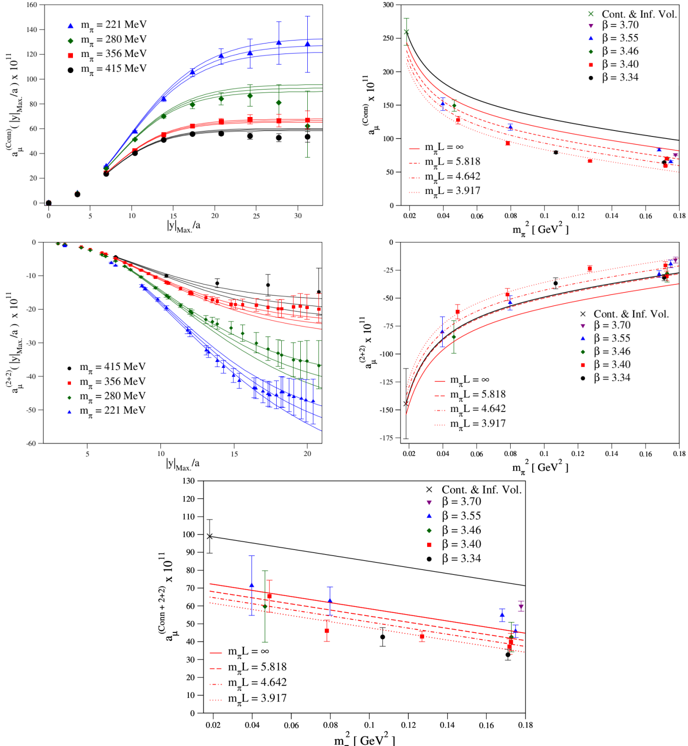



(tot)

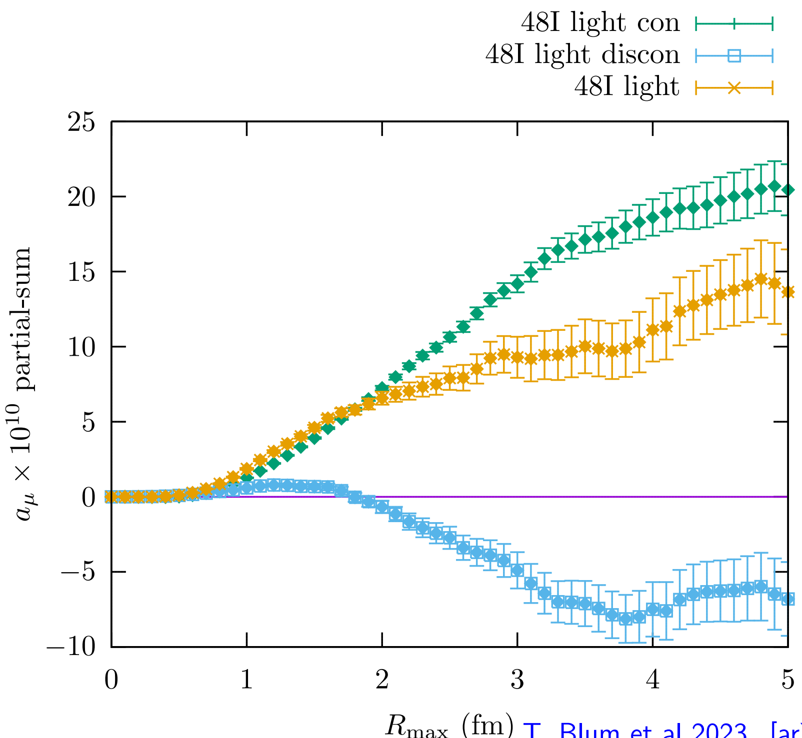
T. Blum et al 2020. (PRL 124, 13, 132002)

# Muon $g-2$ HLbL: Mainz 2021 - Results

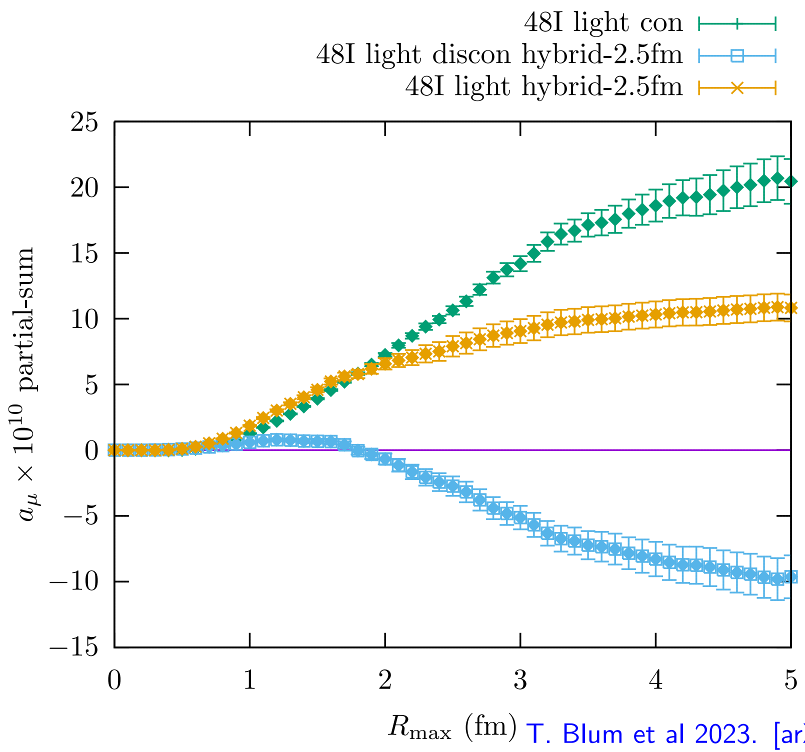





E.H. Chao et al 2021. (EPJC 81, 7, 651)

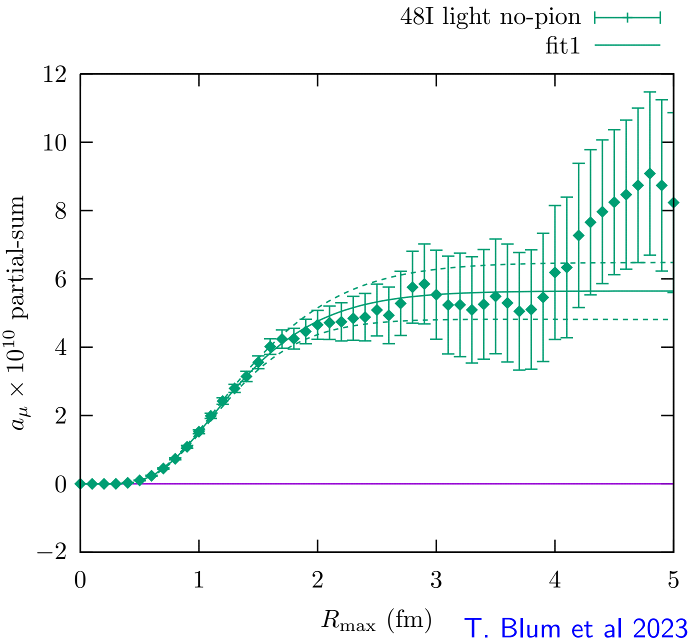



T. Blum et al 2023. [arXiv:2304.04423]




T. Blum et al 2023. [arXiv:2304.04423]

$$f(R_{\text{max}}) = A\frac{R_{\text{max}}^6}{R_{\text{max}}^3 + C^3}e^{-BR_{\text{max}}}$$




T. Blum et al 2023. [arXiv:2304.04423]