Proton mass & physics of gravitational form factors

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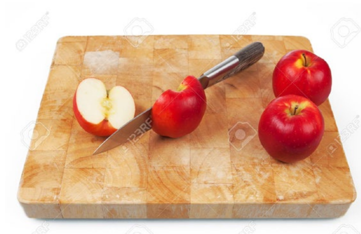
Outline

1. The QCD mass scale
2. The proton mass sum rule
3. Dynamical origin of hadron masses in QCD
4. Physics of gravitational form factors C(D): "pressure" vs. the momentum current distribution
5. Conclusion

The QCD mass scale

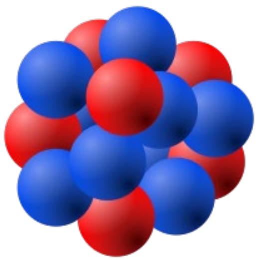
Three sources of mass in GP

1. In classical physics: the mass of an object is the sum of its parts



$$M = \sum m_i$$

2. Mass of atomic nuclei

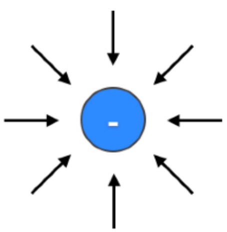


$$M_A = NM_n + ZM_p - B_{A=N+Z}$$

$$-B_{A=N+Z} = \langle \sum_i T_i + \sum_{ij} V_{ij} + \cdots \rangle$$

3. Mass from the energy of fields

Electron carries a static electric field, which produces energy $\varepsilon = \frac{1}{2}\epsilon_0 \vec{E}^2$



Mass from interactions

- The nucleon mass is determined by two different mass scales:

$$\boxed{M_N = \sum_i \alpha_i m_i + \beta \Lambda_{\rm QCD}}$$

  - **Quark masses m_q**
    - Just like the electron mass in atomic physics, determined by Higgs mechanism
    - Electroweak symmetry breaking scale.
  - **QCD scale Λ_{QCD}**
    - QCD scale Λ_{QCD} does not appear directly in the lagrangian: dimensional transmutation
    - Free parameter (imagine increase or decrease by a factor of 10!)

The proton mass sum rule

Criteria

- Physics consideration:
  - Mass is energy, $m = E/c^2$

    What is the individual sources of energy?

    terms in the QCD Hamiltonian
  - Symmetry

  Lorentz symmetry is one of the most important physics constraints.
- The sum rule is unique!

  X. Ji, PRL *Phys. Rev. Lett.* 74 (1995) 1071

Sources of energy

- The effective definition

$$M = E/c^2|_{\vec{P}=0}$$

- What the sources of energy?

$$\mathrm{E}_0 = \sum_i \quad E_i, \quad \mathrm{E}_0 = \int d^3\vec{r}\, E(\vec{r})$$

E_i shall be calculable in theory, measurable in exp.

Mass as internal energy

- Internal mass as a store of energy

$$\mathrm{M}c^2 = \langle N \left| \hat{H}_{QCD} \right| N \rangle |_{\vec{P}=0}$$

This is how the lattice QCD calculate.

- For any relativistic system, the Hamiltonian can be separated into two terms (Ji, PRL,1995),

$$\hat{H}_{QCD} = \hat{H}_T + \hat{H}_S$$

This separation is a fundamental property of special relativity and both parts are scale invariant

Lorentz symmetry

- Energy is related to H$= \int d^3\vec{r}\, T^{00}(\vec{r})$
- $T^{\mu\nu}$ has a mixed symmetry under Lorentz transformation (1,1) + (0,0), and the separate parts are scheme and scale independent.

$$T^{\mu\nu} = T_s^{\mu\nu} + T_T^{\mu\nu}\ ,$$

$$H = H_s + H_T,$$

$$\mathrm{M} = M_s + M_T$$

Tensor and scalar energies

- Tensor energy

$$E_T = \langle H_T \rangle$$

is related to the usual kinetic and potential energy sources.

- Scalar energy

$$E_S = \langle H_S \rangle$$

is related to related to scale-breaking properties of the theory ($\partial^\mu j_{D\mu} \sim H_s$), such as

  - Quark mass m_q
  - Trace anomaly (quantum breaking of scale symmetry).

Does the trace anomaly contribute to the mass?

- Of course! (can also be derived by through time-translation)
- $T^{\mu}_{\mu} = (1+\gamma_m)m\bar{\psi}\psi + \frac{\beta(g)}{2g}F^2$
- In the massless QCD limit, all contribution to the scalar energy (mass) comes from the anomaly. (without trace anomaly, there is no mass!)
- The scalar field from trace anomaly plays the key role for mass generation, like a Higgs mechanism.

Relativistic "virial theorem"

- As an important feature of relativity, one can show

  $E_T = 3E_S$ (virial theorem)

  3 is the dimension of space.

- Scalar energy sets the scale of the tensor energy (kinetic and potential energies of the system).
- In non-relativistic limit of QED & gravity, it reduces

  $\langle V \rangle = -2\langle T \rangle$

  kinetic energy sets the scale for potential energy!

Renormalization: tensor part

- Standard!

$$T^{(\mu\nu)} = T_q^{(\mu\nu)} + T_g^{(\mu\nu)}$$

renormalization of $T_{q,g}^{(\mu\nu)}$ is know to four-loops

$$\begin{pmatrix} T_q^{\mu\nu} \\ T_g^{\mu\nu} \\ T_{\mathrm{gv}}^{\mu\nu} \\ E^{\mu\nu} \end{pmatrix} = \begin{pmatrix} Z_{qq} & Z_{qg} & Z_{qa} & Z_{qe} \\ Z_{gq} & Z_{gg} & Z_{ga} & Z_{ge} \\ 0 & 0 & Z_{aa} & Z_{ae} \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} T_q^{\mu\nu} \\ T_g^{\mu\nu} \\ T_{\mathrm{gv}}^{\mu\nu} \\ E^{\mu\nu} \end{pmatrix}^R$$

$$\frac{d}{d\ln\mu_f^2}\begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix} = \begin{pmatrix} P_{\mathrm{qq}} & P_{\mathrm{qg}} \\ P_{\mathrm{gq}} & P_{\mathrm{gg}} \end{pmatrix} \otimes \begin{pmatrix} q_{\mathrm{s}} \\ g \end{pmatrix}$$

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Low moments of the four-loop splitting functions in QCD

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A B S T R A C T

We have computed the four lowest even-N moments of all four splitting functions for the evolution of flavour-singlet parton densities of hadrons at the fourth order in the strong coupling constant α_s. The perturbative expansion of these moments, and hence of the splitting functions for momentum fractions $x \gtrsim 0.1$, is found to be well behaved with relative α_s-coefficients of order one and sub-percent effects on the scale derivatives of the quark and gluon distributions at $\alpha_s \lesssim 0.2$. More intricate computations, including other approaches such as the operator-product expansion, are required to cover the full x-range relevant to LHC analyses. Our results are presented analytically for a general gauge group for detailed checks and validations of such future calculations.

$$\gamma_{\rm ik}(N,\alpha_{\rm s}) = \sum_{n=0} a_{\rm s}^{n+1}\, \gamma_{\rm ik}^{(n)}(x) \quad \text{with} \quad a_{\rm s} \equiv \frac{\alpha_{\rm s}(\mu_f^2)}{4\pi} \; .$$

$$\begin{aligned}
\gamma_{\rm ps}^{(3)}(N{=}2) \;=\; & n_f\, C_F^3 \left(\frac{227938}{2187} + \frac{1952}{81}\,\zeta_3 + \frac{256}{9}\,\zeta_4 - \frac{640}{3}\,\zeta_5\right) \\
& + n_f\, C_A C_F^2 \left(-\frac{162658}{6561} + \frac{8048}{27}\,\zeta_3 - \frac{1664}{9}\,\zeta_4 + \frac{320}{9}\,\zeta_5\right) \\
& + n_f\, C_A^2\, C_F \left(-\frac{410299}{6561} - \frac{26896}{81}\,\zeta_3 + \frac{1408}{9}\,\zeta_4 + \frac{4480}{27}\,\zeta_5\right) \\
& + n_f\, \frac{d_R^{abcd}\, d_R^{abcd}}{n_c} \left(\frac{1024}{9} + \frac{256}{9}\,\zeta_3 - \frac{2560}{9}\,\zeta_5\right) - n_f^2\, C_F^2 \left(\frac{73772}{6561} + \frac{5248}{81}\,\zeta_3 - \frac{320}{9}\,\zeta_4\right) \\
& + n_f^2\, C_A C_F \left(\frac{160648}{6561} + 48\,\zeta_3 - \frac{320}{9}\,\zeta_4\right) + n_f^3\, C_F \left(-\frac{1712}{729} + \frac{128}{27}\,\zeta_3\right) ,
\end{aligned}$$

$$\gamma_{\rm gq}^{(3)}(N{=}2) \;=\; -\,\gamma_{\rm qq}^{(3)}(N{=}2)\; ,$$

$$\begin{aligned}
\gamma_{\rm qg}^{(3)}(N{=}2) \;=\; & n_f\, C_F^3 \left(\frac{16489}{729} + \frac{736}{81}\,\zeta_3 + \frac{256}{9}\,\zeta_4 - \frac{320}{3}\,\zeta_5\right) \\
& + n_f\, C_A^3 \left(-\frac{88769}{729} + \frac{31112}{81}\,\zeta_3 - 132\,\zeta_4 - \frac{3560}{27}\,\zeta_5\right) - n_f\, C_A C_F^2 \left(\frac{1153727}{13122} - \frac{7108}{81}\,\zeta_3 \right. \\
& \left. + \frac{1136}{9}\,\zeta_4 - \frac{2000}{9}\,\zeta_5\right) + n_f\, C_A^2\, C_F \left(\frac{763868}{6561} - \frac{12808}{27}\,\zeta_3 + \frac{2068}{9}\,\zeta_4 + \frac{40}{9}\,\zeta_5\right) \\
& + n_f\, \frac{d_R^{abcd}\, d_A^{abcd}}{n_a} \left(\frac{368}{9} - \frac{992}{9}\,\zeta_3 - \frac{2560}{9}\,\zeta_5\right) - n_f^2\, C_F^2 \left(\frac{110714}{6561} + \frac{272}{9}\,\zeta_3 - \frac{224}{9}\,\zeta_4\right) \\
& + n_f^2\, C_A\, C_F \left(\frac{249310}{6561} + \frac{5632}{81}\,\zeta_3 - \frac{440}{9}\,\zeta_4\right) + n_f^2\, C_A^2 \left(\frac{48625}{2187} - \frac{3572}{81}\,\zeta_3 \right. \\
& \left. + 24\,\zeta_4 + \frac{160}{27}\,\zeta_5\right) + n_f^2\, \frac{d_R^{abcd}\, d_R^{abcd}}{n_a} \left(-\frac{928}{9} - \frac{640}{9}\,\zeta_3 + \frac{2560}{9}\,\zeta_5\right) \\
& + n_f^3\, C_F \left(-\frac{8744}{2187} + \frac{128}{27}\,\zeta_3\right) + n_f^3\, C_A \left(\frac{3385}{2187} - \frac{176}{81}\,\zeta_3\right) ,
\end{aligned}$$

$$\gamma_{\rm gg}^{(3)}(N{=}2) \;=\; -\,\gamma_{\rm qg}^{(3)}(N{=}2)\; ,$$

Renormalization: scalar part

- Scale invariant!

$$(1+\gamma_m)m\bar{\psi}\psi + \frac{\beta(g)}{2g}F^2$$

- There is no mixing between scalar and tensor in any renormalization method (lattice, DR)!

Unless Lorentz symmetry is broken.

Other schemes breaking Lorentz symmetry is not interesting.

Mass separation

- Independent of dynamics

$$M_s = \frac{1}{4} M, \qquad M_T = \frac{3}{4} M$$

$$M_T = 3\ M_s,$$

- $H_s = \frac{1}{4} T_\mu^\mu$ is the source for dilatation symmetry breaking. Dilaton field.

QCD energies in the nucleon

- Four different types

$$H_{\rm QCD} = H_q + H_m + H_g + H_a \,.$$

$$H_q = \int d^3\vec{x}\; \bar{\psi}(-i\mathbf{D}\cdot\alpha)\psi,$$ ← Quark energy

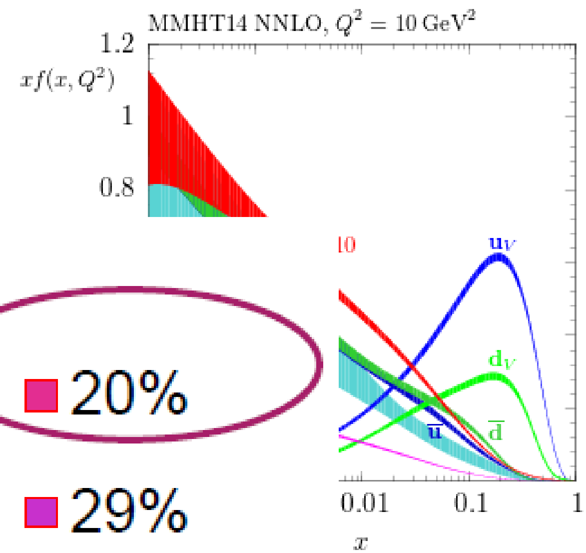
$$H_m = \int d^3\vec{x}\; \bar{\psi} m \psi,$$ ← Quark mass

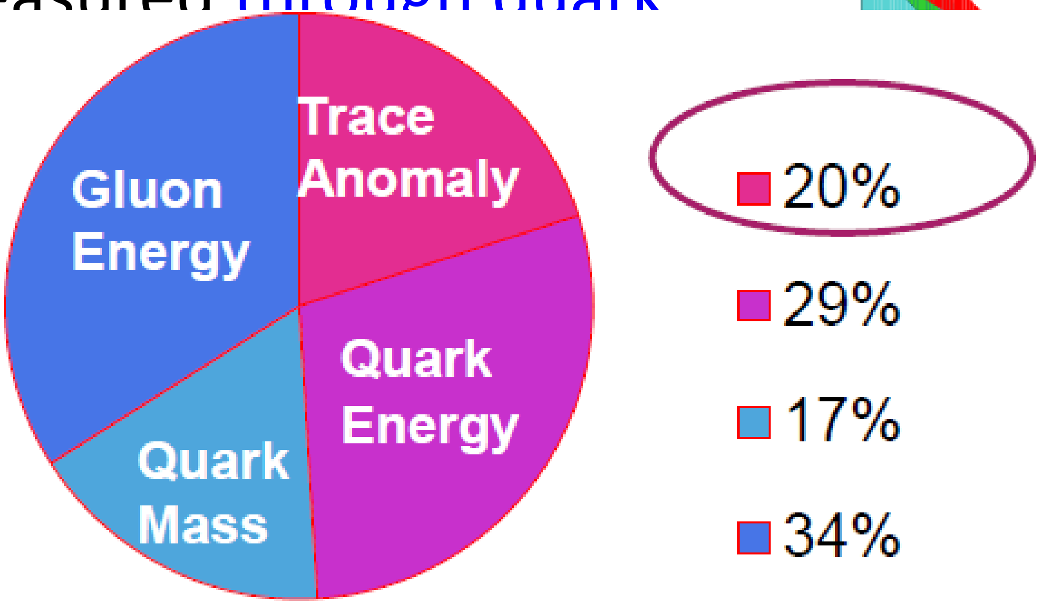
$$H_g = \int d^3\vec{x}\; \frac{1}{2}(\mathbf{E}^2 + \mathbf{B}^2),$$ ← Gluon energy

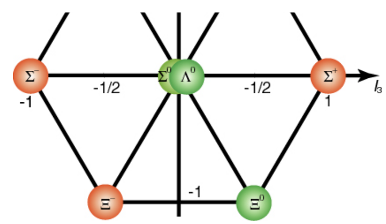
$$\boxed{H_a = \int d^3\vec{x}\; \frac{9\alpha_s}{16\pi}(\mathbf{E}^2 - \mathbf{B}^2).}$$ ← Quantum Anomalous Energy (QAE)

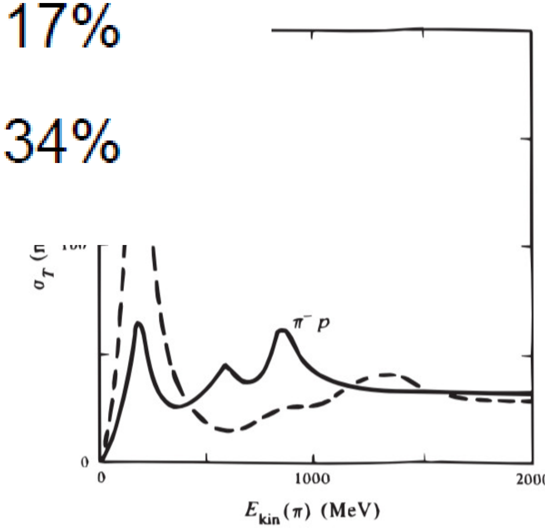
Proton mass content from data (Ji,1995)

- Quark energy and gluon energy can be measured through quark and glu high en
- Quark r measur and thr of strar
- There is information on QAE



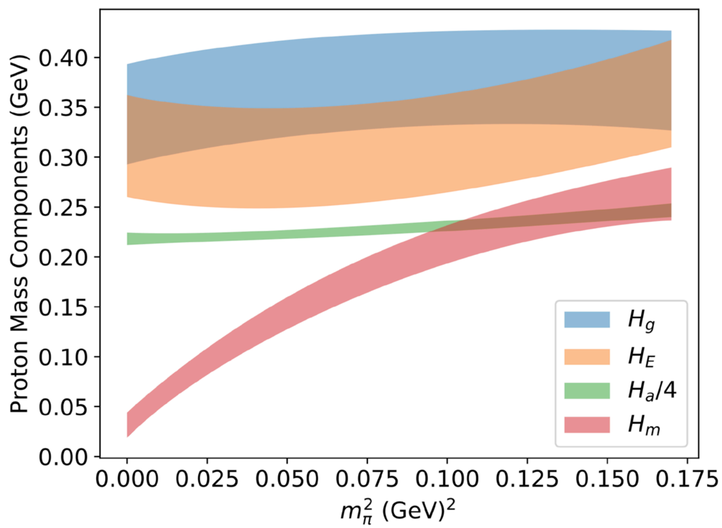




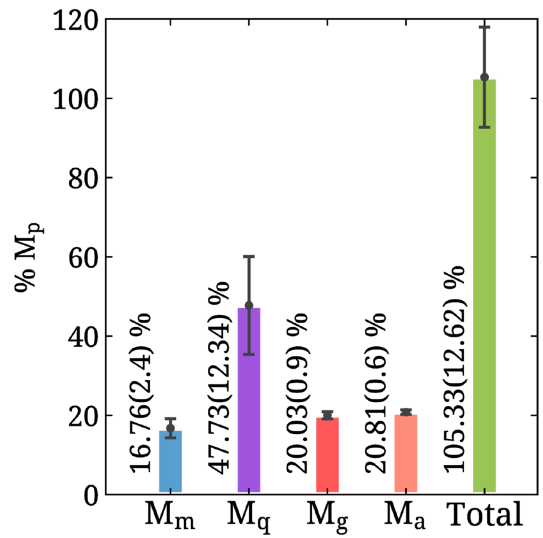


Proton mass on the lattice

To date no direct calculation of the trace anomaly



Y.-B. Yang *et al.*, (χQCD), PRL 121, 212001 (2018)



C. Alexandrou *et al.*, (ETMC), PRL 119, 142002 (2017)
C. Alexandrou *et al.*, (ETMC), PRL 116, 252001 (2016)

Trace anomaly only constrained through sum-rules not calculated directly.

Dynamical origin: scalar field & Higgs mechanism

Dynamical origin

- In the nucleon models, Λ_{QCD} is replaced by some scale related to dynamics.
  - Chiral symmetry breaking

    Scale set by chiral condensate or constituent quark masses
  - Color confinement

    MIT bag constant: energy density of false vacuum
  - Instanton liquid

    Scale related to instanton size and density
  - AdS/CFT…
- In lattice QCD, the scale is related to the lattice spacing, a.

Scalar energy, quantum anomalous energy (QAE)

- Trace relation for mass

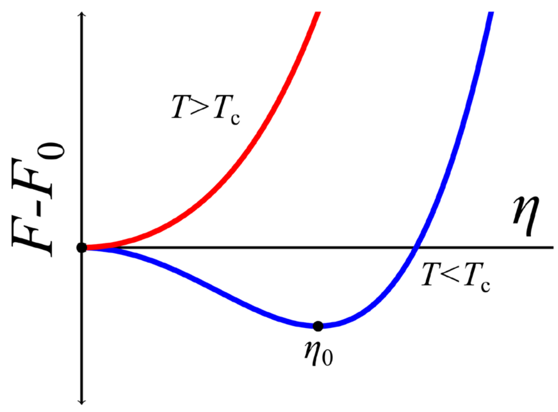
$$2M^2 = \left\langle P \left| (1+\gamma_m) m\bar{\psi}\psi + \frac{\beta(g)}{2g} F^2 \right| P \right\rangle$$

- Not a "total" mass sum rule.
- It is a sum rule for scalar part of the mass

$$E_s = \frac{M}{4} = \alpha\langle p|m\bar{\psi}\psi|p\rangle + \beta\langle p|F^2|p\rangle$$

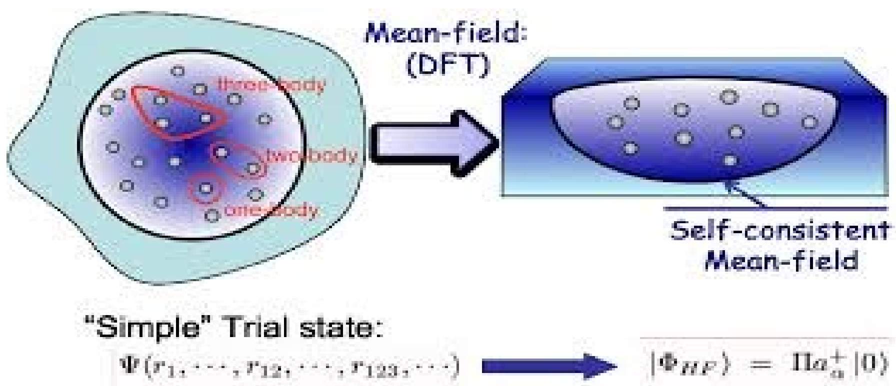
Energy of a scalar field

- Scalar field is special because it can have a vacuum condensate: non-vanishing expectation value in the physical vacuum. $\langle \eta \rangle \neq 0$, which stores the internal energy (latent heat)



Mean field theory for nuclear structure

- Traditional theory for nuclear structure: mean field theory

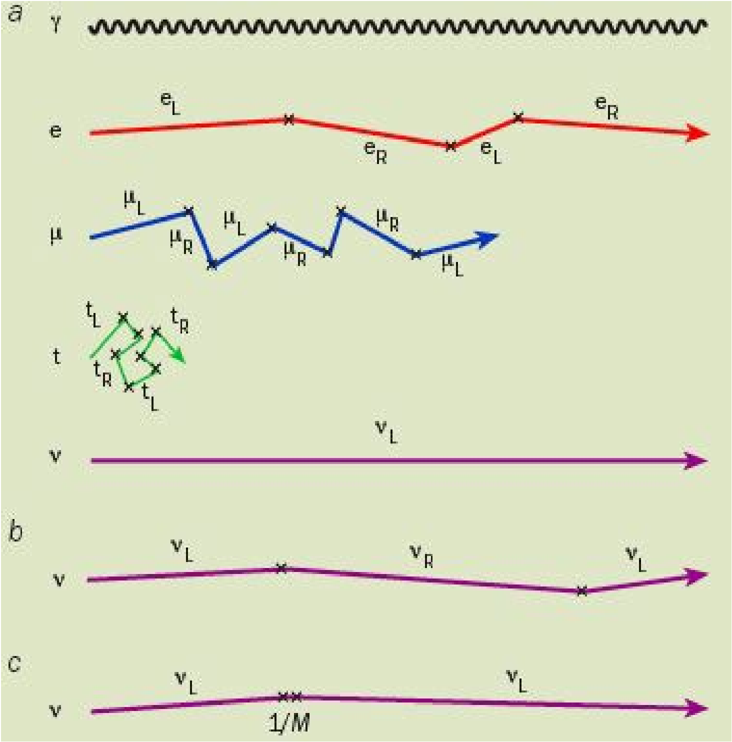


- Two important features:
  a) Mean field depth is about ~40 MeV
  b) Large spin-orbit splitting for nuclear shells.

Masses of electrons and leptons: Higgs mechanism

- There is a scalar field H, which interacts with the fermions $g\bar{\psi}\psi H$. H acquires an expectation value in the vacuum after SSB, $\langle H \rangle = v = 246$ GeV, hence the fermion mass,

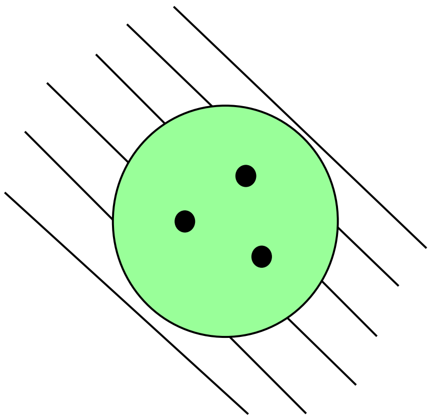
m= gv



Quantum anomalous energy (QAE) contribution to the proton mass:

- The scalar field has a VEV: $\langle 0|F^2|0\rangle$
- QAE comes from the scalar response to the presence of the quarks.

$$\phi = F^2 - \langle 0|F^2|0\rangle$$

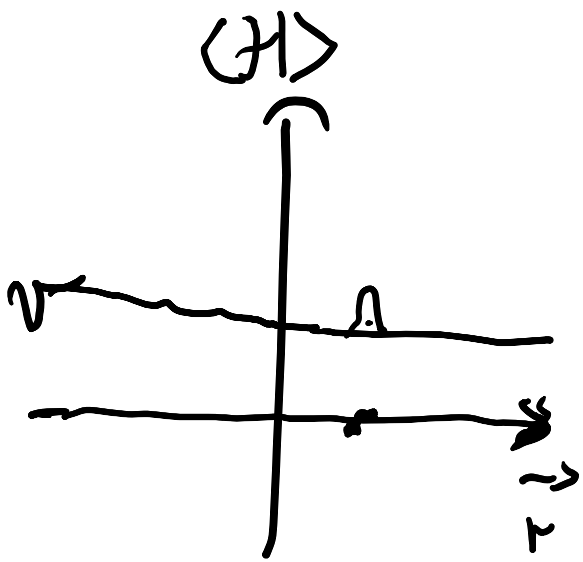


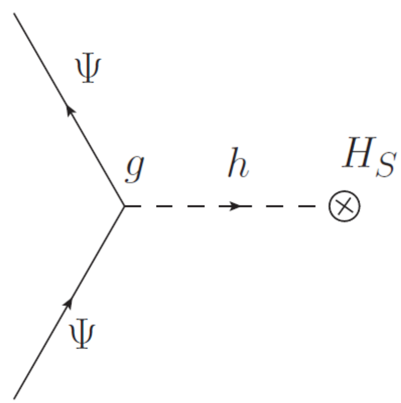
- The contribution is similar to the Higgs mechanism in electroweak theory, with gluon scalar as a dynamical Higgs field.

Dynamical picture of the fermion mass in Higgs mechanism

- Part of the fermion mass comes from the dynamical excitation of the higgs field in the presence of fermion

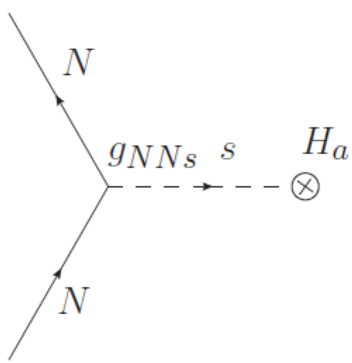
$$m_f \sim \langle f|H_S|f\rangle \sim \langle f|h|f\rangle$$





QAE as a dynamical response

- $E_a \sim \langle N|F^2|N\rangle$
- This matrix element can also be calculated through dynamical scalar excitations



$$\langle N|\phi|N\rangle = \sum_s \frac{g_{NNs} f_s}{m_s^2} \ .$$

Test of the QCD "Higgs mechanism"

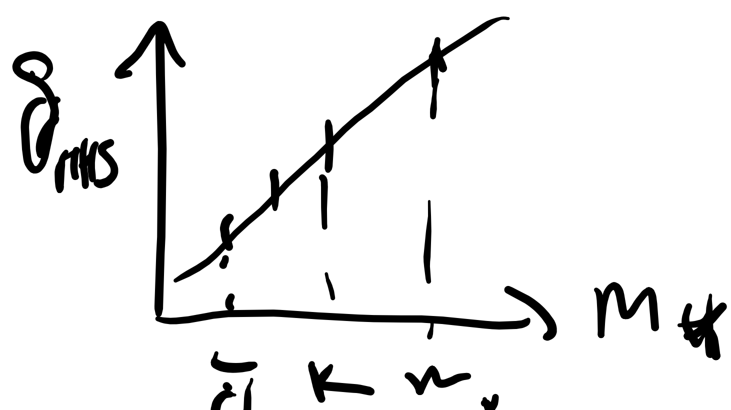
- The couplings of the scalars with the hadrons are proportional to the hadron masses.

$$g_{HHs} \sim m_H$$

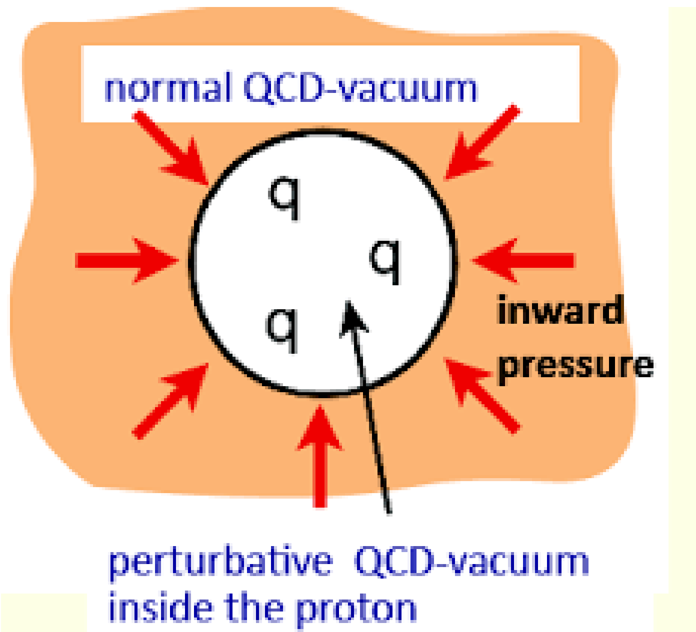
this also works for pion and kaon.

One can do the similar test as one does for Higgs particles at LHC but much more complicated

- Scalar spectrum



Physics of QAE in the MIT bag model



M.I.T. Bag Model

- The boundary condition generates discrete energy eigenvalues.

$$\varepsilon_n = \frac{x_n}{R}$$

R - radius of the Bag

x_1=2.04

$$E_{kin}(R) = N_q \frac{x_n}{R}$$

N_q = # of quarks inside the bag

$$E_{pot}(R) = \frac{4}{3}\pi R^3 B$$

B – bag constant that reflects the bag pressure

26

Mass = quark kinetic energy + B(scalar-field condensate)

Mass structure: gravitations form factors

- Form factors of EMT for quarks and gluons (Ji,1996)

$$\langle P'|T^{\mu\nu}_{q,g}|P\rangle = \overline{U}(P')[A_{q,g}(\Delta^2)\gamma^{(\mu}\overline{P}^{\nu)} + B_{q,g}(\Delta^2)\overline{P}^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha/2M + C_{q,g}(\Delta^2)(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2)/M + \overline{C}_{q,g}(\Delta^2)g^{\mu\nu}M]U(P)\,,$$

- Form factors for the total EMT (Pagels, 1966)

$$\begin{aligned}\langle P'\left|T^{\mu\nu}\right|P\rangle = \bar{u}\left(P'\right)\Big[& A\left(Q^2\right)\gamma^{(\mu}\bar{P}^{\nu)} \\ & + B\left(Q^2\right)\bar{P}^{(\mu}i\sigma^{\nu)\alpha}q_\alpha/2M \\ & + C\left(Q^2\right)\left(q^\mu q^\nu - g^{\mu\nu}q^2\right)/M\Big]u(P)\,,\end{aligned}$$

$A = A_q + A_g,\ B\ \&\ C\ etc.,\ \ \bar{C}_q + \bar{C}_g = 0$

Scalar form factor

- Form factor of the scalar density

$$\langle P' | T_\mu^\mu | P \rangle = \bar{u}(P') u(P) G_s(Q^2) ,$$

where,

$$G_s(Q^2) = \left[MA(Q^2) - B(Q^2)\frac{Q^2}{4M} + C(Q^2)\frac{3Q^2}{M} \right]$$

- Fourier transformation of Gs gives us the scalar field distribution inside the Nucleon
- Dynamical MIT "bag constant".

Scalar radius

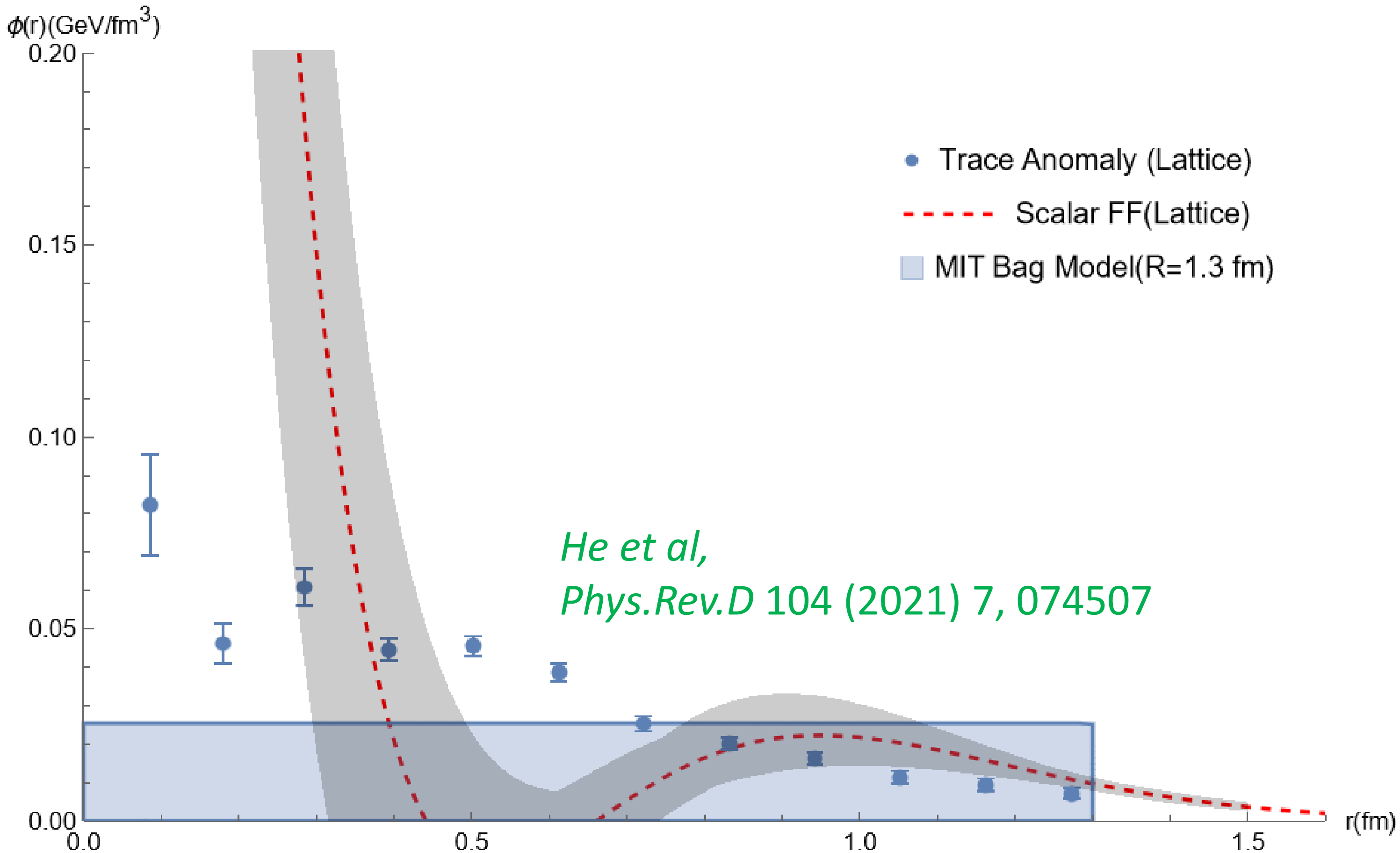
- Scalar field radius might be similar to confinement radius
- The radius

$$\langle r^2 \rangle_s = -6\frac{dA(Q^2)}{dQ^2} - 18\frac{C(0)}{M^2}$$

- MIT bag scalar radius

$$r_s^2 = \frac{3}{5}R^2, \ r_s = 1.3fm$$

Scalar field (QAE) distribution inside the proton



He et al,
Phys.Rev.D 104 (2021) 7, 074507

Mass form factor

$$\langle P' \left| T^{00} \right| P \rangle = \bar{u}\left(P'\right) u(P) G_m(Q^2) \ .$$

where

$$G_m(Q^2) = \left[M A\left(Q^2\right) - B(Q^2) \frac{Q^2}{4M} + C(Q^2) \frac{Q^2}{M} \right]$$

Scalar and mass radii

- Definition:

$$\langle r^2\rangle_{s,m} = -6\frac{dG_{s,m}(Q^2)}{dQ^2} ,$$

$$\begin{aligned}\langle r^2\rangle_s &= -6\frac{dA(Q^2)}{dQ^2} - 18\frac{C(0)}{M^2} \\ \langle r^2\rangle_m &= -6\frac{dA(Q^2)}{dQ^2} - 6\frac{C(0)}{M^2} ,\end{aligned}$$

- The difference

$$\langle r^2\rangle_s - \langle r^2\rangle_m = -12\frac{C(0)}{M^2}$$

- Conjecture $\langle r^2\rangle_s > \langle r^2\rangle_m$ or C(0)<0

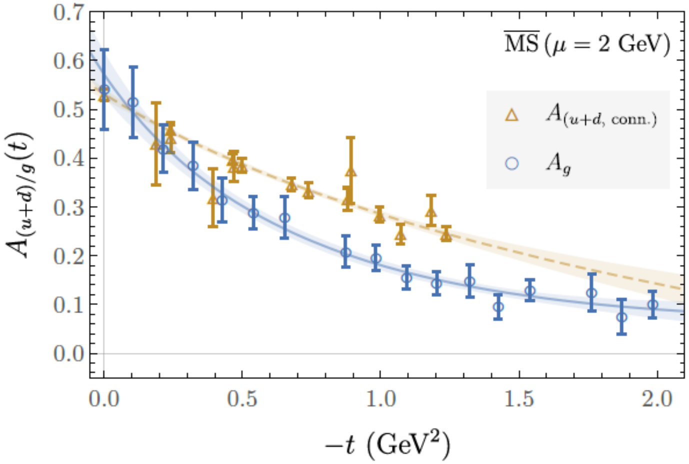
Lattice calculations

- Radius from A-FF:

  Hagler et al (2008)

  Shanahan et al (2018)

  $$\langle r^2 \rangle_A = (0.5\ fm)^2$$

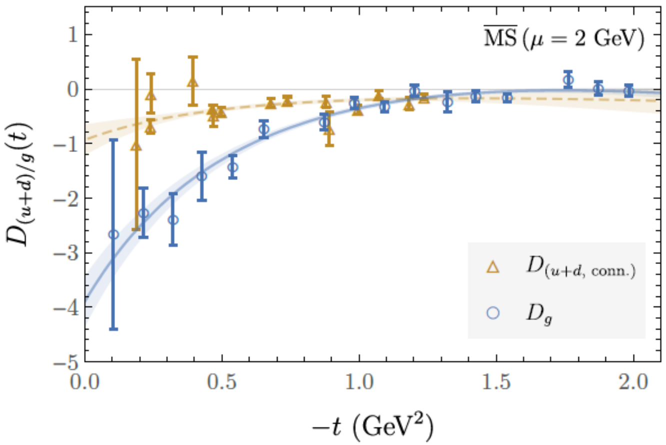


- C-FF contribution

  D = - 5.0

$$\langle r^2 \rangle_s = (1.1\ fm)^2$$

$$\langle r^2 \rangle_m = (0.75\ fm)^2$$



Physics of gravitational form factor C(D)

Momentum current & pressure

- Gravitational form factors

$$\langle P'|T^{\mu\nu}_{q,g}|P\rangle = \overline{U}(P')[A_{q,g}(\Delta^2)\gamma^{(\mu}\overline{P}^{\nu)} + B_{q,g}(\Delta^2)\overline{P}^{(\mu}i\sigma^{\nu)\alpha}\Delta_\alpha/2M + C_{q,g}(\Delta^2)(\Delta^\mu\Delta^\nu - g^{\mu\nu}\Delta^2)/M + \overline{C}_{q,g}(\Delta^2)g^{\mu\nu}M]U(P)\,,$$

- In the Breit frame, C and C-bar are related to the form factor of T^{ij}
- T^{ij} has been originally introduced as a stress tensor of fluids and solids, and is related pressure etc
- However, in relativistic theory, its definition is momentum (density) current

Momentum density current

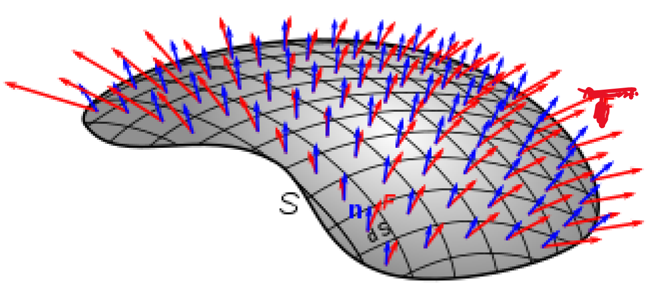
- T^{ij} is actually momentum density current (MC). It describes vector flow (j) of momentum component (i) or vice versa.
- The momentum density $p^i(\vec{\mathrm{r}}) = \mathrm{T}^{\mathrm{i0}}\,(\vec{\mathrm{r}})$ satisfies the conservation law

$$\frac{\partial p^i(\vec{r},t)}{\partial t} + \partial_j T^{ij} = 0$$

The flux of i-momentum following through a surface dS is just

$$F^i = \int T^{ij} dS_j$$

which can be + or -.



Questions about pressure interpretations

1. Is the pressure the same as we understand in a gas or liquid?
2. What does the negative pressure mean?
3. Does a proton need negative and positive pressure region to maintain its stability?
4. What the pressure is acting on?

….

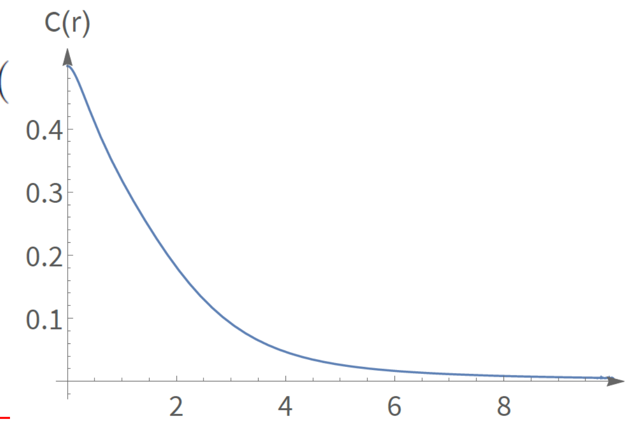
Stress tensor in H atom

By non-relativistic reduction of the Dirac equation, one can construct the following EMT $T^{ij}_{\rm QM}$ which consists of a kinetic term

$$T^{ij}_K = -\frac{1}{4m}\left(\phi^\dagger \partial^i \partial^j \phi - \partial^i \phi^\dagger \partial^j \phi + \mathrm{c.c}\right) \ , \qquad (94)$$

plus a potential term made of interacting electric fields of the proton and electron,

$$T^{ij}_V = \delta^{ij} \nabla V_p \cdot \nabla V_e - \partial^i V_e \partial^j V_p - \partial^i V_p \partial^j V_e \ . \qquad ($$



The trace of $T^{ij}_{\rm QM} = T^{ij}_V + T^{ij}_K$ can be calculated as

$$T^{ij}_{\rm QM}(\vec{r}) = (\delta^{ij}\nabla^2 - \nabla^i \nabla^j)\frac{C_{\rm QM}(r)}{M}$$

$$\boxed{\frac{C_{\rm QM}(r)}{M} = \frac{1}{2\nabla^2} T^{ii}_{\rm QM} = \frac{e^{-2\alpha r}\alpha(2\alpha r+1)}{16\pi r^2} - \frac{\alpha}{16\pi r^2}}$$

Review on multipoles of electric current $\vec{j}$

- For static system, one has the current conservation

$$\partial_i J^i = 0$$

- In momentum space (two D.O.F)

$$q_i J^i = 0$$

Thus, one has many vanishing moments,

$$\int d^3\vec{r}\ r_{(i_1} \ldots r_{i_l} j_{i)}(\vec{r}) = 0$$

Two independent series

- Magnetic multipoles

$$\tilde{V}^{(l)}_{ii_1\ldots i_l} \sim \int d^3\vec{r}\; m_i(\vec{r}) r_{(i_1}\ldots r_{i_{l-1})} \;,$$

$$\vec{m}(\vec{r}) = \vec{r} \times \vec{j}(\vec{r}) \;,$$

Most important: dipole moment or magnetic moment

- Longitudinal multipoles

moments of $\vec{r} \cdot \vec{j}$

which does not contribute to static E&M multipoles

Multipoles of momentum current

- Current conservation (only 3 DOF)

$$\partial_j T^{ij} = 0$$

- One general identity

$$\frac{1}{k!}\sum_P \int d^3\vec{r} T_{ii_{P(1)}} r_{i_{P(2)}} \dots r_{i_{P(k)}} = 0$$

or two vanishing series of moments

$$U^{(l+2)}_{iji_1\dots i_l} \equiv T_{(ij,i_1\dots i_l)} \ ,$$
$$\tilde{U}^{(l+1)}_{iji_1\dots i_l} \equiv \frac{2l}{l+2} T_{i[j,i_1]\dots i_l} \ ,$$

Three non-vanishing moment series:

- Scalar multipoles (“pressure” multipoles)

$$S(r) = T^{ii}(r)$$

$$S^J = \int d^3\vec{r}\, \mathrm{S}(\mathrm{r})\mathrm{r}_{r_1} \ldots \mathrm{r}_{\mathrm{i_j}}$$

($S^{(0)}=0$, however, scalar monopole density does not)

- Tensor multipoles (natural parity, “shear pressure”)

T^J from $\int d^3\vec{r}\, T_{ij}(\mathrm{r})\mathrm{r}_i \quad \ldots \mathrm{r}_{\mathrm{i_j}}$

- Tensor multipoles (unnatural parity)

$\tilde{T}^J$ from $\int d^3\vec{r}\, T_{i[j}(\mathrm{r})\mathrm{r}_{r_1]} \ldots \mathrm{r}_{\mathrm{i_j}}$

Form factors & multipoles: Scalar particle

- Form factor

$$\langle P'|T^{\mu\nu}|P\rangle$$
$$= 2P^\mu P^\nu A(q^2) + 2(q^\mu q^\nu - g^{\mu\nu} q^2)C(q^2) \ ,$$

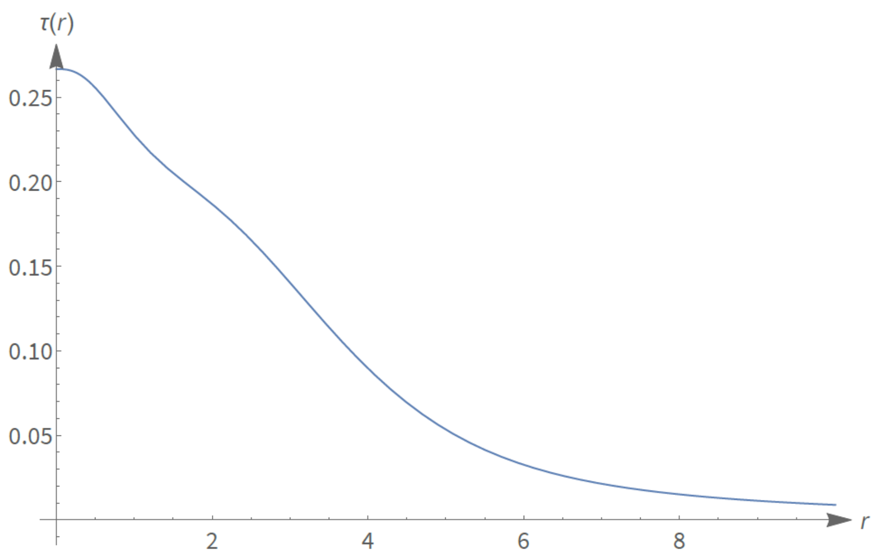
- Tensor monopole T0 ("shear flow")

$$T^{(0)} = \frac{1}{5}\int d^3\vec{r}T_{ij}(\vec{r})\left(r_i r_j - \frac{\delta_{ij}}{3}r^2\right)$$

normalization, $\tau = -T^{(0)}/2$ = D/4M (the D term)

Tensor monopole density

$$\tau(r) = -\frac{2\pi}{5} r^2 \left(r^i r^j - \frac{1}{3} r^2 \delta^{ij} \right) T_{ij}(r)$$



when integrated over, one gets, $\frac{\hbar^2}{4m}\,(1 + O(\alpha))$

Tensor monopole moment τ

- For a free boson

$$\tau_{\text{boson}} = -\frac{\hbar^2}{4M}$$

- It has been argued that the stable system must have τ negative.
- However, for H-atom, we find

$$\tau = \hbar^2/4M\ (1+O(\alpha))$$

Stability of a system shall not depend on momentum current properties. It is quantum mechanics!

Gravitational field from form factor C

- Linearized Einstein equation

$$\Box \bar{h}^{\mu\nu} = \frac{16\pi G}{c^4} T^{\mu\nu}$$

where $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}$ and $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{\eta^{\mu\nu}}{2} h^{\rho}_{\rho}$

- The solution with C form factor is

$$\boxed{\begin{aligned} h_C^{00}(\vec{r}) &= -\frac{8\pi G}{c^4 M} C(r) \\ h_C^{ij}(\vec{r}) &= \frac{8\pi G}{c^4 M} C(r) \delta^{ij} \end{aligned}}$$

- Given C(r) decays exponentially, so does the metric perturbation.

Spin-1/2 particle

- Form factors

$$\langle P' | T^{\mu\nu} | P \rangle = \bar{u}(P'S') \Big[A(q^2)\gamma^{(\mu}\bar{P}^{\nu)} + B(q^2)\frac{\bar{P}^{(\mu} i\sigma^{\nu)\alpha} q_\alpha}{2M} + C(q^2)\frac{q^\mu q^\nu - g^{\mu\nu} q^2}{M} \Big] u(PS) ,$$

- Apart from the angular momentum multipoles, one has the tensor monopole

$$\tau = \frac{C(0)}{M} .$$

C(0) shall be negative for the nucleon from a different reason.

Spin-1 particle

- Six form factors

$$
\begin{aligned}
&\langle P', \epsilon_f | T^{\mu\nu}(0) | P, \epsilon_i \rangle \\
&= -2\bar{P}^\mu \bar{P}^\nu \left[(\epsilon_f^\star \cdot \epsilon_i) A(q^2) + E^{\alpha\beta} q_\alpha q_\beta \frac{\tilde{A}(q^2)}{M^2} \right] \\
&+ J(q^2) \frac{i\bar{P}^{(\mu} S^{\nu)\alpha} q_\alpha}{M} \\
&- 2(q_\mu q_\nu - g_{\mu\nu} q^2) \left[(\epsilon_f^\star \cdot \epsilon_i) C(q^2) + E^{\alpha\beta} q_\alpha q_\beta \frac{\tilde{C}(q^2)}{M^2} \right] \\
&- \left[(E^{\mu\nu} q^2 - E^{\mu\alpha} q^\nu q_\alpha - E^{\alpha\nu} q^\mu q_\alpha + g^{\mu\nu} E^{\alpha\beta} q_\alpha q_\beta \right] D(q^2)
\end{aligned}
$$

- Mass quadrupole, tensor quadrupole, and scalar quadrupole:

$$T_{ij}^{(2)} = -\frac{\tilde{C}(0)}{48M^2} \hat{E}_{ij} \qquad \sigma_{ij} = \frac{D(q^2 = 0)}{M} \hat{E}_{ij}$$

Conclusions

- The mass sum rule with physical definition and symmetry is unique.
- The anomaly or scalar contribution to the mass of the proton acts like Higgs mechanism. It is important to measure mass and scalar radius
- The form factor C can best be characterized by gravtional tensor monopole density which generates specific type of gravity. (pressure and stability are not natural concepts in this context).

Questions

1. Is the pressure the same as we understand in a gas or liquid?
2. What does the negative pressure mean?
3. Does a proton need negative and positive pressure region to maintain its stability?
4. What the pressure is acting on?

....

What is the (usual) pressure?

- Pressures for non-interacting systems

Average kinetic energy either in thermal or quantum state

  - Pressure in ideal gas

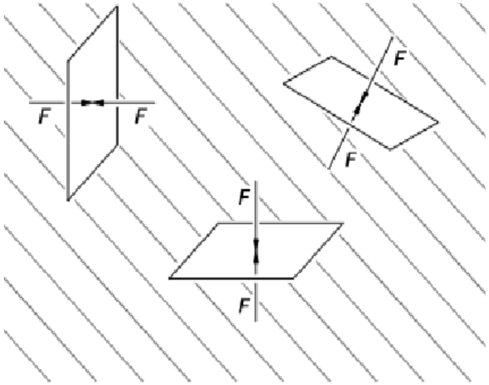
    $P = nk_BT = 2/3$ K.E.

  - Pressure in a quantum fermi gas

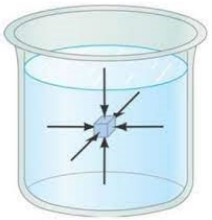
    $P = 2/3$ K.E.

  - Non-directional (locally in equilibrium)
  - Pressure is always non-negative

  unless in unstable phase.



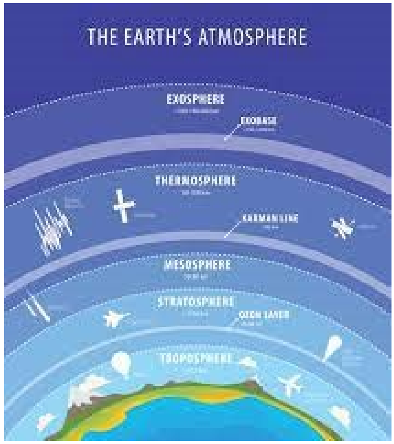
Pressure in a system with interactions : $P = -\frac{\partial F}{\partial V}$



- Consider a small test volume δV in thermal equilibrium $(M.F.P. \ll (\delta V)^{\frac{1}{3}})$, the interaction range λ
- Short-range interaction, $\lambda \ll (\delta V)^{1/3}$

  repulsive interaction: p↑

  attractive interaction: p↓
- Long-range interaction, $\lambda \gg (\delta V)^{1/3}$
  - The interaction is not part of the pressure.
  - For a confining system (atmosphere on earth) the pressure goes to zero!



Energy-momentum tensor and pressure

- What has been calculated or measured is related to the energy-momentum tensor in space

$$T^{\mu\nu}\,(\vec{r}) \quad (\mu,\nu = 0,1,2,3)$$

- It is well-known that in the ideal gas/fluid model

$$T^{\mu\nu} = \begin{pmatrix} \varepsilon & & & 0 \\ & p & & \\ & & p & \\ 0 & & & p \end{pmatrix}$$

where p is the normal pressure in the gas. What has been measured is the analogue:

$$p = \frac{1}{3}\,(T^{11}+T^{22}+T^{33}).$$

This is not the usual pressure

- This pressure does not follow from the usual definition, but from an analogy that the nucleon is some sort of fluid of quarks and gluons.
- There is nothing wrong with a definition, but one needs to be careful (cannot be literal) when interpreting it.
- This pressure takes into account long-range confinement interactions, unlike the example of atmosphere on the earth: It is not just the kinetic energy of quarks and gluons.

The meaning of p(r)

- $p = \frac{1}{3}(T^{11}+T^{22}+T^{33})$

is the average of

the momentum flow (or current) in the x-direction for the momentum component-x, which can be positive or negative

with

the momentum flow (or current) in the y-direction for the momentum component y, which can be positive or negative

and

the momentum flow (or current) in the z-direction for the momentum component z, which can be positive or negative

Answer for Q2: negative pressure?

- Since the momentum flow pattern in a system can be rather arbitrary (no particular constraint), there is no reason that the “pressure” defined as such be positive definite.
- It is just a definition ☺
- Note, however, that in real fluid $T^{ii} > 0$ everywhere.

Answer for Q3: stability

- Laue condition for stability

$$\int d^3\vec{r}\, p(r) = 0$$

p(r) must be positive and negative.

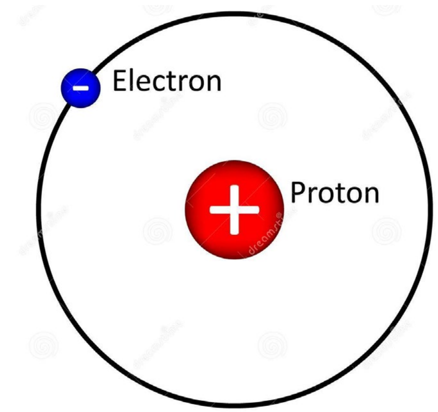
- However, this follows directly from current conservation $\partial_j T^{ij} = 0$:

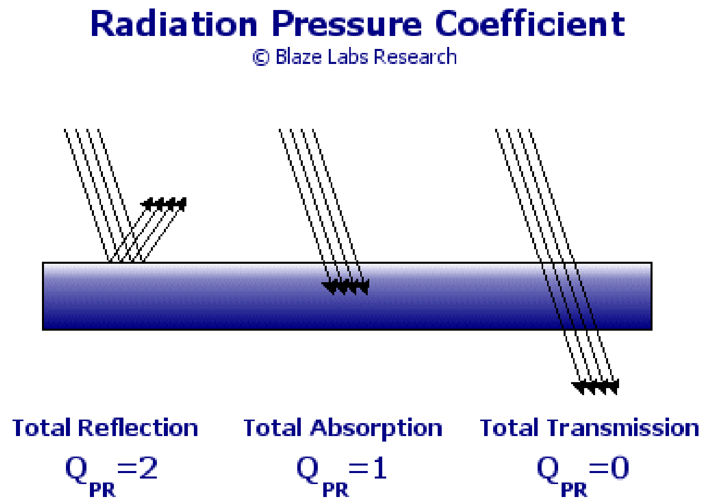
$$\int d^3\vec{r}\, T^{ij} = 0$$
$$\int d^3\vec{r}\, r^{(k} r^l \ldots r^m T^{ij)} = 0$$

- Stability of a quantum system is guaranteed by quantum mechanics (no classical equivalent)

Q4: What is the pressure acting on?

- Consider H-atom, with one electron there is a momentum current flow, the pressure is certainly not on the electron.
- T^{ij} is about the motion pattern, not acting on other parts of the system
- One imagines a fictious surface which intersects the momentum current.
  - Normal definition of the pressure assumes it bounces
  - Here one assumes the current gets absorbed (Q=1)





My current understanding

- The pressure or shear pressure are introduced to characterize a momentum flow pattern.
- One shall not over-interpret them literally (mechanical stability etc.)