Modified Villain formulation of abelian Chern-Simons theory

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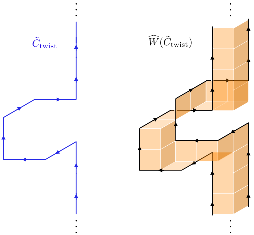


work to appear

with Tin Sulejmanpasic

Ubiquity of (abelian) Chern-Simons theory

- most general abelian 2+1d topological order: $U(1)^K$ (Wen, Zee)
  - integer/fractional quantum hall effects
- chiral/parity anomaly
- level-rank/boson-fermion dualities

In the continuum, $U(1)$ Chern-Simons theory is incredibly simple!

$$“S = \frac{ik}{4\pi} \int a \wedge da”$$

No nontrivial local operators, just topological Wilson lines

Naively a TQFT doesn't need a UV completion

Why discretize Chern-Simons theory?

Better understand subtleties of the continuum

- framing anomaly
- monopoles
- level quantization

literature dating back to late '80s
missed interesting global aspects

Establish exact boson/fermion dualities

- extend exact particle-vortex duality on the lattice
- existing lattice studies (e.g. $\psi \leftrightarrow \mathsf{XY} + U(1)_1$)
  implements CS term with massive fermions

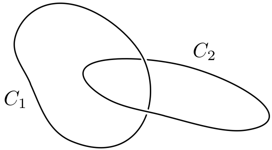
  (Chen, Son, Wang, Raghu '17, Chen, Zimet '18)

Interesting because it is not obvious it can be done

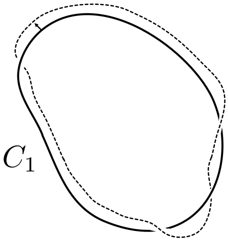
Continuum subtleties: framing

Gauss formula for linking #

$$\langle W(C_1)W(C_2)\rangle = \exp\left[\frac{2\pi i}{2k}\sum_{i,j=1,2}\Phi(C_i,C_j)\right]$$



Ambiguous when $i = j$: self-linking requires point-splitting Polyakov '88 Witten '89



Natural to try to resolve on the lattice!

But can it be done using a **local** action?

Lattice subtleties: extra zero modes

Long history of lattice CS/Maxwell-CS theories Fröhlich + Marchetti '89, Lüscher '89, Müller '90, Eliezer + Semenoff '92

- most constructions feature extra zero modes

Hamiltonian formulation

- detrimental: cannot determine commutation relations

But: can be lifted with a suitable choice of action

Eliezer, Semenoff '92

Euclidean formulation

- generic for local, gauge-invariant, parity-odd action

  Berruto, Diamantini, Sodano '00

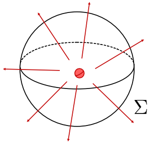
  analogies with Nielson-Ninomiya

Lattice subtleties: monopoles

Older literature did not distinguish between $U(1)$ and $\mathbb{R}$

$\mathbb{R}$ gauge theory: $\frac{1}{2\pi}\int_\Sigma da = 0$ $U(1)$ gauge theory: $\frac{1}{2\pi}\int_\Sigma da \in \mathbb{Z}$

if Σ contractible,
indicates the presence
of a monopole



conventional discretizations of $U(1)$ gauge theory
come with finite-action monopoles

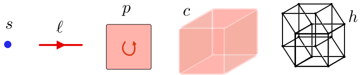
This is problematic: monopoles are **not** gauge-invariant
in the presence of a CS term!

Pisarski '86
Affleck, Harvey, Palla, Semenoff '89

Obstructions to lattice CS?

Issue	Resolution
Wilson loops require framing	Definition of lattice action requires a global 'framing' choice: Wilson loops are literally ribbons
Extra zero modes	Zero modes project out all unframed Wilson lines!
Lattice-scale monopoles spoil gauge-invariance	Systematically remove using 'modified Villain' formalism (Gattringer, Sulejmanpasic '19 Gorantla, Lam, Seiberg, Shao '21)

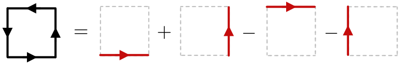
Lattice preliminaries



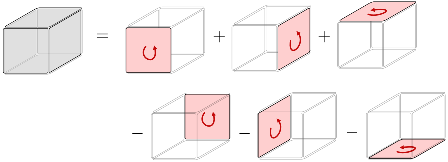
$$\varphi_s, a_\ell, n_p = \text{0-, 1-, 2-forms}, \qquad d = \text{lattice exterior derivative}$$

$(d\varphi)_\ell$

$(da)_p$

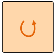


$(dn)_c$



Villain formulation of $U(1)$ gauge theory

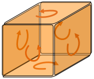
algebra-valued link fields $a_\ell \in \mathbb{R}$

\+ discrete plaquette fields $n_p \in \mathbb{Z}$ 

1-form gauge redundancy makes a compact: $U(1) = \mathbb{R}/2\pi\mathbb{Z}$

$$a_\ell \to a_\ell + (d\lambda)_\ell + 2\pi\, m_\ell,\;\; n_p \to n_p + (dm)_p,$$
$$\lambda \in \mathbb{R},\;\; m_\ell \in \mathbb{Z}$$

$$\frac{1}{2\pi}\int_\Sigma da \in \mathbb{Z} \implies \frac{1}{2\pi}\sum_{p\in\Sigma}\Big[(da)_p - 2\pi n_p\Big] = -\sum_{p\in\Sigma} n_p \in \mathbb{Z}$$

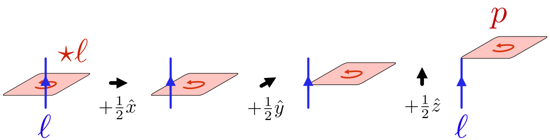
monopole:  $(dn)_c \neq 0$

Lattice building block: cup product

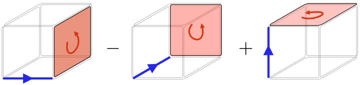
Naive action: $\frac{ik}{4\pi}\int a\wedge da \implies \frac{ik}{4\pi}\sum_c a\cup da$ Fröhlich & Marchetti '89

$\cup$ = lattice analog of $\wedge$ in de Rham cohomology:

$$\alpha^p\cup\beta^q=(\alpha\cup\beta)^{p+q}\,,\quad d(\alpha^p\cup\beta^q)=d\alpha^p\cup\beta^q+(-1)^p\,\alpha^p\cup d\beta^q$$



$(\alpha^{(1)}\cup\beta^{(2)})_c$ =



(cup products break lattice rotational invariance)

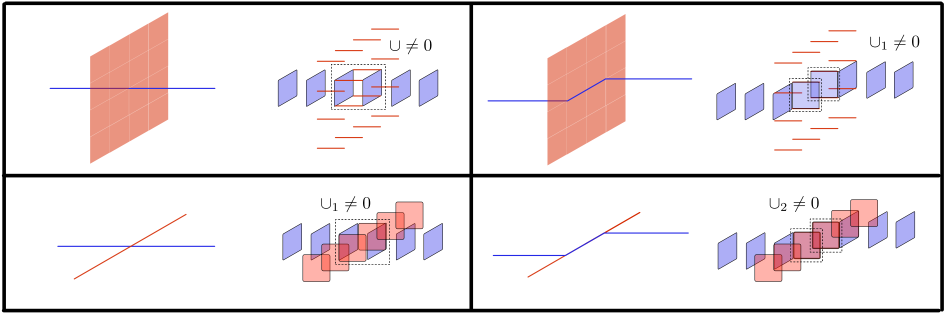
New ingredient: higher cup products

Unlike $\wedge$ product, $\cup$ is not graded commutative

Instead: $$\alpha \cup \beta = \beta \cup \alpha + d\big(\alpha \cup_1 \beta\big) + d\alpha \cup_1 \beta + \alpha \cup_1 d\beta$$

$$\alpha^p \cup_i \beta^q = (\alpha \cup_i \beta)^{p+q-i}$$

On simplicial complexes in algebraic topology (Steenrod, '47)
On the **hypercubic** lattice only constructed recently (Chen, Tata '21)



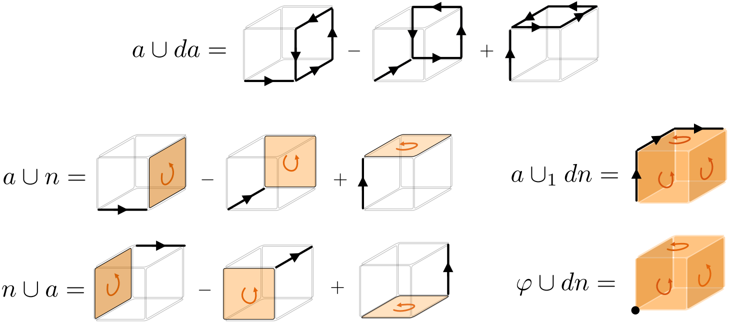
Lattice action

$$S(a,n,\varphi) = \sum_c \frac{ik}{4\pi} a \cup da - \frac{ik}{2}(a \cup n + n \cup a) - \frac{ik}{2} a \cup_1 dn + i\varphi \cup dn$$

naive lattice action

terms which ensure large gauge invariance

terms involving monopoles



Large gauge invariance and level quantization

$$S(a,n,\varphi) = \sum_c \frac{ik}{4\pi} a \cup da - \frac{ik}{2}(a \cup n + n \cup a) - \frac{ik}{2} a \cup_1 dn + i\varphi \cup dn$$

naive lattice action | terms which ensure large gauge invariance | terms involving monopoles

Large gauge transformation: $a \to a + 2\pi m, n \to n + dm$:

$$\delta S = -ik\pi \sum_c (m \cup dm + m \cup n + n \cup m + m \cup_1 dn)$$
$$\in k\pi i\,\mathbb{Z}\,.$$

Level quantization: $k \in 2\mathbb{Z}$

(odd levels require more ingredients)

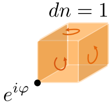
Electric charge of monopoles

$$\sum_c \frac{ik}{4\pi} a \cup da - \frac{ik}{2}(a \cup n + n \cup a) - \frac{ik}{2} a \cup_1 dn + i\varphi \cup dn$$

First two terms only invariant under $a \to a + d\lambda$ if $dn = 0$:

Modified Villain formulation: integrating out φ removes monopoles

$\mathcal{M} = e^{i\varphi}$ is a **monopole operator**:



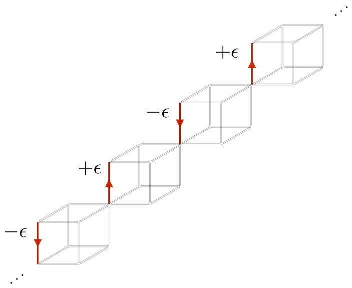
Keep φ explicitly if we assign it gauge charge: $\varphi \to \varphi - k\lambda$

$\Longrightarrow$ Monopole operators have electric charge k:

$$\mathcal{M} \to e^{-ik\lambda}\mathcal{M}$$

Staggered symmetry and zero modes

Action is invariant under staggered shifts $a_\ell \to a_\ell + \epsilon_\ell$



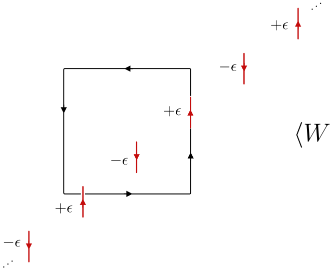
'zero modes' noticed in the older literature

View as symmetry: which operators are **charged?** which are **neutral?**

Selection rules for Wilson loops

Start with an ordinary Wilson loop: $\langle W(C)\rangle = \int D[\text{fields}]\, e^{i\sum_{\ell\in C} a_\ell}\, e^{-S}$

choose staggered shift
to cross C



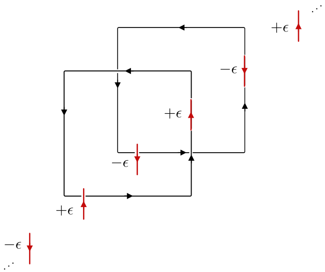
$\langle W(C)\rangle = e^{i\epsilon}\,\langle W(C)\rangle$

Wilson loops of any size identically vanish: $\langle W(C)\rangle = 0$

acts like a gauge redundancy: projects out operators

Selection rules for Wilson loops

Insert a second Wilson loop slightly displaced:



Doubled Wilson loops are neutral and survive!

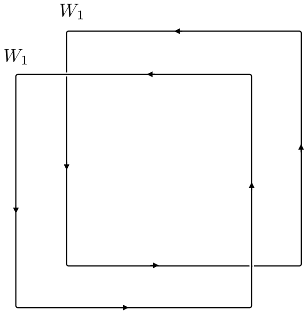
(these are the *only* operators that survive)

Wilson loops are ribbons

$U(1)_k$ CS theory has $\mathbb{Z}_k$ 1-form symmetry:
$$a \to a + \frac{2\pi}{k}\omega,\ d\omega = 0$$

Wilson lines are the charged operators
$$\langle W \rangle \to e^{\frac{2\pi i}{k}} \langle W \rangle$$

Doubled Wilson loops have **twice** the minimal charge at long distances:



Wilson loops are ribbons

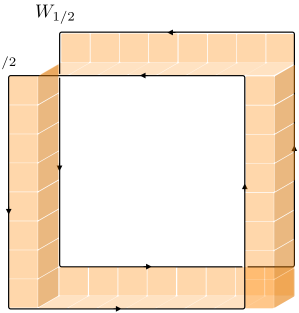
$U(1)_k$ CS theory has $\mathbb{Z}_k$ 1-form symmetry:

$$a \to a + \frac{2\pi}{k}\omega,\ \ d\omega = 0$$

Wilson lines are the charged operators

$$\langle W \rangle \to e^{\frac{2\pi i}{k}} \langle W \rangle$$

Minimal charge Wilson loops are **ribbons:** edges connected by a surface

$\widehat{W}(\tilde{C}) =$ 

$\cup_1$ product is crucial:

$$\widehat{W}(\tilde{C}) = \exp\left(\frac{i}{2}\sum_c a \cup \delta[\tilde{C}]\right)$$

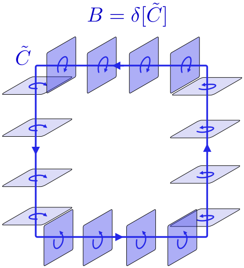
$$\times \exp\left(\frac{i}{2}\sum_c \delta[\tilde{C}] \cup a\right)$$

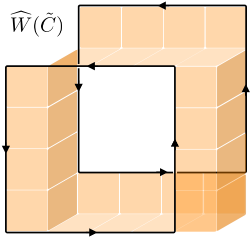
$$\times \exp\left(i\pi \sum_c \delta[\tilde{C}] \cup_1 n\right)$$

Wilson loops are topological

Wilson lines are also the **charge** operators of the 1-form symmetry

2-form background gauge field for 1-form symmetry: $B = \delta[\tilde{C}]$ $\Leftrightarrow$ Topological, framed Wilson loop on curve $\tilde{C}$





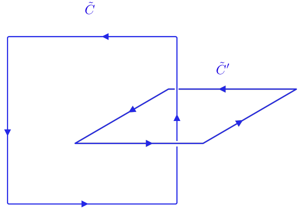
background gauge transformations $B \to B + dV$ $\Leftrightarrow$ deformations of loop $\tilde{C}$

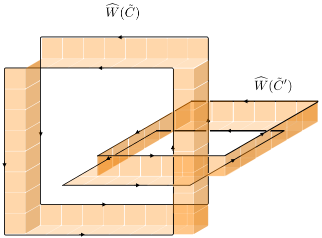
’t Hooft anomaly

Under background gauge transformations

$$B \to B + dV$$

partition function transforms with phase $\Leftrightarrow$ Wilson loops $\tilde{C}$ can be deformed **up to** phases



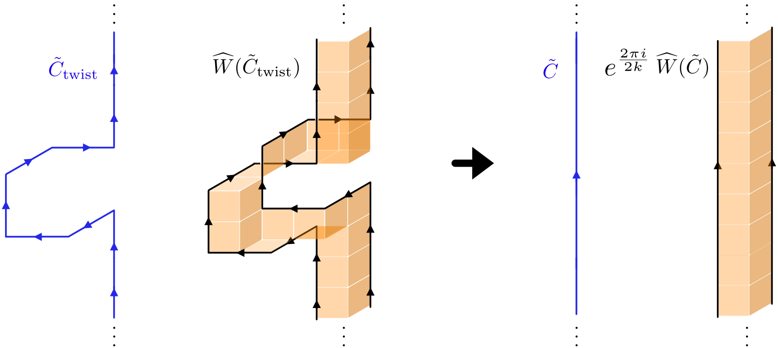


$$\langle \widehat{W}(\tilde{C})\widehat{W}(\tilde{C}')\rangle = e^{\frac{2\pi i}{k}} \langle \widehat{W}(\tilde{C})\rangle\langle \widehat{W}(\tilde{C}')\rangle$$

familiar linking relation

Topological spin

We can identify twisted Wilson loops by their anomalous phase:



'Topological spin' $s = \frac{1}{2k}$ determined by self-linking number

Definition via 4d θ term

$$S_\theta = \sum_h \frac{i\theta}{8\pi^2}(da - 2\pi n) \cup (da - 2\pi n) \qquad \leftarrow \text{naive lattice version of } F \wedge F$$

$$- \frac{i\theta}{4\pi}(da - 2\pi n) \cup_1 dn + \frac{i\theta}{2\pi} a \cup dn + ib \cup dn$$

b = **magnetic** gauge field = Lagrange multiplier removing monopoles

Witten effect: $b \to b - \frac{\theta}{2\pi}(d\lambda + 2\pi m)$

Action *density* is gauge-invariant $\implies$ can consider a lattice X with boundary

$$S_{\theta=2\pi k}(a, n, b = d\varphi)\Big|_X = S_{\mathsf{CS}}(a, n, \varphi)\Big|_{\partial X}$$

φ is a dyonic Higgs field

w/ fixed bulk lattice $X \simeq T^2 \times D$ we must condense dyons in the bulk to reproduce all fluxes on boundary

Anomaly inflow

4d θ term with $\theta = 2\pi k$ is a SPT phase protected by 1-form symmetry

Partition function is $= 1$ on a closed manifold without background fields, but:

$$S_{\theta=2\pi k}(B) = \frac{2\pi i}{2k} \sum_h B \cup B + B \cup_1 dB = \frac{2\pi i}{2k} \sum_h \mathcal{P}(B)$$

$\mathcal{P}$ = ‘Pontryagin square’ maps $H^2(M, \mathbb{Z}_k) \times H^2(M, \mathbb{Z}_k) \to H^4(M, \mathbb{Z}_{2k})$

Provides anomaly inflow for ’t Hooft anomaly for 1-form symmetry

Summary:

We discretized $U(1)_k$ ($k \in 2\mathbb{Z}$) CS theory on a cubic spacetime lattice with global structure manifest:

- small/large gauge invariance and level quantization
- explicit monopole operators with electric charge
- $\mathbb{Z}_k$ 1-form symmetry and 't Hooft anomaly
- framing of Wilson loops: natural observables are ribbons!

(bonus: new form of lattice theta term)

Future directions/questions

Villain Hamiltonians for CS and 4d θ term

incorporate Villain variables into Eliezer-Semenoff construction

formulate odd k theory

requires explicit spin structure dependence

apply to axion-Maxwell theory

$\chi\, F \wedge F$

non-invertible defects using CS construction?

edge modes on lattice with boundary

gravitational anomaly?

compare partition function with different choices of cup products

Monopole operators