Overview: Neutrino Mean Field and Quantum Kinetic Equations in the Early Universe and Compact Objects

Evan Grohs (he/him/his)

North Carolina State University

Astrophysical Neutrinos and the Origin of the Elements

Institute for Nuclear Theory

University of Washington, Seattle, WA

25 July 2023

NC STATE UNIVERSITY

Outline

1. Introduction
2. Deriving the Quantum Kinetic Equations
3. QKE solution in homogeneous and isotropic geometry: the Early Universe
4. Compact Objects and Flavor Instabilities
5. Conclusion

Neutrino Oscillations

1. Early conjecture by Pontecorvo (1950's) that neutrinos had non-zero masses and could mix

2. Consistent with Wolfenstein (1979) and Mikheyev & Smirnov (1985) solution to solar neutrino problem

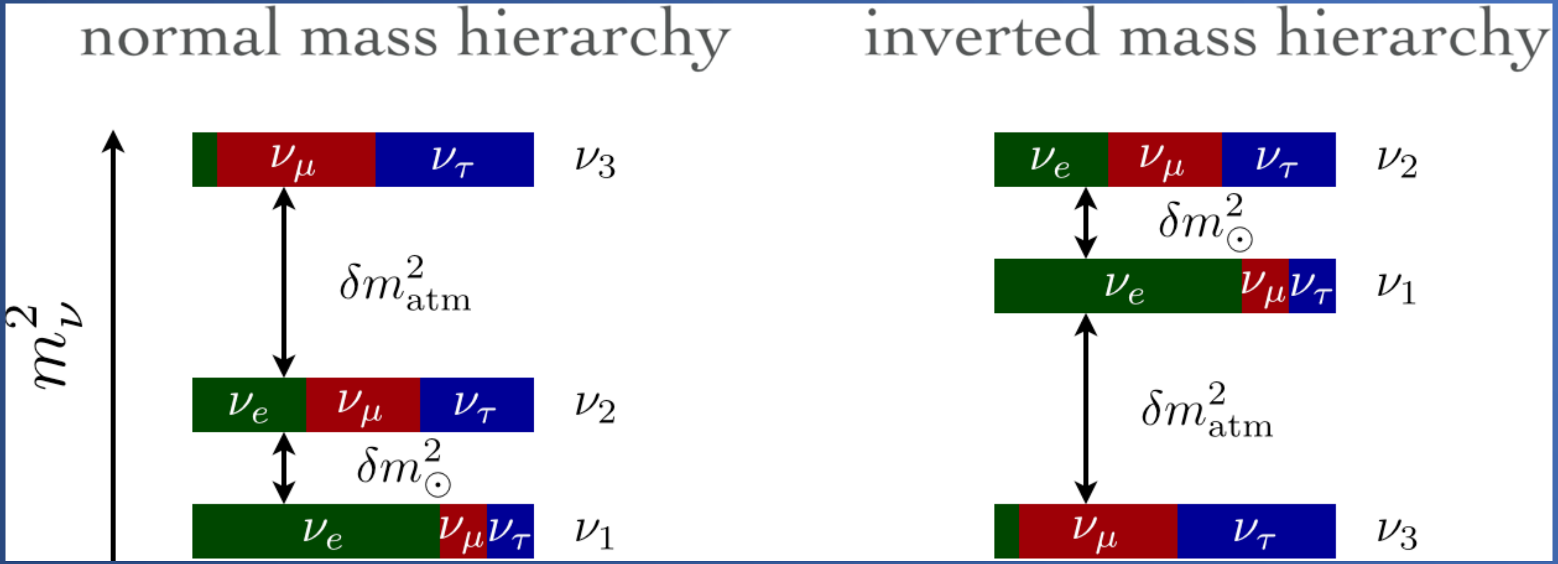
3. Pantaleone (1992) pointed out non-linear effects could be important in dense media

4. Samuel (1993) did early simulations and saw novel effects

Neutrino Mass

Mass eigenbasis is not coincident with *Weak* eigenbasis

1. Unitary Transformation in vacuum: PMNS matrix
2. Neutrinos oscillate between weak eigenstates
3. Density matrix for neutrino distribution



Mass squared differences:

$$\delta m^2_\odot = 7.5 \times 10^{-5}\,\mathrm{eV}^2$$
$$\delta m^2_{\rm atm} = 2.6 \times 10^{-3}\,\mathrm{eV}^2$$

c/o George Fuller

The mixing

undetermined : $\delta; \quad \phi_1, \phi_2$

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \begin{pmatrix} \nu_1 \\ e^{i\phi_1}\nu_2 \\ e^{i\phi_2}\nu_3 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & \\ & c_{23} & s_{23} \\ & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & & s_{13}e^{-i\delta} \\ & 1 & \\ -s_{13}e^{i\delta} & & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & \\ -s_{12} & c_{12} & \\ & & 1 \end{pmatrix} \begin{pmatrix} \nu_1 \\ e^{i\phi_1}\nu_2 \\ e^{i\phi_2}\nu_3 \end{pmatrix}$$

Atmospheric   Reactor (Daya Bay/Reno/Double Chooz)   Solar

$$\sin^2\theta_{23} = \begin{cases} 0.386^{+0.024}_{-0.021}, & \text{NH} \\ 0.392^{+0.039}_{-0.022} & \text{IH} \end{cases}$$

$$\Rightarrow (3\sigma)\begin{cases} 0.331 \leftrightarrow 0.637, & \text{NH} \\ 0.335 \leftrightarrow 0.663 & \text{IH} \end{cases}$$

$$\sin^2\theta_{13} = \begin{cases} 0.0241 \pm 0.0025, & \text{NH} \\ 0.0244^{+0.0023}_{-0.0025}, & \text{IH} \end{cases}$$

$$\sin^2\theta_{12} = 0.307^{+0.018}_{-0.016}$$

Bari global analysis
(Valencia quite similar)

c/o
Wick Haxton

Sigl & Raffelt (1993)

1. Refractive Index Matrix:
   a. $N_f \times N_f$ matrix with off-diagonal terms for neutrino refraction
   b. Both test neutrino and refractive index matrix described by density matrix
2. Evolution of exact M-particle Green's functions can be expanded pertubatively in 1-particle functions

$$\dot{\rho}_{\mathbf{p}} = -i[\Omega^0_{\mathbf{p}}, \rho_{\mathbf{p}}] + i\langle [H^0_{\mathrm{int}}, \mathcal{D}^0_{\mathrm{p}}]\rangle - \frac{1}{2}\int_{-\infty}^{\infty} dt \langle [H^0_{\mathrm{int}}(t)[H^0_{\mathrm{int}}(t), \mathcal{D}^0_{\mathbf{p}}]]\rangle$$

3. Solve QKEs for charge-current, neutral current, and self-interaction terms

Volpe et al (2013)

1. Based upon the Bogoliobov-Born-Green-Kirkwood-Yvon (BBGKY) hierarchy with truncation

$$\hat{H} = \sum_{k} \hat{H}_0(k) + \sum_{k<k'} \hat{V}(k,k') \qquad i\frac{d\hat{D}}{dt} = [\hat{H}, \hat{D}]$$

$$i\frac{d\rho_{1\dots s}}{dt} = [H^{(s)}, \rho_{1\dots s}] + \mathrm{tr}_{s+1}[V_{s+1}^{(1\dots s)}, \rho_{1\dots s+1}]$$

2. Tower of Equations: truncate after ρ_1

$$\rho_{12} = \rho_1\rho_2 + c_{12}$$

3. Consistent with the mean field and collision terms in Sigl & Raffelt (1993)

Blaschke & Cirigliano (2016)

1. Finite-temperature field-theory approach first used in Vlasenko, Fuller, Cirigliano (2013)
2. M-particle Green's functions truncated to single-particle in perturbative expansion akin to Sigl and Raffelt (1993)

Neutrinos: $F = F(x, \vec{p})$

Antineutrinos: $\overline{F} = \overline{F}(x, \vec{p})$

$$F = \begin{bmatrix} f_{LL} & f_{LR} \\ f_{LR}^{\dagger} & f_{RR} \end{bmatrix}$$

Generalized $2N_f \times 2N_f$ density matrices

N_f: number of flavors
2 helicity states

f_{LL}^{ii} : occupation numbers

f_{LL}^{ij} : flavor coherence

f_{LR} : spin coherence

f_{RR} : opposite helicity

Neutrino Density Matrices & QKEs

1. Ignore spin coherence and wrong helicity states

$$f(t,\vec{r},\vec{p}) = \begin{pmatrix} f_{ee} & f_{ex} \\ f_{ex}^{*} & f_{xx} \end{pmatrix} \qquad \overline{f}(t,\vec{r},\vec{p}) = \begin{pmatrix} \overline{f}_{ee} & \overline{f}_{ex} \\ \overline{f}_{ex}^{*} & \overline{f}_{xx} \end{pmatrix}$$

2. Only keep single-particle correlation functions

3. QKEs from Blaschke & Cirigliano (2016):

Drift Term — Force Term — Coherent Term — Collision Term

$$\frac{\partial f}{\partial t} + \dot{\vec{x}} \cdot \frac{\partial f}{\partial \vec{x}} + \dot{\vec{p}} \cdot \frac{\partial f}{\partial \vec{p}} = -i[H, f] + C$$

QKEs in the Early Universe

Change array dimensions from Boltzmann to QKE:

$$\{f_i(\epsilon)\}, \{\overline{f}_i(\epsilon)\} \rightarrow f_{ij}(\epsilon), \overline{f}_{ij}(\epsilon)$$

2 Generalized 3×3 density matrices (no spin coherence)

ϵ: Energy-like variable

Equations of motion for neutrinos:

$$\frac{df}{dt} = -i[H, f] + C[f]$$

H: Hamiltonian-like potential (coherent)

C: Collision term from Blaschke & Cirigliano (2016)

→ Nonlinear coupled ODEs

Coherent Term in the Early Universe

$$H = H_V + H_A + H_S$$

$$H_V = \frac{1}{2\epsilon T_{\rm cm}} U M^2 U^\dagger$$

Vacuum Oscillations

$$M^2 = {\rm diag}(0, \delta m_\odot^2, \pm\delta m_{\rm atm}^2)$$

$$H_A = \sqrt{2} G_F (L + \widetilde{L})$$

Asymmetric Term

$$H_S = -\frac{8\sqrt{2} G_F \epsilon T_{\rm cm}}{3 m_W^2} (E + \cos^2\theta_W \widetilde{E})$$

Symmetric term (proportional to energy density)

Self-Interacting Term (Neglect in EU)

$$\sqrt{2}G_F\widetilde{L} = H_\nu$$

Self-Interacting term (nonlinear in density matrices):

$$H_\nu = \frac{\sqrt{2}G_F}{(2\pi)^3}\int d^3q(1-\cos\vartheta)[\varrho(t,\vec{x},\vec{q}) - \overline{\varrho}^*(t,\vec{x},\vec{q})]$$

$$\cos\vartheta = \frac{\vec{p}\cdot\vec{q}}{pq}$$

Angle between 3-momenta vectors

Collisions

Positron-Electron Annihilation

$$\nu(k)\bar{\nu}(q_3) \rightarrow e^+(q_2)e^-(q_1)$$

Loss Potential (Blaschke & Cirigliano 2016)

$$\Pi_R^+(k) = \frac{-32G_F^2}{|\vec{k}|} \int \tilde{d}q_1 \tilde{d}q_2 \tilde{d}q_3 (2\pi)^4 \sum_{I=L,R} \Big[(1 - f_{e,1})(1 - \bar{f}_{e,2}) \times Y_I \bar{f}_3 \Big(2Y_I \mathcal{M}_I^R(q_1, -q_2, -q_3, k) - Y_{J\neq I} \mathcal{M}_m(q_1, -q_2, -q_3, k) \Big) \Big]$$

Amplitudes

$$\mathcal{M}_I^R(q_1, q_2, q_3, k) = \Big(\delta_I^L (q_3 q_1)(k q_2) + \delta_I^R (q_3 q_2)(k q_1) \Big) \delta^{(4)}(k - q_3 - q_1 + q_2)$$

$$\mathcal{M}_m(q_1, q_2, q_3, k) = m_e^2 (k q_3) \delta^{(4)}(k - q_3 - q_1 + q_2)$$

Collisions (cont.)

Matrices in Weak eigenbasis

$$Y_L = \begin{bmatrix} \frac{1}{2} + \sin^2\theta_W & 0 & 0 \\ 0 & -\frac{1}{2} + \sin^2\theta_W & 0 \\ 0 & 0 & -\frac{1}{2} + \sin^2\theta_W \end{bmatrix}$$

$$Y_R = \sin^2\theta_W \times \mathbb{1}$$

Collision Term

$$C[f(\epsilon)] = \frac{1}{2}\{\Pi_R^+, f(\epsilon)\} - \frac{1}{2}\{\Pi_R^-, \mathbb{1} - f(\epsilon)\}$$

Early Universe work in 3 flavor

Damping procedure for off-diagonal collision terms

A. Dolgov et al (2002)

G. Mangano et al (2005)

Mixed procedures for off-diagonal

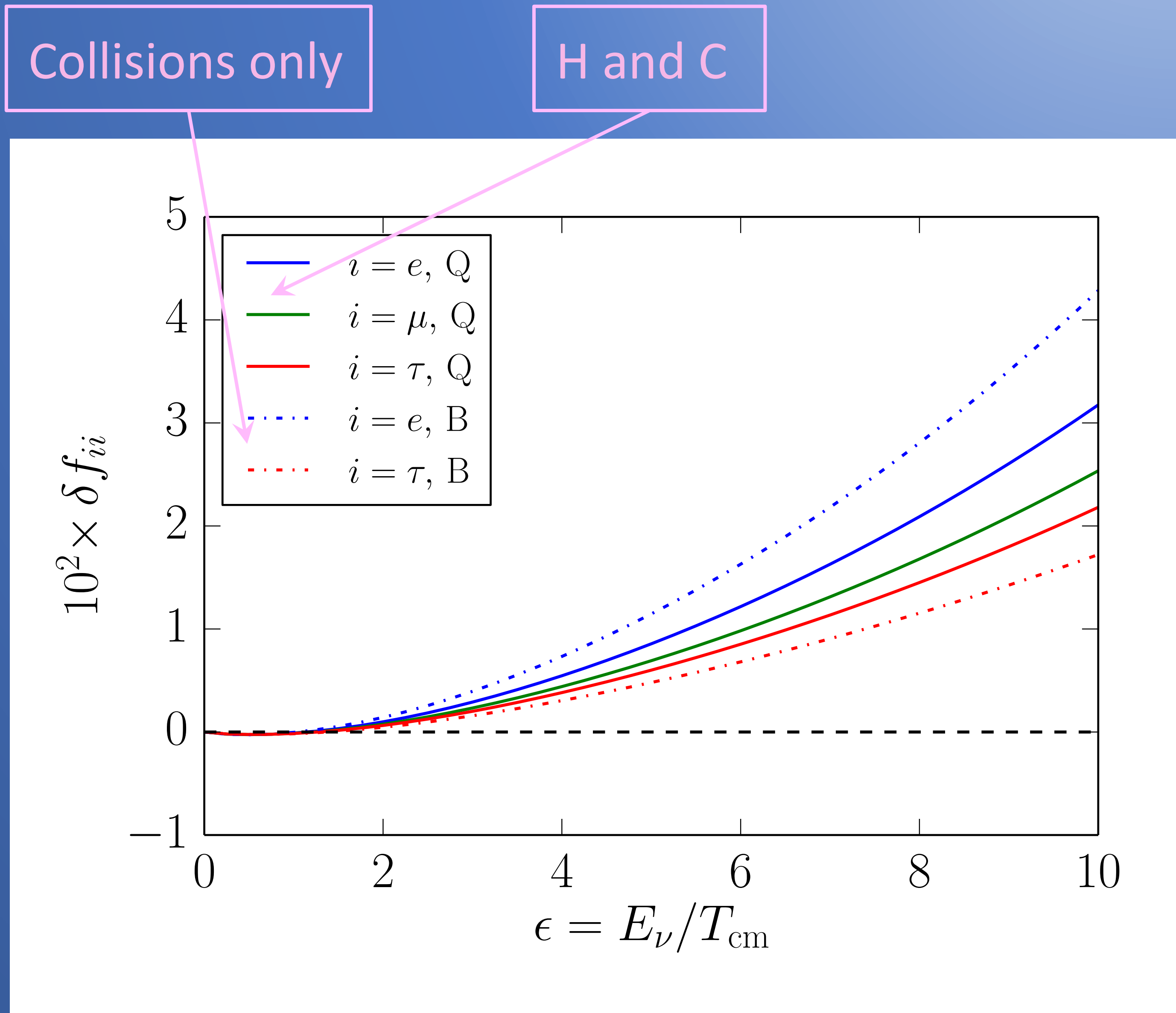
P. de Salas & S. Pastor (2016)

QKE formalism for off-diagonal

Akita & Yamaguchi (2020)

Froustey et al (2020)

Bennet et al (2021)



Logical Progression of Neutrino Decoupling

$$\left\{\begin{matrix}\text{Equil.}\\ T_{\rm cm}=T\\ f\propto\mathbb{1}\end{matrix}\right\} \xrightarrow[e^-e^+\to\nu\overline{\nu}\ \text{(slow)}]{e^-e^+\to 2\gamma\ \text{(fast)}} \{T_{\rm cm}<T\} \xrightarrow[Y_L\not\propto\mathbb{1}]{C[f]\neq 0} \left\{\begin{matrix}f_{ee}\neq f_{xx}\\ f_{ij}=0\end{matrix}\right\}$$

Boltz. or QKE

QKE only

$$\xrightarrow{U\neq\mathbb{1}} \{f_{ij}\neq 0\} \xrightarrow[\theta_{12}\ \text{or}\ \theta_{13}\neq 0]{\theta_{23}\neq\pi/4} \{f_{\mu\mu}\neq f_{\tau\tau}\}$$

Fast Flavor Instability -- Qualitative

1. As neutrino moves through compact object environment:
    a. Neutrinos propagate through field of other neutrinos
    b. Self-energy diagrams have (relatively) large amplitudes $\sim G_F$

1. Non-linear self-interactions sensitive to asymmetry of neutrinos vs. anti-neutrinos $\Longrightarrow$ lepton number

2. Self-interacting term can couple to vacuum and create spectral split: (Slow) Collective Oscillations [see Duan, Fuller, Qian 2010]

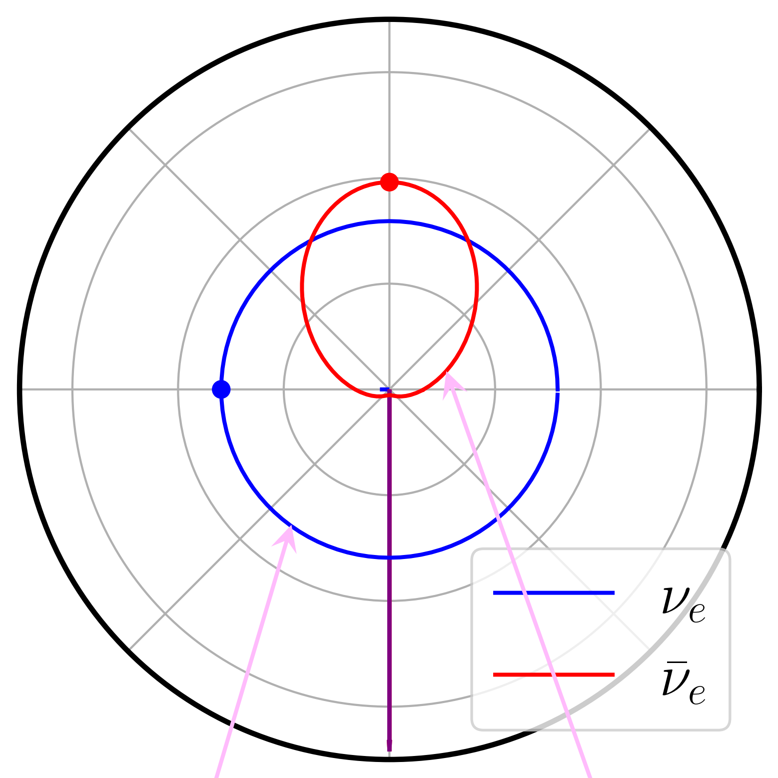
3. Lepton number crossings (in angle) can give rise to rapid flavor transformation (Sawyer, 2005): Fast Flavor Conversion

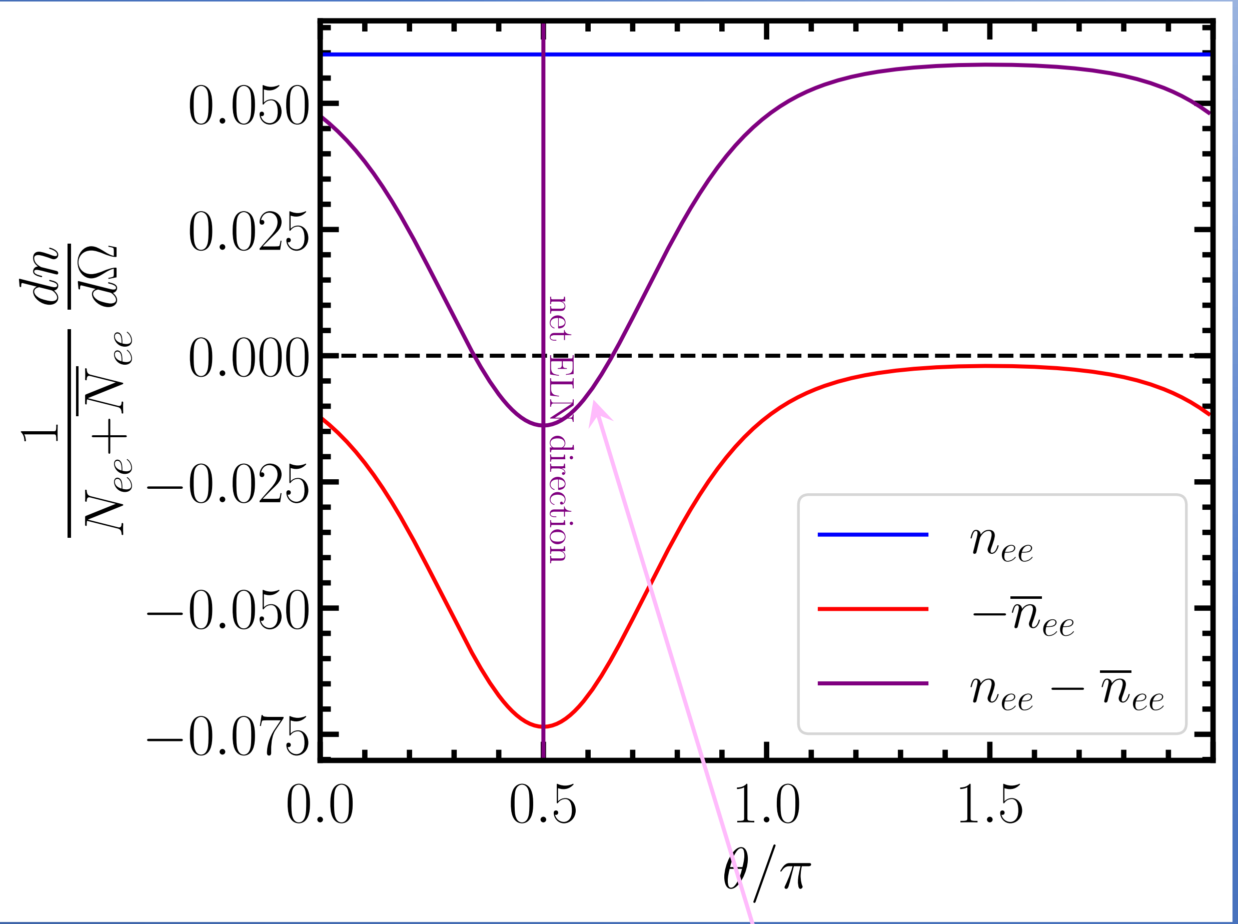
ELN Crossing

$$\overline{N}_{ee} = \frac{N_{ee}}{3}$$

$$\mathbf{F}_{ee} = \mathbf{0}$$

$$\overline{\mathbf{F}}_{ee} \neq \mathbf{0}$$





$Z = 0$ $\quad Z > 0$

Crossing

Maximum Entropy Relation

$$\frac{dn}{d\Omega} = \frac{N}{4\pi} \frac{Z}{\sinh(Z)} e^{Z \cos\theta}$$

Collective FFC

1. Much work has been done on characterizing FFC in dense environments

2. Asymptotic states via conservation law for lepton number and crossing disappearance in ELN-XLN [cf. H. Nagakura]
   a. Characterize quasi-steady state with angular moments

3. Scan over large volume of simulation space using angular moments [cf. J. Froustey]

4. Flavor depolarization in 1D box and sensitivity to initial perturbations [cf. Z. Xiong]

Collisional Instabilities

1. Associated with asymmetries between neutrino and anti-neutrino collision rates – can couple to FFC

2. Instability in homogeneous and isotropic conditions [cf. H. Duan]
   a. Energy dependent scattering
   b. Two kinds of instabilities with resonance-like features

3. FFC energy-independent but collisions are not [cf. H. Nagakura]
   a. Isoenergetic scattering via neutral current
   b. Emission and absorption via charge current

4. Instabilities in spherical symmetry around neutrinosphere [cf. Z. Xiong]

Thermodynamics of Oscillating Neutrinos [cf. L. Johns]

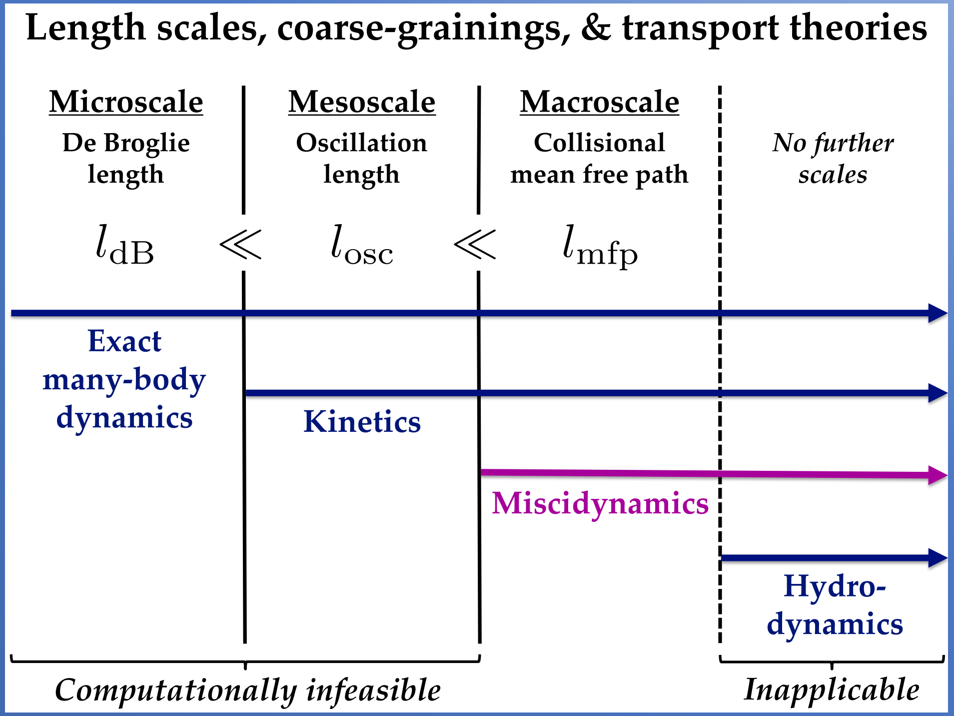
Equilibration of neutrino mixing

Thermalization set by fluctuations in the neutrino system

$$i(\partial_t + \hat{\mathbf{p}} \cdot \partial_{\mathbf{x}})\rho_\nu = [H_\nu, \rho_\nu] + iC_\nu$$

$$\Downarrow$$

$$i(\partial_t + \hat{\mathbf{p}} \cdot \partial_{\mathbf{x}})\rho_\nu^{\mathrm{eq}} = iC_\nu^{\mathrm{eq}}$$



Miscidynamics:
bridge kinetics and hydrodynamics

Conclusion

1. Mean-field/QKEs give the evolution of the neutrino flavor field using single particle correlation functions (reduced density matrices)

2. The Early Universe provides a symmetrical system to solve the QKEs robustly: adiabaticity

3. Instabilities exist in compact object environment: fast flavor and collisional

4. Compact Objects do not exhibit the same symmetry as the Early Universe [cf. S. Richers]

Next INT workshop:

Beyond the QKEs and the standard model – implications on nuclear and dark physics

2026?

Winter? 🙂