Non-invertible symmetries and Goldstone modes

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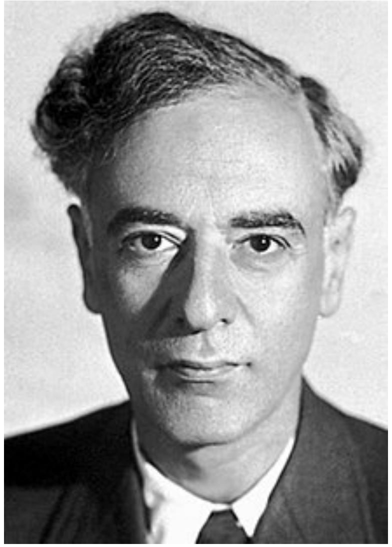
The (old) Landau paradigm

What is the Landau paradigm?



1. Phases of matter are classified by their patterns of broken and unbroken symmetries.
2. Critical points between phases can be studied by universal theories of the "order parameter".

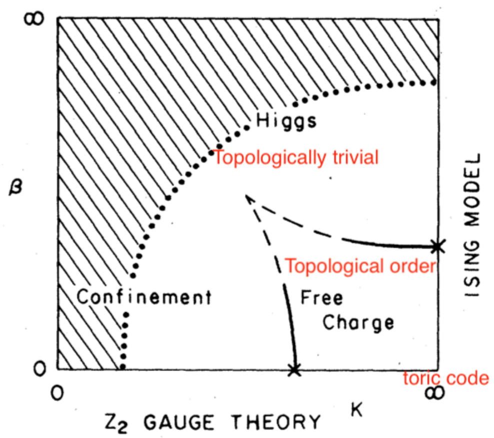
This works spectacularly well for many phases of matter, and is the foundation of textbook many-body physics.

However….

Beyond the (old) Landau paradigm

Much modern work involves phases and transitions between them that do not fit into this paradigm, e.g.

1. Massless QED! (the focus of this talk).
2. Topological order (many other talks in this workshop).
3. Many others… (deconfined criticality; (non)-Fermi liquids; etc. etc…)



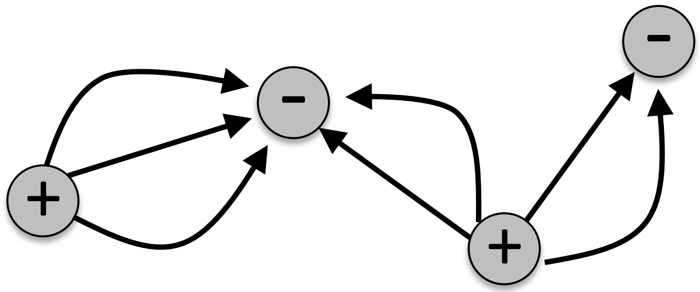
Can we do better?

Symmetries of QED

In this talk, I'll discuss some new ways to think about a very familiar system: massless QED.

$$S = \int d^4x \left[-\frac{1}{g^2} F^2 + \bar{\psi} \left(\partial\!\!\!/ + A\!\!\!/ \right) \psi \right]$$

Consists of massless fermions interacting via electric fields.



What are the symmetries, exactly?

Symmetries of QED

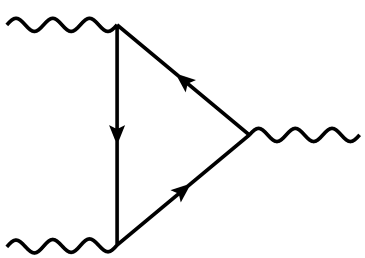
First, let us consider freezing EM; view A as an external gauge field.

$$S = \int d^4x \left[-\frac{1}{g^2} F^2 + \bar{\psi} \left(\partial\!\!\!/ + A\!\!\!/ \right) \psi \right]$$

Global symmetries are very well-understood:

$$j_V^\mu = \bar{\psi} \gamma^\mu \psi \qquad \partial_\mu j_V^\mu = 0$$

$$j_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi \qquad \partial_\mu j_A^\mu = \frac{1}{4\pi^2} F \wedge F$$



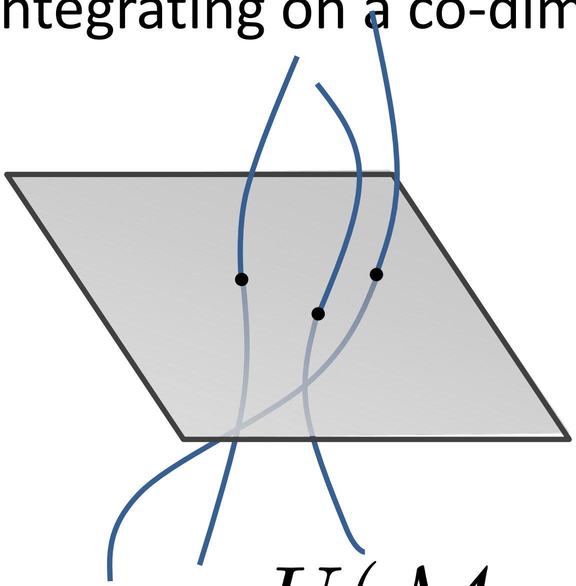
't Hooft anomaly: operator on RHS is a fixed external source. What about dynamical EM?

Generalizations of symmetry

To understand this, let's review ordinary 1-index currents first:

$$\nabla_\mu j^\mu = 0 \qquad d \star j = 0$$

An ordinary current counts particles; "catch them all" by integrating on a co-dimension 1 subspace:



$$Q = \int_{\mathcal{M}_{d-1}} \star j$$

Defines a U(1)-valued topological codimension-1 surface operator.

$$U(\mathcal{M}_{d-1}) = \exp(i\alpha Q(\mathcal{M}_{d-1}))$$

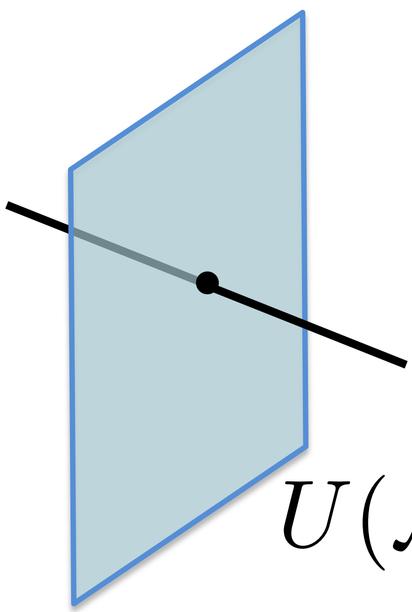
(Fancy way to talk about "conserved charge"; call this a 0-form symmetry)

Higher form symmetries

Now a 2-index current (Gaiotto, Kapustin, Seiberg, Willet):

$$\nabla_\mu J^{\mu\nu} = 0 \qquad d \star J = 0$$

A 2-index current counts strings; as they don't end in space or time, "catch them all" by integrating on a co-dimension 2 subspace:



$$Q = \int_{\mathcal{M}_{d-2}} \star J$$

Local current defines a U(1)-valued topological codimension-2 surface operator.

$$U(\mathcal{M}_{d-2}) = \exp(i\alpha Q(\mathcal{M}_{d-2}))$$

This is called a 1-form symmetry.

Symmetries of QED I

Now consider the situation where A is dynamical.

$$S = \int d^4x \left[-\frac{1}{g^2} F^2 + \bar{\psi} \left(\partial\!\!\!/ + A\!\!\!/ \right) \psi \right]$$

Vector current is gauged; no longer a global symmetry.

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi \qquad \partial_\mu j_V^\mu = 0$$

New 1-form global symmetry associated with conservation of magnetic flux.

$$J^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \qquad \nabla_\mu J^{\mu\nu} = 0$$

No magnetic monopoles means that magnetic field lines don't end. $J^{\mu\nu}$ counts magnetic flux density. (e.g. 4d photon is a Goldstone mode!)

Symmetries of QED II

$$S = \int d^4x \left[-\frac{1}{g^2} F^2 + \bar{\psi} \left(\partial\!\!\!/ + A\!\!\!/ \right) \psi \right]$$

How about the axial current?

$$j_A^\mu = \bar{\psi} \gamma^\mu \gamma^5 \psi$$

$$\partial_\mu j_A^\mu = \frac{1}{4\pi^2} F \wedge F = -\frac{1}{4\pi^2} J \wedge J$$

Now the right hand side of the anomaly equation is an operator, not a fixed source; it appears that the current is simply not conserved?

True situation is somewhat more subtle…

Symmetries of QED II

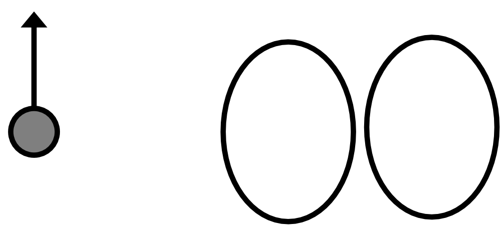
Summary of symmetries in massless QED:

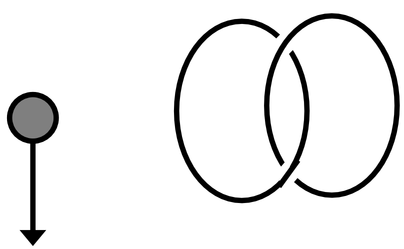
$$\partial_\mu J^{\mu\nu} = 0$$

$$\partial_\mu j_A^\mu = -\frac{1}{4\pi^2} J \wedge J$$

1-form symmetry for magnetic flux conservation.

Non-conservation of axial charge given in terms of 2-form current.



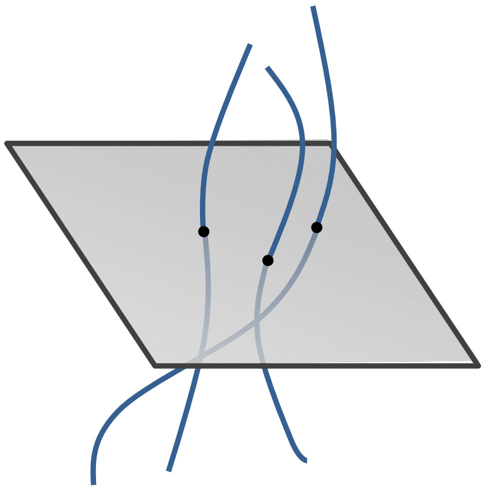


Right way to think about this was explained recently in terms of non-invertible symmetry (Choi, Lam, Shao; Cordova, Ohmori).

Non-invertible symmetry in QED

No conserved gauge-invariant current. Conserved charge?

$$Q(\mathcal{M}_3) = \int_{\mathcal{M}_3} \left(\star j_A - \frac{1}{4\pi^2} A \wedge dA \right)$$



Try and make a topological surface operator like before:

$$U(\mathcal{M}_3) = \exp\left(i\alpha Q(\mathcal{M}_3)\right)$$

This is gauge-invariant under small gauge transformations, but not large ones, unless α is 2π integer – but then the operator is always 1.

Feels like there is no useful conserved charge...

Non-invertible symmetry in QED

Try to make fractional α = 2***π*** /N:

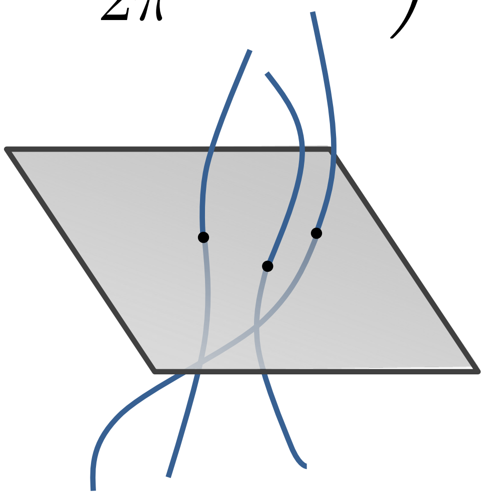
$$\frac{2\pi}{N}Q(\mathcal{M}_3) = \frac{2\pi}{N}\int_{\mathcal{M}_3}\left(\star j_A - \frac{1}{4\pi^2}A\wedge dA\right)$$

Idea (Choi et al; Cordova et. al) : let's introduce a new dynamical field *a* defined only on the defect! Then we can write:

$$\frac{2\pi}{N}Q(\mathcal{M}_3) = \int_{\mathcal{M}_3}\left(\frac{2\pi}{N}\star j_A - \frac{N}{4\pi}a\wedge da + \frac{1}{2\pi}a\wedge dA\right)$$

"Integrating out *a*" gives us the top line: but now everything is gauge-invariant for integer N!

Constructed a topological charge operator, at the cost of introducing a new field *a*.

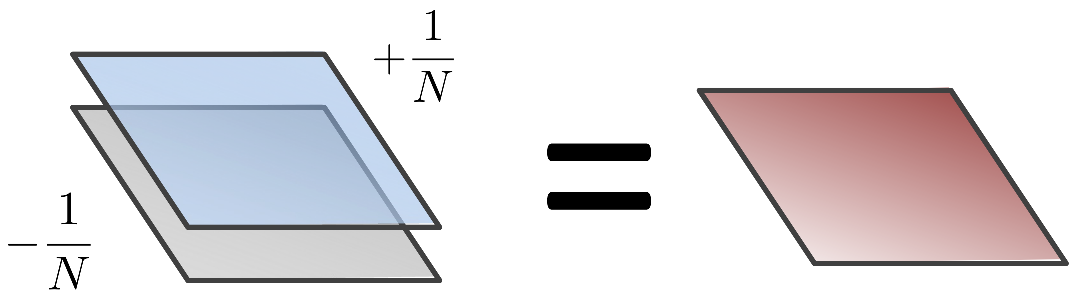


Some remarks

$$\frac{2\pi}{N}Q(\mathcal{M}_3) = \int_{\mathcal{M}_3}\left(\frac{2\pi}{N}\star j_A - \frac{N}{4\pi}a\wedge da + \frac{1}{2\pi}a\wedge dA\right)$$

$$\psi \to e^{\frac{2\pi i}{N}\gamma^5}\psi$$

- Replace axial phase rotation *1/N* with any rational number by using a slightly fancier TQFT. Correct way to think about axial symmetry in QED.
- Called non-invertible symmetry (much recent work!), as it has no inverse; acting with the opposite charge doesn't give a trivial operator.



- No local conserved current! But the topological operators are useful: selection rules, constraints on effective actions, etc.
- Opens up many doors: what is the phase structure of these symmetries? Is there a Goldstone theorem? Can we do hydrodynamics? Etc. etc.

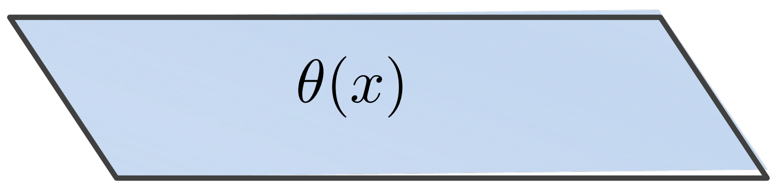
Another way

Another way to construct a topological charge operator (I. García-Etxebarria, NI; A. Karasik):

$$U_\alpha(\mathcal{M}_3) = \int [d\theta] \exp\left(i\alpha \int_{\mathcal{M}_3} \left(\star j_A - \frac{1}{4\pi^2}(A - d\theta) \wedge dA \right) \right)$$

$$\star\tilde{j}$$

Introduce a compact scalar field θ on the defect:



Still topological and non-invertible; difference with previous construction:

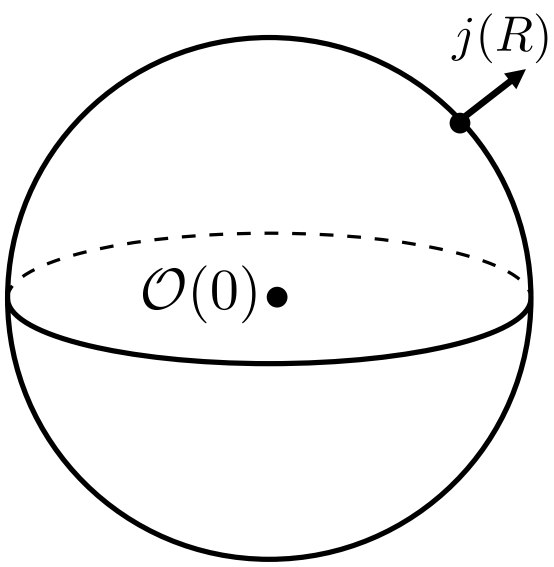
- There is now a gauge-invariant local current operator defined on the defect.
- Relatedly: symmetry transformation now labeled by arbitrary U(1) phase.
- This version will let us prove an analogue of Goldstone's theorem.

Goldstone's theorem I

Let's first formulate Goldstone's theorem for ordinary symmetries in Euclidean path-integral language (Hofman, NI).

Ward identity for U(1) current:

$$d\star j(x)\mathcal{O}(0) = iq\delta^{(4)}(x)\mathcal{O}(0)$$



Integrate both sides over interior of S^3.

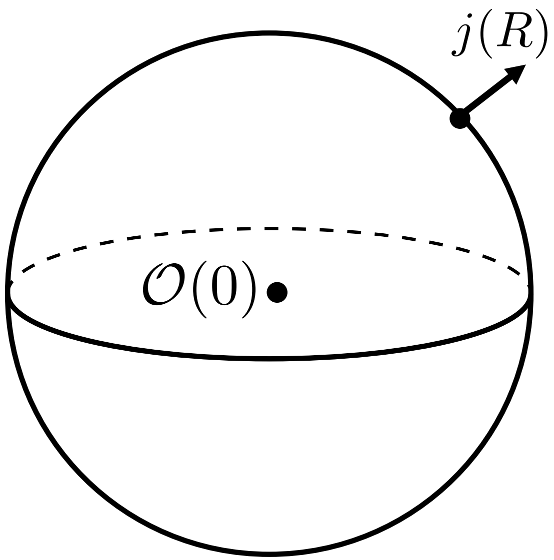
$$\int_{S^3} \star j(R)\mathcal{O}(0) = iq\mathcal{O}(0)$$

$$\langle j(R)\mathcal{O}(0)\rangle \sim \frac{iq\langle\mathcal{O}\rangle}{R^3}$$

If vev of charged operator is nonzero, must be a power-law correlation in theory; this is Goldstone mode.

Goldstone's theorem II: non-invertible

Let's now formulate Goldstone's theorem for these non-invertible symmetries.



Wrap the operator with the charge defect:

$$U_\alpha(S^3)O(0) = e^{iq\alpha}O(0)$$

Take a derivative with respect to symmetry parameter:

$$\int_{S^3} \star\tilde{j}(R)\mathcal{O}(0) = iq\mathcal{O}(0)$$

$$\langle \tilde{j}(R)\mathcal{O}(0)\rangle_c \sim \frac{iq\langle\mathcal{O}\rangle}{R^3}$$

Proof is essentially the same (except current operator is localized on defect). We have Goldstone mode.

Axion effective theory

What is the low-energy theory of the Goldstone mode? Axion theory:

$$S[\phi, A] = \int \left(\frac{1}{2\gamma^2} d\phi^2 + \frac{1}{4g^2} F^2 + i\phi F \wedge F + \cdots \right)$$

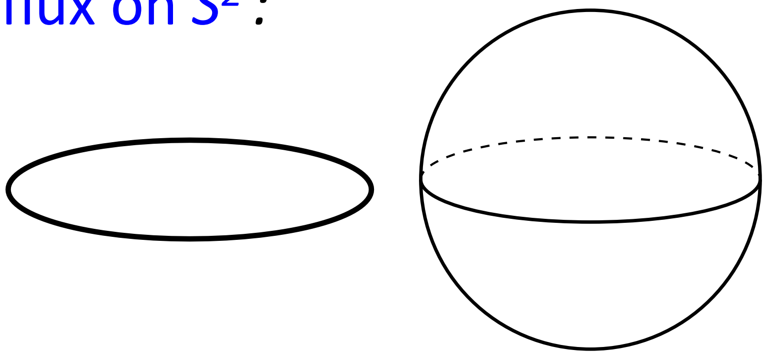
“Why” is this theory massless?

Often some words about ”no U(1) instantons”; from our point of view, it is massless because it is the Goldstone mode of a spontaneously broken non-invertible symmetry.

(Same words apply in application to massless QCD: why is pion massless despite anomaly? Also protects masslessness of many fields in string theory.).

Behavior in non-zero flux sector

Interesting things happen when there is nontrivial topology. E.g consider $R \times S^2 \times S^1$ with defect wrapping $S^2 \times S^1$. There are sectors with flux on S^2 :



$$\int_{S^2} F = 2\pi n$$

$$U_\alpha(\mathcal{M}_3) = \int [d\theta] \exp\left(i\alpha \int_{\mathcal{M}_3} \left(\star j_A - \frac{1}{4\pi^2}(A - d\theta) \wedge dA \right)\right)$$

Sum over windings of θ results in projector of flux! (Example of non-invertibility).

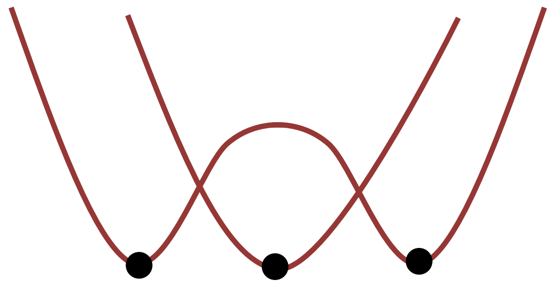
$$\sum_w \exp\left(\frac{i\alpha w n}{2}\right) \sim \delta_{n,0}$$

Future directions:

Physical applications of higher-form and non-invertible symmetries are still in their infancy. Basic questions are open:

Which systems have these symmetries? (Much recent study! See Yuya's talk!) When is there a Goldstone theorem?

What is the phase structure? Phase transitions and critical phenomena?

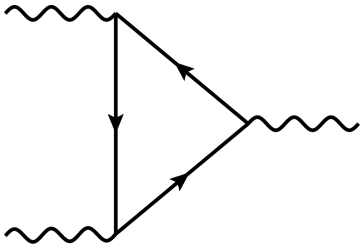


For QED: can we reformulate chiral magnetohydrodynamics with these non-invertible symmetries as a guiding principle? (Das, NI, Poovuttikul).

What is the new Landau paradigm built around these new symmetries?

Summary

- Non-invertible symmetry is a new way to think about systems with an ABJ anomaly.
- There is a Goldstone theorem for (some of) these non-invertible symmetries.
- A new non-invertible Landau paradigm?



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