Bootstrapping QCD

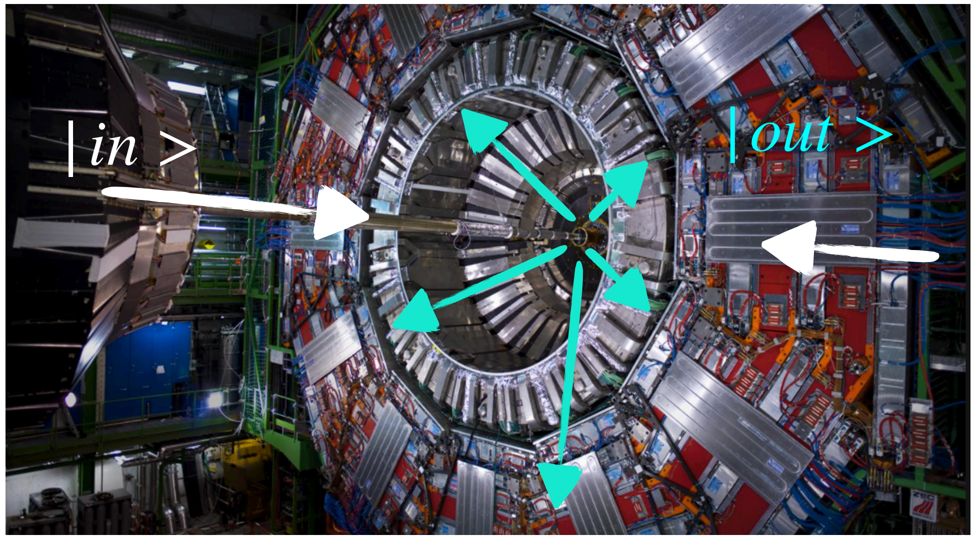
Andrea Guerrieri

Mar 21, 2023

S-matrix and Bootstrap

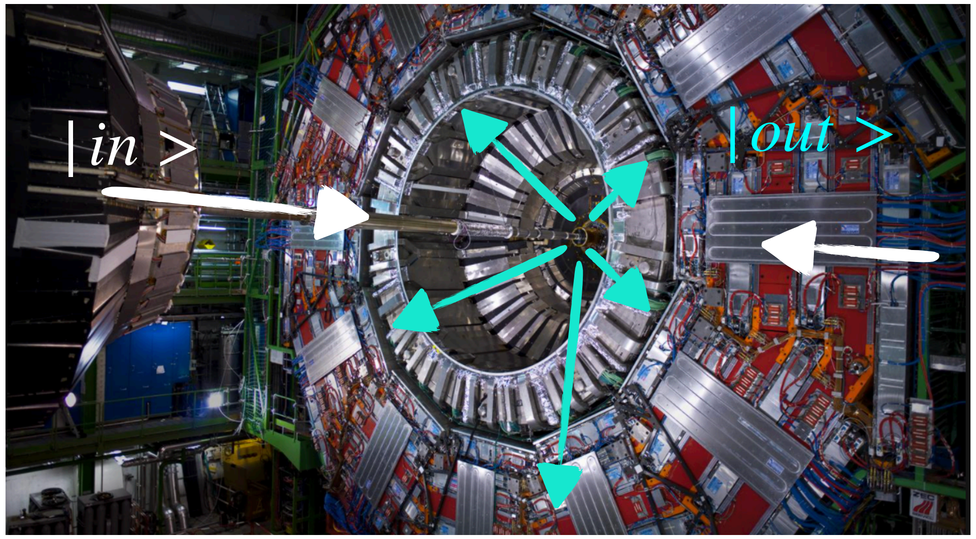
S-matrix or **Scattering Amplitude**

$$\mathcal{S}_{in \to out} \equiv < in | out >$$



The original idea: global symmetries, causality, crossing and unitarity fully constrain the S-matrix

S-matrix and Bootstrap



S-matrix or **Scattering Amplitude**

$$\mathcal{S}_{in\to out} \equiv < in | out >$$

The original idea: global symmetries, causality, crossing and unitarity fully constrain the S-matrix

Too hard, slow progress since the '60s

The modern approach: constrain the space of all possible QFTs and QG theories compatible with causality, crossing, and unitarity

Plan of the Talk

1) **General Bounds [Model Independent]:** Pion Scattering in the Chiral Limit

ALG, Penedones, Vieira, JHEP 06 (2021), arXiv: 2011.02802

2) **Precision Physics? [Data Driven]:** Pion Scattering in the gapped phase

ALG, Penedones, Vieira, PRL 122 (2019), arXiv: 1810.12849

3) **Future predictions?** Glueball Scattering

ALG, Hebbar, van Rees, work in progress

4) **An application to Confinement**

Elias-Miró, ALG, Hebbar, Penedones, Vieira, PRL 123 (2019), arXiv: 1906.08098

General Bounds: Bootstrap Strategy

Low Energy Spectrum: N=3 identical massless scalars

General Bounds: Bootstrap Strategy (crossing and Analyticity)

Low Energy Spectrum: N=3 identical massless scalars

Step 1: define a **crossing symmetric** and **analytic** Ansatz for the 2->2 Amplitude

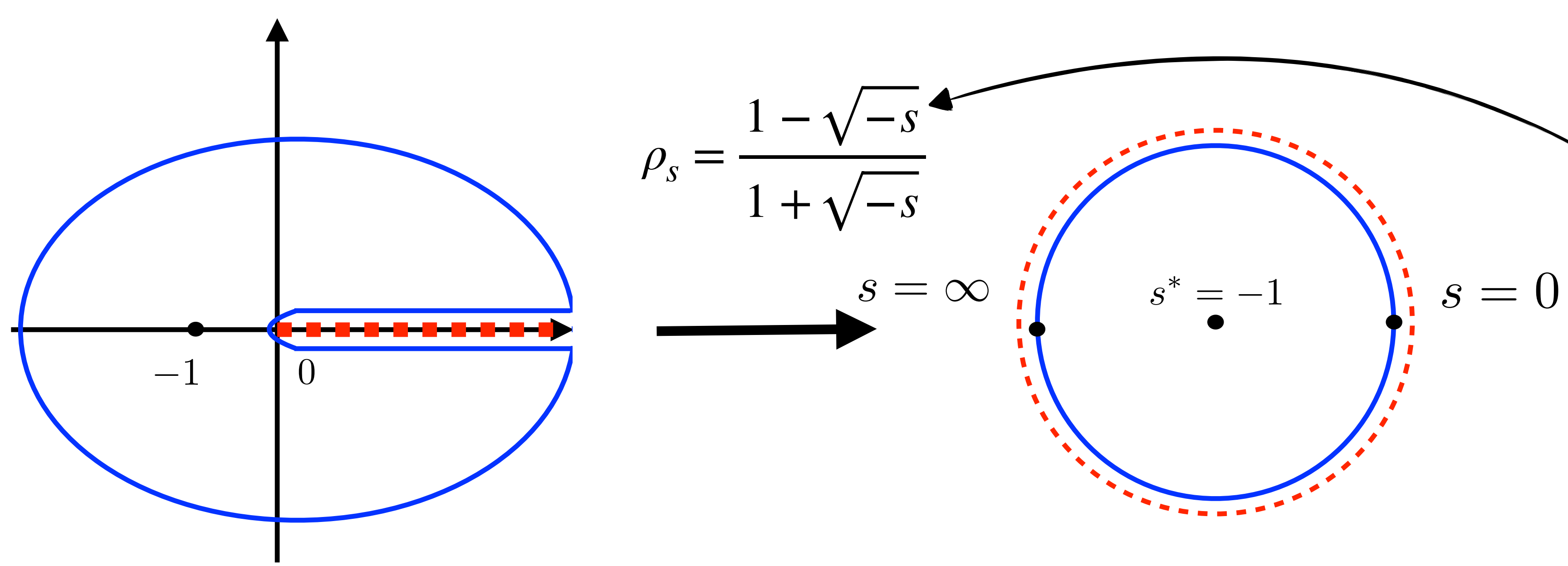
$$\mathcal{T}_{ab}^{cd} = A(s\,|\,t,u)\delta_{ab}\delta^{cd} + A(t\,|\,s,u)\delta_a^c\delta_b^d + A(u\,|\,s,t)\delta_a^d\delta_b^c$$

$$A(s\,|\,t,u) = \sum_{a+b\leq N} c_{a,b}\rho_s^a(\rho_t^b + \rho_u^b) + d_{(a,b)}(\rho_t^a\rho_u^b + \rho_t^b\rho_u^a)$$

Crossing

Analytic Extension

$$\rho_s = \frac{1-\sqrt{-s}}{1+\sqrt{-s}}$$



Maximal Analyticity

General Bounds: Bootstrap Strategy (Soft Theorems)

Low Energy Spectrum: N=3 identical massless scalars

Step 1: define a **crossing symmetric** and **analytic** Ansatz for the 2->2 Amplitude

Step 2: impose **soft behavior** $\mathscr{T}(s=t=u=0)=0$

Low Energy Expansion of our Ansatz

$$A(s|t,u) = \frac{s}{f_\pi^2} + \frac{1}{f_\pi^4}\left[\alpha s^2 + \beta(t^2+u^2) - \frac{s^2}{32\pi^2}\log\frac{-s}{f_\pi^2} - \frac{t-u}{96\pi^2}\left(t\log\frac{-t}{f_\pi^2} - u\log\frac{-u}{f_\pi^2}\right)\right] + \ldots$$

$$\frac{1}{4}f_\pi^2\,\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right)$$

Universal Tree Level

$$\ell_1\left[\mathrm{tr}\left(\partial_\mu U^\dagger \partial^\mu U\right)\right]^2 + \ell_2\,\mathrm{tr}\left(\partial_\mu U^\dagger \partial_\nu U\right)\mathrm{tr}\left(\partial^\mu U^\dagger \partial^\nu U\right)$$

Leading Non universal Corrections

Leading Loops

In practice we fix some of the $c_{a,b}$, $d_{(a,b)}$ free variables to cancel square roots

$$\rho(s) = 1 + 2i\sqrt{s} - 2s - 2is^{3/2} + 2s^2 + \ldots$$

General Bounds: Bootstrap Strategy (Unitarity)

Low Energy Spectrum: N=3 identical massless scalars

Step 1: define a **crossing symmetric** and **analytic** Ansatz for the 2->2 Amplitude

Step 2: impose **soft behavior** $\mathcal{T}(s=t=u=0)=0$

Low Energy Expansion of our Ansatz

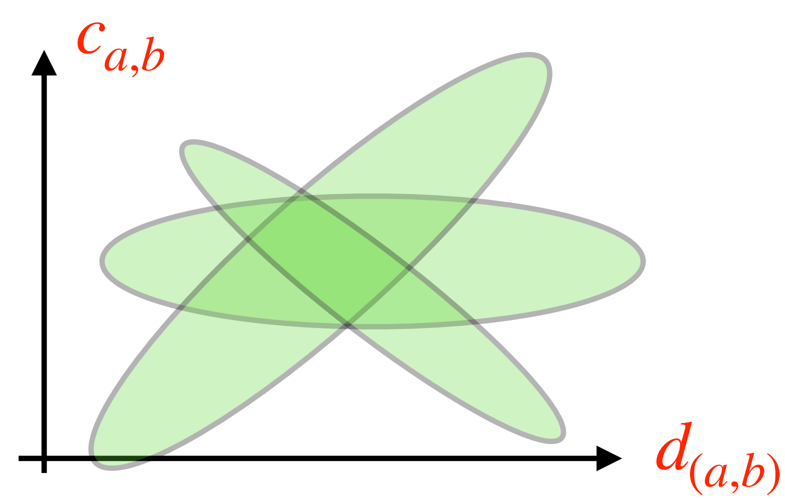
Step 3: impose unitarity numerically as a **constraint** (the $c_{a,b}$, $d_{(a,b)}$ cannot vary arbitrarily)

$$A(s\,|\,t,u)=\sum_{a+b\leq N} c_{a,b}\rho_s^a(\rho_t^b+\rho_u^b)+d_{(a,b)}(\rho_t^a\rho_u^b+\rho_t^b\rho_u^a)$$

Project the ansatz on Irreps of the flavor and Lorenz group: Decompose in Isospin and Partial Waves

$$S_\ell^I(s)\equiv 1+iT_\ell^I(s)=\mathbb{P}_\ell^I[A]$$

$$|\,S_\ell^I(s)\,|^2\leq 1,\quad s>0,\quad \ell=0,2,\ldots,\infty,\quad I=0,1,2$$



General Bounds: Bootstrap Strategy

Low Energy Spectrum: N=3 identical massless scalars

Step 1: define a **crossing symmetric** and **analytic** Ansatz for the 2->2 Amplitude

Step 2: impose **soft behavior** $\mathcal{T}(s = t = u = 0) = 0$

Low Energy Expansion of our Ansatz

Step 3: impose unitarity numerically as a **constraint**
(the $c_{a,b}$, $d_{(a,b)}$ cannot vary arbitrarily)

Step 4: Formulate an **OPTIMIZATION PROBLEM**

E.g. Minimize β at fixed α (in units of f_π) given Unitarity Constraints

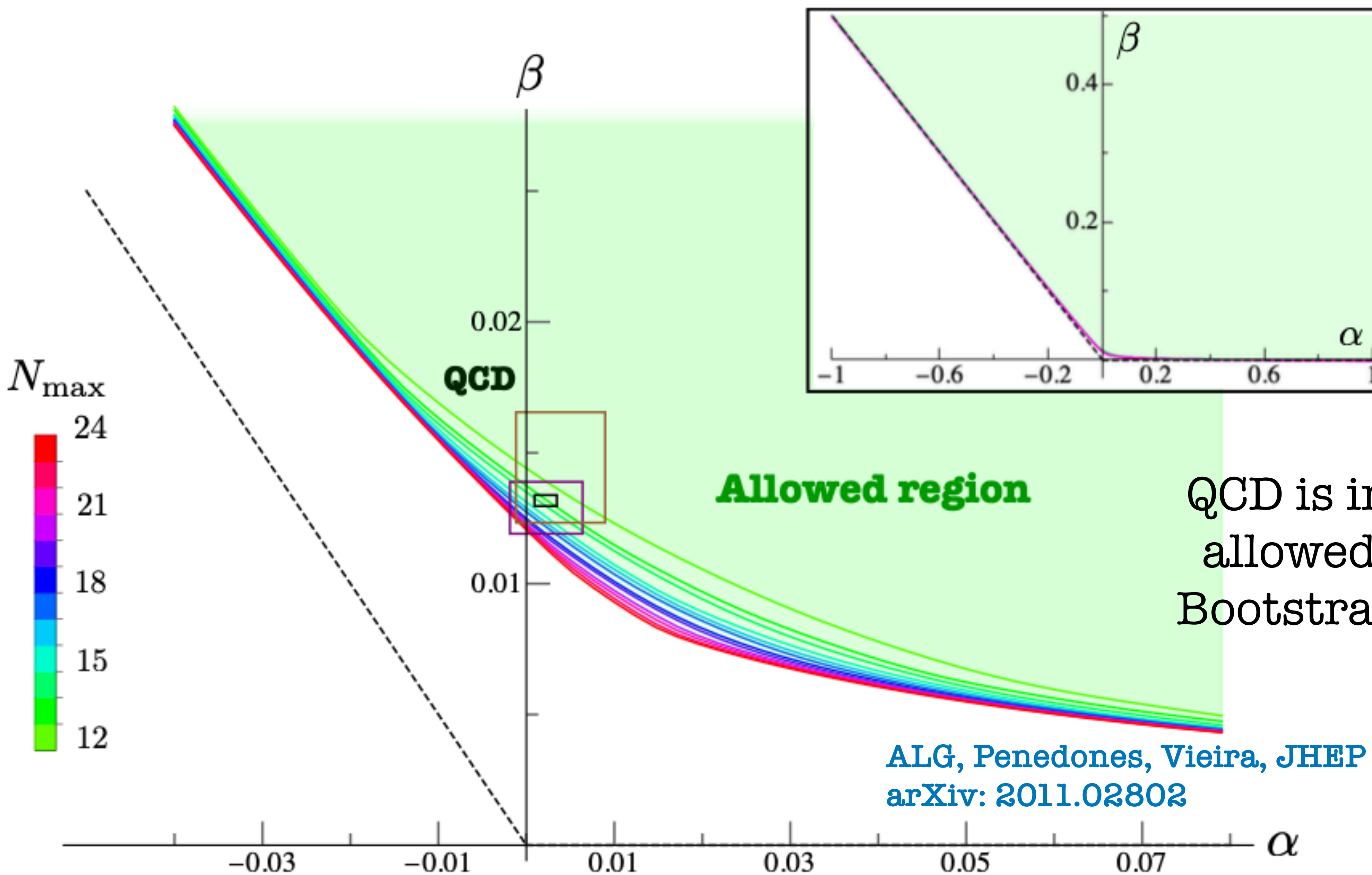
$$A(s|t,u) = \frac{s}{f_\pi^2} + \frac{1}{f_\pi^4}\left[\alpha s^2 + \beta(t^2+u^2) - \frac{s^2}{32\pi^2}\log\frac{-s}{f_\pi^2} - \frac{t-u}{96\pi^2}\left(t\log\frac{-t}{f_\pi^2} - u\log\frac{-u}{f_\pi^2}\right)\right] + \dots$$

General Bounds on α, β

α, β are linear functions of the Ansatz parameters $c_{a,b}, d_{(a,b)}$

$$A(s\,|\,t,u) = \sum_{a+b\leq N} c_{a,b}\rho_s^a(\rho_t^b + \rho_u^b) + d_{(a,b)}(\rho_t^a\rho_u^b + \rho_t^b\rho_u^a)$$

Minimize β at fixed α (in units of f_π) given Unitarity Constraints



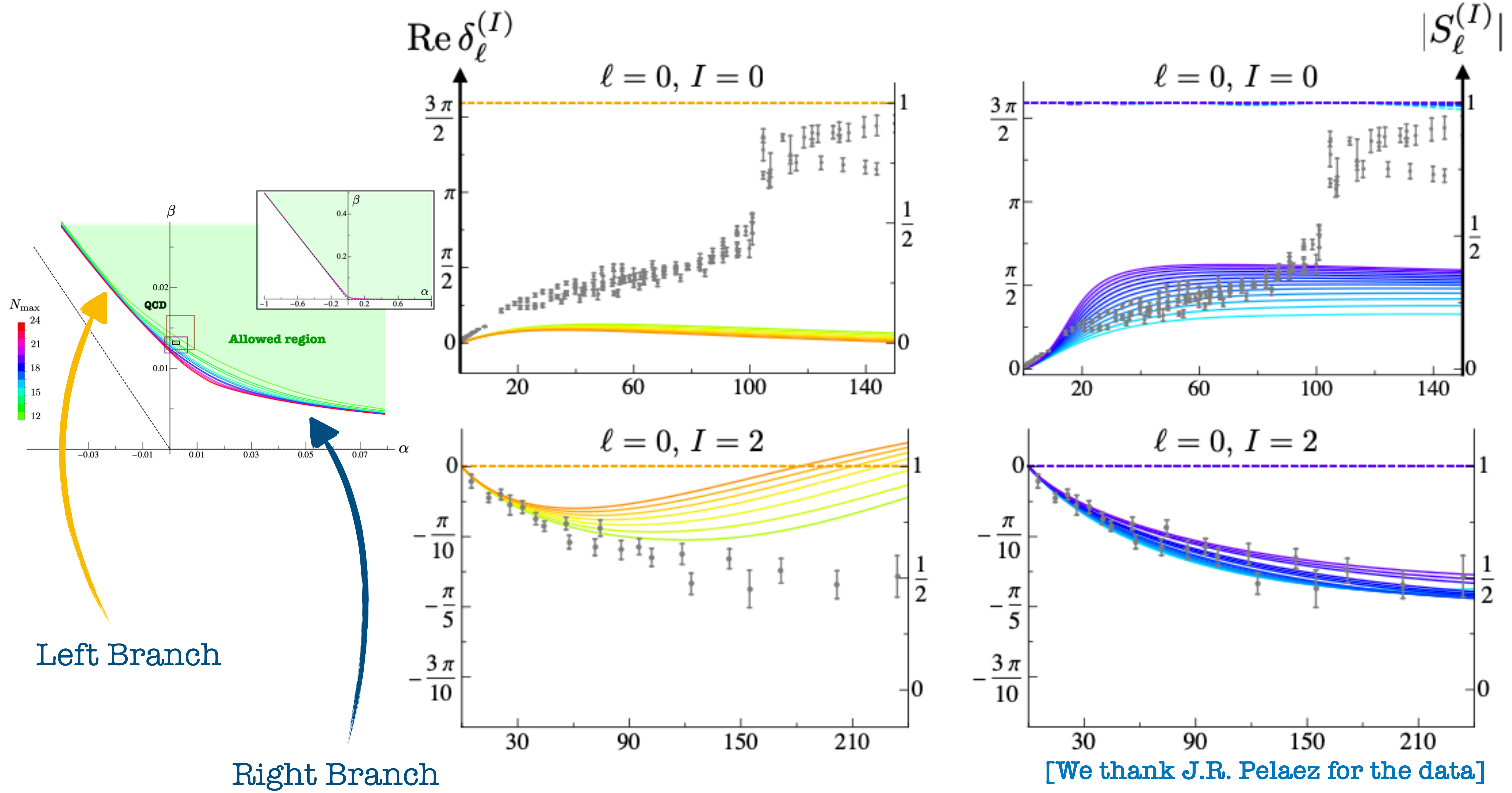
QCD is inside the allowed region! Bootstrap Works!

ALG, Penedones, Vieira, JHEP 06 (2021), arXiv: 2011.02802

General Bounds: Extremal Amplitudes

The optimization procedure fixes the coefficients $c_{a,b}, d_{(a,b)}$!!

Phase Shifts and Resonances as a prediction of the Bootstrap!

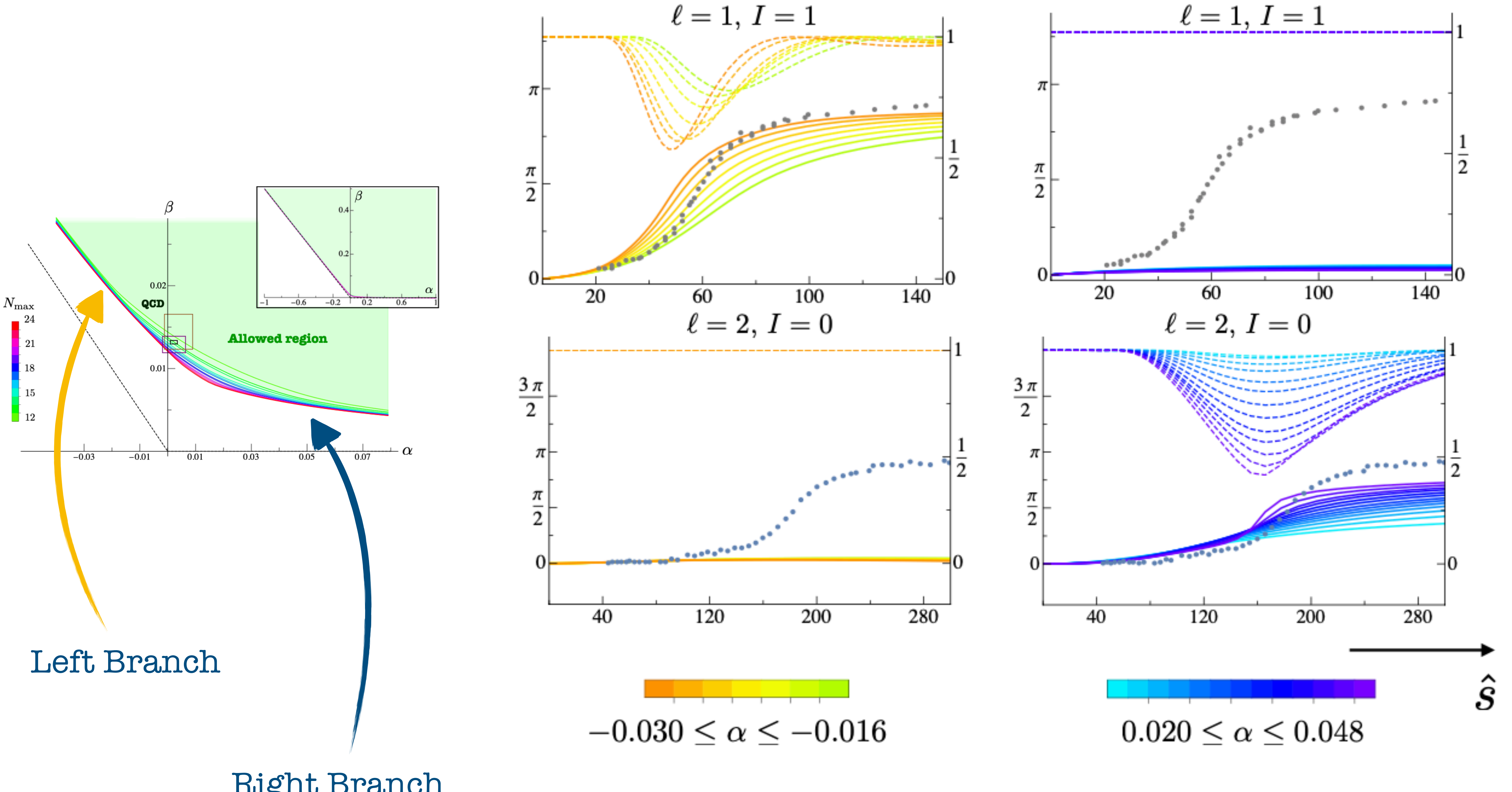


[We thank J.R. Pelaez for the data]

General Bounds: Extremal Amplitudes

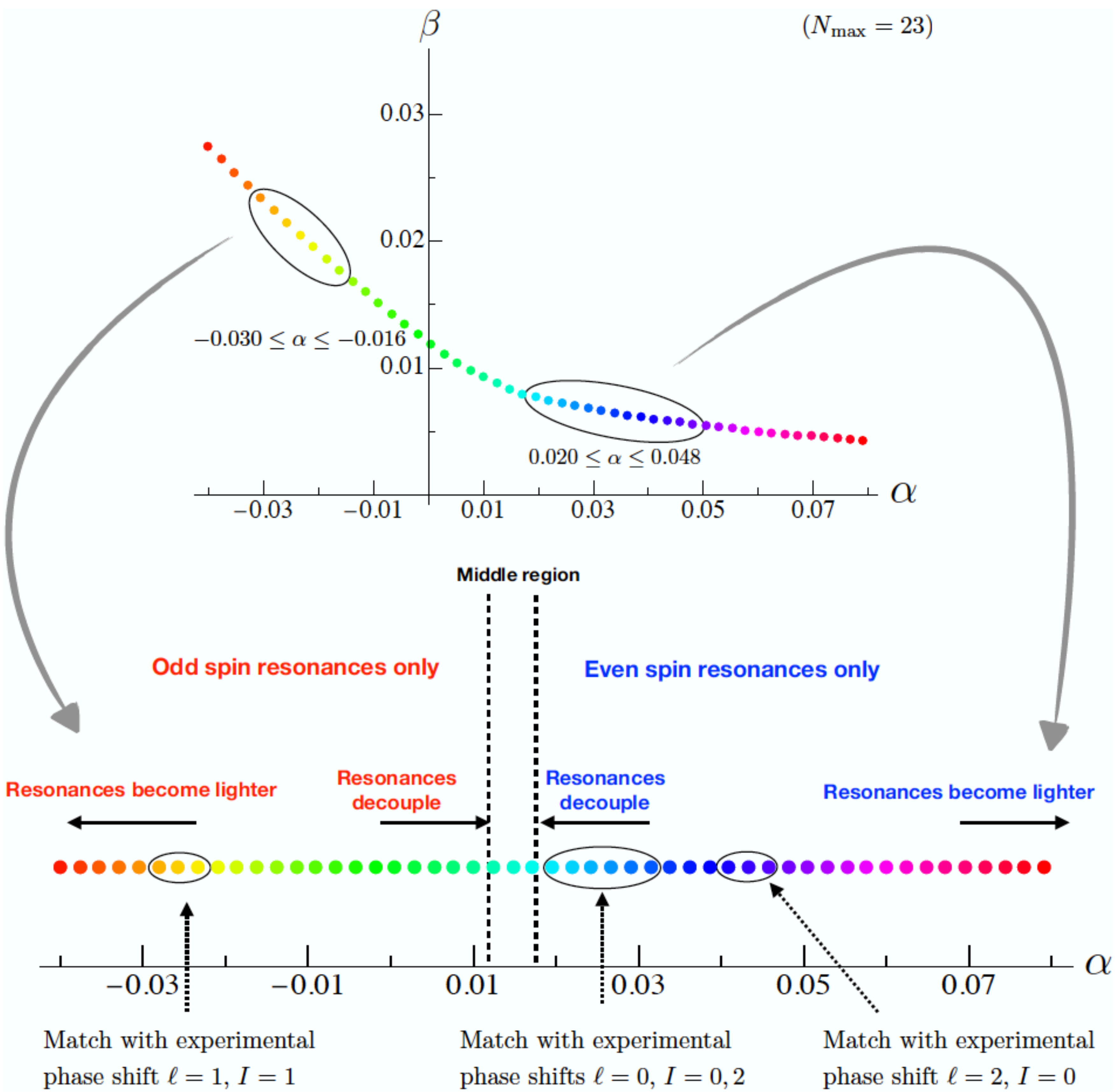
The optimization procedure fixes the coefficients $c_{a,b}, d_{(a,b)}$!!

Phase Shifts and Resonances as a prediction of the Bootstrap!



[We thank J.R. Pelaez for the data]

General Bounds: Summary



Precise Bounds: Gapped Pions

Low Energy Spectrum: N=3 identical **massive** scalars

Precise Bounds: Gapped Pions

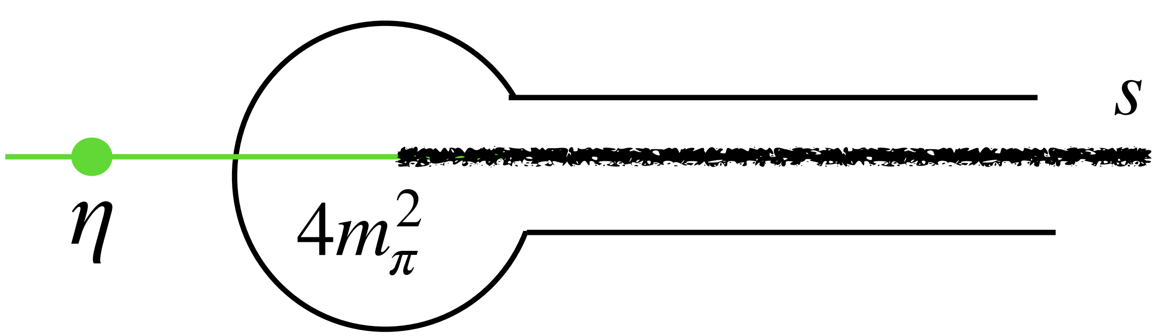
Low Energy Spectrum: N=3 identical **massive** scalars

Step 1: define a **crossing symmetric** and **analytic** Ansatz for the 2->2 Amplitude

Step 3: impose unitarity numerically as a **constraint**
(the $c_{a,b}$, $d_{(a,b)}$ cannot vary arbitrarily)

$$A(s\,|\,t,u) = \sum_{a+b\leq N} c_{a,b}\rho_s^a(\rho_t^b + \rho_u^b) + d_{(a,b)}(\rho_t^a\rho_u^b + \rho_t^b\rho_u^a)$$

$$\rho_s = \frac{\sqrt{4m^2-\eta} - \sqrt{4m^2-s}}{\sqrt{4m^2-\eta} + \sqrt{4m^2-s}}$$



Precise Bounds: Gapped Pions

Low Energy Spectrum: N=3 identical **massive** scalars

Step 1: define a **crossing symmetric** and **analytic** Ansatz for the 2->2 Amplitude

Step 3: impose unitarity numerically as a **constraint**
(the $c_{a,b}, d_{(a,b)}$ cannot vary arbitrarily)

Main difference: Pions are approximate Goldstone bosons, what universal low energy properties?

Weinberg's low energy theorems

Below Threshold Zeros

$$T_0^{(0)}(s_0) = 0, \text{ where } s_0 \simeq \frac{1}{2} m_\pi^2$$

$$T_1^{(1)}(4m^2) = 0$$

$$T_0^{(2)}(s_2) = 0, \text{ where } s_2 \simeq 2 m_\pi^2$$

Precise Bounds: The Pion Peninsula

Main assumption: nonperturbative existence of the Chiral Zeros

Bootstrap Question: allowed Chiral Zeros region **given the experimental input available**?

Input:

1. ρ resonance
2. Scattering lengths

$$|\, a_\ell^I - a_\ell^I(\mathrm{exp}) \,| \leq \Delta a_\ell^I$$

ALG, Penedones, Vieira, PRL 122 (2019), arXiv: 1810.12849

Precise Bounds: The Pion Peninsula

Main assumption: nonperturbative existence of the Chiral Zeros

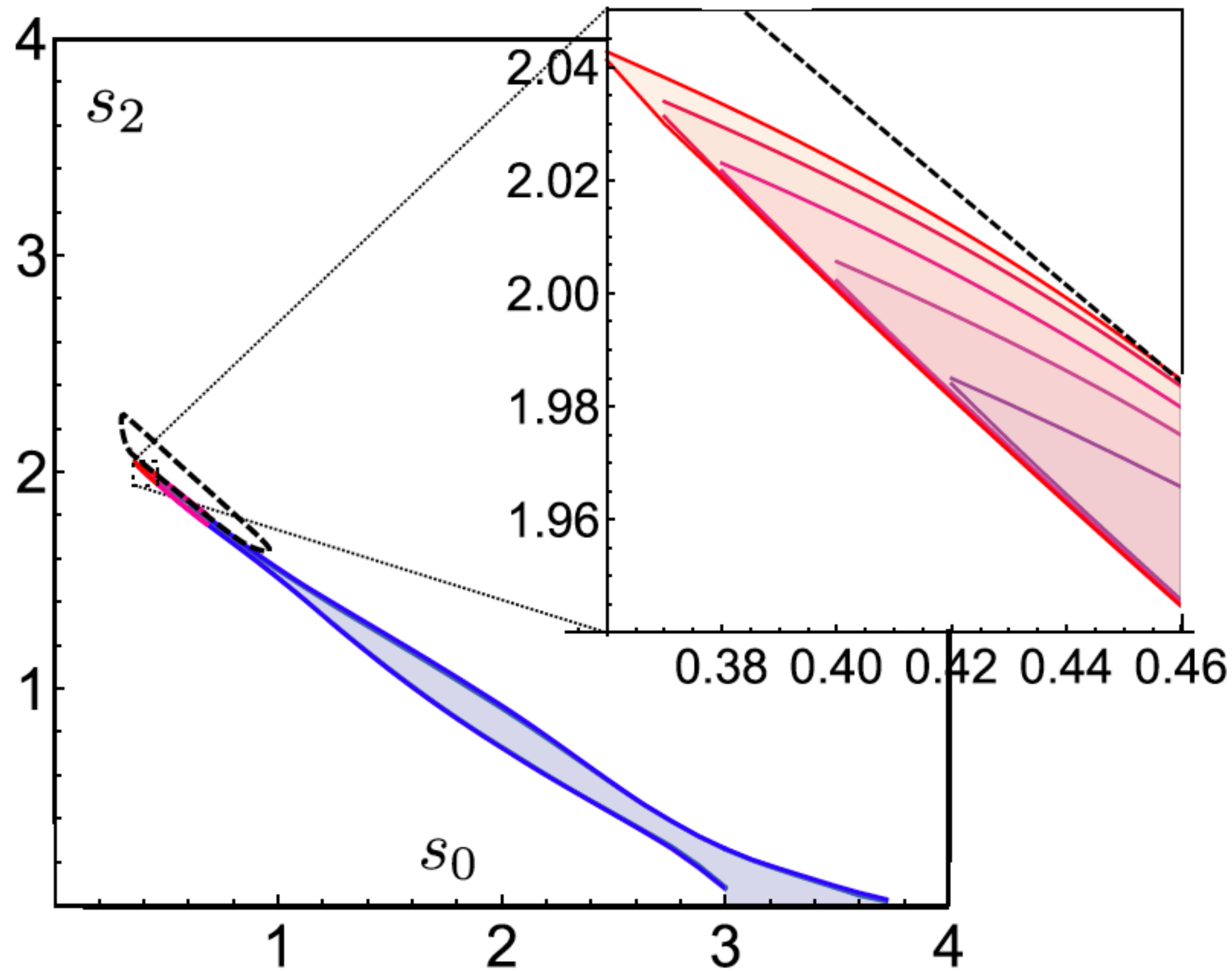
Bootstrap Question: allowed Chiral Zeros region **given the experimental input available**?

Input:

1. ρ resonance
2. Scattering lengths

$$|\,a_\ell^I - a_\ell^I(\exp)\,| \leq \Delta a_\ell^I$$

ALG, Penedones, Vieira, PRL 122 (2019)
arXiv: 1810.12849



Precise Bounds: The Pion Peninsula

Main assumption: nonperturbative existence of the Chiral Zeros

Bootstrap Question: allowed Chiral Zeros region **given the experimental input available**?

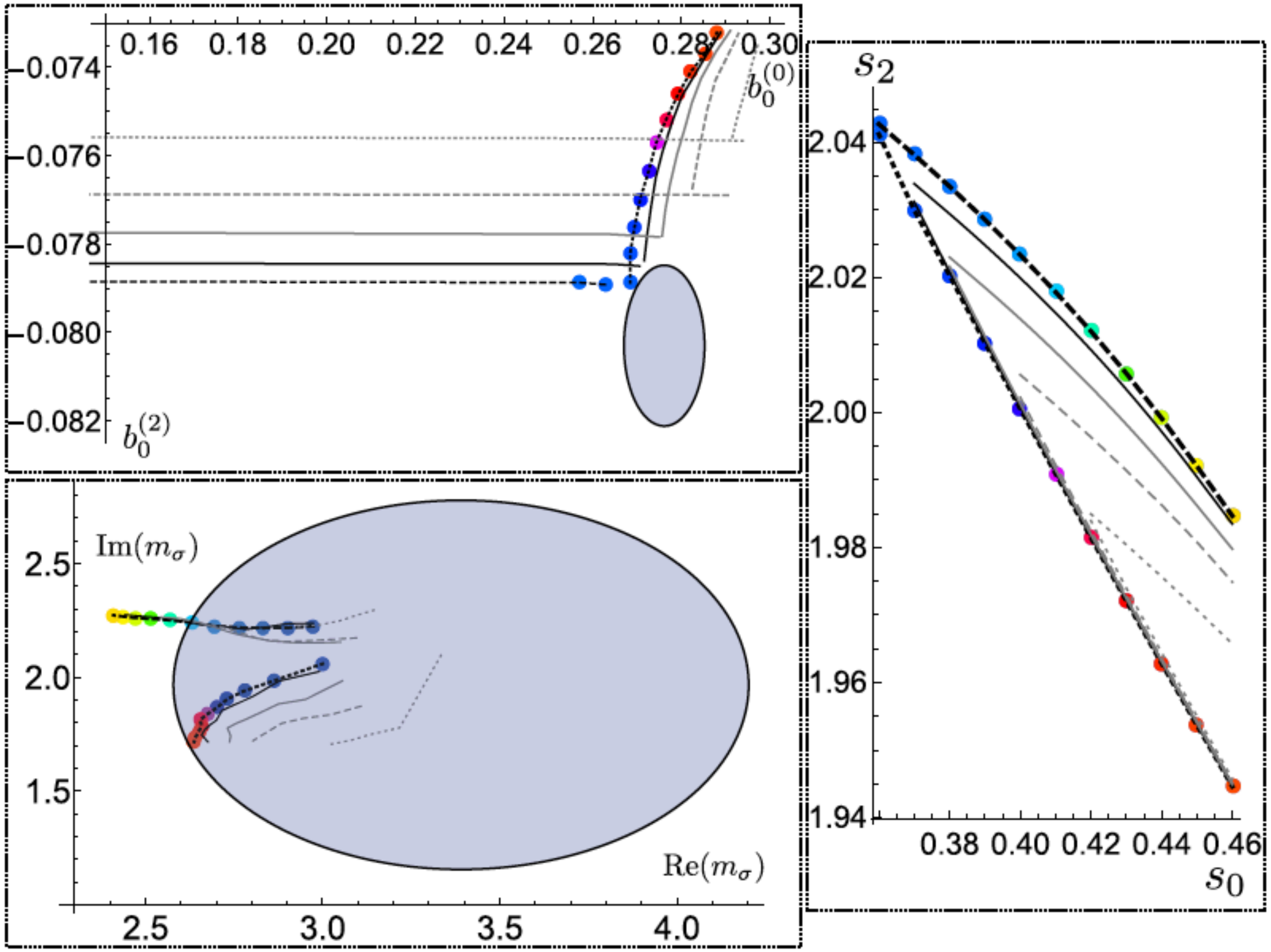
Input:

1. ρ resonance
2. Scattering lengths

$$|a_\ell^I - a_\ell^I(\text{exp})| \leq \Delta a_\ell^I$$

ALG, Penedones, Vieira, PRL 122 (2019), arXiv: 1810.12849

σ mass and width, and effective ranges agree close to the tip!



Future predictions: Glueball Scattering?

Take pure SU(N) YM, the low energy spectrum contains Glueballs

	$M(J^{PC})/\sqrt{\sigma}$ continuum limit				
	J^{PC}	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$
G	0^{++} gs	3.781(23)	3.405(21)	3.271(27)	3.156(31)
G*	0^{++} ex1	6.126(38)	5.855(41)	5.827(62)	5.689(53)
H	2^{++} gs	5.349(20)	4.894(22)	4.742(15)	4.690(20)
H*	2^{++} ex1	$7.22(6)^{+}$	6.788(40)	6.694(40)	6.607(45)
	3^{++} gs	$8.13(8)^{*}$	$7.71(9)^{*}$		7.29(9)
	4^{++} gs		$7.60(12)^{*}$	$7.36(9)^{*}$	7.41(10)

Athenodorou, Teper, arXiv: 2106.00364

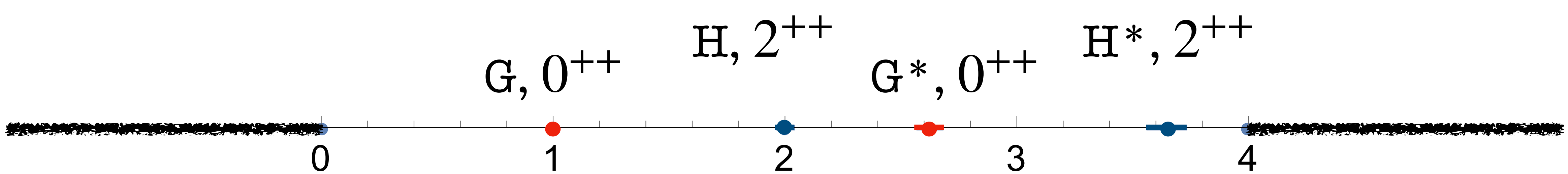
Future predictions:
Glueball Scattering?

Take pure SU(N) YM, the low energy spectrum contains Glueballs

	$M(J^{PC})/\sqrt{\sigma}$ continuum limit				
	J^{PC}	$SU(2)$	$SU(3)$	$SU(4)$	$SU(5)$
G	0^{++} gs	3.781(23)	3.405(21)	3.271(27)	3.156(31)
G*	0^{++} ex1	6.126(38)	5.855(41)	5.827(62)	5.689(53)
H	2^{++} gs	5.349(20)	4.894(22)	4.742(15)	4.690(20)
H*	2^{++} ex1	$7.22(6)^{+}$	6.788(40)	6.694(40)	6.607(45)
	3^{++} gs	$8.13(8)^{*}$	$7.71(9)^{*}$		7.29(9)
	4^{++} gs		$7.60(12)^{*}$	$7.36(9)^{*}$	7.41(10)

Athenodorou, Teper, arXiv: 2106.00364

Consider SU(2) YM, $GG \to GG$ amplitude

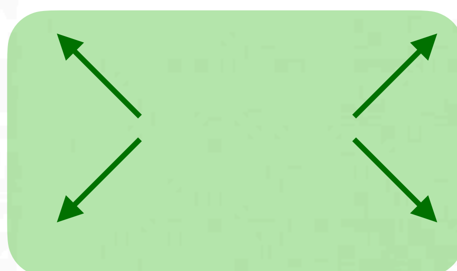


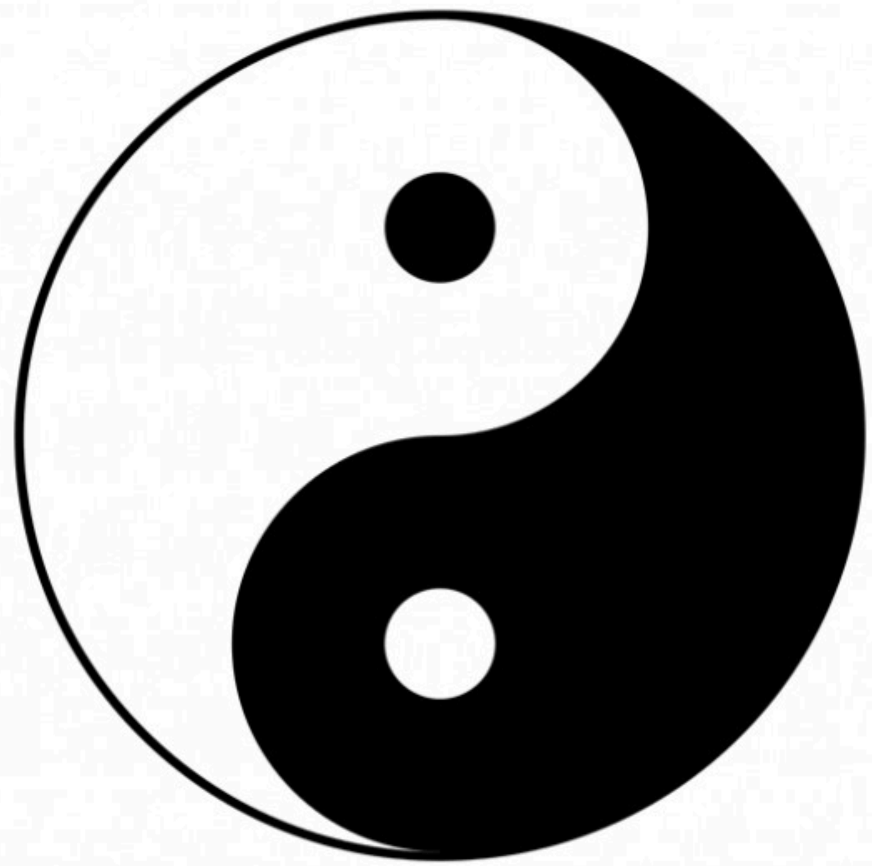
$$-\frac{g_G^2}{s-m_G^2}-\frac{g_{G*}^2}{s-m_{G*}^2}-\frac{g_H^2 P_2((u-t)/(m_H^2-4m_G^2))}{s-m_H^2}-\frac{g_{H*}^2 P_2((u-t)/(m_{H*}^2-4m_G^2))}{s-m_{H*}^2}+\text{crossed poles}$$

Dual Boostrap Methods

Primal Formulation:
Construct an ansatz for the amplitude

Paulos, Penedones, van Rees, Toledo, Vieira 1607.06110 (2D), 1708.06765 (4D)

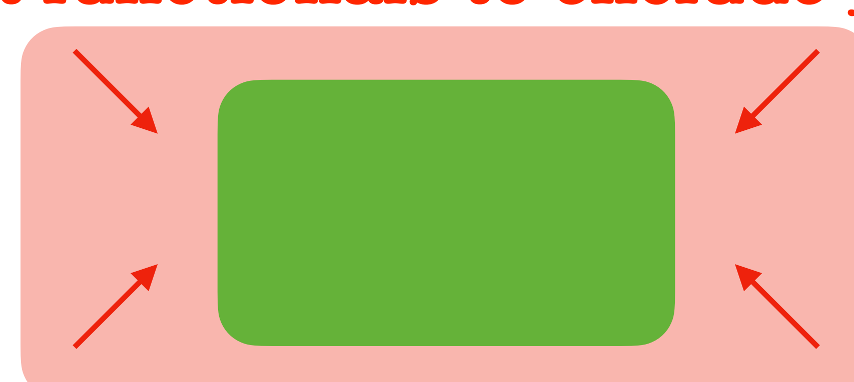




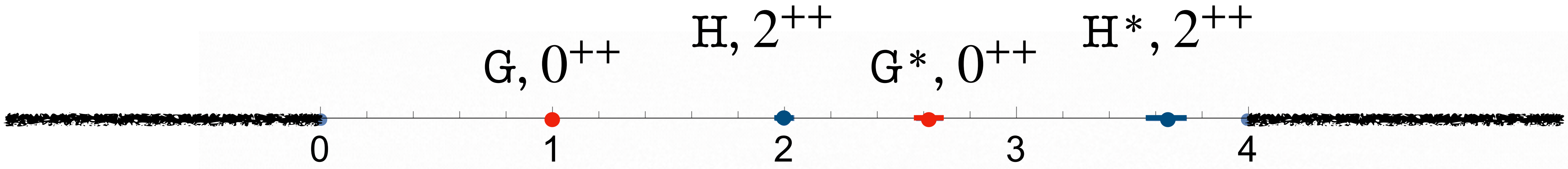
Dual Bootstrap pioneers
Martin, Lukasczuk, Lopez, Mennessier, Bonnier
60s, 70s

Homrich, ALG, Vieira '20 (2D) 2008.02770
Elias-Miró, ALG (2D) 2106.07957
ALG, Sever '21 (4D) 2106.10257

Dual Formulation:
Construct functionals to exclude points

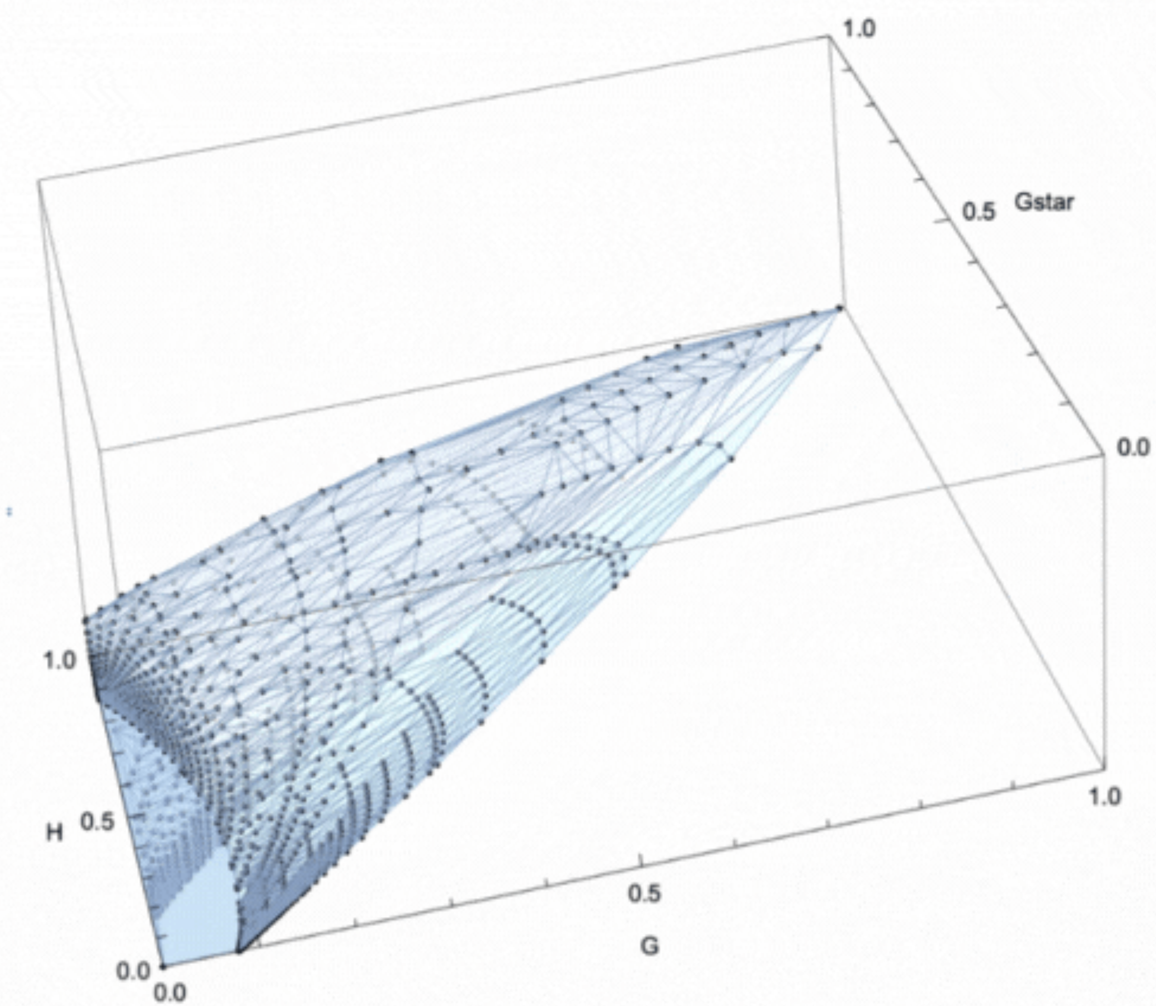


Future predictions:
Coupling Constants Space

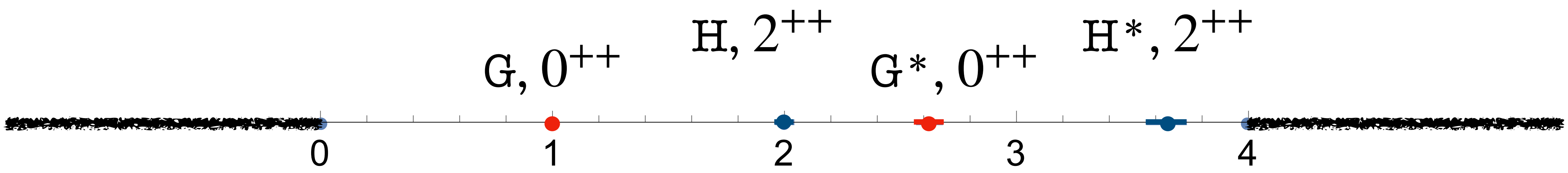


What are the allowed values for $\{g_G, g_{G*}, g_H, g_{H*}\}$?

ALG, Hebbar, van Rees, work in progress

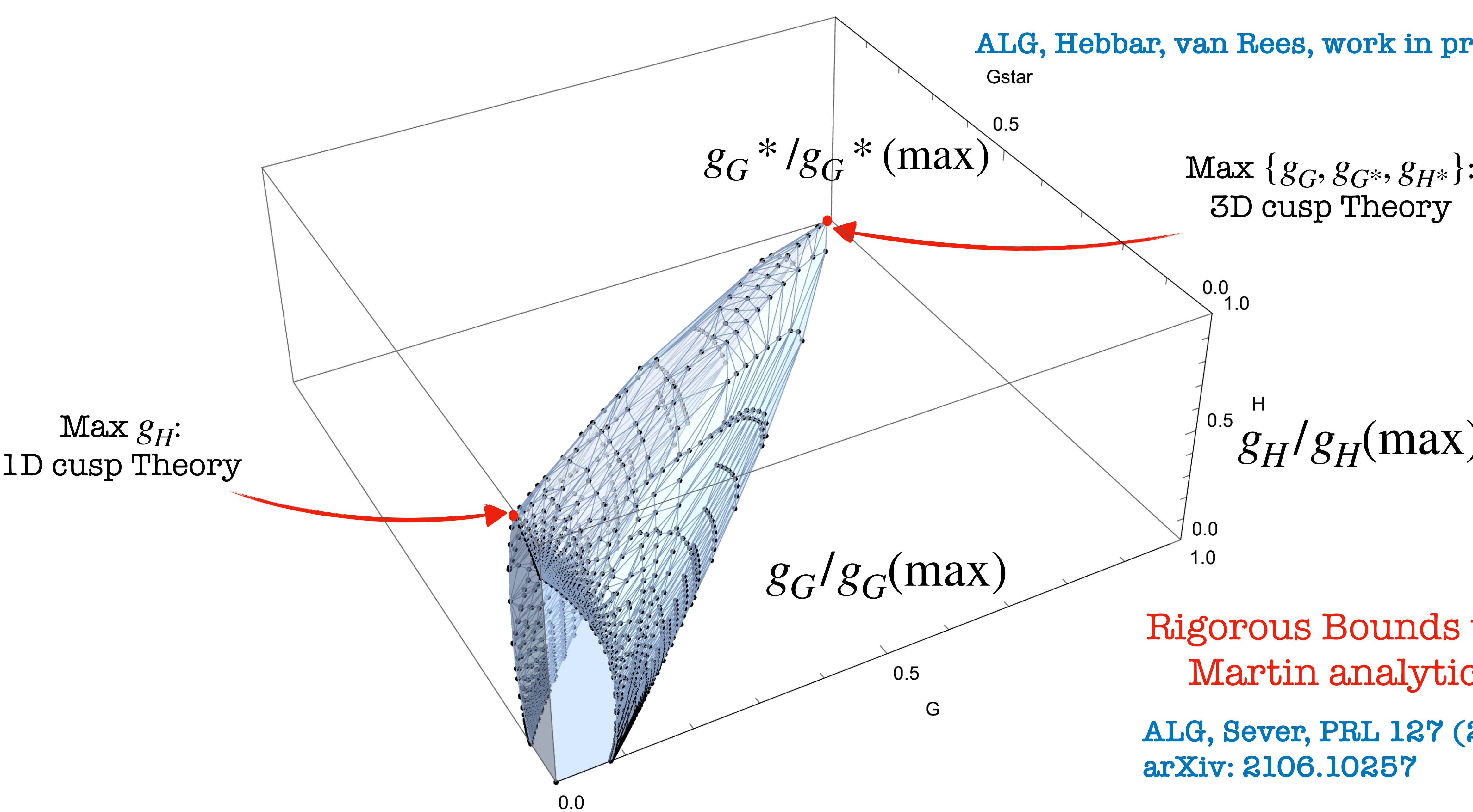


Future predictions: Coupling Constants Space



What are the allowed values for $\{g_G, g_{G^*}, g_H, g_{H^*}\}$?

ALG, Hebbar, van Rees, work in progress



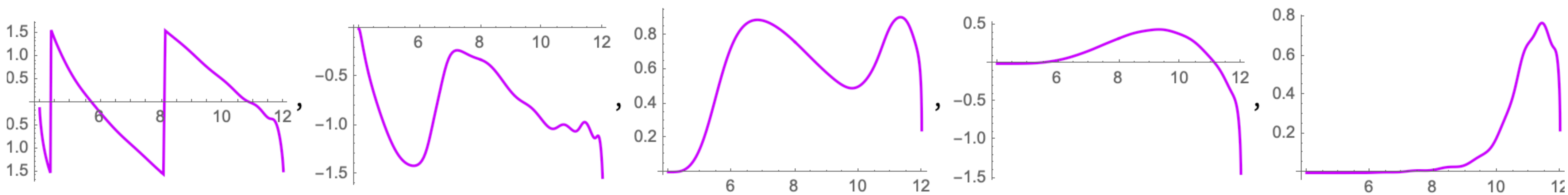
Rigorous Bounds using Martin analyticity

ALG, Sever, PRL 127 (2021)
arXiv: 2106.10257

Future predictions:
Phase Shifts

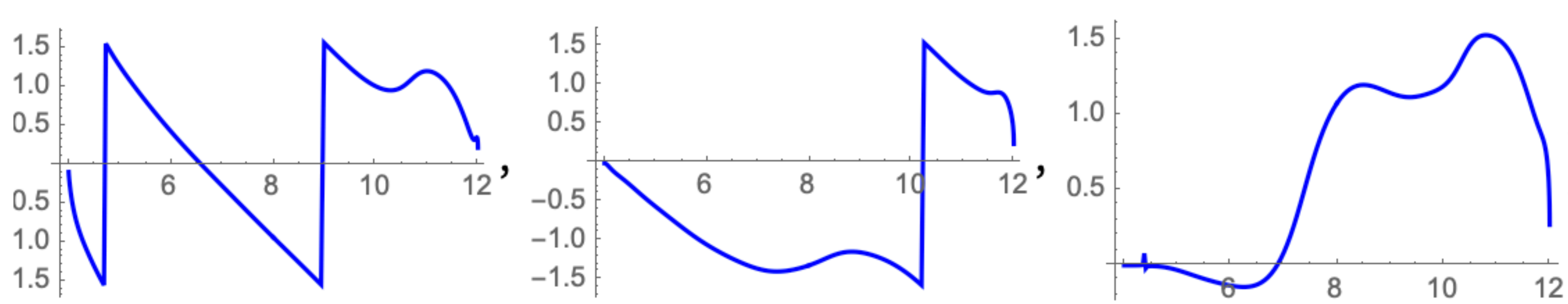
ALG, Hebbar, van Rees, work in progress

$g_G(\text{max}) \simeq 220$ $\quad g_{G^*}(\text{max}) \simeq 380$ $\quad g_{H^*}(\text{max}) \simeq 15$



$\ell = 0$ $\quad \ell = 2$ $\quad \ell = 4$ $\quad \ell = 6$ $\quad \ell = 8$

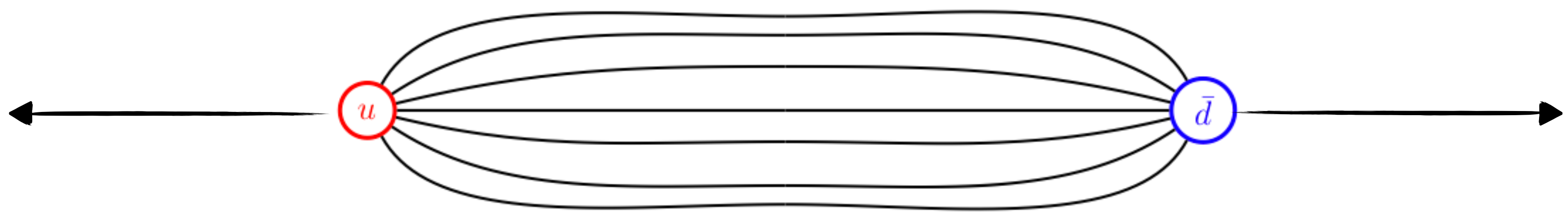
$g_H(\text{max}) \simeq 75$



$\ell = 0$ $\quad \ell = 2$ $\quad \ell = 4$

Applications to Confinement

Distance between quarks $\to \infty$



Large Distance Universality [D=3]

$$E_0(R)\ell_s = \frac{R}{\ell_s} + \frac{\pi}{6}\frac{\ell_s}{R} - \frac{\pi^2}{72}\left(\frac{\ell_s}{R}\right)^3 + \frac{\pi^3}{432}\left(\frac{\ell_s}{R}\right)^5 - \left(\frac{5\pi^4}{10368} + \Delta(\gamma_3)\right)\left(\frac{\ell_s}{R}\right)^7 + \ldots$$

Luscher 80's

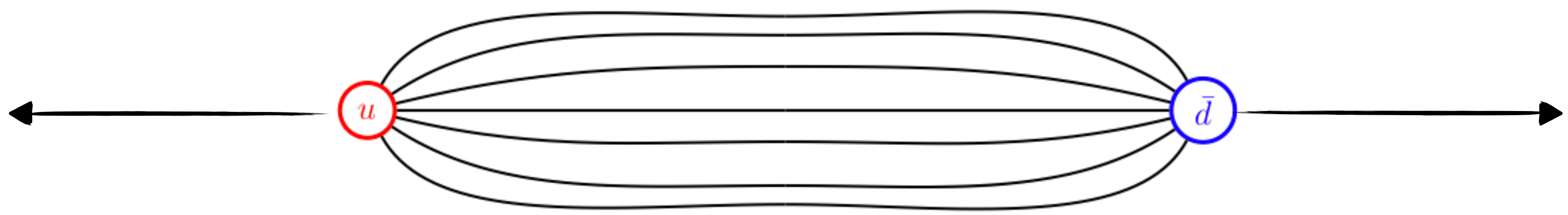
Luscher, Weisz, Drummond

Aharony, Komargodsky, Dubovsky, Flauger, Gorbenko

Elias-Miró, ALG, Hebbar, Penedones, Vieira, PRL 123 (2019), arXiv: 1906.08098

Applications to Confinement

Distance between quarks $\to \infty$



Large Distance Universality [D=3]

$$E_0(R)\ell_s = \frac{R}{\ell_s} + \frac{\pi}{6}\frac{\ell_s}{R} - \frac{\pi^2}{72}\left(\frac{\ell_s}{R}\right)^3 + \frac{\pi^3}{432}\left(\frac{\ell_s}{R}\right)^5 - \left(\frac{5\pi^4}{10368} + \Delta(\gamma_3)\right)\left(\frac{\ell_s}{R}\right)^7 + \dots$$

Luscher 80's

Luscher, Weisz, Drummond

Aharony, Komargodsky, Dubovsky, Flauger, Gorbenko

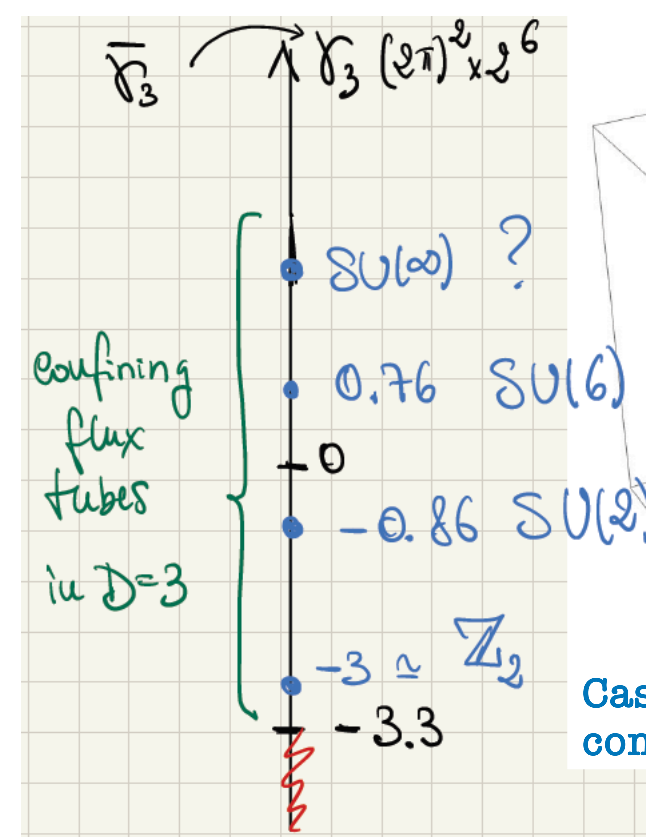
Elias-Miró, ALG, Hebbar, Penedones, Vieira, PRL 123 (2019), arXiv: 1906.08098

Flux-Tube excitations S-matrix

$$S(s) = e^{i\frac{s}{4}\ell_s + i\gamma_3 s^3 \ell_s^6 + \dots}$$

S-matrix Bootstrap: $\gamma_3 \geq -1/768$

$$\boxed{\Delta(\gamma_3) = -\frac{32}{225}\pi^6\gamma_3 \leq \frac{\pi^6}{5400}}$$



Dubovsky, Flauger, Gorbenko

Caristo, Caselle, Magnoli, Panero

Caselle [private communication]

Future directions?

General Bounds
QCD is the UV completion, not very useful…but useful for Quantum Gravity

Precision Physics?
Pion Scattering in the gapped phase including the set of available data, how much can be fixed?

Can it be useful for pheno?

Glueball Scattering (theoretically fun)

Plug the bounds in the Lüsher formula, inclusion of other processes $GG^* \to GG^*$, … , treating Landau singularities rigorously, …