Transitions in (1+1) Light Front $\phi 4$ Theory using a Quantum Computing Method

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Workshop for Tensor Networks in Many Body and Quantum Field Theory

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IOWA STATE UNIVERSITY

Outlines

- **Introduction of Light Front ϕ4 Theory**
- **Discretized light-cone quantization (DLCQ)**
- **What kind of non-perturbative structure can we get?**
- **Critical coupling in continuum limit, the challenge in classical calculation**
- **Variational quantum eigensolver (VQE)**

Scalar ϕ4 Theory

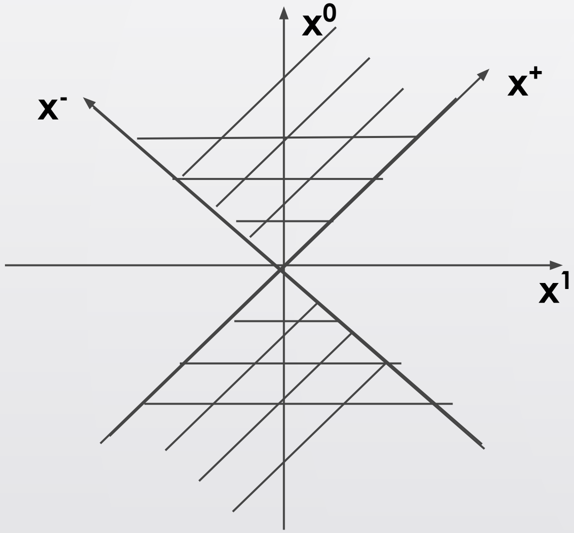
$$\mathcal{L} = \partial^{\mu}\phi\partial_{\mu}\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

$\mu^2 > 0$ (Symmetric phase)
$\mu^2 < 0$ (Broken symmetry phase)

- **The simplest interacting quantum field theory that displays spontaneous symmetry breaking which is the same underlying mechanism of Higgs mechanism**
- **has various non-perturbative structures (e.g. kinks, Instantons, solitons, …)**
- **is a testbed for new methods since it has been studied a lot so that it is convenient to make a comparison between methods**

Discretized light-cone quantization (DLCQ)

H. C. Pauli and S. J. Brodsky, PRD 32, 1993 (1985), T. Eller, H. C. Pauli, and S. J. Brodsky, PRD 35, 1493 (1987),
A. Harindranath and J. P. Vary, PRD 36, 1141 (1987)

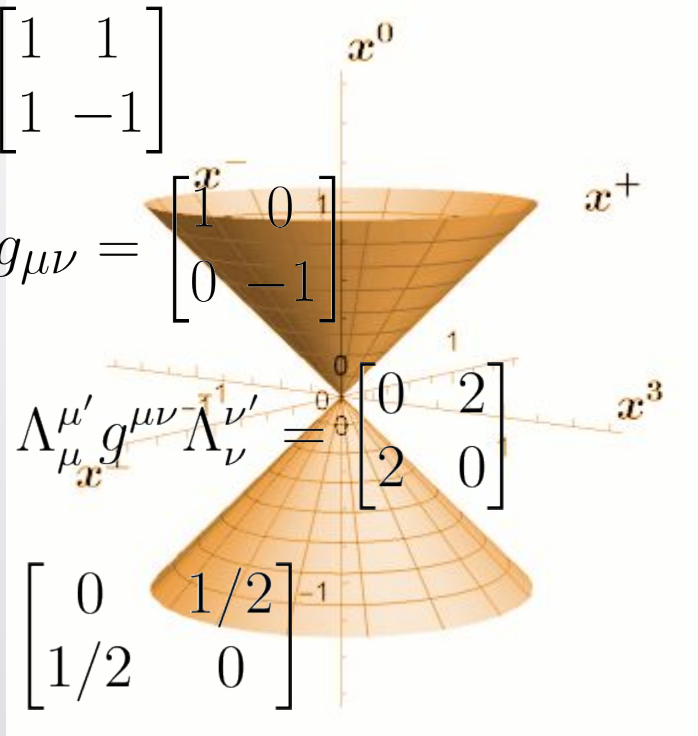


$x^+ = x^0 + x^1$ **light-front time**

$x^- = x^0 - x^1$ **light-front longitudinal coordinate**

$k^+ = k^0 + k^1$ **light-front momentum**

$k^- = k^0 - k^1$ **light-front energy**



$$\Lambda^{\mu'}_{\mu} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$g^{\mu\nu} = g_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$g^{\mu'\nu'} = \Lambda^{\mu'}_{\mu} g^{\mu\nu} \Lambda^{\nu'}_{\nu} = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}$$

$$g_{\mu'\nu'} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}$$

$$k \cdot x = k^0 x^0 - k^1 x^1 = \frac{1}{2} k^+ x^- + \frac{1}{2} k^- x^+$$

$$= \frac{1}{2} k^+ (x^0 - x^1) + \frac{1}{2} k^- (x^0 + x^1) = \frac{1}{2} (k^+ + k^-) x^0 - \frac{1}{2} (k^+ - k^-) x^1$$

Discretized light-cone quantization (DLCQ)

H. C. Pauli and S. J. Brodsky, PRD 32, 1993 (1985), T. Eller, H. C. Pauli, and S. J. Brodsky, PRD 35, 1493 (1987), A. Harindranath and J. P. Vary, PRD 36, 1141 (1987)

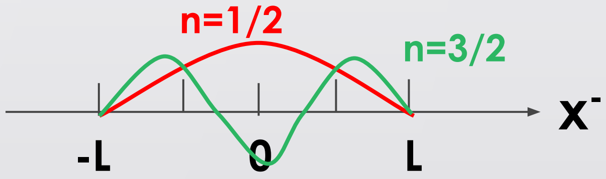
$$\phi(x^+ = 0, x^-) = \frac{1}{2\pi}\int \frac{dk^+}{2k^+}[a(k^+)e^{-i\frac{1}{2}k^+x^-} + a^\dagger(k^+)e^{i\frac{1}{2}k^+x^-}] \qquad \left[a(k^+), a^\dagger(k'^+)\right] = 2\pi 2k^+\delta(k^+ - k'^+)$$

$$-L \le x^- \le +L \qquad k^+ \to k_n^+ = \frac{2\pi}{L}n$$

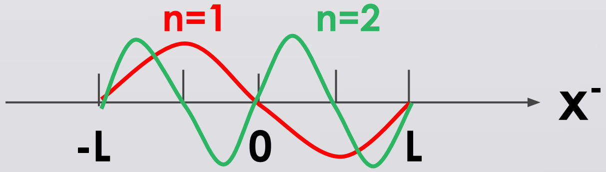
$$\phi = \frac{1}{\sqrt{4\pi}}\sum_n \frac{1}{\sqrt{n}}(a_n e^{-i\frac{n\pi}{L}x^-} + a_n^\dagger e^{i\frac{n\pi}{L}x^-}) \qquad \left[a_n, a_m^\dagger\right] = \delta_{nm}$$

n=1,2,3,... for periodic boundary condition (PBC);
n=1/2, 3/2, 5/2, … for anti-periodic boundary condition (APBC).

Anti-periodic



Periodic



APBC naturally excludes the zero mode, while PBC contains the zero mode. We use PBC omitting the zero mode, based on the discussion [1] that the zero mode does not have significant effect on the critical coupling at infinite resolution. The role of zero mode will be deferred to future explorations.

[1] J. S. Rozowsky and C. B. Thorn, PRL 85, 1614 (2000)

Discretized light-cone quantization (DLCQ)

H. C. Pauli and S. J. Brodsky, PRD 32, 1993 (1985), T. Eller, H. C. Pauli, and S. J. Brodsky, PRD 35, 1493 (1987),
A. Harindranath and J. P. Vary, PRD 36, 1141 (1987)

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi) = \frac{1}{2}\partial^+\phi\partial_+\phi + \frac{1}{2}\partial^-\phi\partial_-\phi - V(\phi) = \frac{1}{2}\partial^+\phi\partial^-\phi - V(\phi)$$

$$\mathcal{H} = \frac{\partial\mathcal{L}(x)}{\partial(\partial^+\varphi(x))}\partial^+\varphi(x) - \mathcal{L}(x) = \frac{1}{2}\partial^-\phi\partial^+\phi - \mathcal{L} = V(x)$$

The gauge invariant symmetry energy momentum tensor $T^{\mu\nu} = (\frac{\partial}{\partial(\partial_\mu\phi)}\partial^\nu\phi - g^{\mu\nu})\mathcal{L}$

$$P^\mu = \frac{1}{2}\int dx^- T^{+\mu}$$

$$H \equiv \frac{2\pi}{L}P^- = \frac{2\pi}{L}\int dx^-\mathcal{H}$$

$$K \equiv \frac{L}{2\pi}P^+ = \sum_n n a_n^\dagger a_n$$

The invariant mass

$$M^2 = P^\mu P_\mu = P^+P^- = KH$$

$$P^- = \frac{1}{2}\int dx^- T^{+-} = \frac{1}{2}\int dx^-(\partial^+\phi\partial^-\phi - 2\mathcal{L}) = \int dx^- V(x) = \int dx^-\mathcal{H}$$

$$\begin{aligned}
P^+ &= \frac{1}{2}\int dx^- T^{++} = \frac{1}{2}\int dx^-\partial^+\phi\partial^+\phi = 2\int dx^-\partial_-\phi\partial_-\phi \\
&= 2\frac{1}{4\pi}\int dx^-\sum_n\frac{1}{\sqrt{n}}[(-i\frac{n\pi}{L})a_n e^{-i\frac{n\pi}{L}x} + (i\frac{n\pi}{L})a_n^\dagger e^{i\frac{n\pi}{L}x}]\sum_m\frac{1}{\sqrt{m}}[(-i\frac{m\pi}{L})a_m e^{-i\frac{m\pi}{L}x} + (i\frac{m\pi}{L})a_m^\dagger e^{i\frac{m\pi}{L}x}] \\
&= 2\frac{1}{4\pi}\sum_{n,m}\frac{1}{\sqrt{nm}}\int dx^-[(-i\frac{n\pi}{L})a_n e^{-i\frac{n\pi}{L}x} + (i\frac{n\pi}{L})a_n^\dagger e^{i\frac{n\pi}{L}x}][(-i\frac{m\pi}{L})a_m e^{-i\frac{m\pi}{L}x} + (i\frac{m\pi}{L})a_m^\dagger e^{i\frac{m\pi}{L}x}] \\
&= 2\frac{1}{4\pi}\sum_{n,m}\frac{1}{\sqrt{nm}}\int dx^-[\frac{n\pi}{L}\frac{m\pi}{L}a_n^\dagger a_m e^{i\frac{(n-m)\pi}{L}x} + \frac{n\pi}{L}\frac{m\pi}{L}a_n a_m^\dagger e^{-i\frac{(n-m)\pi}{L}x}] \\
&= 2\frac{1}{4\pi}\sum_{n,m}\frac{1}{n}\frac{n\pi}{L}\frac{m\pi}{L}[a_n^\dagger a_m\frac{L}{\pi}(2\pi)\delta_{nm} + a_n a_m^\dagger\frac{L}{\pi}(2\pi)\delta_{nm}] \\
&= \sum_n\frac{n\pi}{L}[a_n^\dagger a_n + a_n a_n^\dagger] = \sum_n\frac{n\pi}{L}[2a_n^\dagger a_n + 1] = \frac{2\pi}{L}\sum_n n a_n^\dagger a_n + \cancel{\infty} = \frac{2\pi}{L}K
\end{aligned}$$

$$\mathcal{L} = \frac{1}{2}\partial^+\phi\partial^-\phi - \frac{\mu^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4$$

$$\mathcal{H} = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

- The Hamiltonian exhibits ϕ → -ϕ symmetry, so the even and odd particle sectors are decoupled. ✅
- In the broken symmetric phase: ϕ = 0 is unstable, the two minima at ϕ = ± v break the global symmetry (spontaneous symmetry breaking, i.e. SSB) ✅
- Kinks (topological excitations) at the broken symmetry phase ✅
- The states in the even and odd particle sector might become degenerate in the continuum limit.
- The ground state at weak coupling of the symmetric phase is dominated by single-particle state carrying the full momentum (no co-moving particles).
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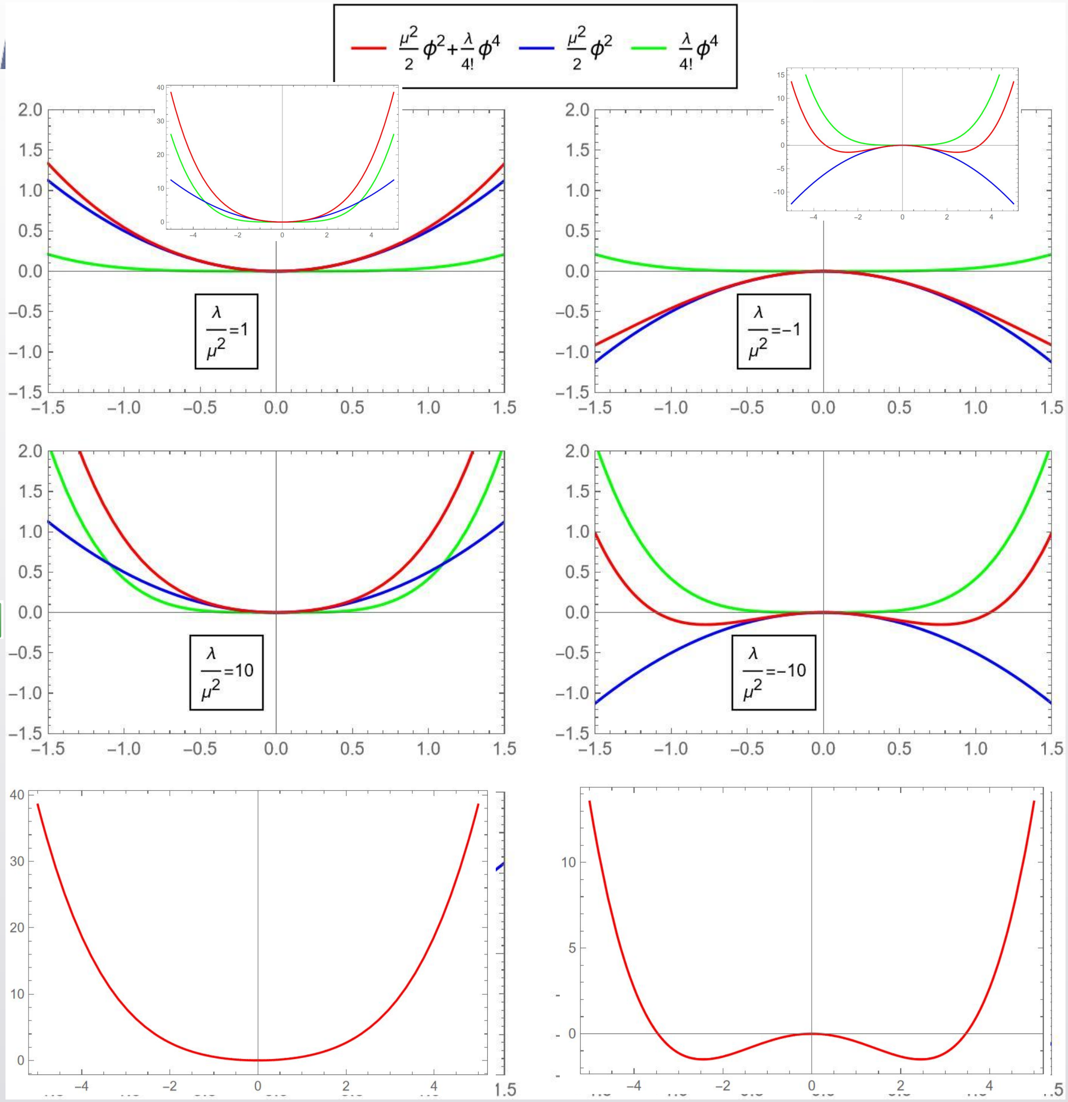


Symmetric phase $\mu^2>0$ Broken symmetry phase $\mu^2<0$

Hamiltonian of (1+1) ϕ4 theory in DLCQ

A. Harindranath and J. P. Vary, PRD 36, 1141 (1987)

$$H = \frac{2\pi}{L}\int dx^- \mathcal{H} = \frac{2\pi}{L}\int dx^- (\frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4)$$

$$\phi = \frac{1}{\sqrt{4\pi}}\sum_n \frac{1}{\sqrt{n}}(a_n e^{-i\frac{n\pi}{L}x^-} + a_n^\dagger e^{i\frac{n\pi}{L}x^-})$$

$$H = \mu^2 \sum_n^K \frac{1}{n} a_n^\dagger a_n + \frac{\lambda}{4\pi}\left(\sum_{k\le l, m\le n}^K \frac{1}{N_{kl}} \frac{1}{N_{mn}} \frac{a_k^\dagger a_l^\dagger a_m a_n}{\sqrt{klmn}} \delta_{m+n,k+l} + \sum_{k,l\le m\le n}^K \frac{1}{N_{lmn}} \frac{a_k^\dagger a_l a_m a_n + a_n^\dagger a_m^\dagger a_l^\dagger a_k}{\sqrt{klmn}} \delta_{k,m+n+l} \right)$$

$$N_{kl} = \begin{cases} 1 & k \neq l \\ 2! & k = l \end{cases}, \quad N_{lmn} = \begin{cases} 1 & l \neq m \neq n \\ 2! & l = m \neq n \text{ or } l \neq m = n \\ 3! & l = m = n \end{cases}.$$

n=1,2,3,... for periodic boundary condition (PBC);
n=1/2, 3/2, 5/2, … for anti-periodic boundary condition (APBC).

Eigenvalue problem

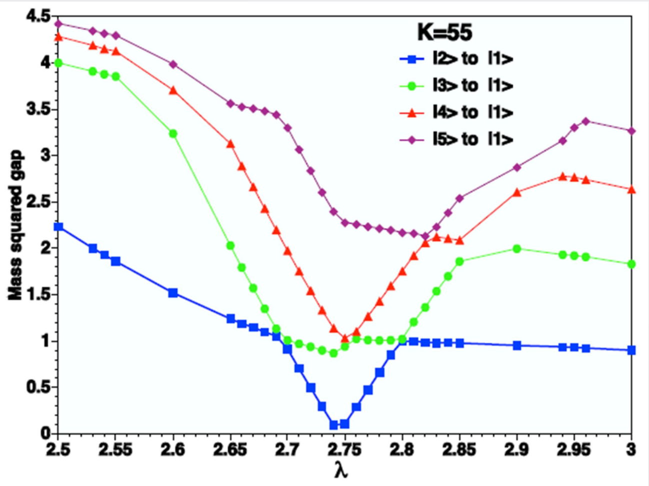
D. Chakrabarti, A. Harindranath, and J. P. Vary, PRD 71, 125012 (2005)

Many body states are represented by Fock-space basis for m_1 quanta with n_1 units of momentum and so on. $|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \cdots, n_i^{m_i}, \cdots\rangle$

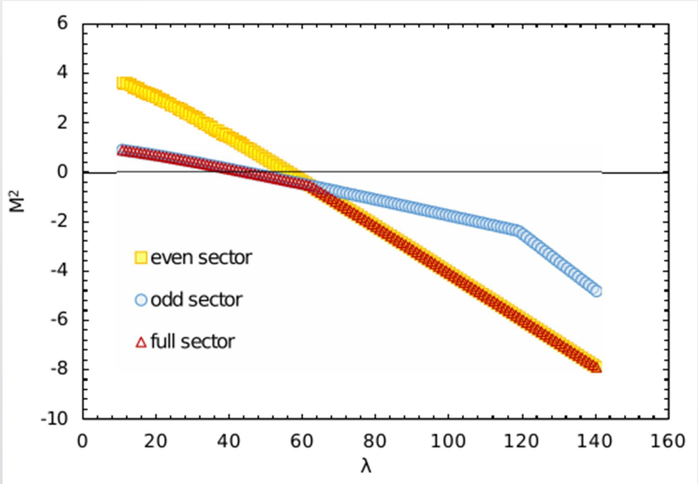
$$a_{n_i}|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \cdots, n_i^{m_i}, \cdots\rangle = \sqrt{m_i}|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \cdots, n_i^{m_i-1}, \cdots\rangle$$

$$a_{n_i}^\dagger|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \cdots, n_i^{m_i}, \cdots\rangle = \sqrt{m_i+1}|n_1^{m_1}, n_2^{m_2}, n_3^{m_3}, \cdots, n_i^{m_i+1}, \cdots\rangle$$

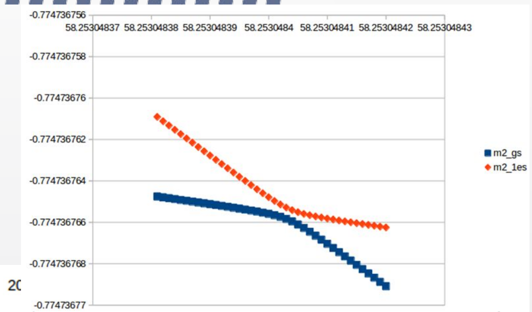
$$K = \sum_n n a_n^\dagger a_n = \sum_n n\hat{N}_n = \sum_i n_i \cdot m_i \qquad \langle M^2\rangle = K\langle H\rangle$$

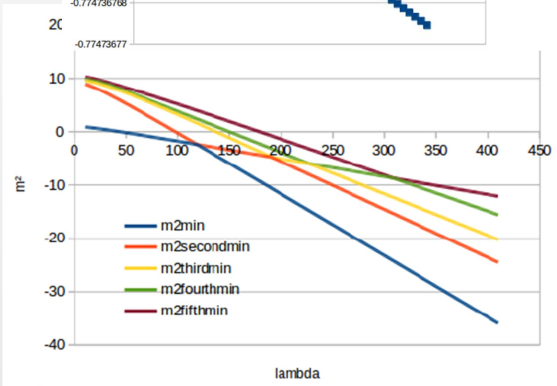


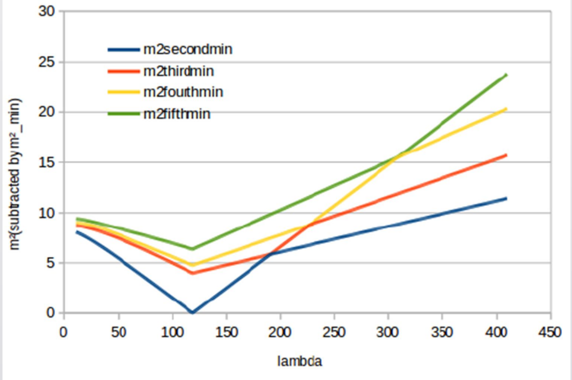
Broken symmetry phase



Symmetric phase, K=16





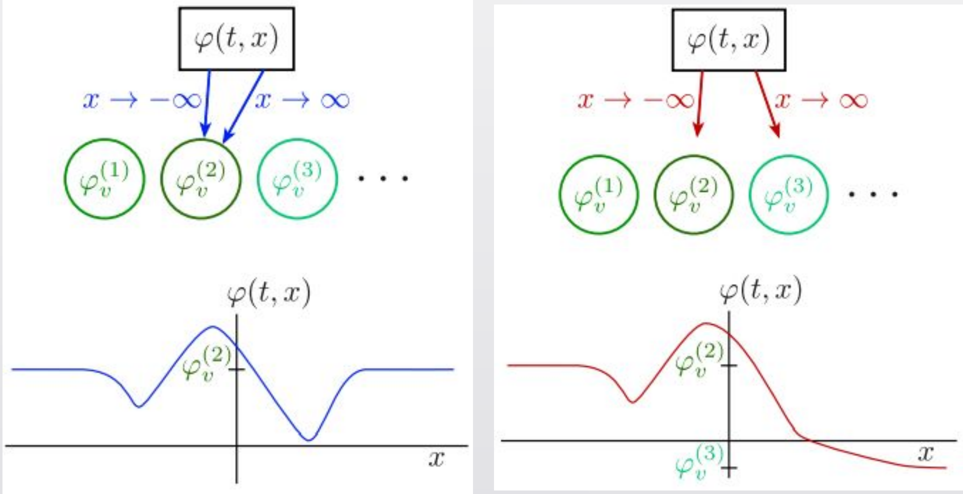


Symmetric phase odd particle sector, K=16

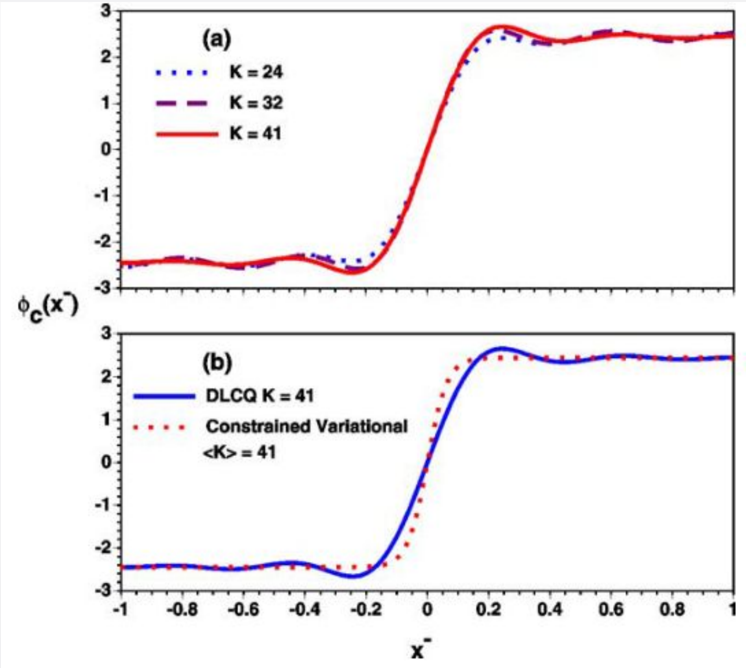
Kinks in (1+1) ϕ4 theory using DLCQ

D. Chakrabarti, A. Harindranath, L. Martinovic, J. P. Vary, PLB 582, 196-202 (2004),
D. Chakrabarti, A. Harindranath, and J. P. Vary, PRD 71, 125012 (2005)

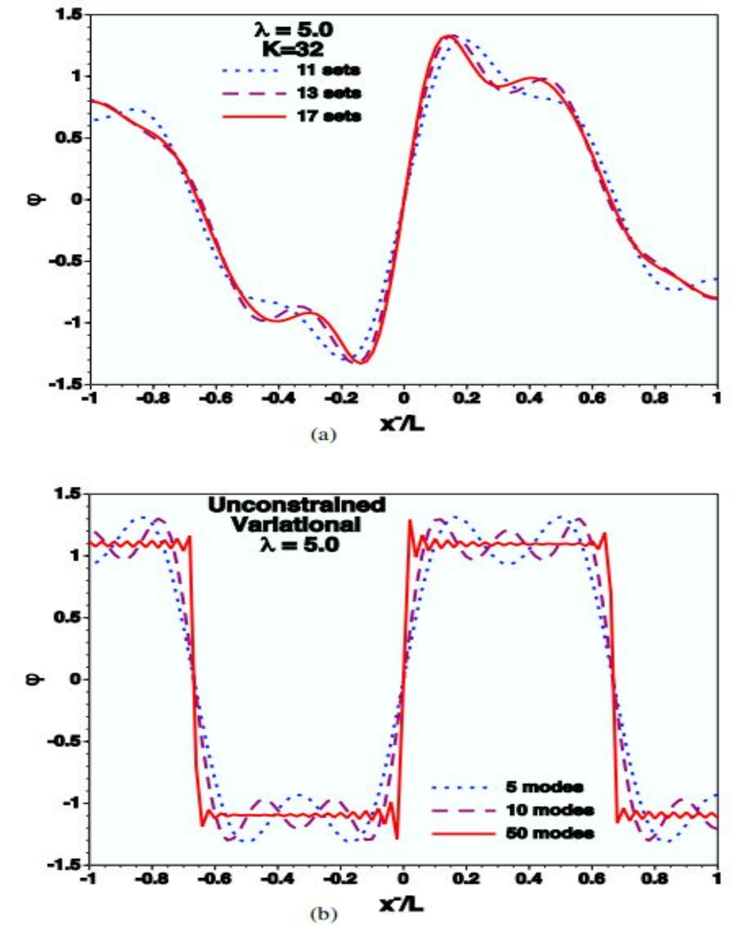
Broken symmetry phase. Fourier transform to coordinate space ϕ_c



M. Lizunova, J. van Wezel, SciPost Phys. Lect. Notes 23 (2021)



Kink



Kink-antikink-kink

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$$\mathcal{H} = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$$

- **The Hamiltonian exhibits ϕ → -ϕ symmetry, so the even and odd particle sectors are decoupled.** ✅
- **In the broken symmetric phase: ϕ = 0 is unstable, the two minima at ϕ = ± v break the global symmetry (spontaneous symmetry breaking, i.e. SSB)** ✅
- **Kinks (topological excitations) at the broken symmetry phase** ✅
- **The states in the even and odd particle sector might become degenerate in the continuum limit.** ✅
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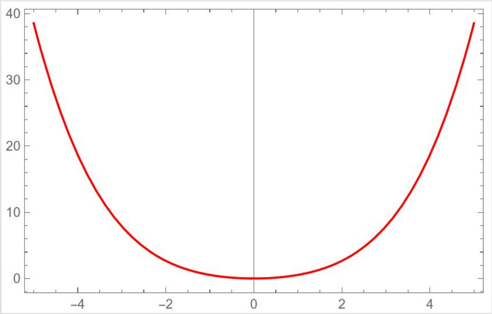
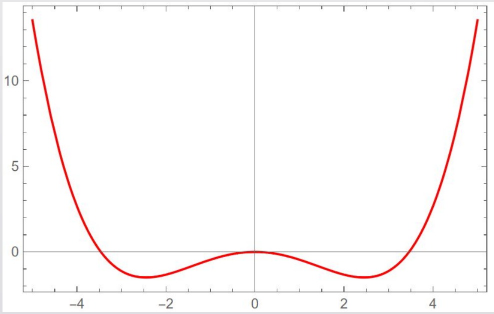


Symmetric phase $\mu^2>0$



Broken symmetry phase $\mu^2<0$

Critical coupling in continuum limit

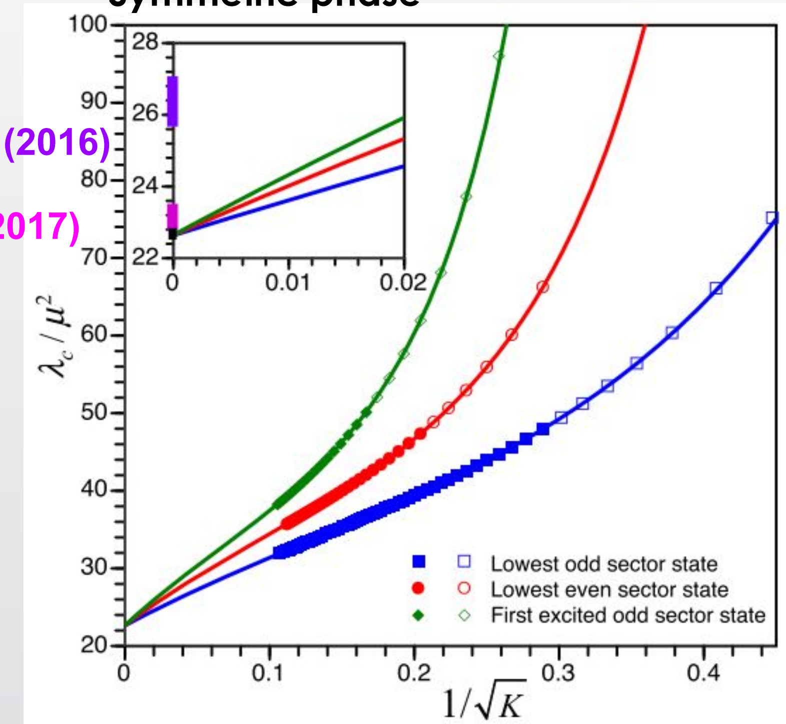
J. P. Vary, M. Huang, S. Jawadekar, M. Sharaf, A. Harindranath, and D. Chakrabarti

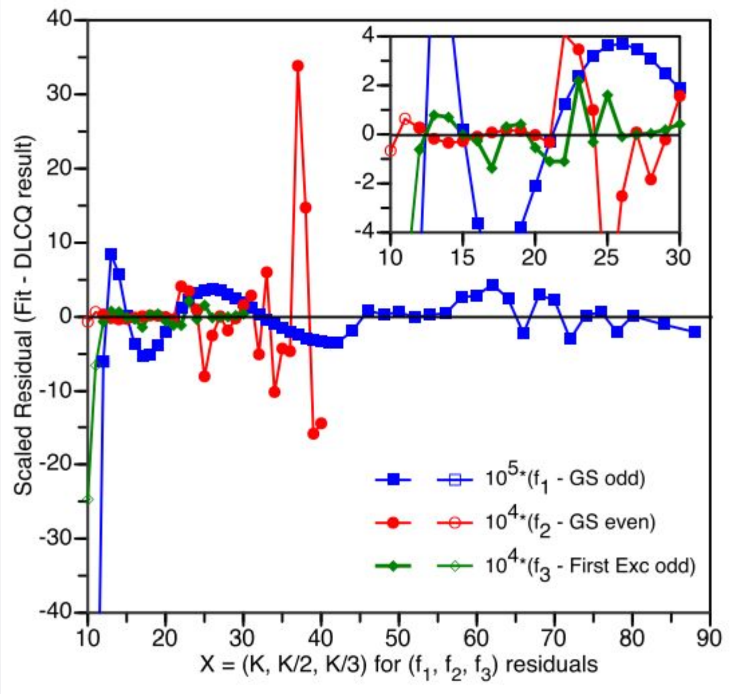
Phys. Rev. D 105, 016020 (2022)

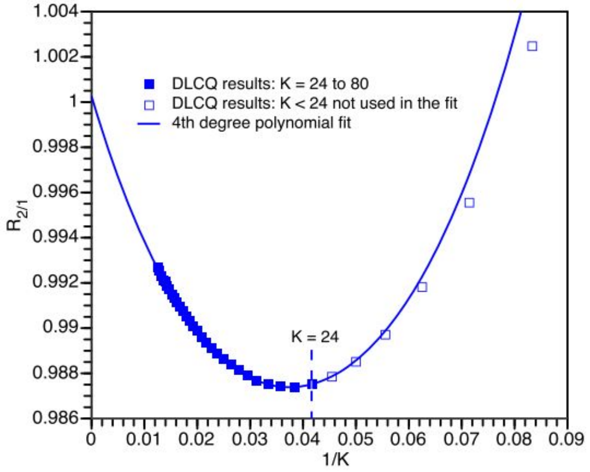
Symmetric phase

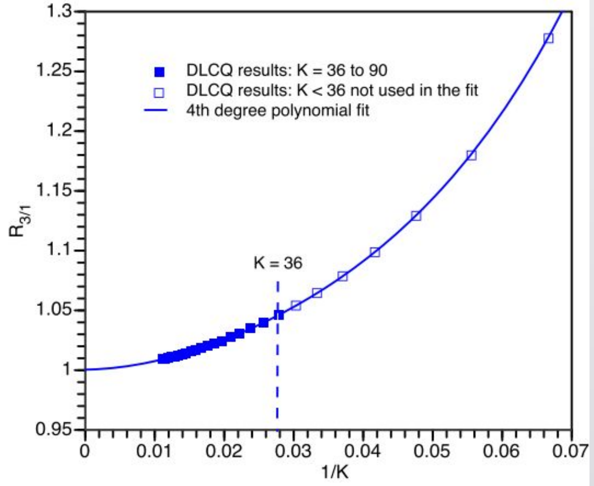
Burkardt et al. (2016)

Anand et al. (2017)









Struggle in fitting (due to the lacking of higher K results): the fitting incorporates (and tested by the goodness of the fit) the assumption that the even particle ground state and the first excited state of the odd particle sector are duplicated by the odd particle ground state multiplied by a R(1/K) factor.

Critical coupling in continuum limit

$$\bar{g} = \frac{\pi}{6} g, \text{ where } g = \frac{\lambda}{4\pi\mu^2}$$

$$\begin{aligned}\mu_{LF}^2 &= \mu_{ET}^2 + \lambda\left[\langle 0|\frac{\phi^2}{2}|0\rangle - \langle 0|\frac{\phi^2}{2}|0\rangle_{\text{free}}\right] \\ &= \mu_{ET}^2 + \lambda(-\frac{1}{4\pi}\Delta) = \mu_{ET}^2 - \frac{\lambda}{4\pi}\Delta.\end{aligned}$$

Method	$\bar{g}_c(LF)$	$\bar{g}_c(ET)$
DLCQ (Huang (2020))	1.26	4.53±2.51
DLCQ (Harindranath and Vary (1987))	1.38	6.59±3.65
Light-front symmetric polynomials (Burkardt et al. (2016))	1.1±0.03	2.98±1.65
Quasiparse eigenvector (Lee et al. (2001))	–	2.5
Density matrix renormalization group (Sugihara (2004))	–	2.4954(4)
Lattice Monte Carlo (Schaich and Loinaz (2009))	–	$2.70^{+0.025}_{-0.013}$
Lattice Monte Carlo (Bosetti et al. (2015))	–	2.79±0.02
Uniform matrix product (Milsted et al. (2013))	–	2.766(5)
Renormalized Hamiltonian truncation (Rychkov and Vitale (2015))	–	2.97(14)

The disagreement between equal-time and light-front critical coupling still needs to be investigated.

Resort to quantum computing?

- Classical computing hits the memory bound in non-perturbative field theory calculations with increasing resolution. Quantum computing is promising in reducing the memory consumption, as N configurations can be encoded by only $log_2 N$ qubits in compact encoding [1].
- Demonstrate that transition in the (1+1) ϕ_4 theory with discretized light-cone quantization (DLCQ) [2] in the strong coupling region can be observed by solving the ground state using the VQE quantum algorithm.
- DLCQ Hamiltonian formulation naturally fits the applications for quantum computing

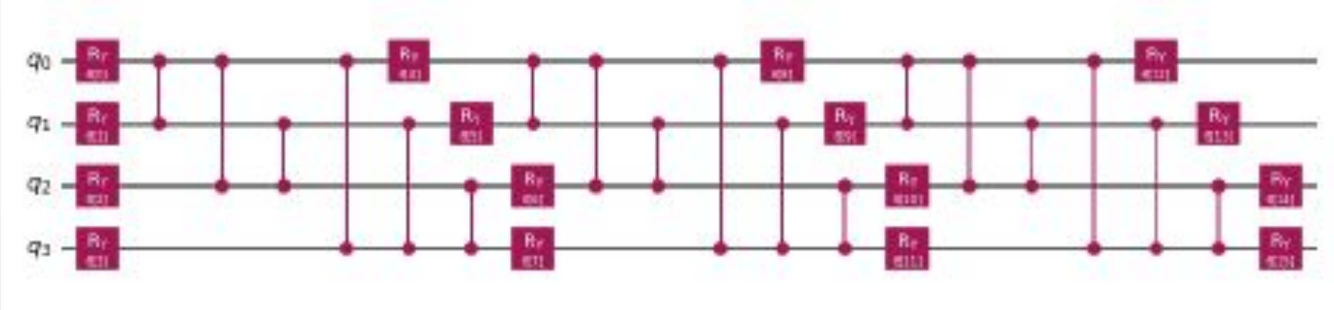
[1] M. Kreshchuk, S. Jia, W. M. Kirby, G. Goldstein, J. P. Vary, and P. J. Love, "Simulating hadronic physics on noisy intermediate-scale quantum devices using basis light-front quantization," *Physical Review A*, 103, 062601 (2021)

[2] J. P. Vary, M. Huang, S. Jawadekar, M. Sharaf, A. Harindranath, and D. Chakrabarti, "Critical coupling for two-dimensional ϕ4 theory in discretized light-cone quantization," *Physical Review D*, 105, 016020 (2022)

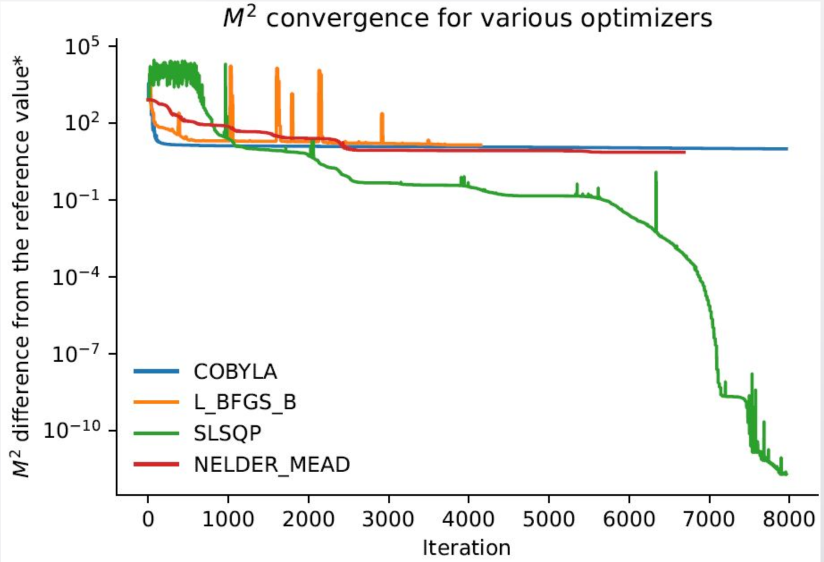
Variational Quantum Eigensolver (VQE) using IBM Qiskit

$$H \to H_q = \frac{1}{N}\sum_{i=1}^{N^2} \langle H|P_i\rangle P_i = \frac{1}{N}\sum_{i=1}^{N^2} Tr(P_i H) P_i \qquad P_i \in \{I, X, Y, Z\}^{\otimes log_2 N}, i = 1, ..., N^2$$

$$E_0 \le \tilde{E}_0 = \frac{\langle\phi|H|\phi\rangle}{\langle\phi|\phi\rangle}, \quad |\phi\rangle = |\phi(\theta_1, ..., \theta_m)\rangle$$



"TwoLocal" with 16 parameters as an ansatze for K = 9 odd sector (Hamiltonian matrix of 16×16 dimension encoded by 4 qubits)



ideal simulation ("statevector")

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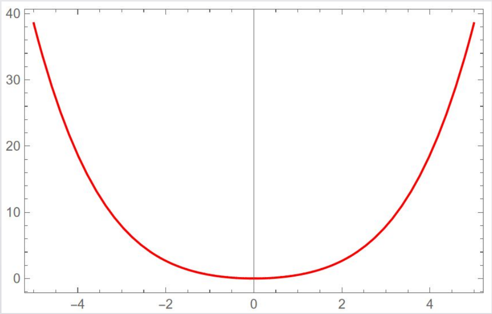
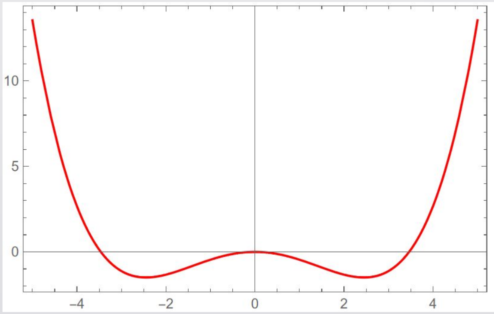


Symmetric phase μ^2>0

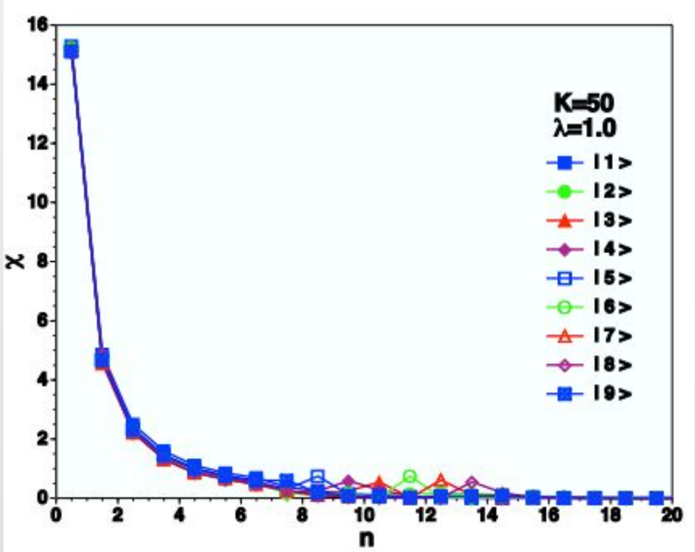


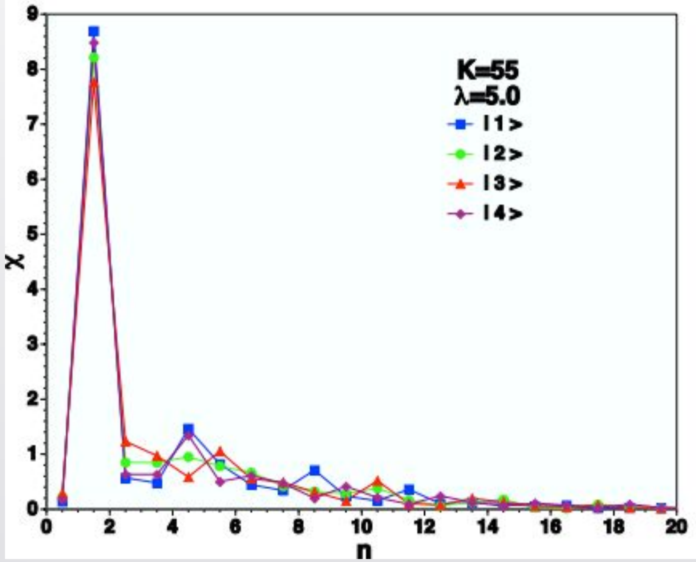
Broken symmetry phase μ^2<0

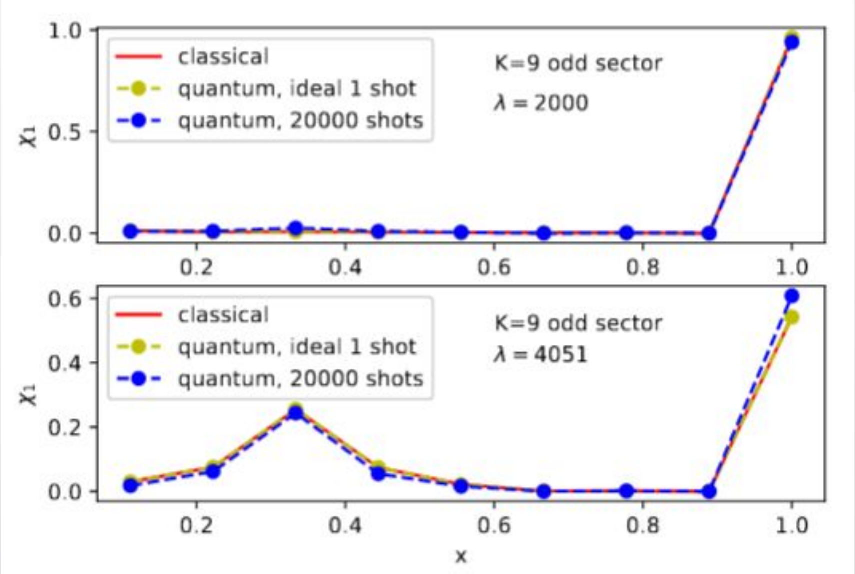
Parton Distribution Function

$$\chi_i(x_k = \frac{k}{K}) = \sum_j |\alpha_{ji}|^2 \frac{\langle \phi_j | a_k^\dagger a_k | \phi_j \rangle}{N_j}$$

Preliminary results







D. Chakrabarti, A. Harindranath, and J. P. Vary, PRD 71, 125012 (2005)

classical, "statevector" and "qasm" simulations

Broken symmetry phase

Symmetric phase

More on (1+1) ϕ4 theory

- The Hamiltonian exhibits $\phi \rightarrow -\phi$ symmetry, so the even and odd particle sectors are decoupled.✅
- In the broken symmetric phase: $\phi = 0$ is unstable, the two minima at $\phi = \pm v$ break the global symmetry (spontaneous symmetry breaking, i.e. SSB)✅
- Kinks (topological excitations) at the broken symmetry phase✅
- The states in the even and odd particle sector might become degenerate in the continuum limit.✅
- The ground state at weak coupling of the symmetric phase is dominated by single-particle state carrying the full momentum (no co-moving particles).✅
- The ground state at weak coupling of the broken symmetry phase is dominated by the maximum number of particles having the lowest allowed momentum (maximize the number of co-moving particles).✅
- The role of zero mode with the periodic boundary condition❓
- Connection to the Ising model, extracting the critical exponent❓
- Symmetry restoring phase transition (kink condensation)❓
- Broken symmetry phase and symmetric phase have the same or different critical coupling❓
- Transformation between the equal-time critical coupling with the light-front critical coupling❓
- Other methods for solving (1+1) ϕ4 theory that can decrease the numerical calculation burden❓
- Other observables to calculate, such as the vacuum expectation value❓